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Normal forms for semilinear equations with non-dense domain with applications to age structured models

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Abstract

Normal form theory is very important and useful in simplifying the forms of equations restricted on the center manifolds in studying nonlinear dynamical problems. In this paper, using the center manifold theorem associated with the integrated semigroup theory, we develop a normal form theory for semilinear Cauchy problems in which the linear operator is not densely defined and is not a Hille–Yosida operator and present procedures to compute the Taylor expansion and normal form of the reduced system restricted on the center manifold. We then apply the main results and computation procedures to determine the direction of the Hopf bifurcation and stability of the bifurcating periodic solutions in a structured evolutionary epidemiological model of influenza A drift and an age structured population model.

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1. Introduction

1.1. Normal form theory

To determine the qualitative behavior of a nonlinear system in the neighborhood of a nonhyperbolic equilibrium point, the center manifold theorem implies that it could be reduced to the problem of determining the qualitative behavior of the nonlinear system restricted on the center manifold, which reduces the dimension of a local bifurcation problem near the nonhyperbolic equilibrium point. The normal form theory provides a way of finding a nonlinear analytic transformation of coordinates in which the nonlinear system restricted on the center manifold takes the "simplest" form, called normal form. These two methods, one reduces the dimension of the original system and the other eliminates the nonlinearity of the reduced system, are conjunctly used to study bifurcations in nonlinear dynamical systems. A normal form theorem was obtained first by Poincaré [54] and later by Siegel [56] for analytic differential equations. Simpler proofs of Poincaré's theorem and Siegel's theorem were given in Arnold [5], Meyer [46], Moser [48], and Zehnder [67]. For more results about normal form theory and its applications see, for example, the monographs by Arnold [5], Chow and Hale [8], Guckenheimer and Holmes [26], Meyer and Hall [47], Siegel and Moser [57], Chow et al. [9], Kuznetsov [34], and others.

Normal form theory has been extended to various classes of partial differential equations. In the context of autonomous partial differential equations we refer to Ashwin and Mei [6] (PDEs on the square), Eckmann et al. [18] (abstract parabolic equations), Faou et al. [20,21] (Hamiltonian PDEs), Hassard, Kazarinoff and Wan [28] (functional differential equations), Faria [22,23] (PDEs with delay), Foias et al. [25] (Navier–Stokes equation), Kokubu [33] (reaction–diffusion equations), McKean and Shatah [45] (Schrödinger equation and heat equations), Nikolenko [51] (abstract semi-linear equations), Shatah [55] (Klein–Gordon equation), Zehnder [68] (abstract parabolic equations), etc. We refer to Chow et al. [10] (and references therein) for a normal form theory in quasiperiodic partial differential equations.

In this paper, we develop a normal form theory for the following abstract Cauchy problem with non-dense domain

$$\begin{cases} \dfrac{du(t)}{dt} = Au(t) + F\big(u(t)\big), & t \geq 0, \\ u(0) = x \in \overline{D(A)}, \end{cases} \tag{1.1}$$

where $A : D(A) \subset X \to X$ is a non-densely defined linear operator on a Banach space X, and $F : \overline{D(A)} \to X$ is a k-time continuously differentiable function for some $k \geq 2$. The Cauchy problem (1.1) is said to be non-densely defined if

$$\overline{D(A)} \neq X.$$

Therefore A is not the infinitesimal generator of a strongly continuous semigroup of bounded linear operators on X, and in general, $F(x)$ does not belong to $\overline{D(A)}$. So the solution of system (1.1) is not a mild solution derived from the classical semi-linear formulation.

1.2. *Motivation – age structured models*

The main motivation comes from investigating the nonlinear dynamics of structured (age, size, space, etc.) population models described by various types of equations including partial differential equations with nonlinear and nonlocal boundary conditions (Cantrell and Cosner [7], Diekmann and Heesterbeek [14], Iannelli [29], Magal and Ruan [41], Murray [49], Perthame [53], Thieme [61], Webb [64], etc.). It is well-known that several types of differential equations, such as functional differential equations (Adimy [1], Diekmann et al. [15], Hale and Verduyn Lunel [27], Liu et al. [35]), age structured models (Magal [39], Magal and Ruan [42], Perthame [53], Thieme [58,60], and Webb [64]), parabolic partial differential equations (Chu et al. [11], Ducrot et al. [16]), and partial differential equations with delay (Ducrot et al. [17] and Wu [66]), can be formulated as non-densely defined Cauchy problems in the form of (1.1). Here we present two examples of structured models and refer to Da Prato and Sinestrari [13], Thieme [58,59], and Magal and Ruan [40,42] for more examples.

(a) A structured model of influenza A drift. Suppose that the total host population size N is a constant. Let $I(t)$ be the number of infected individuals at time t. Let $a \geq 0$ be the time since the last infection, that is, the duration of time since an individual has been susceptible. Assume that the average number of amino acid substitutions is a continuous variable. More precisely, let $k > 0$ be the average number of amino acid substitutions per unit of time (that is, the mutation rate). Then the number of substitutions after a period of time a in the susceptible class is given by ka. Let $s(t,a)$ be the density of uninfected hosts (structured with respect to a), so that $\int_{a_0}^{a_1} s(t,a)da = \int_{ka_0}^{ka_1} s(t,k^{-1}l)k^{-1}dl$ is the number of uninfected hosts that were last infected by a virus which differed by more than ka_0 and less than ka_1 amino acid substitutions from the virus strain prevailing at time t. $\nu > 0$ is the recovery rate of the infected hosts. $\gamma \in L^\infty_+(0,+\infty)$ describes how amino acid substitutions affect the probability of reinfection and satisfies $\liminf_{a\to+\infty}\gamma(ka)>0$. Define $\rho:=\tau/k$ as the threshold of sensitivity, which is the time necessary to be re-infected after one infection. This is also equivalent to assuming that it is necessary to reach a threshold value τ for the average number of amino acid substitutions before re-infection. Assume that

$$\gamma(a)=\delta\chi(a):=\begin{cases}\delta & \text{if } a\geq\tau,\\ 0 & \text{if } a\in(0,\tau),\end{cases}$$

where $\delta>0$. Note that $\int_0^{+\infty}s(t,a)(a)da+I(t)=N$, $\forall t\geq 0$. Assume without loss of generality that $k=1$ and $N=1$. Then we have a structured evolutionary epidemiological model of influenza A drift (see Pease [52], Inaba [30,31], Magal and Ruan [44])

$$\begin{cases}\dfrac{\partial s(t,a)}{\partial t}+\dfrac{\partial s(t,a)}{\partial a}=-\delta\chi(a)s(t,a)\left(1-\displaystyle\int_0^{+\infty}s(t,l)dl\right)^+, & t\geq 0,\ a\geq 0,\\ s(t,0)=\nu\left(1-\displaystyle\int_0^{+\infty}s(t,l)dl\right)^+, & \\ s(0,.)=s_0\in L^1_+(0,+\infty)\quad\text{with } \displaystyle\int_0^{+\infty}s_0(l)dl\leq 1, & \end{cases}\tag{1.2}$$

where $x^+=\max(x,0)$. Consider the Banach space

$$X=\mathbb{R}\times L^1(0,+\infty)$$

endowed with the usual product norm

$$\left\|\begin{pmatrix}\alpha\\ \varphi\end{pmatrix}\right\|=|\alpha|+\|\varphi\|_{L^1}.$$

Consider the linear operator $A:D(A)\subset X\to X$ defined by

$$A\begin{pmatrix}0\\ \varphi\end{pmatrix}=\begin{pmatrix}-\varphi(0)\\ -\varphi'\end{pmatrix}\tag{1.3}$$

with

$$D(A)=\{0\}\times W^{1,1}(0,+\infty).$$

Then

$$X_0:=\overline{D(A)}=\{0\}\times L^1(0,+\infty).$$

Set

$$X_+:=\mathbb{R}_+\times L^1_+(0,+\infty)\quad\text{and}\quad X_{0+}:=X_0\cap X_+=\{0\}\times L^1_+(0,+\infty).$$

We also consider the nonlinear operator $F : (0, +\infty) \times (0, +\infty) \times \overline{D(A)} \to X$ defined by

$$F\left(\nu, \delta, \begin{pmatrix} 0 \\ \varphi \end{pmatrix}\right) = \begin{pmatrix} \nu(1 - \int_0^{+\infty} \varphi(l)dl)^+ \\ -(1 - \int_0^{+\infty} \varphi(l)dl)^+ \delta\chi\varphi \end{pmatrix}. \tag{1.4}$$

By identifying $s(t, .)$ with $u(t) = \begin{pmatrix} 0 \\ s(t,.) \end{pmatrix}$, we can rewrite the system as the following abstract Cauchy problem

$$\frac{du(t)}{dt} = Au(t) + F\big(\nu, \delta, u(t)\big) \quad \text{for } t \geq 0 \quad \text{and} \quad u(0) = \begin{pmatrix} 0 \\ s_0 \end{pmatrix} \in \overline{D(A)}. \tag{1.5}$$

(b) An age structured population model. Let $u(t, a)$ denote the density of a population at time t with age a. Consider the following age structured model with nonlinear boundary conditions (see Magal and Ruan [43])

$$\begin{cases} \dfrac{\partial u(t, a)}{\partial t} + \dfrac{\partial u(t, a)}{\partial a} = -\mu u(t, a), \quad a \in (0, +\infty), \\ u(t, 0) = \alpha h\left(\displaystyle\int_0^{+\infty} \gamma(a)u(t, a)da\right), \\ u(0, .) = \varphi \in L^1_+\big((0, +\infty); \mathbb{R}\big), \end{cases} \tag{1.6}$$

where $\mu > 0$ is the mortality rate of the population, $\alpha\gamma(a)$ is the fertility rate at age a, $\alpha \geq 0$ is a parameter, the function $h(\cdot)$ describes some limitation for the reproduction of the population. Similarly as in (a) for Eq. (1.2), consider the same space $X = \mathbb{R} \times L^1(0, +\infty)$ with the usual product norm. Let $A : D(A) \subset X \to X$ be the linear operator on X defined by

$$A\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi(0) \\ -\varphi' - \mu\varphi \end{pmatrix} \tag{1.7}$$

with the same domain $D(A) = \{0\} \times W^{1,1}(0, +\infty)$ and $\overline{D(A)} = X_0$. Let $H : X_0 \to X$ be the map defined by

$$H\left(\begin{pmatrix} 0 \\ \varphi \end{pmatrix}\right) = \begin{pmatrix} h(\int_0^{+\infty} \gamma(a)\varphi(a)da) \\ 0_{L^1} \end{pmatrix}. \tag{1.8}$$

Then by identifying $u(t, .)$ to $v(t) = \begin{pmatrix} 0 \\ u(t,.) \end{pmatrix} \in X_0$, the system (1.6) can be reformulated as the following non-densely defined abstract Cauchy problem

$$\frac{dv(t)}{dt} = Av(t) + \alpha H\big(v(t)\big), \quad \text{for } t \geq 0, \qquad v(0) = \begin{pmatrix} 0 \\ \varphi_0 \end{pmatrix} \in \overline{D(A)}. \tag{1.9}$$

It has been shown that Hopf bifurcation can occur in age-structured models such as (1.2) and (1.6) (see Magal and Ruan [44,43] and references cited therein) and a Hopf bifurcation theory has been recently developed for general age structured models in Liu et al. [36]. But up to now, the stability of the bifurcating periodic orbits is unknown for such systems. To address this issue, a normal form theory for age structured models, and more generally for the non-densely defined abstract Cauchy problem, needs to be developed. This is a first motivation for this paper. Another motivation is coming from the analysis of Bogdanov–Takens bifurcation in the context of age structured models. By using the normal theory presented in this article Liu, Magal and Xiao [38] obtain recently a first example of Bogdanov–Takens for an age structured model.

1.3. Nonlinear dynamics of semilinear equations with non-dense domain

For the semilinear Cauchy problem (1.1), if A is a Hille–Yosida operator and is densely defined (i.e., $\overline{D(A)} = X$), given an initial datum in the phase space integrating the equation yields a corresponding trajectory through that point (the constant of variation formula), so the semilinear Cauchy problem is well-posed and has been extensively studied, see Engel and Nagel [19]. When A is non-densely defined (i.e., $\overline{D(A)} \neq X$), the constant of variation formula may be not well-defined and one may be able to integrate the equation twice to recover the well-posedness (this is how integrated semigroups are introduced). When A is a Hille–Yosida operator but its domain is non-densely defined, Da Prato and Sinestrari [13] investigated the existence of several types of solutions for non-densely defined Cauchy problems. Thieme [58] studied the semilinear Cauchy problem with a Lipschitz perturbation of the closed linear operator A by using integrated semigroup theory.

We have been interested in studying the nonlinear dynamics, such as stability, bifurcations, periodic solutions, and invariant manifolds, in the semilinear Cauchy problem (1.1) when A is non-densely defined and is not Hille–Yosida, and have made some progress on this subject. Magal and Ruan [40] presented some techniques and results for integrated semigroups, obtained necessary and sufficient conditions for the existence of mild solutions for non-densely defined non-homogeneous Cauchy problems, and applied the results to study age-structured models. Magal and Ruan [42] extended the results of Thieme [58] to the case when the operator A is not Hille–Yosida. Namely, we studied the positivity of solutions to the semilinear problem (1.1), the Lipschitz perturbation of the problem, differentiability of the solutions with respect to the state variable, time differentiability of the solutions, and the stability of equilibria. Magal and Ruan [43] established a center manifold theory for the semilinear Cauchy problem (1.1) with non-dense domain. Center-unstable manifolds for non-densely defined semilinear Cauchy problems were studied in Liu et al. [37]. Employing the center manifold theory in [43], Liu et al. [36] established a Hopf bifurcation theorem for abstract non-densely defined Cauchy problems.

In this paper we use the integrated semigroup theory, the semilinear Cauchy problem theory, and the center manifold theory (see [40,42,43]) to establish a normal form theory for the non-densely defined Cauchy problem (1.1) when $\overline{D(A)}$ is not dense in X and A is not a Hille–Yosida operator. The goal is to provide a method for computing the required lower order terms of the Taylor expansion and the normal form of the reduced equations. The main difficulty comes from the fact that the center manifold is defined by using implicit formulae in general. Here we will show that it is possible to find some appropriate changes of variables (in Banach spaces) to compute the Taylor expansion at any order and the normal form of the reduced system. The main results and computation procedures will be used to Hopf bifurcation in the structured evolutionary epidemiological model of influenza A drift (1.2) and the age structured population model (2.2).

