Asymptotic Stability for Volterra Intergrodifferential Systems

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ABSTRACT

In this paper, the successive overrelaxation iteration and the variation of parameters formula are used to discuss the stability of linear and nonlinear Volterra integrodifferential equations. Some sufficient conditions are obtained, such that the trivial solution of the Volterra integrodifferential equations is asymptotically stable; these stability conditions are given directly from the coefficients of the equations.

1. INTRODUCTION

It is well known that the variation of parameters formula for linear and nonlinear differential equations is an important tool in the study of qualitative properties of perturbed equations, cf. Alekseev [1], Lakshmikantham and Leela [11], Dannan and Elaydi [5, 6], etc. A corresponding variation of the parameters formula for linear integrodifferential equations was developed and utilized to investigate the perturbed integrodifferential equations by Grossman and Miller [9].

Recently, Beesack [2] and Bernfeld and Lord [3] established a kind of nonlinear variation of parameters formula to unperturbed systems of differential systems, which contains no integral terms. Hu, Lakshmikantham and Rao [10] and Lakshmikantham and Rao [12] developed a nonlinear variation of

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parameters formula for perturbed integrodifferential equations. By means of the formula established in [10] and [12], Elaydi and Rao [7] and Elaydi and Sivasundaram [8] investigated Liapunov stabilities and Lipschitz stabilities for the integrodifferential systems. For the Liapunov stabilities of integrodifferential equations, many interesting results can be found in Burton [4], Lakshmikantham and Leela [11].

In numerical analysis, iteration is one of the most important methods to solve algebraic equations; see, for example, Ortega and Rheinbadt [13], Vorga [16] and Young [17]. Recently in [14], from the view of large scale systems, we used the successive overrelaxation iteration and the variation of parameters formula of ordinary differential equations to investigate the stability of nonlinear perturbed differential equations.

In the present paper, we use the successive overrelaxation iteration of [14] and the variation of parameters formula of integrodifferential equations developed in [10] to discuss the stability of linear and nonlinear Volterra integrodifferential equations. Some sufficient conditions are obtained, such that the trivial solution of the Volterra integrodifferential equations is asymptotically stable. Differing from the Liapunov functions method, we give stability conditions directly from the coefficients of the systems.

2. LINEAR SYSTEMS

We consider the linear Volterra integrodifferential system

$$x'(t) = A(t)x(t) + \int_{t_0}^{t} B(t,s)x(s)\,ds, \qquad x(t_0) = x_0,\ t_0 \geqslant 0, \quad (2.1)$$

and the perturbed system of (2.1)

$$y'(t) = A(t)y(t) + \int_{t_0}^{t} B(t,s)y(s)\,ds + F(t, y(t), Sy(t)),$$

$$y(t_0) = x_0, \qquad (2.2)$$

in which $A(t)$ and $B(t,s)$ are $n \times n$ matrices continuous on R_+ and $R_+ \times R_+$, respectively, $F \in C[R_+ \times R^n \times R^n, R^n]$ and

$$Sy(t) = \int_{t_0}^{t} K(t,s,y(s))\,ds, \qquad K \in C[R_+ \times R_+ \times R^n, R^n].$$

Firstly, we recall the following well-known result, which was established originally by Grossman and Miller [9].

LEMMA 2.1. *The unique solution $y(t)$ of system (2.2) is given by*

$$y(t, t_0, x_0) = R(t, t_0) x_0 + \int_{t_0}^{t} R(t, s) F(s, y(s), Sy(s))\, ds, \qquad t \geqslant t_0, \tag{2.3}$$

where $R(t, s)$ is the solution of the initial value problem

$$\frac{\partial R}{\partial s}(t, s) + R(t, s) A(s) + \int_{s}^{t} R(t, \sigma) B(t, \sigma)\, d\sigma = 0, \qquad R(t, t) = I_{nn} \tag{2.4}$$

for $t_0 \leqslant s \leqslant t < \infty$, in which I_{nn} is an $n \times n$ identity matrix.

