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Successive Overrelaxation Iteration for the Stability of Large Scale Systems

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The successive overrelaxation iteration is used to investigate the stability of large scale systems, and some sufficient conditions such that the zero solution of an overall system is asymptotically stable are obtained, which only depend upon the fundamental matrices of the variational isolated subsystems and the interconnected matrices.

1. INTRODUCTION

During the past two decades, the stability of large scaler systems has been studied by numerous authors; many results can be found in the books of Michel and Miller [12] and Siljak [14]. Recently, Liao [7–10] used the lumped iterations to investigate stabilities of large scale systems and interval matrices.

Iterations are the most important methods used to solve algebraic equations in number analysis (cf. Ortega and Rheinbadt [13], Vorga [15], Young [16]). In this article, following Liao [9], we employ the successive overrelaxation iteration and the variation of constants formula (cf. Alekseev [1], Athanassov [2], Brauer [3, 4, 6], Brauer and Strauss [5]) to discuss the stability of large scale systems. Some simple algebraic criterion are obtained directly from coefficients of the system, and it is possible to circumvent the difficulty of finding Liapunov functions.

2. MAIN RESULTS

Consider the large scale system

$$\dot{x}_i = f_i(t, x_i) + g_i(t, x) \tag{1}$$

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in which $x_i \in R^{n_i}$, $\sum_{i=1}^{r} n_i = n$, $t \in J = [t_0, +\infty)$, $f_i: J \times R^{n_i} \to R^{n_i}$, $g_i: J \times R^n \to R^{n_i}$, $i = 1, 2, ..., r$. We also consider the isolated subsystems

$$\dot{x}_i = f_i(t, x_i), \qquad i = 1, 2, ..., r. \tag{2}$$

Assume that f_i, g_i are continuous, $f_{ix_i} = \partial f_i / \partial x_i$ exists and is continuous, $f_i(t, 0) = 0$, and $g_i(t, 0) = 0$, $i = 1, 2, ..., r$.

Denoting $x_i(t; t_0, x_i^0)$ as the solution of (2) through $(t_0, x_i^0) \in J \times R^{n_i}$, obviously $x_i(t; t_0, x_i^0)$ is differentiable about (t, t_0, x_i^0) on $J \times J \times D_i$, $D_i = \{x_i \in R^{n_i}: \|x_i\| < H \leqslant \infty\}$.

Consider the variational system of (2):

$$\dot{z}_i = f_{ix_i}(t, x_i(t; t_0, x_i^0)) z_i, \qquad i = 1, 2, ..., r. \tag{3}$$

The fundamental matrix solution of (3) is given by Brauer [3, 4]:

$$\Phi_i(t; t_0, x_i^0) = \frac{\partial}{\partial x_i^0} (x_i(t; t_0, x_i^0)). \tag{4}$$

The connection between the stability of the zero solution of the ordinary differential equation and the zero solution of its variational equation has been extensively investigated (cf. [2–6]). In the following we will use the "logarithmic norm" of Lozinskii [11]

$$\mu(A) = \lim_{h \to 0^+} 1/h(\|I + hA\| - 1) \tag{5}$$

for a matrix A, where I denotes the identity matrix. As pointed out by Brauer [4], the logarithmic norm depends on the particular norm used for vectors and matrices. For example, the Euclidean norm $\|A\|$ is the square root of the largest eigenvalue of $A^T A$, while $\mu(A)$ is the largest eigenvalue of $\frac{1}{2}(A^T + A)$. From [3] we know

$$\|\Phi_i(t; t_0, x_i^0)\| \leqslant \exp \int_{t_0}^{t} \lambda_i(s)\, ds, \qquad \lambda_i(s) \overset{\text{def}}{=} \mu[f_{ix_i}(t, x_i(t; t_0, x_i^0))]. \tag{6}$$

Using the variation of constants formula of Brauer [4], we have

$$x_i(t) = \Phi_i(t; t_0, x_i^0)\, x_i^0 + \int_{t_0}^{t} \Phi_i(s; t_0, x_i)\, g_i(s, x(s))\, ds, \quad i = 1, 2, ..., r. \tag{7}$$