1.4. An outline

The plan of the paper is as follows. In Section 2, we will present some preliminary results on integrated semigroups and the center manifold theorem for the non-densely defined Cauchy problem (1.1) and give an outline on the computation of the reduced system restricted on the center manifold. In Section 3, we introduce some notions, justify the change of variables that establishes the equivalence between the original and reduced systems and present the normal form theory of the nonresonance type. In Section 4, we provide computational procedures for the Taylor expansion and normal form of the reduced system on the center manifold. Section 5 is devoted to the application of the normal theory to the two structured models (1.2) and (1.6); namely, we calculate the Taylor expansion of the reduced system of the structured evolutionary epidemiological model (1.2) of influenza A drift on the center manifold and present the normal form of the age structured population model (1.6) restricted on the center manifold, respectively, from which we are able to study stability and direction of the Hopf bifurcation in these two structured models.

2. Preliminaries and the sketchy computation procedure

2.1. Semiflows generated by nondensely defined Cauchy problems

A mild solution of Eq. (1.1) (or integrated solution) is a solution of the integral equation

$$u(t) = x + A\int_0^t u(s)ds + \int_0^t F\big(u(s)\big)ds \quad \text{for each } t \geq 0,$$

wherein

$$\int_0^t u(s)ds \in D(A) \quad \text{for each } t \geq 0.$$

This last inclusion implies in particular that $u(t) \in \overline{D(A)}$ for each $t \geq 0$.

From hereon, set

$$X_0 := \overline{D(A)}$$

and consider A_0 as the part of A in X_0. That is, A_0 is the linear operator on X_0 defined by

$$A_0 = A \quad \text{on } D(A_0) := \big\{x \in D(A) : Ax \in X_0\big\}.$$

Throughout this article, we will make the following assumption:

Assumption 2.1. We assume that:

(i) The resolvent set $\rho(A)$ of A is non-empty;
(ii) A_0 is the infinitesimal generator of a strongly continuous semigroup $\{T_{A_0}(t)\}_{t\geq 0}$ of bounded linear operators on X_0.

Since $\rho(A)$ is non-empty, we have $\rho(A) = \rho(A_0)$ and can define the integrated semigroup $\{S_A(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ generated by A as

$$S_A(t) := (\lambda I - A_0) \int_0^t T_{A_0}(s)ds(\lambda I - A)^{-1}$$

whenever $\lambda \in \rho(A)$. Then it is equivalent to considering a mild solution or a solution of

$$u(t) = T_{A_0}(t)x + \big(S_A \diamond F(u)\big)(t) \quad \text{for } t \geq 0,$$

where

$$(S_A \diamond f)(t) := \frac{d}{dt}(S_A * f)(t),$$

whenever $t \to (S_A * f)(t) := \int_0^t S_A(t-s)f(s)ds$ is differentiable.

We refer to Arendt [2,3], Arendt et al. [4], Da Prato and Sinestrari [13], Kellermann and Hieber [32], Magal and Ruan [40,43], Neubrander [50], and Thieme [59,60] for more results and references on integrated semigroups. When A is a Hille–Yosida operator (see Kellermann and Hieber [32]), the convolution $(S_A * f)(t)$ is differentiable with respect to the time variable t as long as $f \in L^1((0,\tau), X)$ (with $\tau < +\infty$), and one has the following estimation

$$\big\|(S_A \diamond f)(t)\big\| \leq M \int_0^t e^{\omega(t-s)} \big\|f(s)\big\|ds, \quad \forall t \geq 0,$$

for some constant $M \geq 1$, and $\omega \in \mathbb{R}$.

For the age structured model (1.2), the linear operator A is Hille–Yosida if and only if $p = 1$. When $p > 1$, the time differentiability of $(S_A * f)(t)$ becomes more difficult. Similar difficulties arise in the context of parabolic equations with nonhomogeneous boundary conditions (see Ducrot et al. [16]). Therefore, here we will consider the most general case in which A is not a Hille–Yosida operator. In this case, the time differentiability of $(S_A * f)(t)$ becomes an issue. In practice, this question is related to the existence of solutions for PDE problems with nonhomogeneous boundary conditions which are only $L^{\widehat{p}}$ integrable in time. This question has been studied recently in Magal and Ruan [40] and Thieme [60] for the general case and in [16] for the almost sectorial case.

Motivated by the above discussions, in this paper we will make the following assumption.

Assumption 2.2. The map $t \to (S_A * f)(t)$ is differentiable whenever $t \to f(t)$ is continuous, and there exists a map $\delta : [0,+\infty) \to [0,+\infty)$ such that

$$\big\|(S_A \diamond f)(t)\big\| \leq \delta(t) \sup_{s\in[0,t]} \big\|f(s)\big\|, \quad \forall t \geq 0,$$

where

$$\delta(t) \to 0 \quad \text{as } t \searrow 0.$$

Now we need to consider the linear bounded perturbation of A and have the following perturbation theorem which was proved in Magal and Ruan [40, Theorem 3.1].

Theorem 2.3. *Let Assumptions 2.1 and 2.2 be satisfied. Let $L \in \mathcal{L}(X_0, X)$. Then $A + L : D(A) \subset X \to X$ satisfies Assumptions 2.1 and 2.2.*

Recall that the equilibrium solutions of system (1.1) must satisfy

$$A\bar{u} + F(\bar{u}) = 0 \quad \text{with } \bar{u} \in D(A).$$

So, by using the change of variables $v(t) = u(t) - \bar{u}$, we obtain

$$v(t) = x + A \int_0^t v(s)ds + \int_0^t G\big(v(t)\big)ds$$

with $G(x) = F(x + \bar{u}) - F(\bar{u})$. Therefore, without loss of generality, we can assume that $\bar{u} = 0$. Furthermore, due the perturbation theorem (Theorem 2.3), we can replace A by $A + DG(0)$ and G by $G - DG(0)$, and assume that $F : X_0 \to X$ is C^k-smooth with $k \geq 1$ in some neighborhood of 0 and satisfies

$$F(0) = 0 \quad \text{and} \quad DF(0) = 0.$$

Now, $\bar{u} = 0$ is an equilibrium and the linearized equation around 0 is

$$\frac{du}{dt} = A_0 u(t) \quad \text{for } t \geq 0, \qquad u(0) = x \in X_0.$$

In this setting, one can construct a semi-linear theory, we refer to [42] for more results on this topic. In particular, when F is Lipschitz on bounded sets of X_0, the Cauchy problem (1.1) generates a unique maximal nonlinear semiflow $\{U(t)\}_{t\geq 0}$ generated on the subspace $X_0 := \overline{D(A)}$.

2.2. Spectral decomposition of the state space

By analogy to ordinary differential equations, we assume that X_0 has the following spectral decomposition

$$X_0 := X_{0s} \oplus X_{0c} \oplus X_{0u}$$

in terms of the spectrum of A_0. Namely, we consider

$$\begin{aligned}
\sigma_s(A_0) &= \big\{\lambda \in \sigma(A_0) : \mathrm{Re}(\lambda) < 0\big\}, \\
\sigma_c(A_0) &= \big\{\lambda \in \sigma(A_0) : \mathrm{Re}(\lambda) = 0\big\}, \\
\sigma_u(A_0) &= \big\{\lambda \in \sigma(A_0) : \mathrm{Re}(\lambda) > 0\big\},
\end{aligned}$$

and assume that for each $k = s, c, u$,

$$(\lambda I - A_0)^{-1} X_{0k} \subset X_{0k}$$

for each λ in $\rho(A_0)$, the resolvent set of A_0, and $\sigma_k(A_0)$ is equal to the spectrum of $A_{0k} : D(A_0) \cap X_{0k} \to X_{0k}$, the part of A_0 in X_{0k}. It follows that

$$T_{A_0}(t) X_{0k} \subset X_{0k}, \quad \forall t \geq 0.$$

Here in order to obtain a smooth center manifold, we need to assume that

$$\dim(X_{0s}) = +\infty, \qquad \dim(X_{0c}) < +\infty, \quad \text{and} \quad \dim(X_{0u}) < +\infty.$$

Actually since the proof for the smoothness of the center manifold uses some smooth truncation function on X_{0c}, this assumption is needed (in general) to derive a smooth local center manifold theory. Since X_{0s} is an infinite dimensional space, it is well known that one needs an extra condition to derive the growth rate of $T_{A_{0s}}(t) = T_{A_0}(t)|_{X_{0s}}$. Namely we need to assume that

$$\omega_0(A_{0s}) := \lim_{t \to +\infty} \frac{\ln(\|T_{A_{0s}}(t)\|_{\mathcal{L}(X_{0s})})}{t} < 0.$$

Since X_{0c} and X_{0u} are finite dimensional spaces, this condition is also equivalent to the fact that the essential growth rate of $\{T_{A_0}(t)\}_{t \geq 0}$ is negative, i.e.,

$$\omega_{\text{ess}}(A_0) := \lim_{t \to +\infty} \frac{\ln(\|T_{A_0}(t)\|_{\text{ess}})}{t} < 0.$$

See Webb [64,65] and Engel and Nagel [19] for more discussions on this topic.

For clarity, we summarize the above assumptions on A_0 in the following statement.

Assumption 2.4. There exist two bounded linear projectors with finite rank, $\Pi_{0c} \in \mathcal{L}(X_0) \setminus \{0\}$ and $\Pi_{0u} \in \mathcal{L}(X_0)$, such that

$$\Pi_{0c} \Pi_{0u} = \Pi_{0u} \Pi_{0c} = 0$$

and

$$\Pi_{0k} T_{A_0}(t) = T_{A_0}(t) \Pi_{0k}, \quad \forall t \geq 0,\ \forall k = \{c, u\}.$$

In addition,

(a) If $\Pi_{0u} \neq 0$, then $\omega_0(-A_0|_{\Pi_{0u}(X_0)}) < 0$.
(b) $\sigma(A_0|_{\Pi_{0c}(X_0)}) \subset i\mathbb{R}$.
(c) If $\Pi_{0s} := I - (\Pi_{0c} + \Pi_{0u}) \neq 0$, then $\omega_0(A_0|_{\Pi_{0s}(X_0)}) < 0$.

In order to obtain an ordinary differential equation for the reduced system, we need the following theorem which was proved in Magal and Ruan [43, Proposition 3.5].

Theorem 2.5. *Let Assumption 2.1 be satisfied. Let $\Pi_0 : X_0 \to X_0$ be a bounded linear operator of projection satisfying the following properties*

$$\Pi_0(\lambda I - A_0)^{-1} = (\lambda I - A_0)^{-1}\Pi_0, \quad \forall\lambda > \omega,$$

and

$$\Pi_0(X_0) \subset D(A_0) \quad \textit{and} \quad A_0|_{\Pi_0(X_0)} \textit{ is bounded.}$$

Then there exists a unique bounded linear operator of projection Π on X satisfying the following properties:

(i) $\Pi|_{X_0} = \Pi_0$.
(ii) $\Pi(X) \subset X_0$.
(iii) $\Pi(\lambda I - A)^{-1} = (\lambda I - A)^{-1}\Pi$, $\forall\lambda > \omega$.

Since $\dim(X_{0c}) < +\infty$ and $\dim(X_{0u}) < +\infty$, the linear operators A_{0c} and A_{0u} are bounded. So we can apply the above theorem to obtain Π_c (respectively Π_u), a unique extension of the projectors on Π_{0c} (respectively Π_{0u}). Define

$$\Pi_s := I - (\Pi_c + \Pi_u).$$

Then we obtain a decomposition of the larger state space

$$X = X_s \oplus X_c \oplus X_u$$

with

$$X_c = X_{0c}, \qquad X_u = X_{0u}, \qquad X_{0s} \subsetneq X_s,$$

and such that

$$(\lambda I - A)^{-1} X_k \subset X_k, \quad \forall\lambda \in \rho(A).$$

Moreover, for $k = s, c, u$, we have $\sigma(A_k) = \sigma_k(A_0)$, wherein A_k is the part of A in X_k.

Now, for $k = c, u$, we can project (1.1) on X_{0k}, and $u_k(t) := \Pi_k u(t)$ satisfies an ordinary differential equation

$$\frac{du_k(t)}{dt} = A_k u_k(0) + \Pi_k F\big(u(t)\big)$$

while when we project (1.1) on X_{0s}, $u_s(t) = \Pi_s u(t)$ is a solution of a new non-densely defined Cauchy problem on X_s:

$$\frac{du_s(t)}{dt} = A_s u_s(0) + \Pi_s F\big(u(t)\big).$$

At this level, in order to construct a comprehensive center manifold theory, one needs an evaluation of $\|\Pi_s(S_A \diamond F(u))(t)\| = \|(S_{A_s} \diamond \Pi_s F(u))(t)\|$ expressed as a function of the growth rate of $T_{A_{0s}}(t) = T_{A_0}(t)|_{X_{0s}}$. So the following result which was proved in Magal and Ruan [42, Proposition 2.14] plays a particularly important role in this context.

Proposition 2.6. *Let Assumptions 2.1 and 2.2 be satisfied. Let $\omega_A \in \mathbb{R}$ and $M_A \geq 1$ be such that*

$$\left\|T_A(t)\right\| \leq M_A e^{\omega_A t}, \quad \forall t \geq 0.$$

Let $\varepsilon > 0$ be fixed. Then for each $\tau_\varepsilon > 0$ satisfying $M_A \delta(\tau_\varepsilon) \leq \varepsilon$, we have

$$\left\|(S_A \diamond f)(t)\right\| \leq C(\varepsilon, \gamma) \sup_{s \in [0,t]} e^{\gamma(t-s)} \left\|f(s)\right\|, \quad \forall t \geq 0,$$

whenever $\gamma \in (\omega_A, +\infty)$, $f \in C(\mathbb{R}_+, X)$, and with

$$C(\varepsilon, \gamma) := \frac{2\varepsilon \max(1, e^{-\gamma \tau_\varepsilon})}{1 - e^{(\omega_A - \gamma)\tau_\varepsilon}}.$$

2.3. Center manifold theorem

Set

$$X_h := X_s \oplus X_u.$$

Let $\Pi_c \in \mathcal{L}(X)$ be the projector satisfying

$$\Pi_c(X) = X_c \quad \text{and} \quad (I - \Pi_c)(X) = X_h.$$

Define

$$\Pi_h := I - \Pi_c.$$

The following result is based on the approach developed by Vanderbauwhede [62] and Vanderbauwhede and Iooss [63, Theorem 3]. In the context of integrated semigroups the following result was proved in Magal and Ruan [43, Theorem 4.21].