As in Ruan [14], we suppose that system (2.1) can be decomposed as follows:

$$x_i'(t) = A_{ii}(t) x_i(t) + \int_{t_0}^{t} B_{ii}(t, s) x_i(s)\, ds + \sum_{j=1, j \neq i}^{r} A_{ij}(t) x_j(t)$$
$$+ \sum_{j=1, j \neq i}^{r} \int_{t_0}^{t} B_{ij}(t, s) x_j(s)\, ds, \qquad x_i(t_0) = x_i^0, \tag{2.5}$$

where

$$x_i \in R_+^{n_i}, \qquad i = 1, 2, \ldots, r, \ \sum_{i=1}^{r} n_i = n,$$

and $A_{ij}(t)$, $B_{ij}(t, s)$ are $n_i \times n_j$ matrices continuous on R_+ and $R_+ \times R_+$, respectively. We also consider the subsystems

$$x_i'(t) = A_{ii}(t) x_i(t) + \int_{t_0}^{t} B_{ii}(t, s) x_i(s)\, ds, \qquad i = 1, 2, \ldots, r. \tag{2.6}$$

Letting

$$F_i(t, x(t), Sx(t)) = \sum_{j=1, j\neq i}^{r} \left[A_{ij}(t) x_j(t) + \int_{t_0}^{t} B_{ij}(t, s) x_j(s)\, ds \right], \quad (2.7)$$

we can rewrite system (2.3) as the following:

$$\begin{aligned} x_i'(t) &= A_{ii}(t) x_i(t) + \int_{t_0}^{t} B_{ii}(t, s) x_i(s)\, ds + F_i(t, x(t), Sx(t)), \\ x_i(t_0) &= x_i^0. \end{aligned} \quad (2.8)$$

Now system (2.8) can be considered as the perturbed system of system (2.7); hence, by Lemma 2.1, the solution of system (2.8) is given by

$$x_i(t) = R_i(t, t_0) x_i^0 + \int_{t_0}^{t} R_i(t, s) F_i(s, x(s), Sx(s))\, ds, \quad (2.9)$$

where $R_i(t, t_0)$ is the solution of the initial value problem

$$\begin{gathered} \frac{\partial R_i}{\partial s}(t, s) + R_i(t, s) A_{ii}(s) + \int_{s}^{t} R_i(t, \sigma) B_{ii}(\sigma, s)\, d\sigma = 0, \\ R_i(t, t) = I_{ii} \end{gathered}$$

for $t_0 \leqslant s \leqslant t < \infty$, $i = 1, 2, \dots, r$. As in [17], using the successive overrelaxation iteration on (2.9), we obtain that

$$\begin{aligned} x_i^{(m)}(t) = {} & (1 - \omega) x_i^{(m-1)}(t) + \omega R_i(t, t_0) x_i^0 \\ & + \int_{t_0}^{t} R_i(t, s) F_i\big(s, x_1^{(m)}(s), \dots, x_{i-1}^{(m)}(s), \\ & \qquad x_i^{(m-1)}(s), \dots, x_r^{(m-1)}(s), Sx_1^{(m)}(s), \dots, Sx_{i-1}^{(m)}(s), \\ & \qquad Sx_i^{(m-1)}(s), \dots, Sx_r^{(m-1)}(s)\big)\, ds, \end{aligned} \quad (2.10)$$

where $m = 1, 2, \dots$; $i = 1, 2, \dots, r$; $0 < \omega < 2$, and we choose

$$x_i^{(0)}(t) = \omega R_i(t, t_0) x_i^0, \; i = 1, 2, \dots, r. \quad (2.11)$$

Then we have the following result.

THEOREM 2.1. *Suppose that*

(*i*) *there exist positive constants* α *and* M_i $(i = 1, 2, \dots, r)$, *such that*

$$\|R_i(t, t_0)\| \leqslant M_i e^{-\alpha(t-t_0)}, \qquad t \geqslant t_0; \tag{2.12}$$

(*ii*) *there exist functions* $\lambda_{ij} \in C[R_+, R_+]$, $\beta_{ij} \in C[R_+ \times R_+, R_+]$, $i, j = 1, 2, \dots, r$, *such that*

$$\|F_i(t, x(t), Sx(t))\| \leqslant \sum_{j=1}^{r} \left[\lambda_{ij}(t)\|x_j\| + \int_{t_0}^{t} \beta_{ij}(t, s)\|x_j(s)\|\, ds \right]; \tag{2.13}$$

(*iii*) *there exist positive constants* b_{ij} $(i, j = 1, 2, \dots, r)$, *such that*

$$M_i \int_{t_0}^{t} \left[\lambda_{ij}(s) + \int_{s}^{t} e^{\alpha(\sigma - s)} \beta_{ij}(\sigma, s)\, d\sigma \right] ds \leqslant b_{ij}; \tag{2.14}$$

(*iv*) *the spectral radius* $\rho(D_\omega) < 1$, *where*

$$D_\omega = \left(I_{rr} - \omega B_*\right)^{-1}\left((1 - \omega) I_{rr} + \omega B^*\right),$$

and $0 < \omega < 2$, I_{rr} *is an* $r \times r$ *identity matrix*, $B = (b_{ij})_{r \times r}$, $B^* = B - B_*$, B_* *is the left lower triangle matrix of* B.