For (7), we use the successive overrelaxation iteration

$$x_i^{(m)}(t) = (1-\omega)\, x_i^{(m-1)}(t) + \omega \Phi_i(t; t_0, x_i^0) x_i^0$$
$$+ \omega \int_{t_0}^{t} \Phi_i(t; s, x_i(s))\, g_i(s, x_1^m(s), ..., x_{i-1}^{(m)}(s),$$
$$x_i^{(m-1)}(s), ..., x_i^{(m-1)}(s))\, ds, \tag{8}$$

$m = 1, 2, ..., i = 1, 2, ..., r, 0 < \omega < 2$, and choose

$$x_1^{(0)}(t) \overset{\text{def}}{=} \omega \Phi_1(t; t_0, x_1^0)\, x_1^0 \tag{9}$$

$$x_i^{(0)}(t) \overset{\text{def}}{=} \omega \Phi_i(t; t_0, x_i^0)\, x_i^0$$
$$+ \omega \int_{t_0}^{t} \Phi_i(t; s, x_i(s))\, g_i(s, x_1^{(0)}(s), ..., x_{i-1}^{(0)}(s), 0, ..., 0)\, ds,$$
$$i = 1, 2, ..., r. \tag{10}$$

THEOREM 1. *Suppose that*

(i) *there exist continuous functions* $l_{ij}(t) \in C[t_0, +\infty)$, $i, j = 1, 2, ..., r$, *such that*

$$\| g_i(t, x) - g_i(t, y) \| \leqslant \sum_{j=1}^{r} l_{ij}(t)\, \|x_j - y_j\|$$

for all $(t, x_i) \in J \times R^{n_i}$, $(t, y_i) \in J \times R^{n_i}$, $i, j = 1, 2, ..., r$;

(ii) *there is a continuous function* $\alpha(t) \in C[t_0, +\infty)$ *and a constant* $M_i > 0$ *such that* $x_i^{(0)}(t)$ *defined by* (9), (10) *satisfies*

$$\|x_i^{(0)}(t)\| \leqslant \omega M_i \|x_i^0\| \exp \int_{t_0}^{t} \lambda_i(s)\, ds \leqslant \omega M_i \|x_i^0\| \exp \int_{t_0}^{t} \alpha(s)\, ds \tag{11}$$

and $\lim_{t \to +\infty} \exp \int_{t_0}^{t} \alpha(s)\, ds = 0$;

(iii) $\int_{t_0}^{t} M_i l_{ij}(s) \exp \int_s^t (\lambda_i(u) - \alpha(u))\, du\, ds \leqslant b_{ij} = \text{const.}$ *and the spectral radius* $\rho(D_\omega) < 1$, *where* $D_\omega = (I - \omega B_*)^{-1}((1-\omega) I + \omega B^*)$, $0 < \omega < 2$, $B^* = B - B_*$, $B = (b_{ij})_{r \times r}$, *and* B_* *is the left lower triangular matrix of* B.

Then the zero solution of system (1) *is asymptotically stable.*

Proof. By (6), (9), (10), (11) we have

$$\|x_i^{(0)}(t)\|$$
$$\leqslant \omega M_i \|x_i^0\| \exp \int_{t_0}^{t} \lambda_i(u)\, du$$
$$\leqslant \omega M_i \|x_i^0\| \exp \int_{t_0}^{t} \alpha(u)\, du \overset{\text{def}}{=} \| y_i^{(0)}(t) \|, \qquad i = 1, 2, ..., r \tag{12}$$

$$
\begin{aligned}
&\|x_1^{(1)}(t)-x_1^{(0)}(t)\| \\
&\quad \leqslant (1-\omega)\,\|x_1^0\| + \omega \int_{t_0}^{t} \\
&\qquad \times\left(\exp\int_s^t \lambda_1(u)\,du \sum_{j=1}^{r} M_j l_{1j}(s)\,\omega\,\|x_j^0\| \exp\int_{t_0}^{s}\alpha(u)\,du\right) ds \\
&\quad \leqslant (1-\omega)\,\|x_1^{(0)}\| \\
&\qquad +\omega \sum_{j=1}^{r}\int_{t_0}^{t}\left(\exp\int_s^t (\lambda_1(u)-\alpha(u))\,du\, M_j l_{1j}(s)\,\omega\,\|x_j^0\|\right) ds \\
&\qquad \times \exp\int_{t_0}^{t}\alpha(u)\,du \\
&\quad \leqslant (1-\omega)\,\|y_1^{(0)}(t)\| + \omega \sum_{j=1}^{r} b_{1j}\,\|y_j^{(0)}(t)\| \\
&\quad \overset{\text{def}}{=} \|y_1^{(1)}(t)-y_1^{(0)}(t)\|
\end{aligned}
\tag{13$_1$}
$$