Theorem 2.7. *Let Assumptions 2.1, 2.2 and 2.4 be satisfied. Let $r > 0$ and let $F : B_{X_0}(0, r) \to X$ be a map. Assume that there exists an integer $k \geq 1$ such that F is k-time continuously differentiable in $B_{X_0}(0, r)$ with $F(0) = 0$ and $DF(0) = 0$. Assume that $F : X_0 \to X$ is C^k-smooth with $k \geq 1$ in some neighborhood of 0. Then in the setting described above, there exists a C^k-smooth function $\Psi : X_{0c} \to X_{0h}$ which satisfies $\Psi(0) = 0$, $D\Psi(0) = 0$, and*

$$M = \left\{x_c + \Psi(x_c) : x_c \in X\right\} \quad (\textbf{\textit{center manifold}})$$

is a locally invariant center manifold for the semiflow generated by (1.1). This means that there exists $\Omega \subset X_0$, a bounded neighborhood of 0 in X_0, such that

$$\textit{if } u(0) \in M \cap \Omega \textit{ and } u(t) \in \Omega, \ \forall t \in [0, \tau], \quad \textit{then } u(t) \in M.$$

Moreover, as Π_c is defined on X, we can project (1.1) on X_c and obtain an ordinary differential equation

$$\frac{du_c(t)}{dt} = A_{0c}u_c(t) + \Pi_c F\left[u_c(t) + \Psi\left(u_c(t)\right)\right] \quad (\textbf{\textit{reduced system}}). \tag{2.1}$$

Furthermore, if $t \to u_c(t)$ is a solution on an interval I of the reduced system (2.1) and

$$u_c(t) + \Psi\left(u_c(t)\right) \in \Omega, \quad \forall t \in I,$$

then $t \to u(t) := u_c(t) + \Psi(u_c(t))$ is a mild solution of system (1.1); namely,

$$u(t) = u(s) + A \int_s^t u(l)dl + \int_s^t F\left(u(s)\right)ds, \quad \forall t, s \in I \textit{ with } t \geq s.$$

Conversely, if $u : \mathbb{R} \to X_0$ is a complete orbit of (1.1) and if

$$u(t) \in \Omega, \quad \forall t \in \mathbb{R},$$

then

$$u(t) \in M, \quad \forall t \in \mathbb{R},$$

and $t \to \Pi_c u(t)$ is a solution of the reduced system (2.1).

2.4. A sketchy procedure of computing the reduced system

Our paper is devoted to the computation of the Taylor expansion and normal form of the reduced system (2.1). First, one needs to realize that the center manifold Ψ is known only through an implicit fixed point procedure. Of course, when

$$D^l\Psi(0) = 0 \quad \text{for each } l = 1, \ldots, k,$$

the Taylor expansion of the reduced system (2.1) is simply given by

$$\frac{du_c(t)}{dt} = A_{0c}u_c(t) + \sum_{l=1}^{k} \frac{1}{l!}\Pi_c D^l F(0)\left(u_c(t), \ldots, u_c(t)\right) + h.o.t.$$

In general, the only information available to compute the Taylor expansion and normal form of the reduced system is the following result (see Magal and Ruan [43, Lemma 4.20]).

Lemma 2.8. *Let Assumptions 2.1, 2.2 and 2.4 be satisfied. Let $r>0$ and let $F: B_{X_0}(0,r) \to X$ be a k-time continuously differentiable map (with $k \geq 1$) in $B_{X_0}(0,r)$ with*

$$F(0)=0 \quad \text{and} \quad DF(0)=0$$

and

$$\Pi_h D^j F(0)|_{X_c \times X_c \times \cdots \times X_c} = 0 \quad \text{for each } j=2,\ldots,k.$$

Then

$$D^j \Psi(0) = 0 \quad \text{for each } j=1,\ldots,k.$$

Description of the method at the order 3. Assume first that

$$\Pi_h D^2 F(0)|_{X_c \times X_c} \neq 0.$$

Let $G \in V^2(X_c, D(A) \cap X_h)$ be a homogeneous polynomial of degree 2 (see Section 3 for a precise definition). Consider the following global change of variable

$$v := u - G(\Pi_c u) \quad \Leftrightarrow \quad \begin{cases} \Pi_c v = \Pi_c u, \\ \Pi_h v = \Pi_h u - G(\Pi_c u) \end{cases} \quad \Leftrightarrow \quad u = v + G(\Pi_c v). \tag{2.2}$$

Then formally (since the solution is not time differentiable), we obtain

$$\begin{aligned} v' &= u' - DG(\Pi_c u)\big(\Pi_c u'\big) \\ &= Au + F(u) - DG(\Pi_c u)\big(\Pi_c\big[Au + F(u)\big]\big). \end{aligned}$$

Thus

$$\begin{aligned} v' = {} & A\big[v + G(\Pi_c v)\big] + F\big(v + G(\Pi_c v)\big) \\ & - DG(\Pi_c v)\big(\Pi_c\big[Av + F\big(v + G(\Pi_c v)\big)\big]\big). \end{aligned}$$

We naturally introduce the Lie bracket

$$[A,G](x_c) = DG(x_c)(A_c x_c) - AG(x_c), \quad \forall x_c \in X_c.$$

Then we obtain a new non-densely defined Cauchy problem

$$\frac{dv(t)}{dt} = Av(t) + H\big(v(t)\big) \quad \text{for } t \geq 0 \quad \text{and} \quad v(0) = x \in \overline{D(A)}, \tag{2.3}$$

where

$$H(v) = F\big[v + G(\Pi_c v)\big] - [A,G](\Pi_c v) - DG(\Pi_c v)\big(\Pi_c F\big(v + G(\Pi_c v)\big)\big).$$

We can rewrite H as

$$H(v) = F(v) - [A, G](\Pi_c v) + \big[F\big(v + G(\Pi_c v)\big) - F(v)\big]$$
$$- DG(\Pi_c v)\big(\Pi_c F\big(v + G(\Pi_c v)\big)\big),$$

and since $DF(0) = 0$, we obtain

$$\frac{1}{2!}\Pi_h D^2 H(0)(x_c, x_c) = \frac{1}{2!}\Pi_h D^2 F(0)(x_c, x_c) - [A, G](x_c).$$

Therefore, in order to cancel out the second order term we need to solve

$$[A, G](x_c) = \frac{1}{2!}\Pi_h D^2 F(0)(x_c, x_c) \quad \text{with } G \in V^2\big(X_c, D(A) \cap X_h\big). \tag{2.4}$$

By applying Theorem 2.7 and Lemma 2.8 to system (2.3), we deduce that the reduced system of (2.3) has the following form

$$\frac{dv_c}{dt} = A_c v_c + \Pi_c F\big[v_c + G(v_c)\big] + R(v_c),$$

where

$$R(v_c) = \Pi_c F\big[v_c + G(v_c) + \Psi(v_c)\big] - \Pi_c F\big[v_c(t) + G\big(v_c(t)\big)\big]$$

and $\Psi : X_{0c} \to X_{0h}$ is a local center manifold of the new system (2.3) satisfying

$$\Psi(0) = 0, \qquad D\Psi(0) = 0, \quad \text{and} \quad D^2\Psi(0) = 0.$$

Now assume that the F is C^4-smooth locally around 0 (so is Ψ). Then we see that $R(v_c)$ is of order 4 and the Taylor expansion of the reduced system (2.1) at the order 3 is given by

$$\begin{aligned}\frac{dv_c}{dt} &= A_c v_c + \frac{1}{2!}\Pi_c D^2 F(0)[v_c, v_c] \\ &\quad + \frac{1}{2!}\big\{\Pi_c D^2 F(0)\big[G(v_c), v_c\big] + \Pi_c D^2 F(0)\big[v_c, G(v_c)\big]\big\} \\ &\quad + \frac{1}{3!}\Pi_c D^3 F(0)[v_c, v_c, v_c] + h.o.t.\end{aligned} \tag{2.5}$$

Therefore, in order to compute the Taylor expansion of the reduced system at the order 3, we (only) need to compute G at the order 2. Then we can apply the normal form theory to the reduced ODE system.

An alternative approach, to compute both the normal form and the reduced system, would be to use the following change of variables

$$u := v + G(\Pi_c v)$$

wherein

$$G \in V^2\big(X_c, D(A)\big).$$

In this case

$$u := v + G(\Pi_c v) \quad \Leftrightarrow \quad \begin{cases} \Pi_c u = \Pi_c v + \Pi_c G(\Pi_c v), \\ \Pi_h u = \Pi_h v + \Pi_h G(\Pi_c v). \end{cases} \tag{2.6}$$

Then the map $\xi_c(x_c) = x_c + \Pi_c G(x_c)$ from X_c into itself is only locally invertible around 0. This type of change of variables leads to an infinite dimensional normal form theory for non-densely defined Cauchy problem.

3. Normal form theory – nonresonant type results

Let $m \geq 1$. Let Y be a closed subspace of X. Let $\mathcal{L}_s(X_0^m, Y)$ be the space of bounded m-linear symmetric maps from $X_0^m = X_0 \times X_0 \times \cdots \times X_0$ into Y and $\mathcal{L}_s(X_c^m, D(A))$ be the space of bounded m-linear symmetric maps from $X_c^m = X_c \times X_c \times \cdots \times X_c$ into $D(A)$. That is, for each $L \in \mathcal{L}_s(X_c^m, D(A))$,

$$L(x_1, \ldots, x_m) \in D(A), \quad \forall (x_1, \ldots, x_m) \in X_c^m,$$

and the maps $(x_1, .., x_m) \to L(x_1, \ldots, x_m)$ and $(x_1, .., x_m) \to AL(x_1, \ldots, x_m)$ are m-linear bounded from X_c^m into X. Let $\mathcal{L}_s(X_c^m, X_h \cap D(A))$ be the space of bounded m-linear symmetric maps from $X_c^m = X_c \times X_c \times \cdots \times X_c$ into $D(A_h) = X_h \cap D(A)$ which belongs to $\mathcal{L}_s(X_c^m, D(A))$.

Let $k = \dim(X_c)$ and Y be a subspace of X. We define $V^m(X_c, Y)$ the linear space of homogeneous polynomials of degree m. More precisely, given a basis $\{b_j\}_{j=1,\ldots,k}$ of X_c, $V^m(X_c, Y)$ is the space of finite linear combinations of maps of the form

$$x_c = \sum_{j=1}^{k} x_j b_j \in X_c \to x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k} V$$

with

$$n_1 + n_2 + \cdots + n_k = m \quad \text{and} \quad V \in Y.$$

Define a map $\mathcal{G}$: $\mathcal{L}_s(X_c^m, Y) \to V^m(X_c, Y)$ by

$$\mathcal{G}(L)(x_c) = L(x_c, \ldots, x_c), \quad \forall L \in \mathcal{L}_s\big(X_c^m, Y\big).$$

Let $G \in V^m(X_c, Y)$, we have $G(x_c) = \frac{1}{m!} D^m G(0)(x_c, \ldots, x_c)$. So

$$\mathcal{G}^{-1}(G) = \frac{1}{m!} D^m G(0).$$

In other words, we have

$$L = \frac{1}{m!} D^m G(0) \quad \Leftrightarrow \quad G(x_c) = L(x_c, \ldots, x_c), \quad \forall x_c \in X_c.$$

It follows that $\mathcal{G}$ is a bijection from $\mathcal{L}_s(X_c^m, Y)$ into $V^m(X_c, Y)$. So we can also define $V^m(X_c, D(A))$ as

$$V^m\big(X_c, D(A)\big) := \mathcal{G}\big(\mathcal{L}_s\big(X_c^m, D(A)\big)\big).$$

In order to use the usual formalism in the context of normal form theory, we now define the Lie bracket (Guckenheimer and Holmes [26, page 141]). Recall that

$$X_c = X_{0c} \subset D(A_0) \subset D(A),$$

so the following definition makes sense.

Definition 3.1. Let Assumptions 2.1, 2.2 and 2.4 be satisfied. Then for each $G \in V^m(X_c, D(A))$, we define the *Lie bracket*

$$[A, G](x_c) := DG(x_c)(Ax_c) - AG(x_c), \quad \forall x_c \in X_c. \tag{3.1}$$

Recalling that $A_c \in \mathcal{L}(X_c)$ is the part of A in X_c, we obtain

$$[A, G](x_c) = DG(x_c)(A_c x_c) - AG(x_c), \quad \forall x_c \in X_c.$$

Set $L := \frac{1}{m!} D^m G(0) \in \mathcal{L}_s(X_c^m, D(A) \cap X_h)$. We also have

$$DG(x_c)(y) = mL(y, x_c, \ldots, x_c), \qquad DG(x_c)A_c x_c = mL(A_c x_c, x_c, \ldots, x_c),$$

and

$$[A, G](x_c) = \frac{d}{dt}\big[L\big(e^{A_c t}x_c, \ldots, e^{A_c t}x_c\big)\big](0) - AL(x_c, \ldots, x_c). \tag{3.2}$$

We consider two cases when G belongs to different subspaces, namely, $G \in V^m(X_c, D(A) \cap X_h)$ and $G \in V^m(X_c, D(A))$, respectively.

3.1. $G \in V^m(X_c, D(A) \cap X_h)$

We consider the change of variables (2.2), i.e.,

$$u = v + G(\Pi_c v).$$

Then

$$G(x_c) := L(x_c, x_c, \ldots, x_c), \quad \forall x_c \in X_c.$$

The map $x_c \to AG(x_c)$ is differentiable and

$$D(AG)(x_c)(y) = ADG(x_c)(y) = mAL(y, x_c, \ldots, x_c).$$

Define a map $\xi : X \to X$ by

$$\xi(x) := x + G(\Pi_c x), \quad \forall x \in X.$$

Since the range of G is included in X_h, we obtain the following equivalence

$$y = \xi(x) \quad \Leftrightarrow \quad x = \xi^{-1}(y),$$

where

$$\xi^{-1}(y) := y - G(\Pi_c y), \quad \forall y \in X,$$

and

$$\Pi_c \xi^{-1}(x) = \Pi_c x, \quad \forall x \in X.$$

Finally, since $G(x) \in D(A)$, we have

$$\xi\big(\overline{D(A)}\big) \subset \overline{D(A)} \quad \text{and} \quad \xi^{-1}\big(\overline{D(A)}\big) \subset \overline{D(A)}.$$

The following result justifies the change of variables (2.2).

Lemma 3.2. *Let Assumptions 2.1, 2.2 and 2.4 be satisfied. Let $L \in \mathcal{L}_s(X_c^m, X_h \cap D(A))$. Assume that $u \in C([0,\tau], X)$ is an integrated solution of the Cauchy problem*

$$\frac{du(t)}{dt} = Au(t) + F\big(u(t)\big), \quad t \in [0,\tau], \qquad u(0) = x \in \overline{D(A)}. \tag{3.3}$$

Then $v(t) = \xi^{-1}(u(t))$ is an integrated solution of the system

$$\frac{dv(t)}{dt} = Av(t) + H\big(v(t)\big), \quad t \in [0,\tau], \qquad v(0) = \xi^{-1}(x) \in \overline{D(A)}, \tag{3.4}$$

where $H : \overline{D(A)} \to X$ is the map defined by

$$H\big(\xi(x)\big) = F\big(\xi(x)\big) - [A, G](\Pi_c x) - DG(\Pi_c x)\big[\Pi_c F\big(\xi(x)\big)\big].$$

Conversely, if $v \in C([0,\tau], X)$ is an integrated solution of (3.4), then $u(t) = \xi(v(t))$ is an integrated solution of (3.3).