Then the trivial solution of system (2.1) *is asymptotically stable.*

PROOF. From (2.11) and hypothesis (i), we have

$$\|x_i^{(0)}(t)\| \leqslant \omega M_i \|x_i^0\| e^{-\alpha(t-t_0)} \overset{def}{=} \|y_i^{(0)}(t)\|, \qquad t \geqslant t_0, \ i = 1, 2, \dots, r. \tag{2.15}$$

Writing (2.15) in the vector form, we get

$$col\left(\|x_1^{(0)}(t)\|, \dots, \|x_r^{(0)}(t)\|\right) \leqslant I_{rr} col\left(M_1\|x_1^0\|, \dots, M_r\|x_r^0\|\right) e^{-\alpha(t-t_0)}. \tag{2.16}$$

By (2.10) and (2.11) we have

$$
\begin{aligned}
&\|x_1^{(1)}(t) - x_1^{(0)}(t)\| \\
&\quad \leqslant (1-\omega)\|x_1^{(0)}(t)\| \\
&\qquad + \omega\int_{t_0}^{t} M_1 e^{-\alpha(t-s)} \sum_{j=1}^{r} \left[\lambda_{1j}(s)\left\| x_j^{(0)}(s) \right\| \right. \\
&\qquad\qquad \left. + \int_{t_0}^{t} \beta_{1j}(t,u)\left\| x_j^{(0)}(u) \right\| du \right] ds \\
&\quad \leqslant (1-\omega)\|x_1^{(0)}(t)\| + \omega\int_{t_0}^{t} M_1 e^{-\alpha(t-s)} \sum_{j=1}^{r} \omega M_j \|x_j^0\| \\
&\qquad \cdot \left[\lambda_{1j}(s) e^{-\alpha(t-t_0)} + \int_{t_0}^{t} \beta_{1j}(s,u) e^{-\alpha(u-t_0)}\, du \right] ds \\
&\quad \leqslant (1-\omega)\|x_1^{(0)}\| \\
&\qquad + \omega M_1 \sum_{j=1}^{r} \omega M_j \|x_j^0\| \int_{t_0}^{t} e^{-\alpha(t-s)} \lambda_{1j}(s) e^{-\alpha(s-t_0)}\, ds \\
&\qquad + \omega M_1 \sum_{j=1}^{r} \omega M_j \|x_j^0\| \int_{t_0}^{t} e^{-\alpha(t-s)} \int_{t_0}^{s} \beta_{1j}(s,u) e^{-\alpha(u-t_0)}\, du\, ds.
\end{aligned}
$$

Using Fubini's Theorem, we obtain that

$$
\begin{aligned}
&\|x_1^{(1)}(t) - x_1^{(0)}(t)\| \\
&\quad \leqslant (1-\omega)\|x_1^{(0)}(t)\|
\end{aligned}
$$

$$+ \omega M_1 \sum_{j=1}^{r} \omega M_j \|x_j^0\| e^{-\alpha(t-t_0)} \int_{t_0}^{t} \lambda_{1j}(\tau)\, d\tau$$

$$+ \omega M_1 \sum_{j=1}^{r} \omega M_j \|x_j^0\| e^{-\alpha(t-t_0)} \int_{t_0}^{t} \int_{s}^{t} e^{-\alpha(u-\tau)} \beta_{1j}(u,\tau)\, du\, d\tau$$

$$= (1-\omega)\|x_1^{(0)}(t)\| + \omega \sum_{j=1}^{r} M_1$$

$$\cdot \int_{t_0}^{t} \left[\lambda_{1j}(s) + \int_{s}^{t} e^{-\alpha(u-s)} \beta_{1j}(u,s)\, du \right] ds \cdot \omega M_j \|x_j^0\| e^{-\alpha(t-t_0)}$$

$$\leqslant (1-\omega)\|y_1^{(0)}(t)\| + \omega \sum_{j=1}^{r} b_{1j} \|y_j^{(0)}(t)\|$$