$$
\begin{aligned}
&\|x_2^{(1)}(t)-x_2^{(0)}(t)\| \\
&\quad \leqslant (1-\omega)\|\,x_2^{(0)}(t)\| \\
&\qquad +\omega\left[\int_{t_0}^{t} M_2 \exp\int_s^t (\lambda_2(u)-\alpha(u))\,du\, l_{2j}(s) \sum_{j=1}^{r} b_{1j}\omega M_j \|x_j^0\|\,ds\right] \\
&\qquad \times \exp\int_{t_0}^{t}\alpha(u)\,du \\
&\qquad +\omega \sum_{j=2}^{r}\left[\int_{t_0}^{t} M_2 \exp\int_s^t (\lambda_2(u)-\alpha(u))\,du\, l_{2j}(s)\,\omega M_j \|x_j^0\|\,ds\right] \\
&\qquad \times \exp\int_{t_0}^{t}\alpha(u)\,du \\
&\quad \leqslant (1-\omega)\,\|y_2^{(0)}(t)\| + \omega b_{21}\,\|y_1^{(1)}(t)-y_1^{(0)}(t)\| + \omega\sum_{j=2}^{r} b_{2j}\,\|y_j^{(0)}(t)\| \\
&\quad \overset{\text{def}}{=} \|y_2^{(1)}(t)-y_2^{(0)}(t)\| \;\ldots
\end{aligned}
\tag{13$_2$}
$$

$$
\begin{aligned}
&\|x_r^{(1)}(t)-x_r^{(0)}(t)\| \\
&\quad \leqslant (1-\omega)\,\|x_r^{(0)}(t)\| \\
&\qquad +\omega b_{r1}\left(\sum_{j=1}^{r} b_{1j}\omega M_j\,\|x_j^0\|\right)\exp\int_{t_0}^{t}\alpha(u)\,du
\end{aligned}
$$

$$+\omega b_{r2}\left(b_{21}\sum_{j=1}^{r} b_{1j}\omega M_j\|x_j^0\| + \sum_{j=2}^{r} b_{2j}\,\omega M_j\,\|x_j^0\|\right)\exp\int_{t_0}^{t}\alpha(u)\,du$$

$$+\cdots+\omega b_{rr}\omega M_r\,\|x_r^0\|\exp\int_{t_0}^{t}\alpha(u)\,du$$

$$\leqslant (1-\omega)\,\|y_r^{(0)}(t)\| + \omega b_{r1}\|y_1^{(1)}(t)-y_1^{(0)}(t)\| + \cdots + \omega b_{rr}\,\|y_r^{(0)}(t)\|$$

$$\stackrel{\text{def}}{=} \|y_r^{(1)}(t)-y_r^{(0)}(t)\|. \qquad (13)_r$$

Writing $(13)_1$, $(13)_2$, ..., $(13)_r$ in the vector form

$$\begin{aligned} I\operatorname{col}(&\|y_1^{(1)}(t)-y_1^{(0)}(t)\|, ..., \|y_r^{(1)}(t)-y_r^{(0)}(t)\|)\\ &\leqslant \omega B_*\operatorname{col}(\|y_1^{(1)}(t)-y_1^{(0)}(t)\|, ..., \|y_r^{(1)}(t)-y_r^{(0)}(t)\|)\\ &\quad+((1-\omega)I+\omega B^*)\operatorname{col}(\|y_1^{(0)}(t)\|, ..., \|y_r^{(0)}(t)\|) \end{aligned} \qquad (14)$$

then we have

$$\begin{aligned} \operatorname{col}(&\|x_1^{(1)}(t)-x_1^{(0)}(t)\|, ..., \|x_r^{(1)}(t)-x_r^{(0)}(t)\|)\\ &\leqslant \operatorname{col}(\|y_1^{(1)}(t)-y_1^{(0)}(t)\|, ..., \|y_r^{(1)}(t)-y_r^{(0)}(t)\|)\\ &=(1-\omega B_*)^{-1}((1-\omega)\,I+\omega B^*)\operatorname{col}(\|y_1^{(0)}(t)\|, ..., \|y_r^{(0)}(t)\|)\\ &\stackrel{\text{def}}{=} D_\omega\operatorname{col}(\|y_1^{(0)}(t)\|, ..., \|y_r^{(0)}(t)\|). \end{aligned} \qquad (15)$$