Proof. Assume that $u \in C([0,\tau], X)$ is an integrated solution of the system (3.3), that is,

$$\int_0^t u(l)dl \in D(A), \quad \forall t \in [0,\tau],$$

and

$$u(t) = x + A\int_0^t u(l)dl + \int_0^t F\big(u(l)\big)dl, \quad \forall t \in [0,\tau].$$

Set

$$v(t) = \xi^{-1}\big(u(t)\big), \quad \forall t \in [0,\tau].$$

We have

$$\begin{aligned}
A\int_0^t v(l)dl &= A\int_0^t u(l)dl - \int_0^t AG\big(\Pi_c u(l)\big)dl \\
&= u(t) - x - \int_0^t F\big(u(l)\big)dl - \int_0^t AG\big(\Pi_c u(l)\big)dl \\
&= u(t) - G\big(\Pi_c u(t)\big) - \big(x - G(\Pi_c x)\big) \\
&\quad + \big(G\big(\Pi_c u(t)\big) - G(\Pi_c x)\big) \\
&\quad - \int_0^t F\big(u(l)\big)dl - \int_0^t AG\big(\Pi_c u(l)\big)dl \\
&= v(t) - \xi^{-1}(x) + \big(G\big(\Pi_c u(t)\big) - G(\Pi_c x)\big) \\
&\quad - \int_0^t F\big(u(l)\big)dl - \int_0^t AG\big(\Pi_c u(l)\big)dl.
\end{aligned}$$

Since $\dim(X_c) < +\infty$, $t \to \Pi_c u(t)$ satisfies the following ordinary differential equations

$$\frac{d\Pi_c u(t)}{dt} = A_{0c}\Pi_c u(t) + \Pi_c F\big(u(t)\big).$$

By integrating both sides of the above ordinary differential equations, we obtain

$$\begin{aligned}
G\big(\Pi_c u(t)\big) - G(\Pi_c x) &= \int_0^t DG\big(\Pi_c u(l)\big)\left(\frac{d\Pi_c u(l)}{dl}\right)dl \\
&= \int_0^t DG\big(\Pi_c u(l)\big)\big(A_{0c}\Pi_c u(l)\big) + DG\big(\Pi_c u(l)\big)\big(\Pi_c F\big(u(l)\big)\big)dl.
\end{aligned}$$

It follows that

$$A\int_0^t v(l)dl = v(t) - \xi(x)$$
$$+ \int_0^t DG\big(\Pi_c u(l)\big)\big[A_{0c}\Pi_c u(l) + \Pi_c F\big(u(l)\big)\big]dl$$
$$- \int_0^t F\big(u(l)\big)dl - \int_0^t AG\big(\Pi_c u(l)\big)dl.$$

Thus

$$v(t) = \xi(x) + A\int_0^t v(l)dl + \int_0^t H\big(v(l)\big)dl,$$

in which

$$H\big(\xi(x)\big) = F\big(\xi(x)\big) + AG\big(\Pi_c\xi(x)\big)$$
$$- DG\big(\Pi_c\xi(x)\big)\big[A_c\Pi_c\xi(x) + \Pi_c F\big(\xi(x)\big)\big].$$

Since $\Pi_c\xi = \Pi_c$, the first implication follows. The converse follows from the first implication by replacing F by H and ξ by ξ^{-1}. $\square$

Set for each $\eta > 0$,

$$BC^\eta(\mathbb{R}, X) := \Big\{f \in C(\mathbb{R}, X) : \sup_{t\in\mathbb{R}} e^{-\eta|t|}\big\|f(t)\big\| < +\infty\Big\}.$$

The following lemma was proved in Magal and Ruan [43, Lemma 4.6].

Lemma 3.3. *Let Assumptions 2.1, 2.2 and 2.4 be satisfied. For each $\eta \in (0, -\omega_0(A_{0s}))$, the following properties are satisfied:*

(i) *For each $f \in BC^\eta(\mathbb{R}, X)$ and each $t \in \mathbb{R}$,*

$$K_s(f)(t) := \lim_{\tau\to-\infty} \Pi_s\big(S_A \diamond f(\tau + .)\big)(t - \tau) \quad \textit{exists}.$$

(ii) *K_s is a bounded linear operator from $BC^\eta(\mathbb{R}, X)$ into itself.*

(iii) *For each $t, s \in \mathbb{R}$ with $t \ge s$,*

$$K_s(f)(t) = T_{A_{0s}}(t - s)K_s(f)(s) + \Pi_s\big(S_A \diamond f(s + .)\big)(t - s).$$

The following lemma was proved in Magal and Ruan [43, Lemma 4.7].

Lemma 3.4. *Let Assumptions 2.1, 2.2 and 2.4 be satisfied. Let $\eta \in (0, \inf_{\lambda \in \sigma(A_{0u})} \operatorname{Re}(\lambda))$ be fixed. Then we have the following:*

(i) *For each $f \in BC^\eta(\mathbb{R}, X)$ and each $t \in \mathbb{R}$,*

$$K_u(f)(t) := -\int_0^{+\infty} e^{-A_{0u}l} \Pi_u f(l+t)dl = -\int_t^{+\infty} e^{-A_{0u}(l-t)} \Pi_u f(l)dl$$

exists.

(ii) *K_u is a bounded linear operator from $BC^\eta(\mathbb{R}, X)$ into itself.*

(iii) *For each $f \in BC^\eta(\mathbb{R}, X)$ and each $t, s \in \mathbb{R}$ with $t \geq s$,*

$$K_u(f)(t) = e^{A_{0u}(t-s)} K_u(f)(s) + \Pi_{0u}\big(S_A \diamond f(s+.)\big)(t-s).$$

We now prove the following lemma.

Lemma 3.5. *Let Assumptions 2.1, 2.2 and 2.4 be satisfied. If*

$$f(t) = t^k e^{\lambda t} x$$

for some $k \in \mathbb{N}$, $\lambda \in i\mathbb{R}$, and $x \in X$, then

$$(K_u + K_s)(\Pi_h f)(0) = (-1)^k k!(\lambda I - A_h)^{-(k+1)} \Pi_h x \in D(A_h) \subset D(A).$$

Proof. We have

$$\begin{aligned}
K_u(f)(0) &= -\int_0^{+\infty} e^{\lambda l} l^k e^{-A_{0u}l} \Pi_u x dl \\
&= -\frac{d^k}{d\lambda^k} \int_0^{+\infty} e^{\lambda l} e^{-A_{0u}l} \Pi_u x dl \\
&= -\frac{d^k}{d\lambda^k} (-\lambda I + A_{0u})^{-1} \Pi_u x \\
&= \frac{d^k}{d\lambda^k} (\lambda I - A_{0u})^{-1} \Pi_u x \\
&= (-1)^k k!(\lambda I - A_{0u})^{-(k+1)} \Pi_u x.
\end{aligned}$$

Similarly, we have for $\mu > \omega_A$ that

$$(\mu I - A)^{-1} K_s(f)(0) = \lim_{\tau \to -\infty} (\mu I - A)^{-1} \Pi_s\big(S_A \diamond f(\tau + .)\big)(-\tau)$$

$$= \lim_{\tau \to -\infty} \int_0^{-\tau} T_{A_{0s}}(-\tau - s)(\mu I - A)^{-1} \Pi_s f(s+\tau) ds$$

$$= \lim_{r \to +\infty} \int_0^{r} T_{A_{0s}}(r - s)(\mu I - A)^{-1} \Pi_s f(s-r) ds$$

$$= \int_0^{+\infty} T_{A_{0s}}(l)(\mu I - A)^{-1} \Pi_s f(-l) dl.$$

So we obtain that

$$\begin{aligned}
(\mu I - A)^{-1} K_s(f)(0) &= \int_0^{+\infty} (-l)^k e^{-\lambda l} T_{A_0}(l)(\mu I - A)^{-1} \Pi_s x dl \\
&= \frac{d^k}{d\lambda^k} (\lambda I - A_0)^{-1} (\mu I - A)^{-1} \Pi_s x \\
&= (-1)^k k! (\lambda I - A_0)^{-(k+1)} (\mu I - A)^{-1} \Pi_s x \\
&= (\mu I - A)^{-1} (-1)^k k! (\lambda I - A_s)^{-(k+1)} \Pi_s x.
\end{aligned}$$

Since $(\mu I - A)^{-1}$ is one-to-one, we deduce that

$$K_s(f)(t) = (-1)^k k! (\lambda I - A_s)^{-(k+1)} \Pi_s x$$

and the result follows. $\square$

The first result of this section is the following proposition which is related to nonresonant normal forms for ordinary differential equations (see Guckenheimer and Holmes [26], Chow and Hale [8], and Chow et al. [9]).

Proposition 3.6. *Let Assumptions 2.1, 2.2 and 2.4 be satisfied. For each $R \in V^m(X_c, X_h)$, there exists a unique map $G \in V^m(X_c, X_h \cap D(A))$ such that*

$$[A, G](x_c) = R(x_c), \quad \forall x_c \in X_c. \tag{3.5}$$

Moreover, (3.5) is equivalent to

$$G(x_c) = (K_u + K_s)\big(R\big(e^{A_c \cdot} x_c\big)\big)(0),$$

or

$$L(x_1, \dots, x_m) = (K_u + K_s)\big(H\big(e^{A_c \cdot} x_1, \dots, e^{A_c \cdot} x_m\big)\big)(0),$$

with $L := \frac{1}{m!} D^m G(0)$ and $H := \frac{1}{m!} D^m R(0)$.

Proof. Assume first that $G \in V^m(X_c, X_h \cap D(A))$ satisfies (3.5). Then $L = \frac{1}{m!} D^m G(0) \in \mathcal{L}_s(X_c^m, X_h \cap D(A))$ satisfies

$$\frac{d}{dt}\big[L\big(e^{A_c t}x_1, \dots, e^{A_c t}x_m\big)\big](0) = A_h L(x_1, \dots, x_m) + H(x_1, \dots, x_m),$$

where $H = \frac{1}{m!} D^m R(0) \in \mathcal{L}_s(X_c^m, X_h)$. Then (3.5) is satisfied if and only if for each $(x_1, \dots, x_m) \in X_c^m$ and each $t \in \mathbb{R}$,

$$\frac{d}{dt}\big[L\big(e^{A_c t}x_1, \dots, e^{A_c t}x_m\big)\big](t) = A_h L\big(e^{A_c t}x_1, \dots, e^{A_c t}x_m\big) + H\big(e^{A_c t}x_1, \dots, e^{A_c t}x_m\big). \tag{3.6}$$

Set

$$v(t) := L\big(e^{A_c t}x_1, \dots, e^{A_c t}x_m\big), \quad \forall t \in \mathbb{R},$$

and

$$w(t) := H\big(e^{A_c t}x_1, \dots, e^{A_c t}x_m\big), \quad \forall t \in \mathbb{R}.$$

The Cauchy problem (3.6) can be rewritten as

$$\frac{dv(t)}{dt} = A_h v(t) + w(t), \quad \forall t \in \mathbb{R}. \tag{3.7}$$

Since L and H are bounded multilinear maps and $\sigma(A_{0c}) \subset i\mathbb{R}$, it follows that for each $\eta > 0$,

$$v \in BC^\eta(\mathbb{R}, X) \quad \text{and} \quad w \in BC^\eta(\mathbb{R}, X).$$

Let $\eta \in (0, \min(-\omega_0(A_{0s}), \inf_{\lambda \in \sigma(A_{0u})} \mathrm{Re}(\lambda)))$. By projecting (3.7) on X_u, we have

$$\frac{d\Pi_u v(t)}{dt} = A_u \Pi_u v(t) + \Pi_u w(t),$$

or equivalently, $\forall t, s \in \mathbb{R}$ with $t \geq s$,

$$\Pi_u v(t) = e^{A_u(t-s)} \Pi_u v(s) + \int_s^t e^{A_u(t-l)} \Pi_u w(l)\,dl,$$

$$\Pi_u v(s) = e^{-A_u(t-s)} \Pi_u v(t) - \int_s^t e^{-A_u(l-s)} \Pi_u w(l)\,dl.$$

By using the fact that $v \in BC^\eta(\mathbb{R}, X)$, we obtain when t goes to $+\infty$ that

$$\Pi_u v(s) = K_u(\Pi_u w)(s), \quad \forall s \in \mathbb{R}.$$

Thus, for $s = 0$ we have

$$\Pi_u L(x_1, \ldots, x_m) = K_u\big(\Pi_u H\big(e^{A_c \cdot} x_1, \ldots, e^{A_c \cdot} x_m\big)\big)(0). \tag{3.8}$$

By projecting (3.7) on X_s, we obtain

$$\frac{d\Pi_s v(t)}{dt} = A_s \Pi_s v(t) + \Pi_s w(t),$$

or equivalently, $\forall t, s \in \mathbb{R}$ with $t \geq s$,

$$\Pi_s v(t) = T_{A_s}(t-s)\Pi_s v(s) + \big(S_{A_s} \diamond \Pi_s w(.+s)\big)(t-s).$$

By using the fact that $v \in BC^\eta(\mathbb{R}, X)$, we have when s goes to $-\infty$ that

$$\Pi_s v(t) = K_s(\Pi_s w)(t), \quad \forall t \in \mathbb{R}.$$

Thus, for $t = 0$ it follows that

$$\Pi_s L(x_1, \ldots, x_m) = K_s\big(\Pi_s H\big(e^{A_c \cdot} x_1, \ldots, e^{A_c \cdot} x_m\big)\big)(0). \tag{3.9}$$

Summing up (3.8) and (3.9), we deduce that

$$L(x_1, \ldots, x_m) = (K_u + K_s)\big(H\big(e^{A_c \cdot} x_1, \ldots, e^{A_c \cdot} x_m\big)\big)(0). \tag{3.10}$$

Conversely, assume that $L(x_1, \ldots, x_m)$ is defined by (3.10) and set

$$v(t) := (K_u + K_s)\big(H\big(e^{A_c(t+.)} x_1, \ldots, e^{A_c(t+.)} x_m\big)\big)(0), \quad \forall t \in \mathbb{R}.$$

Then we have

$$v(t) = L\big(e^{A_c t} x_1, \ldots, e^{A_c t} x_m\big), \quad \forall t \in \mathbb{R}.$$