$$\overset{def}{=} \|y_1^{(1)}(t) - y_1^{(0)}(t)\|. \tag{2.17.1}$$

Similar to the above procedure and the proof of Theorem 1 in [15], we have

$$\|x_2^{(1)}(t) - x_2^{(0)}(t)\|$$

$$\leqslant (1-\omega)\|y_2^{(0)}(t)\| + \omega b_{21} \|y_1^{(1)}(t) - y_1^{(0)}(t)\|$$

$$+ \omega \sum_{j=2}^{r} b_{2j} \|y_j^{(0)}(t)\|$$

$$\overset{def}{=} \|y_2^{(1)}(t) - y_2^{(0)}(t)\| \cdots \tag{2.17.2}$$

and

$$\|x_r^{(1)}(t) - x_r^{(0)}(t)\|$$

$$\leqslant (1-\omega)\|y_r^{(0)}(t)\|$$

$$+ \omega b_{r1} \|y_1^{(1)}(t) - y_1^{(0)}(t)\| + \cdots + \omega b_{rr} \|y_r^{(0)}(t)\|$$

$$\overset{def}{=} \|y_r^{(1)}(t) - y_r^{(0)}(t)\|. \tag{2.17.r}$$

Rewriting (2.17.1), (2.17.2), ..., and (2.17.r) in the vector form

$$
\begin{aligned}
I_{rr} col\big(\|y_1^{(1)}(t) - y_1^{(0)}(t)\|, \dots, \|y_r^{(1)}(t) - y_r^{(0)}(t)\|\big) \\
\leqslant \omega B_* col\big(\|y_1^{(1)}(t) - y_1^{(0)}(t)\|, \dots, \|y_r^{(1)}(t) - y_r^{(0)}(t)\|\big) \\
+ \big((1-\omega) I_{rr} + \omega B^*\big) col\big(\|y_1^{(0)}(t)\|, \dots, \|y_r^{(0)}(t)\|\big), \quad (2.18)
\end{aligned}
$$

then we have

$$
\begin{aligned}
& col\big(\|x_1^{(1)}(t) - x_1^{(0)}(t)\|, \dots, \|x_r^{(1)}(t) - x_r^{(0)}(t)\| \\
& \leqslant col\big(\|y_1^{(1)}(t) - y_1^{(0)}(t)\|, \dots, \|y_r^{(1)}(t) - y_r^{(0)}(t)\|\big) \\
& = \big(1 - \omega B_*\big)^{-1}\big((1-\omega) I_{rr} + \omega B^*\big) col\big(\|y_1^{(0)}(t)\|, \dots, \|y_r^{(0)}(t)\|\big) \\
& \overset{def}{=} D_\omega col\big(\|y_1^{(0)}(t)\|, \dots, \|y_r^{(0)}(t)\|\big). \quad (2.19)
\end{aligned}
$$

Let

$$
\begin{aligned}
& col\big(\|x_1^{(1)}(t) - x_1^{(0)}(t)\|, \dots, \|x_r^{(1)}(t) - x_r^{(0)}(t)\|\big) \\
& \leqslant col\big(\|y_1^{(1)}(t) - y_1^{(0)}(t)\|, \dots, \|y_r^{(1)}(t) - y_r^{(0)}(t)\|\big) \\
& \leqslant D_\omega^m col\big(\|y_1^{(0)}(t)\|, \dots, \|y_r^{(0)}(t)\|\big),
\end{aligned}
$$

where $D_\omega^m = (d_{ij}^{(m)})_{r \times r}$, $m = 1, 2, \dots,$ we have the following inequalities:

$$
\begin{aligned}
& \|x_1^{(m+1)}(t) - x_1^{(m)}(t)\| \\
& \leqslant \left((1-\omega) + \sum_{j=1}^{r} \omega b_{1j}\right) \sum_{k=1}^{m} d_{jk}^{(m)} \|y_k^{(0)}(t)\| \\
& \overset{def}{=} \|y_1^{(m+1)}(t) - y_1^{(m)}(t)\|. \quad (2.20.1)
\end{aligned}
$$

$$
\begin{aligned}
& \|x_2^{(m+1)}(t) - x_2^{(m)}(t)\| \\
& \leqslant \omega b_{21} \|y_1^{(m+1)}(t) - y_1^{(m)}(t)\| \\
& \quad + \sum_{j=2}^{r} \big((1-\omega) + \omega b_{2j}\big) \|y_j^{(m)}(t) - y_j^{(m-1)}(t)\| \\
& \overset{def}{=} \|y_2^{(m+1)}(t) - y_2^{(m)}(t)\| \cdots \quad (2.20.2)
\end{aligned}
$$