Let

$$\begin{aligned} \operatorname{col}(&\|x_1^{(m)}(t)-x_1^{(m-1)}(t)\|, ..., \|x_r^{(m)}(t)-x_r^{(m-1)}(t)\|)\\ &\leqslant \operatorname{col}(\|y_1^{(m)}(t)-y_1^{(m-1)}(t)\|, ..., \|y_r^{(m)}(t)-y_r^{(m-1)}(t)\|)\\ &\leqslant D_\omega^m\operatorname{col}(\|y_1^{(0)}(t)\|, ..., \|y_r^{(0)}(t)\|) \end{aligned} \qquad (16)$$

$$D_\omega^m=(d_{ij}^{(m)})_{r\times r}, \qquad m=1, 2, ... \qquad (17)$$

We have that

$$\begin{aligned} &\|x_1^{(m+1)}(t)-x_1^{(m)}(t)\|\\ &\quad\leqslant\left((1-\omega)+\sum_{j=1}^{r}\omega b_{1j}\right)\sum_{s=1}^{m} d_{js}^{(m)}\,\|y_s^{(0)}(t)\|\\ &\quad\stackrel{\text{def}}{=} \|y_2^{(m+1)}(t)-y_2^{(m)}(t)\| \end{aligned} \qquad (18)_1$$

$$\begin{aligned}
&\|x_2^{(m+1)}(t)-x_2^{(m)}(t)\| \\
&\quad \leqslant \omega b_{21}\,\|y_1^{(m+1)}(t)-y_1^{(m)}(t)\| \\
&\qquad + \sum_{j=2}^{r}((1-\omega)+\omega b_{2j})\,\|y_j^{(m)}(t)-y_j^{(m-1)}(t)\| \\
&\quad \stackrel{\text{def}}{=} \|y_2^{(m+1)}(t)-y_2^{(m)}(t)\| \ldots
\end{aligned} \tag{18$_2$}$$

$$\begin{aligned}
&\|x_r^{(m+1)}(t)-x_r^{(m)}(t)\| \\
&\quad \leqslant \omega b_{r1}\,\|y_1^{(m+1)}(t)-y_1^{(m)}(t)\|+\omega b_{r2}\,\|y_2^{(m+1)}(t)-y_2^{(m)}(t)\| \\
&\qquad + \cdots + ((1-\omega)+\omega b_{rr})\,\|y_r^{(m+1)}(t)-y_r^{(m)}(t)\| \\
&\quad \stackrel{\text{def}}{=} \|y_r^{(m+1)}(t)-y_r^{(m)}(t)\|.
\end{aligned} \tag{18$_r$}$$

Then we have

$$\begin{aligned}
&\operatorname{col}(\|x_1^{(m+1)}(t)-x_1^{(m)}(t)\|, \ldots, \|x_r^{(m+1)}(t)-x_r^{(m)}(t)\|) \\
&\quad \leqslant D_\omega D_\omega^m \operatorname{col}(\|y_1^{(0)}(t)\|, \ldots, \|y_r^{(0)}(t)\|) \\
&\quad \leqslant D^{m+1} \operatorname{col}(M_1\,\|x_1^0\|, \ldots, M_1\,\|x_r^0\|) \exp \int_{t_0}^{t} \alpha(u)\,du
\end{aligned} \tag{19}$$

$$\begin{aligned}
&\operatorname{col}(\|x_1^{(m+1)}(t)\|, \ldots, \|x_r^{(m+1)}(t)\|) \\
&\quad \leqslant \operatorname{col}(\|x_1^{(m+1)}(t)-x_1^{(m)}(t)\|, \ldots, \|x_r^{(m+1)}(t)-x_r^{(m)}(t)\|) \\
&\qquad + \cdots + \operatorname{col}(\|x_1^{(1)}(t)-x_1^{(0)}(t)\|, \ldots, \|x_r^{(1)}(t)-x_r^{(0)}(t)\|) \\
&\qquad + \operatorname{col}(\|x_1^{(0)}(t)\|, \ldots, \|x_r^{(0)}(t)\|) \\
&\quad \leqslant (D_\omega^{m+1}+D_\omega^m+\cdots+D_\omega+I) \operatorname{col}(M_1\,\|x_1^0\|, \ldots, M_r\,\|x_r^0\|) \\
&\qquad \times \exp \int_{t_0}^{t} \alpha(u)\,du \\
&\quad \leqslant (1-D_\omega)^{-1} \max_{1 \leqslant i \leqslant r} M_i \operatorname{col}(\|x_1^0\|, \ldots, \|x_r^0\|) \exp \int_{t_0}^{t} \alpha(u)\,dy.
\end{aligned} \tag{20}$$