Moreover, using Lemma 3.3(iii) and Lemma 3.4(iii), we deduce that for each $t, s \in \mathbb{R}$ with $t \geq s$,

$$v(t) = T_{A_0}(t-s)v(s) + \big(S_A \diamond w(.+s)\big)(t-s),$$

or equivalently,

$$v(t) = v(s) + A\int_s^t v(l)dl + \int_s^t w(l)dl.$$

Since $t \to v(t)$ is continuously differentiable and A is closed, we deduce that

$$v(t) \in D(A), \quad \forall t \in \mathbb{R},$$

and

$$\frac{dv(t)}{dt} = Av(t) + w(t), \quad \forall t \in \mathbb{R}.$$

The result follows. $\square$

Remark 3.7 *(An explicit formula for L).* Since $n := \dim(X_c) < +\infty$, we can find a basis $\{e_1, \dots, e_n\}$ of X_c such that the matrix of A_c (with respect to this basis) is reduced to the Jordan form. Then for each $x_c \in X_c$, $e^{A_c t} x_c$ is a linear combination of elements of the form

$$t^k e^{\lambda t} x_j$$

for some $k \in \{1, \dots, n\}$, some $\lambda \in \sigma(A_c) \subset i\mathbb{R}$, and some $x_j \in \{e_1, \dots, e_n\}$. Let $\lambda_1, \dots, \lambda_m \in \sigma(A_c) \subset i\mathbb{R}$, $x_1, \dots, x_m \in \{e_1, \dots, e_n\}$, $k_1, \dots, k_m \in \{1, \dots, n\}$. Define

$$f(t) := H\big(t^{k_1} e^{\lambda_1 t \cdot} x_1, \dots, t^{k_m} e^{\lambda_m t \cdot} x_m\big), \quad \forall t \in \mathbb{R}.$$

Since H is m-linear, we obtain

$$f(t) = t^k e^{\lambda} y$$

with

$$k = k_1 + k_2 + \cdots + k_m, \qquad \lambda = \lambda_1 + \cdots + \lambda_m,$$

and

$$y = H(x_1, \dots, x_m).$$

Now by using Lemma 3.5, we obtain the explicit formula

$$(K_u + K_s)\big(H\big((.)^{k_1} e^{\lambda_1 \cdot} x_1, \dots, (.)^{k_m} e^{\lambda_m \cdot} x_m\big)\big)(0) = (-1)^k k! (\lambda I - A_h)^{-(k+1)} \Pi_h y \in D(A).$$

3.2. $G \in V^m(X_c, D(A))$

From (3.2), for each $H \in V^m(X_c, X)$, to find $G \in V^m(X_c, D(A))$ satisfying

$$[A, G] = H, \tag{3.11}$$

is equivalent to finding $L \in \mathcal{L}_s(X_c^m, D(A))$ satisfying

$$\frac{d}{dt}\big[L\big(e^{A_c t} x_1, \dots, e^{A_c t} x_m\big)\big]_{t=0} = AL(x_1, \dots, x_m) + \widehat{H}(x_1, \dots, x_m) \tag{3.12}$$

for each $(x_1, \dots, x_m) \in X_c^m$ with

$$\mathcal{G}(\widehat{H}) = H.$$

Define $\Theta_m^c : V^m(X_c, X_c) \to V^m(X_c, X_c)$ by

$$\Theta_m^c(G_c) := [A_c, G_c], \quad \forall G_c \in V^m(X_c, X_c), \tag{3.13}$$

and $\Theta_m^h : V^m(X_c, X_h \cap D(A)) \to V^m(X_c, X_h)$ by

$$\Theta_m^h(G_h) := [A, G_h], \quad \forall G_h \in V^m\big(X_c, X_h \cap D(A)\big).$$

We decompose $V^m(X_c, X_c)$ into the direct sum

$$V^m(X_c, X_c) = \mathcal{R}_m^c \oplus \mathcal{C}_m^c, \tag{3.14}$$

where

$$\mathcal{R}_m^c := R\big(\Theta_m^c\big)$$

is the range of Θ_m^c, and $\mathcal{C}_m^c$ is some complementary space of $\mathcal{R}_m^c$ into $V^m(X_c, X_c)$.

The range of the linear operator Θ_m^c can be characterized by using the so-called non-resonance theorem. The second result of this section is the following theorem.

Proposition 3.8. *Let Assumptions 2.1, 2.2 and 2.4 be satisfied. Let $H \in R_m^c \oplus V^m(X_c, X_h)$. Then there exists $G \in V^m(X_c, D(A))$ (non-unique in general) satisfying*

$$[A, G] = H. \tag{3.15}$$

Furthermore, if $N(\Theta_m^c) = \{0\}$ (the null space of Θ_m^c), then G is uniquely determined.

Proof. By projecting on X_c and X_h and using the fact that $X_c \subset D(A)$, it follows that solving system (3.11) is equivalent to finding $G_c \in V^m(X_c, X_c)$ and $G_h \in V^m(X_c, X_h \cap D(A))$ satisfying

$$[A_c, G_c] = \Pi_c H \tag{3.16}$$

and

$$[A, G_h] = \Pi_h H. \tag{3.17}$$

Now it is clear that we can solve (3.16). Moreover, by using the equivalence between (3.11) and (3.12), we can apply Proposition 3.6 and deduce that (3.17) can be solved. □

Remark 3.9. In practice, we often have

$$N\big(\Theta_m^c\big) \cap R\big(\Theta_m^c\big) = \{0\}.$$

In this case, a natural splitting of $V^m(X_c, X_c)$ will be

$$V^m(X_c, X_c) = R\big(\Theta_m^c\big) \oplus N\big(\Theta_m^c\big).$$

Define $P_m : V^m(X_c, X) \to V^m(X_c, X)$ the bounded linear projector satisfying

$$\mathcal{P}_m\big(V^m(X_c, X)\big) = \mathcal{R}_m^c \oplus V^m(X_c, X_h), \quad \text{and} \quad (I - \mathcal{P}_m)\big(V^m(X_c, X)\big) = \mathcal{C}_m^c.$$

Again consider the Cauchy problem (3.3). Assume that $DF(0) = 0$. Without loss of generality we also assume that for some $m \in \{2, \dots, k\}$,

$$\Pi_h D^j F(0)|_{X_c \times X_c \times \cdots \times X_c} = 0, \qquad \mathcal{G}\big(\Pi_c D^j F(0)|_{X_c \times X_c \times \cdots \times X_c}\big) \in \mathcal{C}_j^c, \qquad (C_{m-1})$$

for each $j = 1, \dots, m-1$.

Consider the change of variables

$$u(t) = w(t) + G\big(\Pi_c w(t)\big) \tag{3.18}$$

and the map $I + \frac{1}{m!} G \circ \Pi_c : \overline{D(A)} \to \overline{D(A)}$ is locally invertible around 0. We will show that we can find $G \in V^m(X_c, D(A))$ such that after the change of variables (3.18) we can rewrite the system (3.3) as

$$\frac{dw(t)}{dt} = Aw(t) + H\big(w(t)\big), \quad \text{for } t \geq 0, \quad \text{and} \quad w(0) = (I + G \circ \Pi_c)x \in \overline{D(A)}, \tag{3.19}$$

where H satisfies the condition (C_m). This will provide a normal form method which is analogous to the one proposed by Faria and Magalhães [24].

Lemma 3.10. *Let Assumptions 2.1, 2.2 and 2.4 be satisfied. Let $G \in V^m(X_c, D(A))$. Assume that $u \in C([0, \tau], X)$ is an integrated solution of the Cauchy problem (3.3). Then $w(t) = (I + G \circ \Pi_c)^{-1}(u(t))$ is an integrated solution of the system (3.19), where $H : \overline{D(A)} \to X$ is the map defined by*

$$H\big(w(t)\big) = F\big(w(t)\big) - [A, G]\big(\Pi_c w(t)\big) + O\big(\|w(t)\|^{m+1}\big).$$

Conversely, if $w \in C([0, \tau], X)$ is an integrated solution of (3.19), then $u(t) = (I + G \circ \Pi_c)w(t)$ is an integrated solution of (3.3).

Lemma 3.10 can be proved similarly as Lemma 3.2, here we omit it.

Proposition 3.11. *Let Assumptions 2.1, 2.2 and 2.4 be satisfied. Let $r > 0$ and let $F : B_{X_0}(0, r) \to X$ be a map. Assume that there exists an integer $k \geq 1$ such that F is k-time continuously differentiable in $B_{X_0}(0, r)$ with $F(0) = 0$ and $DF(0) = 0$. Let $m \in \{2, \dots, k\}$ be such that F satisfies the condition (C_{m-1}). Then there exists a map $G \in V^m(X_c, D(A))$ such that after the change of variables*

$$u(t) = w(t) + G\big(\Pi_c w(t)\big),$$

we can rewrite system (3.3) as (3.19) and H satisfies the condition (C_m), where

$$H\big(w(t)\big) = F\big(w(t)\big) - [A, G]\big(\Pi_c w(t)\big) + O\big(\|w(t)\|^{m+1}\big).$$

Proof. Let $x_c \in X_c$. We have

$$H(x_c) = F(x_c) - [A, G](\Pi_c x_c) + O\big(\|x_c\|^{m+1}\big).$$

It follows that

$$\begin{aligned} H(x_c) = {} & \frac{1}{2!} D^2 F(0)(x_c, x_c) + \cdots + \frac{1}{(m-1)!} D^{m-1} F(0)(x_c, \dots, x_c) \\ & + \mathcal{P}_m \left[\frac{1}{m!} D^m F(0)(x_c, \dots, x_c) \right] + (I - \mathcal{P}_m) \left[\frac{1}{m!} D^m F(0)(x_c, \dots, x_c) \right] \\ & - [A, G](x_c) + O\big(\|x_c\|^{m+1}\big) \end{aligned}$$

since $DF(0) = 0$. Moreover, by using Proposition 3.8 we obtain that there exists a map $G \in V^m(X_c, D(A))$ such that

$$[A, G](x_c) = \mathcal{P}_m \left[\frac{1}{m!} D^m F(0)(x_c, \dots, x_c) \right].$$

Hence,

$$\begin{aligned} H(x_c) = {} & \frac{1}{2!} D^2 F(0)(x_c, x_c) + \cdots + \frac{1}{(m-1)!} D^{m-1} F(0)(x_c, \dots, x_c) \\ & + (I - \mathcal{P}_m) \left[\frac{1}{m!} D^m F(0)(x_c, \dots, x_c) \right] + O\big(\|x_c\|^{m+1}\big). \end{aligned} \tag{3.20}$$

By the assumption, we have for all $j = 1, \dots, m-1$ that

$$\Pi_h D^j H(0)|_{X_c \times X_c \times \cdots \times X_c} = \Pi_h D^j F(0)|_{X_c \times X_c \times \cdots \times X_c} = 0$$

and

$$\mathcal{G}\big(\Pi_c D^j H(0)|_{X_c \times X_c \times \cdots \times X_c}\big) = \mathcal{G}\big(\Pi_c D^j F(0)|_{X_c \times X_c \times \cdots \times X_c}\big) \in \mathcal{C}_j^c.$$

Now by using (3.20), we have

$$\frac{1}{m!} \Pi_h D^m H(0)|_{X_c \times X_c \times \cdots \times X_c} = \Pi_h \mathcal{G}^{-1} \left[(I - \mathcal{P}_m) \left(\frac{1}{m!} D^m F(0)(x_c, \dots, x_c) \right) \right] = 0$$

and

$$\mathcal{G}\big(\Pi_c D^m H(0)|_{X_c \times X_c \times \cdots \times X_c}\big) = \mathcal{G}\big\{\Pi_c \mathcal{G}^{-1}\big[(I - \mathcal{P}_m)\big(D^m F(0)(x_c, \dots, x_c)\big)\big]\big\} \in \mathcal{C}_m^c.$$

The result follows. $\square$

4. Normal form computation

In this section we provide the method to compute the Taylor expansion at any order and normal form of the reduced system of a system topologically equivalent to the original system:

$$\begin{cases} \dfrac{du(t)}{dt} = Au(t) + F\big(u(t)\big), & t \geq 0, \\ u(0) = x \in \overline{D(A)}. \end{cases} \tag{4.1}$$

Assumption 4.1. Assume that $F \in C^k(\overline{D(A)}, X)$ for some integer $k \geq 2$ with

$$F(0) = 0 \quad \text{and} \quad DF(0) = 0.$$

Set

$$F_1 := F.$$

Once again we consider two cases, namely, $G \in V^m(X_c, D(A) \cap X_h)$ and $G \in V^m(X_c, D(A))$, respectively.

4.1. $G \in V^m(X_c, D(A) \cap X_h)$

For $j = 2, \ldots, k$, we apply Proposition 3.6. Then there exists a unique function $G_j \in V^j(X_c, X_h \cap D(A))$ satisfying

$$[A, G_j](x_c) = \frac{1}{j!} \Pi_h D^j F_{j-1}(0)(x_c, \ldots, x_c), \quad \forall x_c \in X_c. \tag{4.2}$$

Define $\xi_j : X \to X$ and $\xi_j^{-1} : X \to X$ by

$$\xi_j(x) := x + G_j(\Pi_c x) \quad \text{and} \quad \xi_j^{-1}(x) := x - G_j(\Pi_c x), \quad \forall x \in X.$$

Then

$$F_j(x) := F_{j-1}\big(\xi_j(x)\big) - [A, G_j](\Pi_c x) - DG_j(\Pi_c x)\big[\Pi_c F_{j-1}\big(\xi_j(x)\big)\big].$$

Moreover, we have for $x \in X_0$ that

$$\Pi_c F_j(x) = \Pi_c F_{j-1}\big(\xi_j(x)\big) = \Pi_c F_{j-1}\big(x + G_j(\Pi_c x)\big).$$

Since the range of G_j is included in X_h, by induction we have

$$\Pi_c F_j(x) = \Pi_c F\big(x + G_2(\Pi_c x) + G_3(\Pi_c x) + \cdots + G_j(\Pi_c x)\big).$$

Now, we obtain

$$\Pi_h D^j F_k(0)|_{X_c \times X_c \times \cdots \times X_c} = 0 \quad \text{for all } j = 1, \ldots, k.$$

Setting

$$\begin{aligned} u_k(t) &= \xi_k^{-1} \circ \xi_{k-1}^{-1} \circ \cdots \circ \xi_2^{-1}\big(u(t)\big) \\ &= u(t) - G_2\big(\Pi_c u(t)\big) - G_3\big(\Pi_c u(t)\big) - \cdots - G_k\big(\Pi_c u(t)\big), \end{aligned}$$

we deduce that $u_k(t)$ is an integrated solution of the system

$$\begin{cases} \dfrac{du_k(t)}{dt} = Au_k(t) + F_k\big(u_k(t)\big), & t \geq 0, \\ u_k(0) = x_k \in \overline{D(A)}. \end{cases} \tag{4.3}$$

Applying Theorem 2.7 and Lemma 2.8 to system (4.3), we obtain the following result which is one of the main results of this paper.