and

$$
\begin{aligned}
&\|x_r^{(m+1)}(t) - x_r^{(m)}(t)\| \\
&\qquad \leqslant \omega b_{r1}\|y_1^{(m+1)}(t) - y_1^{(m)}(t)\| + \omega b_{r2}\|y_2^{(m+1)}(t) - y_2^{(m)}(t)\| + \cdots \\
&\qquad\quad + ((1-\omega) + \omega b_{rr})\|y_r^{(m+1)}(t) - y_r^{(m)}(t)\| \\
&\overset{def}{=} \|y_r^{(m+1)}(t) - y_r^{(m)}(t)\|. \qquad (2.20.r)
\end{aligned}
$$

Writing (2.20.1), (2.20.2), ..., and (2.20.r) in the vector form

$$
\begin{aligned}
&col\big(\|x_1^{(m+1)}(t) - x_1^{(m)}(t)\|, \dots, \|x_r^{(m+1)}(t) - x_r^{(m)}(t)\|\big) \\
&\qquad \leqslant D_\omega D_\omega^m col\big(\|y_1^{(0)}(t)\|, \dots, \|y_r^{(0)}(t)\|\big) \\
&\qquad \leqslant \omega D_\omega^{m+1} col\big(M_1\|x_1^0\|, \dots, M_r\|x_r^0\|\big)e^{-\alpha(t-t_0)}, \qquad (2.21)
\end{aligned}
$$

which follows from (2.16), we have that

$$
\begin{aligned}
&col\big(\|x_1^{(m+1)}(t)\|, \dots, \|x_r^{(m+1)}(t)\|\big) \\
&\qquad \leqslant col\big(\|x_1^{(m+1)}(t) - x_1^{(m)}(t)\|, \dots, \|x_r^{(m+1)}(t) - x_r^{(m)}(t)\|\big) + \cdots \\
&\qquad \leqslant col\big(\|x_1^{(1)}(t) - x_1^{(0)}(t)\|, \dots, \|x_r^{(1)}(t) - x_r^{(0)}(t)\|\big) \\
&\qquad\quad + col\big(\|x_1^{(0)}(t)\|, \dots, \|x_r^{(0)}(t)\|\big) \\
&\qquad \leqslant \omega\big(D_\omega^{m+1} + D_\omega^m + \cdots + D_\omega + I_{rr}\big) \\
&\qquad\quad \times col\big(M_1\|x_1^0\|, \dots, M_r\|x_r^0\|\big)e^{-\alpha(t-t_0)} \\
&\qquad \leqslant \omega(1 - D_\omega)^{-1} \max_{1\leqslant i\leqslant r} M_i col\big(\|x_1^0\|, \dots, \|x_r^0\|\big)e^{-\alpha(t-t_0)}. \qquad (2.22)
\end{aligned}
$$

Since the spectral radius $\rho(D_\omega) < 1$ and $\sum_{m=0}^{\infty} D_\omega^m = (1 - D_\omega)^{-1}$ is convergent, by the mathematical induction, (2.21) and (2.22) hold for any natural number m.

On any finite interval $(t_0, T]$, the uniform convergence of the series

$$\sum_{m=0}^{\infty} D_{\omega}^{m} col\left(\|x_1^0\|, \dots, \|x_r^0\|\right) e^{-\alpha(t-t_0)}$$

implies the uniform convergence of the series

$$\sum_{m=0}^{\infty} col\left(\|x_1^{(m)}(t) - x_1^{(m-1)}(t)\|, \dots, \|x_r^{(m)}(t) - x_r^{(m-1)}(t)\|\right)$$
$$+ col\left(\|x_1^0\|, \dots, \|x_r^{(0)}(t)\|\right)$$

and hence, also implies the uniform convergence of the series

$$\sum_{m=0}^{\infty} col\left(x_1^{(m)}(y) - x_1^{(m-1)}(t), \dots, x_r^{(m)}(t) - x_r^{(m-1)}(t)\right)$$
$$+ col\left(x_1^{(0)}(t), \dots, x_r^{(0)}(t)\right).$$