By the mathematical induction, we know (19), (20) are true for any natural number m, since the spectral radius $\rho(D_\omega) < 1$, and $\sum_{m=0}^{\infty} D_\omega^m = (1-D_\omega)^{-1}$ is convergent.

On any finite interval $(t_0. T]$, the uniform convergence of

$$\sum_{m=0}^{\infty} D_\omega^m \operatorname{col}(\|x_1^0\|, \ldots, \|x_r^0\|) \exp \int_{t_0}^{t} \alpha(u)\,du \tag{21}$$

implies the uniform convergence of

$$\sum_{m=0}^{\infty} \operatorname{col}(\|x_1^{(m)}(t)-x_1^{(m-1)}(t)\|, \ldots, \|x_r^{(m)}(t)-x_r^{(m-1)}(t)\|) + \operatorname{col}(\|x_1^{(0)}(t)\|, \ldots, \|x_r^{(0)}(t)\|) \tag{22}$$

and hence also implies the uniform convergence of

$$\sum_{m=0}^{\infty} \operatorname{col}(x_1^{(m)}(t)-x_1^{(m-1)}(t), \ldots, x_r^{(m)}(t)-x_r^{(m-1)}(t)) + \operatorname{col}(x_1^{(0)}(t), \ldots, x_r^{(0)}(t)),$$

that is,

$$\operatorname{col}(x_1^{(m)}(t), \ldots, x_r^{(m)}(t)) \underset{m\to\infty}{\rightrightarrows} \operatorname{col}(x_1(t), \ldots, x_r(t)) \tag{23}$$

and

$$\operatorname{col}(\|x_1(t)\|, \ldots, \|x_r(t)\|) \leqslant (1-D_\omega)^{-1} \max_{,1\leqslant i\leqslant r} M_i \operatorname{col}(\|x_1^0\|, \ldots, \|x_r^0\|) \exp \int_{t_0}^{t} \alpha(u)\, du \xrightarrow[t\to+\infty]{} 0 \tag{24}$$

following from condition (ii). So the zero solution of large scale system (1) is asymptotically stable. This completes the proof of the problem.

THEOREM 2. *Suppose that*

(i) *condition* (i) *of Theorem* 1 *holds with* $l_{ij}(t)=l_{ij}=\text{const.}$, $i, j=1, 2, \ldots, r$;

(ii) *there exist constants* $M_i>0$, $\alpha>0$ *such that*

$$\|x_i^{(0)}(t)\| \leqslant \omega M_i \|x_i^0\| \exp(\alpha(t-t_0)), \qquad t\in J;$$

(iii) $\int_{t_0}^{t} M_i l_{ij} \exp(\alpha(t-s))\, ds \leqslant M_i l_{ij} \alpha^{-1} \overset{\text{def}}{=} \tilde{b}_{ij}$, *and the spectral radius* $\rho(\tilde{D}_\omega)<1$, *where* $\tilde{D}_\omega \overset{\text{def}}{=} (I-\omega\tilde{B}_*)^{-1}((1-\omega)I+\omega\tilde{B}^*)$, $0<\omega<2$, $\tilde{B}=(\tilde{b}_{ij})_{r\times r}$, *and* $\tilde{B}_*$, $\tilde{B}^*$ *are similiar to* B_* *and* B^* *in Theorem* 1.

Then the zero solution of system (1) *is exponentially stable.*

The proof is exactly similar to that of Theorem 1; here we omit it.

Remark 1. If $\omega=1$, then the successive overrelaxation iteration (8) reduces to the Gauss–Seidel iteration (Liao [9], Ortega and Rheinbadt [13]).

Remark 2. The successive overrelaxation iteration used here can be used to discuss the connective stability of large scale systems (Siljak [14]), the stabilities of discrete large scale systems, and differential-difference large scale systems (Michel and Miller [12]).

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