Theorem 4.2. *Let Assumptions 2.1, 2.2, 2.4, and 4.1 be satisfied. Then by using the change of variables*

$$\begin{cases} u_k(t) = u(t) - G_2\big(\Pi_c u(t)\big) - G_3\big(\Pi_c u(t)\big) - \cdots - G_k\big(\Pi_c u(t)\big) \\ \quad \Leftrightarrow \\ u(t) = u_k(t) + G_2\big(\Pi_c u_k(t)\big) + G_3\big(\Pi_c u_k(t)\big) + \cdots + G_k\big(\Pi_c u_k(t)\big), \end{cases}$$

the map $t \to u(t)$ is an integrated solution of the Cauchy problem (4.1) if and only if $t \to u_k(t)$ is an integrated solution of the Cauchy problem (4.3). Moreover, the reduced system of Cauchy problem (4.3) is given by the ordinary differential equations on X_c:

$$\frac{dx_c(t)}{dt} = A_c x_c(t) + \Pi_c F\begin{bmatrix} x_c(t) + G_2(x_c(t)) \\ \quad + G_3(x_c(t)) + \cdots + G_k(x_c(t)) \end{bmatrix} + R_c\big(x_c(t)\big), \tag{4.4}$$

where the remainder term $R_c \in C^k(X_c, X_c)$ satisfies

$$D^j R_c(0) = 0 \quad \textit{for each } j = 1, \ldots, k,$$

or in other words $R_c(x_c(t))$ is a remainder term of order k.

If we assume in addition that $F \in C^{k+2}(\overline{D(A)}, X)$, then the map $R_c \in C^{k+2}(X_c, X_c)$ and $R_c(x_c(t))$ is a remainder term of order $k+2$, that is

$$R_c(x_c) = \|x_c\|^{k+2} O(x_c), \tag{4.5}$$

where $O(x_c)$ is a function of x_c which remains bounded when x_c goes to 0, or equivalently,

$$D^j R_c(0) = 0 \quad \textit{for each } j = 1, \ldots, k+1.$$

Proof. By Theorem 2.7 and Lemma 2.8, there exists $\Psi_k \in C^k(X_c, X_h)$ such that the reduced system of (4.3) is given by

$$\frac{dx_c(t)}{dt} = A_c x_c(t) + \Pi_c F\big[x_c(t) + G_2\big(x_c(t)\big) + G_3\big(x_c(t)\big) + \cdots + G_k\big(x_c(t)\big) + \Psi_k\big(x_c(t)\big)\big]$$

and

$$D^j \Psi_k(0) = 0 \quad \text{for } j = 1, \ldots, k.$$

By setting

$$\begin{aligned} R_c(x_c) = \Pi_c F\big[&x_c + G_2(x_c) + G_3(x_c) + \cdots + G_k(x_c) + \Psi_k(x_c)\big] \\ &- \Pi_c F\big[x_c + G_2(x_c) + G_3(x_c) + \cdots + G_k(x_c)\big], \end{aligned}$$

we obtain the first part of the theorem. If we assume in addition that $F \in C^{k+2}(\overline{D(A)}, X)$, then $\Psi_k \in C^{k+2}(X_c, X_h)$. Thus,

$$R_c \in C^{k+2}(X_c, X_c).$$

Set

$$h(x_c) := x_c + G_2(x_c) + G_3(x_c) + \cdots + G_k(x_c).$$

We have

$$\begin{aligned} R_c(x_c) &= \Pi_c\big\{F\big[h(x_c) + \Psi_k(x_c)\big] - F\big[h(x_c)\big]\big\} \\ &= \Pi_c \int_0^1 DF\big(h(x_c) + s\Psi_k(x_c)\big)\big(\Psi_k(x_c)\big)ds. \end{aligned}$$

Define

$$\widehat{h}(x_c) := h(x_c) + s\Psi_k(x_c).$$

Since $DF(0) = 0$, we have

$$\begin{aligned} DF\big(\widehat{h}(x_c)\big)\big(\Psi_k(x_c)\big) &= DF(0)\big(\Psi_k(x_c)\big) + \int_0^1 D^2F\big(l\widehat{h}(x_c)\big)\big(\widehat{h}(x_c), \Psi_k(x_c)\big)dl \\ &= \int_0^1 D^2F\big(l\widehat{h}(x_c)\big)\big(\widehat{h}(x_c), \Psi_k(x_c)\big)dl. \end{aligned}$$

Hence,

$$R_c(x_c) = \Pi_c \int_0^1 \int_0^1 D^2F\big(l\big(h(x_c) + s\Psi_k(x_c)\big)\big)\big(h(x_c) + s\Psi_k(x_c), \Psi_k(x_c)\big)dl\,ds$$

and $h(x_c)$ is a term of order 1, $\Psi_k(x_c)$ is a term of order $k+1$, it follows that (4.5) holds. This completes the proof. $\square$

Remark 4.3. In order to apply the above approach, we first need to compute Π_c and A_c, then $\Pi_h := I - \Pi_c$ can be derived. The point to apply the above procedure is to solve system (4.2). To do this, one may compute

$$(\lambda I - A_h)^{-k} \frac{1}{j!} \Pi_h D^j F(0) \tag{4.6}$$

for each $\lambda \in i\mathbb{R}$ and each $k \geq 1$ by using Remark 3.7, or one may directly solve system (4.2) by computing $\Pi_h \frac{1}{j!} D^j F_{j-1}$. This last approach will involve the computation of (4.6) for some specific values of $\lambda \in i\mathbb{R}$ and some specific values of $k \geq 1$. This turns out to be the main difficulty in applying the above method.

In Section 5, we will use the last part of Theorem 4.2 to avoid some unnecessary computations. We will apply this theorem for $k = 2$, F in C^4, and the remainder term $R_c(x_c)$ of order 4. This means that if we want to compute the Taylor expansion of the reduced system to the order 3 (which is very common in such a context), we only need to compute G_2. So in application the last part of Theorem 4.2 will help to avoid a lot of computations.

4.2. $G \in V^m(X_c, D(A))$

Now we apply Proposition 3.11 recursively to (4.1). Set

$$u_1 := u.$$

For $m = 2, \ldots, k$, let $G_m \in V^m(X_c, D(A))$ be defined such that

$$[A, G_m](x_c) = \mathcal{P}_m \left[\frac{1}{m!} D^m F_{m-1}(0)(x_c, \ldots, x_c) \right] \quad \text{for each } x_c \in X_c.$$

We use the change of variables

$$u_{m-1} = u_m + G_m(\Pi_c u_m).$$

Then we consider F_m given by Proposition 3.11 and satisfying

$$F_m(u_m) = F_{m-1}(u_m) - [A, G_m](\Pi_c u_m) + O\big(\|u_m\|^{m+1}\big).$$

By applying Proposition 3.11, we have

$$\Pi_h D^j F_m(0)|_{X_c \times X_c \times \cdots \times X_c} = 0, \quad \text{for all } j = 1, \ldots, m,$$

and

$$\mathcal{G}\big(\Pi_c D^j F_m(0)|_{X_c \times X_c \times \cdots \times X_c}\big) \in \mathcal{C}_j^c, \quad \text{for all } j = 1, \ldots, m.$$

Thus by using the change of variables locally around 0

$$u_k(t) = (I + G_k\Pi_c)^{-1} \dots (I + G_3\Pi_c)^{-1}(I + G_2\Pi_c)^{-1}u(t),$$

we deduce that $u_k(t)$ is an integrated solution of system (4.3). Applying Theorem 2.7 and Lemma 2.8 to the above system, we obtain the following result which is the main result of this paper and indicates that systems (4.1) and (4.3) are locally topologically equivalent around 0.

Theorem 4.4. *Let Assumptions 2.1, 2.2, 2.4, and 4.1 be satisfied. Then by using the change of variables locally around* 0

$$\begin{cases} u_k(t) = (I + G_k\Pi_c)^{-1} \dots (I + G_3\Pi_c)^{-1}(I + G_2\Pi_c)^{-1}u(t) \\ \quad \Leftrightarrow \\ u(t) = (I + G_2\Pi_c)(I + G_3\Pi_c)\dots(I + G_k\Pi_c)u_k(t), \end{cases}$$

the map $t \to u(t)$ is an integrated solution of the Cauchy problem (4.1) if and only if $t \to u_k(t)$ is an integrated solution of the Cauchy problem (4.3). Moreover, the reduced equation of Cauchy problem (4.3) is given by the ordinary differential equations on X_c:

$$\frac{dx_c(t)}{dt} = A_c x_c(t) + \sum_{m=2}^{k} \frac{1}{m!}\Pi_c D^m F_k(0)\big(x_c(t), \dots, x_c(t)\big) + R_c\big(x_c(t)\big),$$

where

$$\mathcal{G}\left(\frac{1}{m!}\Pi_c D^m F_k(0)|_{X_c \times X_c \times \dots \times X_c}\right) \in \mathcal{C}_m^c, \quad \text{for all } m = 1, \dots, k,$$

and the remainder term $R_c \in C^k(X_c, X_c)$ satisfies

$$D^j R_c(0) = 0 \quad \text{for each } j = 1, \dots, k,$$

or in other words $R_c(x_c(t))$ is a remainder term of order k. If we assume in addition that $F \in C^{k+2}(\overline{D(A)}, X)$. Then the reduced equation of Cauchy problem (4.3) is given by the ordinary differential equations on X_c:

$$\frac{dx_c(t)}{dt} = A_c x_c(t) + \sum_{m=2}^{k+1} \frac{1}{m!}\Pi_c D^m F_k(0)\big(x_c(t), \dots, x_c(t)\big) + R_c\big(x_c(t)\big),$$

the map $R_c \in C^{k+2}(X_c, X_c)$, and $R_c(x_c(t))$ is a remainder term of order $k + 2$, that is

$$R_c(x_c) = \|x_c\|^{k+2} O(x_c),$$

where $O(x_c)$ is a function of x_c which remains bounded when x_c goes to 0, or equivalently,

$$D^j R_c(0) = 0 \quad \text{for each } j = 1, \dots, k + 1.$$

Proof. By Theorem 2.7 and Lemma 2.8, there exists $\Psi_k \in C^k(X_c, X_h)$ such that the reduced system of (4.3) is given by

$$\frac{dx_c(t)}{dt} = A_c x_c(t) + \Pi_c F_k\big[x_c(t) + \Psi_k\big(x_c(t)\big)\big]$$

and

$$D^j \Psi_k(0) = 0 \quad \text{for } j = 1, \dots, k.$$

By setting

$$R_c(x_c) = \Pi_c F_k\big[x_c + \Psi_k(x_c)\big] - \Pi_c F_k(x_c),$$

we obtain the first part of the theorem. If we assume in addition that $F \in C^{k+2}(\overline{D(A)}, X)$, then $\Psi_k \in C^{k+2}(X_c, X_h)$. Thus, $R_c \in C^{k+2}(X_c, X_c)$ and

$$\begin{aligned} R_c(x_c) &= \Pi_c\big\{F_k\big[x_c + \Psi_k(x_c)\big] - F_k(x_c)\big\} \\ &= \Pi_c \int_0^1 DF_k\big(x_c + s\Psi_k(x_c)\big)\big(\Psi_k(x_c)\big)ds. \end{aligned}$$

Set

$$h(x_c) := x_c + s\Psi_k(x_c).$$

Since $DF(0) = 0$, we have

$$\begin{aligned} DF_k\big(h(x_c)\big)\big(\Psi_k(x_c)\big) &= DF_k(0)\big(\Psi_k(x_c)\big) + \int_0^1 D^2 F_k\big(lh(x_c)\big)\big(h(x_c), \Psi_k(x_c)\big)dl \\ &= \int_0^1 D^2 F_k\big(lh(x_c)\big)\big(h(x_c), \Psi_k(x_c)\big)dl. \end{aligned}$$

Hence,

$$R_c(x_c) = \Pi_c \int_0^1 \int_0^1 D^2 F_k\big(l\big(x_c + s\Psi_k(x_c)\big)\big)\big(x_c + s\Psi_k(x_c), \Psi_k(x_c)\big)dlds$$

and $\Psi_k(x_c)$ is a term of order $k+1$, it follows that

$$R_c(x_c) = \|x_c\|^{k+2} O(x_c).$$

The result follows. □

5. Applications

In this section we apply the normal form theory developed in the previous sections to the two examples of structured models (1.2) and (1.6) introduced in Section 1. Namely, we will compute the Taylor expansion of the reduced system of model (1.2) on the center manifold and the normal form of model (1.6) on the center manifold, respectively, from which we will be able to determine the direction of the Hopf bifurcation and stability of the bifurcating periodic solutions in these two models.

5.1. A structured model of influenza A drift

We first recall the results in Magal and Ruan [44] on the existence of Hopf bifurcation in the structured evolutionary epidemiological model (1.2) of influenza A drift. The following theorem was proven in Magal and Ruan [44].

Proposition 5.1.

(i) *Consider the curves defined by*

$$\delta_\nu = \nu + \frac{1+\nu}{\nu}\frac{(\frac{\nu}{c})^2}{1-\sqrt{1-\frac{1}{c^2}}}$$

in the (ν, δ)*-plane for some* $c \geq 1$*. Then for each* $n \geq 0$*,*

$$\nu_n = c\left(\arcsin\left(-\frac{1}{c}\right) + 2(n+1)\pi\right)$$

is a Hopf bifurcation point for system (1.5) around the branch of equilibrium points $\bar{s}_\nu$*, where*

$$\bar{s}_\nu(a) = \begin{cases} \nu(1-\overline{S}_\nu)e^{-\delta_\nu(1-\overline{S}_\nu)(a-1)}, & \text{if } a \geq 1, \\ \nu(1-\overline{S}_\nu), & \text{if } a \in [0,1], \end{cases}$$

$$\overline{S}_\nu = \int_0^{+\infty} \bar{s}_\nu(l)dl = \frac{1+\delta_\nu^{-1}}{1+\nu^{-1}} < 1.$$

Moreover, the period of the bifurcating periodic orbits is close to

$$\omega_n = c\arcsin\left(-\frac{1}{c}\right) + \pi + 2n\pi.$$

(ii) *Consider the curves defined by*

$$\delta_\nu = \nu + \frac{1+\nu}{\nu}\frac{(\frac{\nu}{c})^2}{1+\sqrt{1-\frac{1}{c^2}}}$$

in the (ν, δ)-plane for some $c \geq 1$. Then for each $n \geq 0$,

$$\widehat{\nu}_n = c\left(\arcsin\left(\frac{1}{c}\right) + \pi + 2n\pi\right)$$

is a Hopf bifurcation point for system (1.5) around the branch of equilibrium points $\bar{s}_\nu$. Moreover, the period of the bifurcating periodic orbits is close to

$$\widehat{\omega}_n = c \arcsin\left(\frac{1}{c}\right) + \pi + 2n\pi.$$

In order to apply the normal form theory to system (1.5), we include the parameter ν into the state variable in system (1.5) and consider the system

$$\begin{cases} \dfrac{d\nu(t)}{dt} = 0, \\ \dfrac{du(t)}{dt} = Au(t) + F\big(\nu(t), \delta_{\nu(t)}, u(t)\big) = Au(t) + F\big(\nu(t), u(t)\big), \\ \nu(0) = \nu_0 \in \mathbb{R}, \quad u(0) = u_0 = \begin{pmatrix} 0 \\ s_0 \end{pmatrix} \in \overline{D(A)}, \end{cases} \tag{5.1}$$

where A is defined in (1.3) and F is given by (1.4). Making the change of variables

$$u = \widehat{u} + \bar{u}_\nu \quad \text{and} \quad \nu = \widehat{\nu} + \nu_k$$

with $\bar{u}_\nu = \binom{0}{\bar{s}_\nu}$, we obtain the system

$$\begin{cases} \dfrac{d\widehat{\nu}(t)}{dt} = 0, \\ \dfrac{d\widehat{u}(t)}{dt} = A\widehat{u}(t) + \widehat{H}\big(\widehat{\nu}(t), \widehat{u}(t)\big), \end{cases} \tag{5.2}$$

where

$$\widehat{H}(\widehat{\nu}, \widehat{u}) = F(\widehat{\nu} + \nu_k, \widehat{u} + \bar{u}_{\widehat{\nu}+\nu_k}) - F(\widehat{\nu} + \nu_k, \bar{u}_{\widehat{\nu}+\nu_k}).$$