It follows that $col(x_1^{(m)}(t), \dots, x_r^{(m)}(t))$ converges uniformly to $col(x_1(t), \dots, x_r(t))$ as $m \to \infty$. Hence, we have

$$col\left(\|x_1(t)\|, \dots, \|x_r(t)\|\right)$$
$$\leqslant \omega(1 - D_{\omega})^{-1} \max_{1 \leqslant i \leqslant r} M_i col\left(\|x_1^0\|, \dots, \|x_r^0\|\right) e^{-\alpha(t-t_0)}. \quad (2.23)$$

The right hand side of (2.23) goes to zero as t approaches infinity, so the zero solution of system (2.1) is asymptotically stable. This completes the proof. ■

3. NONLINEAR SYSTEMS

In this section we consider the nonlinear Volterra intergrodifferential system

$$x'(t) = h(t, x(t)) + \int_{t_0}^{t} g(t, s, x(s))\, ds, \qquad x(t_0) = x_0, \quad (3.1)$$

where $h \in C[R_+ \times R^n, R^n]$, $g \in C[R_+ \times R_+ \times R^n, R^n]$, and the perturbed system

$$y'(t) = h(t, y(t)) + \int_{t_0}^{t} g(t, s, y(s))\, ds + F(t, y(t), Sy(t)),$$

$$y(t_0) = x_0, \tag{3.2}$$

where $F \in C[R_+ \times R^n \times R^n, R^n]$, $Sy(t)$ is the same as in Section 2.

We need the following lemma, which is Theorem 1.4 of Hu, Lashmikantham and Rao [10].

LEMMA 3.1. *Suppose $\partial h/\partial x$, $\partial g/\partial x$ exist and are continuous on $R_+ \times R^n$, $R_+ \times R_+ \times R^n$, respectively. Let $x(t) = x(t, t_0, x_0)$ be the unique solution of system* (3.1); *then any solution $y(t) = y(t, t_0, x_0)$ of system* (3.2) *satisfies the integral equation*

$$y(t) = x(t) + \int_{t_0}^{t} \Phi(t, s, y(s)) F(s, y(s), Sy(s))\, ds$$

$$+ \int_{t_0}^{t} \int_{s}^{t} [\Phi(t, \sigma, y(\sigma)) - R(t, \sigma; s, y(s))] g(\sigma, s, y(s))\, d\sigma\, ds \tag{3.3}$$

for $t \geqslant t_0$, where $\Phi(t, t_0, x_0) = (\partial x/\partial x_0)(t, t_0, x_0)$ and $R(t, s; t_0, x_0)$ is the solution of the initial value problem

$$\frac{\partial R}{\partial s}(t, s; t_0, x_0) + R(t, s; t_0 x_0) \frac{\partial h}{\partial x}(s, x(s, t_0, x_0))$$

$$+ \int_{s}^{t} R(t, \sigma; t_0, x_0) \frac{\partial g}{\partial x}(\sigma, s, x(s, t_0, x_0))\, d\sigma = 0,$$

$$R(t, t; t_0, x_0) = I_{nn}, \quad R(t, t_0; t_0, x_0) = \Phi(t, t_0, x_0), \qquad t \geqslant s \geqslant t_0. \tag{3.4}$$

Suppose equation (3.1) can be decomposed as

$$x_i'(t) = h_i(t, x_i(t)) + \int_{t_0}^{t} g_i(t, s, x_i(s))\, ds$$

$$+H_i(t, x(t)) + \int_{t_0}^{t} G_i(t, s, x(s))\, ds, \qquad x_i(t_0) = x_i^0, \quad (3.5)$$

where

$$h_i \in C[R_+ \times R_+^{n_i}, R_+^{n_i}], \quad g_i \in C[R_+ \times R_+ \times R_+^{n_i}, R_+^{n_i}],$$

$$H_i \in C[R_+ \times R_+^{n}, R_+^{n_i}], \quad G_i \in C[R_+ \times R_+^{n}, R_+^{n_i}], \quad \sum_{i=1}^{\gamma} n_i = n.$$

Suppose that

$$h_i(t,0) = g_i(t,0) = 0, \quad H_i(t,0) = G_i(t,0) = 0, \qquad i = 1,2,\dots,r.$$

We also consider the subsystems

$$x_i'(t) = h_i(t, x_i(t)) + \int_{t_0}^{t} g_i(t, s, x_i(s))\, ds, \qquad x_i(t_0) = x_i^0, i = 1,2,\dots,r. \quad (3.6)$$

Let

$$F_i(t, x(t), Sx(t)) = H_i(t, x(t)) + \int_{t_0}^{t} G_i(t, s, x(s))\, ds, \qquad i = 1,2,\dots,r;$$

then (3.5) can be considered as the perturbed system of (3.6) and can be written as

$$x_i'(t) = h_i(t, x_i(t)) + \int_{t_0}^{t} g_i(t, s, x_i(s))\, ds + F_i(t, x(t), Sx(t)),$$

$$x_i(t_0) = x_i^0. \quad (3.7)$$

Suppose that $\partial h_i/\partial x_i$ and $\partial g_i/\partial x_i$ exist and are continuous on $R_+ \times R^{n_i}$, $R_+ \times R_+ \times R^{n_i}$, respectively. Let $\Phi_i(t, t_0, x_i^0) = (\partial x_i/\partial x_i^0)(t, t_0, x_i^0)$ and let $R_i(t, s; t_0, x_i^0)$ be the solution of the equation

$$\frac{\partial R_i}{\partial s}(t, s; t_0, x_i^0) + R_i(t, s; t_0, x_i^0)\frac{\partial h_i}{\partial x_i}(s, t_0, x_i^0)$$

$$+ \int_s^t R_i(t, \sigma; t_0, x_i^0)\frac{\partial g_i}{\partial x_i}(t, s; t_0, x_i^0)\, d\sigma = 0$$

with the initial value $R_i(t, t; t_0, x_i^0) = I_{ii}$ on $t_0 \leqslant s \leqslant t$, and $R_i(t, t_0; t_0, x_i^0) = \Phi_i(t, t_0, x_i^0)$. Let $x_i(t) = x_i(t, t_0, x_0)$ be the solution of system (3.7) and $x_i(t, t_0, x_i^0)$ be the solution of the subsystem (3.6). Then by Lemma 3.1, we have

$$\begin{aligned} x_i(t) &= x_i(t, t_0, x_i^0) + \int_{t_0}^t \Phi_i(t, s, x_i(s)) F_i(s, x(s), Sx(s))\, ds \\ &\quad + \int_{t_0}^t \int_s^t [\Phi_i(t, \sigma, x_i(\sigma)) - R_i(t, \sigma; s, x_i(s))] \\ &\quad\quad g_i(\sigma, s, x_i(s))\, d\sigma\, ds \end{aligned} \tag{3.8}$$

for $t \geqslant t_0$. Using the successive overrelaxation iteration on (3.8), we obtain that

$$\begin{aligned} x_t^{(m)}(t) &= (1 - \omega) x_i^{(m-1)}(t) + \omega x_i(t, t_0, x_i^0) \\ &\quad + \omega \int_{t_0}^t \int_s^t [\Phi_i(t, \sigma, x_i(\sigma)) - R_i(t, \sigma; s, x_i(s))] \\ &\quad\quad g_i(\sigma, s, x_i(s))\, d\sigma\, ds \\ &\quad + \omega \int_{t_0}^t \Phi_i(t, s, x_i(s)) F_i(s, x_1^{(m)}(s), \ldots, x_{i-1}^{(m)}(s), \\ &\quad\quad x_i^{(m-1)}(s), \ldots, x_r^{(m-1)}(s), Sx_1^{(m)}(s), \ldots, \\ &\quad\quad Sx_{i-1}^{(m)}(s), Sx_i^{(m-1)}(s), \ldots, Sx_r^{(m-1)}(s))\, ds, \end{aligned} \tag{3.9}$$

where $m = 1, 2, \ldots;\ i = 1, 2, \ldots, r;\ 0 < \omega < 2$. Then we have

THEOREM 3.1. *Suppose that*

(*i*) *there exist positive constants* α, M_i *and* ρ_i $(i = 1, 2, \ldots, r)$, *such that if* $\|x_i^0\| \leqslant \rho_i$,

$$\|R_i(t, s; t_0, x_i^0)\| \leqslant M_i e^{-\alpha(t-s)}, \quad \|\Phi_i(t, t_0, x_i^0)\| \leqslant M_i e^{\alpha(t-t_0)},$$

$$t \geqslant s \geqslant t_0;$$

(*ii*) *there exist functions* $\lambda_{ij} \in C[R_+, R_+]$ *and* $\beta_{ij} \in C[R_+ \times R_+, R_+]$ $(i, j = 1, 2, \ldots, r)$ *and positive constants* K_i, *such that if* $\|x_i\| \leqslant \rho_i$,

$$\|g_i(t, s, x_i)\| \leqslant K_i \|x_i\| e^{-2\alpha t},$$

$$\|F_i(t, x(t), Sx(t))\| \leqslant \sum_{j=1}^{r} \left[\lambda_{ij}(t)\|x_j\| + \int_{t_0}^{t} \beta_{ij}(t, s)\|x_j(s)\|\, ds \right];$$