Since $\overline{S}_{\nu_k} \in (0, 1)$, the map $u \to F(\nu, u)$ is differentiable in a neighborhood of $\bar{u}_{\nu_k}$. We have

$$\begin{aligned} \partial_{\widehat{\nu}} \widehat{H}(\widehat{\nu}, \widehat{u})(\widetilde{\nu}) = {} & \partial_\nu F(\widehat{\nu} + \nu_k, \widehat{u} + \bar{u}_{\widehat{\nu}+\nu_k})(\widetilde{\nu}) + \partial_u F(\widehat{\nu} + \nu_k, \widehat{u} + \bar{u}_{\widehat{\nu}+\nu_k}) \frac{d(\bar{u}_{\widehat{\nu}+\nu_k})}{d\widehat{\nu}}(\widetilde{\nu}) \\ & - \partial_\nu F(\widehat{\nu} + \nu_k, \bar{u}_{\widehat{\nu}+\nu_k})(\widetilde{\nu}) - \partial_u F(\widehat{\nu} + \nu_k, \bar{u}_{\widehat{\nu}+\nu_k}) \frac{d(\bar{u}_{\widehat{\nu}+\nu_k})}{d\widehat{\nu}}(\widetilde{\nu}) \end{aligned}$$

and

$$\partial_{\widehat{u}} \widehat{H}(\widehat{\nu}, \widehat{u})(\widetilde{u}) = \partial_u F(\widehat{\nu} + \nu_k, \widehat{u} + \bar{u}_{\widehat{\nu}+\nu_k})(\widetilde{u})$$

with

$$\partial_u F(\nu, u)(\widetilde{u}) = \begin{pmatrix} -\nu \int_0^{+\infty} \widetilde{\varphi}(l)dl \\ -\delta_\nu \chi[\widetilde{\varphi}(1 - \int_0^{+\infty} \varphi(l)dl) - \varphi \int_0^{+\infty} \widetilde{\varphi}(l)dl] \end{pmatrix},$$

here $u = \binom{0}{\varphi}$, $\widetilde{u} = \binom{0}{\widetilde{\varphi}} \in \overline{D(A)}$. Hence, $\partial_{\widehat{\nu}} \widehat{H}(0,0)(\widetilde{\nu}) = 0$ and $\partial_{\widehat{u}} \widehat{H}(0,0)(\widetilde{u}) = \partial_u F(\nu_k, \overline{u}_{\nu_k})(\widetilde{u})$.

Set

$$Y = \mathbb{R} \times X, \qquad Y_0 = \mathbb{R} \times \overline{D(A)}$$

and

$$\widehat{A}_\nu := A + \partial_u F(\nu, \overline{u}_\nu).$$

The following lemma is obtained in Magal and Ruan [42].

Lemma 5.2. *The linear operator $\widehat{A}_\nu : D(A) \subset X \to X$ is a Hille–Yosida operator and*

$$\omega_{0,\mathrm{ess}}\big((\widehat{A}_\nu)_0\big) \leq -\delta_\nu(1 - \overline{S}_\nu) < 0.$$

Consider the linear operator $L : D(L) \subset Y \to Y$ defined by

$$L\begin{pmatrix} \widehat{\nu} \\ \widehat{u} \end{pmatrix} = \begin{pmatrix} 0 \\ (A + \partial_u F(\nu_k, \overline{u}_{\nu_k}))\widehat{u} \end{pmatrix} = \begin{pmatrix} 0 \\ \widehat{A}_{\nu_k}\widehat{u} \end{pmatrix}$$

with $D(L) = \mathbb{R} \times D(A)$ and the map $H : \overline{D(L)} \to Y$ defined by

$$H\begin{pmatrix} \widehat{\nu} \\ \widehat{u} \end{pmatrix} = \begin{pmatrix} 0 \\ W\binom{\widehat{\nu}}{\widehat{u}} \end{pmatrix},$$

where $W : \overline{D(L)} \to X$ is defined by

$$W = F(\widehat{\nu} + \nu_k, \widehat{u} + \overline{u}_{\widehat{\nu}+\nu_k}) - F(\widehat{\nu} + \nu_k, \overline{u}_{\widehat{\nu}+\nu_k}) - \partial_u F(\nu_k, \overline{u}_{\nu_k})\widehat{u}.$$

Then we have

$$H\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \quad \text{and} \quad DH\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0.$$

Now we can reformulate system (4.2) as the following system

$$\frac{dw(t)}{dt} = Lw(t) + H\big(w(t)\big), \qquad w(0) = w_0 \in \overline{D(L)}. \tag{5.3}$$

The following lemma is a consequence of the results proved in Liu et al. [35, Section 3.1].

Lemma 5.3. *The linear operator L is a Hille–Yosida operator and the essential growth rate of L_0 satisfies*

$$\omega_{0,\mathrm{ess}}(L_0) \le -\delta_{\nu_k}(1-\overline{S}_{\nu_k}) < 0.$$

Set

$$\Omega = \left\{\lambda \in \mathbb{C} \,\middle|\, \mathrm{Re}(\lambda) > -\delta_{\nu_k}(1-\overline{S}_{\nu_k})\right\}.$$

The characteristic function takes the form

$$\Delta(\nu,\lambda) = 1 + \nu \int_0^{+\infty} e^{-\int_0^a [\lambda+\delta_\nu(1-\overline{S}_\nu)\chi(l)]dl} da - \int_0^{+\infty}\int_0^a e^{-\int_s^a [\lambda+\delta_\nu(1-\overline{S}_\nu)\chi(l)]dl} \delta_\nu \chi(s)\overline{s}_\nu(s) ds da.$$

From Lemma 4.6 in Magal and Ruan [44], we know that

$$(\lambda I - \widehat{A}_{\nu_k})^{-1}\begin{pmatrix}\widehat{\alpha}\\ \widehat{\varphi}\end{pmatrix} = \begin{pmatrix}0\\ \varphi\end{pmatrix} \tag{5.4}$$

is equivalent to

$$\begin{aligned}\varphi(a) = \Delta(\nu_k,\lambda)^{-1}&\left[\begin{array}{l}((1-C_1)\widehat{\alpha} - \nu_k \widehat{I})e^{-\int_0^a [\lambda+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\\ \quad + (C_2\widehat{\alpha}+\widehat{I})\int_0^a e^{-\int_s^a [\lambda+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\delta_{\nu_k}\chi(s)\overline{s}_{\nu_k}(s)ds\end{array}\right]\\ &+ \int_0^a e^{-\int_s^a [\lambda+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\widehat{\varphi}(s)ds,\end{aligned}$$

where

$$\widehat{I} = \int_0^{+\infty}\int_0^a e^{-\int_s^a [\lambda+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\widehat{\varphi}(s) ds da,$$

$$C_1 = \int_0^{+\infty}\int_0^a e^{-\int_s^a [\lambda+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\delta_{\nu_k}\chi(s)\overline{s}_{\nu_k}(s) ds da,$$

$$C_2 = \int_0^{+\infty} e^{-\int_0^a [\lambda+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl} da.$$

By using (4.4) and Lemma 4.6 in [44], we obtain that the projector on the generalized eigenspace of $\widehat{A}_{\nu_k}$ associated to $\widehat{\lambda}$ is given by

$$\Pi_{\widehat{\lambda}}^{\widehat{A}_{\nu_k}}\begin{pmatrix}\widehat{\alpha}\\ \widehat{\varphi}\end{pmatrix}=\begin{pmatrix}0\\ \varphi\end{pmatrix}$$

$$\Leftrightarrow \quad \varphi(a)=\left(\frac{d\Delta(\nu_k,\widehat{\lambda})}{d\lambda}\right)^{-1}$$
$$\times\left[\begin{array}{l}((1-C_1|_{\lambda=\widehat{\lambda}})\widehat{\alpha}-\nu_k\widehat{I}|_{\lambda=\widehat{\lambda}})e^{-\int_0^a[\widehat{\lambda}+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\\ +(C_2|_{\lambda=\widehat{\lambda}}\widehat{\alpha}+\widehat{I}|_{\lambda=\widehat{\lambda}})\int_0^a e^{-\int_s^a[\widehat{\lambda}+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\delta_{\nu_k}\chi(s)\overline{s}_{\nu_k}(s)ds\end{array}\right],$$

where $\widehat{\lambda}=i\omega$ with $\omega\neq 0$. Define $\Pi_c^{\widehat{A}_{\nu_k}}: X\to X$ by

$$\Pi_c^{\widehat{A}_{\nu_k}}\begin{pmatrix}\widehat{\alpha}\\ \widehat{\varphi}\end{pmatrix}=\Pi_{i\omega_k}^{\widehat{A}_{\nu_k}}\begin{pmatrix}\widehat{\alpha}\\ \widehat{\varphi}\end{pmatrix}+\Pi_{-i\omega_k}^{\widehat{A}_{\nu_k}}\begin{pmatrix}\widehat{\alpha}\\ \widehat{\varphi}\end{pmatrix},\quad \forall\begin{pmatrix}\widehat{\alpha}\\ \widehat{\varphi}\end{pmatrix}\in X.$$

Hence

$$\begin{aligned}\Pi_c^{\widehat{A}_{\nu_k}}\begin{pmatrix}1\\0\end{pmatrix}&=\Pi_{i\omega_k}^{\widehat{A}_{\nu_k}}\begin{pmatrix}1\\0\end{pmatrix}+\Pi_{-i\omega_k}^{\widehat{A}_{\nu_k}}\begin{pmatrix}1\\0\end{pmatrix}\\ &=\begin{pmatrix}0\\ (\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda})^{-1}b_1+(\frac{d\Delta(\nu_k,-i\omega_k)}{d\lambda})^{-1}b_2\end{pmatrix},\end{aligned}\tag{5.5}$$

where

$$\begin{aligned}b_1(a)&=(1-C_1|_{\lambda=i\omega_k})e^{-\int_0^a[i\omega_k+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\\ &\quad+C_2|_{\lambda=i\omega_k}\int_0^a e^{-\int_s^a[i\omega_k+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\delta_{\nu_k}\chi(s)\overline{s}_{\nu_k}(s)ds,\\ b_2(a)&=(1-C_1|_{\lambda=-i\omega_k})e^{-\int_0^a[-i\omega_k+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\\ &\quad+C_2|_{\lambda=-i\omega_k}\int_0^a e^{-\int_s^a[-i\omega_k+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\delta_{\nu_k}\chi(s)\overline{s}_{\nu_k}(s)ds.\end{aligned}$$

Set

$$\widehat{e_1}=\begin{pmatrix}0\\ b_1\end{pmatrix},\qquad \widehat{e_2}=\begin{pmatrix}0\\ b_2\end{pmatrix}\quad\text{and}\quad \Pi_h^{\widehat{A}_{\nu_k}}:=I-\Pi_c^{\widehat{A}_{\nu_k}}.$$

Lemma 5.4. *We have*

$$\Pi_h^{\widehat{A}_{\nu_k}}\begin{pmatrix}1\\0\end{pmatrix}=\left(I-\Pi_c^{\widehat{A}_{\nu_k}}\right)\begin{pmatrix}1\\0\end{pmatrix}.$$

For each $\lambda\in i\mathbb{R}\setminus\{-i\omega_k,i\omega_k\}$,

$$(\lambda I-\widehat{A}_{\nu_k}|_{\Pi_h^{\widehat{A}_{\nu_k}}(X)})^{-1}\Pi_h^{\widehat{A}_{\nu_k}}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0\\ \phi\end{pmatrix}$$

with

$$\begin{aligned}\phi(a) = &-\left(\frac{d\Delta(\nu_k, i\omega_k)}{d\lambda}\right)^{-1} \frac{b_1(a)}{(\lambda - i\omega_k)} - \left(\frac{d\Delta(\nu_k, -i\omega_k)}{d\lambda}\right)^{-1} \frac{b_2(a)}{(\lambda + i\omega_k)} \\ &+ \Delta(\nu_k, \lambda)^{-1} \left[\begin{array}{l}(1 - C_1)e^{-\int_0^a [\lambda + \delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl} \\ \quad + C_2 \int_0^a e^{-\int_s^a [\lambda + \delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl} \delta_{\nu_k} \chi(s) \overline{s}_{\nu_k}(s) ds\end{array}\right].\end{aligned}$$

Moreover, if $\lambda = i\omega_k$, *we have*

$$\begin{aligned}&(i\omega_k I - \widehat{A}_{\nu_k}|_{\Pi_h^{\widehat{A}_{\nu_k}}(X)})^{-1} \Pi_h^{\widehat{A}_{\nu_k}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -(\frac{d\Delta(\nu_k, -i\omega_k)}{d\lambda})^{-1} \frac{b_2}{2i\omega_k} - (\frac{d\Delta(\nu_k, i\omega_k)}{d\lambda})^{-2} (\frac{1}{2} \frac{d^2\Delta(\nu_k, i\omega_k)}{d\lambda^2} b_1 + M) \end{pmatrix}\end{aligned} \tag{5.6}$$

with

$$\begin{aligned}M(a) = &\frac{d\Delta(\nu_k, i\omega_k)}{d\lambda} \\ &\times \left[\begin{array}{l} ab_1(a) - C_2|_{\lambda = i\omega_k} \int_0^a s e^{-\int_s^a [i\omega_k + \delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl} \delta_{\nu_k} \chi(s) \overline{s}_{\nu_k}(s) ds \\ - e^{-\int_0^a [i\omega_k + \delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl} \\ \times (\int_0^{+\infty} \int_0^a (a-s) e^{-\int_s^a [i\omega_k + \delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl} \delta_{\nu_k} \chi(s) \overline{s}_{\nu_k}(s) ds da) \\ + (\int_0^{+\infty} a e^{-\int_0^a [i\omega_k + \delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl} da) \\ \\ \times (\int_0^a e^{-\int_s^a [i\omega_k + \delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl} \delta_{\nu_k} \chi(s) \overline{s}_{\nu_k}(s) ds) \end{array}\right].\end{aligned}$$