(*iii*) *there exist positive constants* b_{ij} $(1, j = 1, 2, \ldots, r)$, *such that*

$$M_i \int_{t_0}^{t} \left[\lambda_{ij}(s) + \frac{2K_i \delta_{ij}}{\alpha} e^{-\alpha(t_0 - s)} + \int_{s}^{t} e^{\alpha(u-s)} \beta_{ij}(u, s)\, du \right] ds \leqslant b_{ij},$$

where

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j; \end{cases}$$

(*iv*) *hypothesis* (*iii*) *of Theorem* 2.1 *holds.*

Then the trivial solution of system (3.1) *is asymptotically stable.*

PROOF. By theorem 1.5 of [10] and hypothesis (i), we have

$$\|x_i^{(0)}(t)\| \leqslant M_i \|x_i^0\| e^{-\alpha(t-t_0)} \overset{def}{=} \|y_i^{(0)}(t)\|, \qquad t \geqslant t_0, \tag{3.10}$$

and

$$\begin{aligned}
&\|x_1^{(1)}(t) - x_1^{(0)}(t)\| \\
&\quad \leqslant (1-\omega)\|x_1^{(0)}(t)\| \\
&\qquad + \omega\int_{t_0}^{t}\int_{s}^{t} 2M_1K_1 e^{-\alpha(t-u)}e^{-2\alpha u}\|x_1^{(0)}(s)\|\,du\,ds \\
&\qquad + \omega\int_{t_0}^{t} M_1 e^{-\alpha(t-s)}\sum_{j=1}^{r}\lambda_{1j}(s)\|x_j^{(0)}(s)\|\,ds \\
&\qquad + \omega\int_{t_0}^{t} M_1 e^{-\alpha(t-s)}\sum_{j=1}^{r}\int_{t_0}^{s}\beta_{1j}(s,u)\|x_j^{(0)}(u)\|\,du\,ds \\
&\quad \leqslant (1-\omega)\|x_1^{(0)}(t)\| \\
&\qquad + \omega\int_{t_0}^{t}\int_{s}^{t} 2M_1K_1 e^{-\alpha(t-u)}e^{-2\alpha u}\|x_1^{(0)}(s)\|\,du\,ds \\
&\qquad + \omega\int_{t_0}^{t} M_1 e^{-\alpha(t-s)}\sum_{j=1}^{r}\lambda_{1j}(s)\,\omega M_j\|x_j^0\|e^{-\alpha(t-t_0)}\,ds \\
&\qquad + \omega\int_{t_0}^{t} M_1 e^{-\alpha(t-s)}\sum_{j=1}^{r}\int_{t_0}^{t}\beta_{1j}(s,u)\,\omega M_j\|x_j^0\|e^{-\alpha(t-t_0)}\,du\,ds.
\end{aligned}$$

By Fubini's Theorem and hypothesis (iii), we have

$$\begin{aligned}
&\|x_1^{(1)}(t) - x_1^{(0)}(t)\| \\
&\quad \leqslant (1-\omega)\|x_1^{(0)}(t)\| \\
&\qquad + \omega M_1\int_{t_0}^{t}\Bigg[\sum_{j=1}^{n}\lambda_{1j}(s) + \frac{2K_1}{\alpha}e^{-\alpha(t_0-s)} \\
&\qquad\qquad + \omega\int_{s}^{t} e^{\alpha(u-s)}\sum_{j=1}^{r}\beta_{1j}(u,s)\,du\Bigg]ds\cdot\omega M_j\|x_j^0\|e^{-\alpha(t-t_0)} \\
&\quad \leqslant (1-\omega)\|y_1^{(0)}(t)\| + \omega\sum_{j=1}^{r}b_{1j}\|y_j^{(0)}(t)\| \\
&\quad \stackrel{def}{=} \|y_1^{(1)}(t) - y_1^{(0)}(t)\|.
\end{aligned}$$

The rest of the proof is similar to the proof of Theorem 2.1; here we omit it. ■

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