If $\lambda = -i\omega_k$, *we have*

$$\begin{aligned}&(-i\omega_k I - \widehat{A}_{\nu_k}|_{\Pi_h^{\widehat{A}_{\nu_k}}(X)})^{-1} \Pi_h^{\widehat{A}_{\nu_k}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ (\frac{d\Delta(\nu_k, i\omega_k)}{d\lambda})^{-1} \frac{b_1}{2i\omega_k} - (\frac{d\Delta(\nu_k, -i\omega_k)}{d\lambda})^{-2} (\frac{1}{2} \frac{d^2\Delta(\nu_k, -i\omega_k)}{d\lambda^2} b_2 + N) \end{pmatrix}\end{aligned} \tag{5.7}$$

with

$$\begin{aligned}N(a) = &\frac{d\Delta(\nu_k, -i\omega_k)}{d\lambda} \\ &\times \left[\begin{array}{l} ab_2(a) - C_2|_{\lambda = -i\omega_k} \int_0^a s e^{-\int_s^a [-i\omega_k + \delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl} \delta_{\nu_k} \chi(s) \overline{s}_{\nu_k}(s) ds \\ - e^{-\int_0^a [-i\omega_k + \delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl} \\ \times \int_0^{+\infty} \int_0^a (a-s) e^{-\int_s^a [-i\omega_k + \delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl} \delta_{\nu_k} \chi(s) \overline{s}_{\nu_k}(s) ds da \\ + \int_0^{+\infty} a e^{-\int_0^a [-i\omega_k + \delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl} da \\ \times \int_0^a e^{-\int_s^a [-i\omega_k + \delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl} \delta_{\nu_k} \chi(s) \overline{s}_{\nu_k}(s) ds \end{array}\right].\end{aligned}$$

Proof. Since

$$(\lambda I - \widehat{A}_{\nu_k})^{-1}\begin{pmatrix}0\\ b_1\end{pmatrix} = \frac{1}{(\lambda - i\omega_k)}\begin{pmatrix}0\\ b_1\end{pmatrix}$$

and

$$(\lambda I - \widehat{A}_{\nu_k})^{-1}\begin{pmatrix}0\\ b_2\end{pmatrix} = \frac{1}{(\lambda + i\omega_k)}\begin{pmatrix}0\\ b_2\end{pmatrix},$$

for each $\lambda \in i\mathbb{R} \setminus \{-i\omega_k, i\omega_k\}$,

$$\begin{aligned}&(\lambda I - \widehat{A}_{\nu_k}|_{\Pi_h^{\widehat{A}_{\nu_k}}(X)})^{-1}\Pi_h^{\widehat{A}_{\nu_k}}\begin{pmatrix}1\\ 0\end{pmatrix}\\ &\quad = (\lambda I - \widehat{A}_{\nu_k})^{-1}\Pi_h^{\widehat{A}_{\nu_k}}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}0\\ \phi\end{pmatrix}.\end{aligned}$$

If $\lambda = i\omega_k$, we have

$$\begin{aligned}&(i\omega_k I - \widehat{A}_{\nu_k}|_{\Pi_h^{\widehat{A}_{\nu_k}}(X)})^{-1}\Pi_h^{\widehat{A}_{\nu_k}}\begin{pmatrix}1\\ 0\end{pmatrix}\\ &\quad = \lim_{\substack{\lambda\to i\omega_k\\ \lambda\in\rho(\widehat{A}_{\nu_k})}}(\lambda I - \widehat{A}_{\nu_k}|_{\Pi_h^{\widehat{A}_{\nu_k}}(X)})^{-1}\Pi_h^{\widehat{A}_{\nu_k}}\begin{pmatrix}1\\ 0\end{pmatrix} = \lim_{\substack{\lambda\to i\omega_k\\ \lambda\in\rho(\widehat{A}_{\nu_k})}}\begin{pmatrix}0\\ \phi\end{pmatrix}.\end{aligned}$$

Note that

$$\begin{aligned}&-\left(\frac{d\Delta(\nu_k, i\omega_k)}{d\lambda}\right)^{-1}\frac{b_1(a)}{(\lambda - i\omega_k)}\\ &\qquad + \Delta(\nu_k,\lambda)^{-1}\begin{bmatrix}(1-C_1)e^{-\int_0^a[\lambda+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\\ + C_2\int_0^a e^{-\int_s^a[\lambda+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\delta_{\nu_k}\chi(s)\bar{s}_{\nu_k}(s)ds\end{bmatrix}\\ &\quad = \frac{-\Delta(\nu_k,\lambda)b_1(a) + \frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}(\lambda - i\omega_k)\begin{bmatrix}(1-C_1)e^{-\int_0^a[\lambda+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\\ +C_2\int_0^a e^{-\int_s^a[\lambda+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\delta_{\nu_k}\chi(s)\bar{s}_{\nu_k}(s)ds\end{bmatrix}}{\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}(\lambda - i\omega_k)\Delta(\nu_k,\lambda)}\\ &\quad = \frac{(\lambda - i\omega_k)^2}{\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}(\lambda - i\omega_k)\Delta(\nu_k,\lambda)}\\ &\qquad \times\frac{\left\{-\Delta(\nu_k,\lambda)b_1(a) + \frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}(\lambda - i\omega_k)\begin{bmatrix}(1-C_1)e^{-\int_0^a[\lambda+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\\ +C_2\int_0^a e^{-\int_s^a[\lambda+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\delta_{\nu_k}\chi(s)\bar{s}_{\nu_k}(s)ds\end{bmatrix}\right\}}{(\lambda - i\omega_k)^2}\end{aligned}$$

and

$$\lim_{\lambda\to i\omega_k}\frac{(\lambda-i\omega_k)^2}{\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}(\lambda-i\omega_k)\Delta(\nu_k,\lambda)}=\lim_{\lambda\to i\omega_k}\frac{1}{\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\frac{\Delta(\nu_k,\lambda)}{(\lambda-i\omega_k)}}=\left(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\right)^{-2}.$$

Therefore,

$$\Delta(\nu,\lambda)=(\lambda-i\omega_k)\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}+\frac{(\lambda-i\omega_k)^2}{2}\frac{d^2\Delta(\nu_k,i\omega_k)}{d\lambda^2}+(\lambda-i\omega_k)^3g(\lambda-i\omega_k)$$

with $g(0)=\frac{1}{3!}\frac{d^3\Delta(\nu_k,i\omega_k)}{d\lambda^3}$. Hence

$$\begin{aligned}
&\lim_{\lambda\to i\omega_k}\frac{\left\{-\Delta(\nu_k,\lambda)b_1(a)+\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}(\lambda-i\omega_k)\begin{bmatrix}(1-C_1)e^{-\int_0^a[\lambda+\delta_{\nu_k}(1-\bar{S}_{\nu_k})\chi(l)]dl}\\ +C_2\int_0^a e^{-\int_s^a[\lambda+\delta_{\nu_k}(1-\bar{S}_{\nu_k})\chi(l)]dl}\delta_{\nu_k}\chi(s)\bar{s}_{\nu_k}(s)ds\end{bmatrix}\right\}}{(\lambda-i\omega_k)^2}\\
&=\lim_{\lambda\to i\omega_k}\frac{\left\{-(\lambda-i\omega_k)\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\begin{bmatrix}b_1(a)-(1-C_1)e^{-\int_0^a[\lambda+\delta_{\nu_k}(1-\bar{S}_{\nu_k})\chi(l)]dl}\\ -C_2\int_0^a e^{-\int_s^a[\lambda+\delta_{\nu_k}(1-\bar{S}_{\nu_k})\chi(l)]dl}\delta_{\nu_k}\chi(s)\bar{s}_{\nu_k}(s)ds\end{bmatrix}\right\}}{(\lambda-i\omega_k)^2}\\
&\quad+\lim_{\lambda\to i\omega_k}\frac{-(\frac{(\lambda-i\omega_k)^2}{2}\frac{d^2\Delta(\nu_k,i\omega_k)}{d\lambda^2})b_1(a)}{(\lambda-i\omega_k)^2}\\
&=-M(a)-\frac{1}{2}\frac{d^2\Delta(\nu_k,i\omega_k)}{d\lambda^2}b_1(a),
\end{aligned}$$

which implies (5.6). Similarly, if $\lambda=-i\omega_k$, we can prove (5.7). □

Lemma 5.5. *We have*

$$\begin{aligned}
\Pi_h^{\widehat{A}_{\nu_k}}\begin{pmatrix}0\\ \psi\end{pmatrix}&=\left(I-\Pi_c^{\widehat{A}_{\nu_k}}\right)\begin{pmatrix}0\\ \psi\end{pmatrix}=\begin{pmatrix}0\\ \psi\end{pmatrix}-\Pi_c^{\widehat{A}_{\nu_k}}\begin{pmatrix}0\\ \psi\end{pmatrix}\\
&=\begin{pmatrix}0\\ \psi-\varphi_1-\varphi_2\end{pmatrix}
\end{aligned}$$

with

$$\begin{aligned}
\varphi_1(a)={}&\left(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\right)^{-1}\\
&\times\begin{bmatrix}(-\nu_k\widehat{I}|_{\lambda=i\omega_k,\widehat{\varphi}=\psi})e^{-\int_0^a[i\omega_k+\delta_{\nu_k}(1-\bar{S}_{\nu_k})\chi(l)]dl}\\ +\widehat{I}|_{\lambda=i\omega_k,\widehat{\varphi}=\psi}\int_0^a e^{-\int_s^a[i\omega_k+\delta_{\nu_k}(1-\bar{S}_{\nu_k})\chi(l)]dl}\delta_{\nu_k}\chi(s)\bar{s}_{\nu_k}(s)ds\end{bmatrix}
\end{aligned}\tag{5.8}$$

and

$$\varphi_2(a)=\left(\frac{d\Delta(\nu_k,-i\omega_k)}{d\lambda}\right)^{-1}$$

$$\times \begin{bmatrix} (-\nu_k \widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\psi}) e^{-\int_0^a [-i\omega_k+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl} \\ +\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\psi} \int_0^a e^{-\int_s^a [-i\omega_k+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl} \delta_{\nu_k}\chi(s)\overline{s}_{\nu_k}(s)ds \end{bmatrix}. \tag{5.9}$$

For each $\lambda \in i\mathbb{R} \setminus \{-i\omega_k, i\omega_k\}$,

$$(\lambda I - \widehat{A}_{\nu_k}|_{\Pi_h^{\widehat{A}_{\nu_k}}(X)})^{-1} \Pi_h^{\widehat{A}_{\nu_k}} \begin{pmatrix} 0 \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \phi_1 \end{pmatrix}$$

with

$$\begin{aligned} \phi_1(a) = & -\frac{\varphi_1(a)}{(\lambda - i\omega_k)} - \frac{\varphi_2(a)}{(\lambda + i\omega_k)} + \int_0^a e^{-\int_s^a [\lambda+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl} \psi(s)ds \\ & + \Delta(\nu_k,\lambda)^{-1} \begin{bmatrix} (-\nu_k \widehat{I}|_{\widehat{\varphi}=\psi}) e^{-\int_0^a [\lambda+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl} \\ +\widehat{I}|_{\widehat{\varphi}=\psi} \int_0^a e^{-\int_s^a [\lambda+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl} \delta_{\nu_k}\chi(s)\overline{s}_{\nu_k}(s)ds \end{bmatrix}. \end{aligned}$$

Moreover, if $\lambda = i\omega_k$, we have

$$\begin{aligned} & (i\omega_k I - \widehat{A}_{\nu_k}|_{\Pi_h^{\widehat{A}_{\nu_k}}(X)})^{-1} \Pi_h^{\widehat{A}_{\nu_k}} \begin{pmatrix} 0 \\ \psi \end{pmatrix} \\ & = \begin{pmatrix} 0 \\ -\frac{\varphi_2}{2i\omega_k} + M_2 - (\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda})^{-1}(M_1 + \frac{1}{2}\frac{d^2\Delta(\nu_k,i\omega_k)}{d\lambda^2}\varphi_1) \end{pmatrix} \end{aligned} \tag{5.10}$$

with

$$\begin{aligned} M_1(a) = {} & a\varphi_1(a)\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda} - \widehat{I}|_{\widehat{\varphi}=\psi,\lambda=i\omega_k} \int_0^a s e^{-\int_s^a [i\omega_k+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl} \delta_{\nu_k}\chi(s)\overline{s}_{\nu_k}(s)ds \\ & - \nu_k \left(\int_0^{+\infty}\int_0^a (a-s) e^{-\int_s^a [i\omega_k+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl} \psi(s)ds da \right) e^{-\int_0^a [i\omega_k+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl} \\ & + \left(\int_0^{+\infty}\int_0^a (a-s) e^{-\int_s^a [i\omega_k+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl} \psi(s)ds da \right) \\ & \times \left(\int_0^a e^{-\int_s^a [i\omega_k+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl} \delta_{\nu_k}\chi(s)\overline{s}_{\nu_k}(s)ds \right) \end{aligned}$$

and

$$M_2(a) = \int_0^a e^{-\int_s^a [i\omega_k+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl} \psi(s)ds.$$

If $\lambda = -i\omega_k$, we have

$$(-i\omega_k I - \widehat{A}_{\nu_k}|_{\Pi_h^{\widehat{A}_{\nu_k}}(X)})^{-1}\Pi_h^{\widehat{A}_{\nu_k}}\begin{pmatrix}0\\ \psi\end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ \frac{\varphi_1}{2i\omega_k} + N_2 - (\frac{d\Delta(\nu_k,-i\omega_k)}{d\lambda})^{-1}(N_1 + \frac{1}{2}\frac{d^2\Delta(\nu_k,-i\omega_k)}{d\lambda^2}\varphi_2)\end{pmatrix} \tag{5.11}$$

with

$$N_1(a) = a\varphi_2(a)\frac{d\Delta(\nu_k,-i\omega_k)}{d\lambda} - \widehat{I}|_{\widehat{\varphi}=\psi,\lambda=-i\omega_k}\int_0^a se^{-\int_s^a[-i\omega_k+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\delta_{\nu_k}\chi(s)\bar{s}_{\nu_k}(s)ds$$
$$-\nu_k\left(\int_0^{+\infty}\int_0^a(a-s)e^{-\int_s^a[-i\omega_k+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\psi(s)dsda\right)e^{-\int_0^a[-i\omega_k+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}$$
$$+\left(\int_0^{+\infty}\int_0^a(a-s)e^{-\int_s^a[-i\omega_k+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\psi(s)dsda\right)$$
$$\times\left(\int_0^a e^{-\int_s^a[-i\omega_k+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\delta_{\nu_k}\chi(s)\bar{s}_{\nu_k}(s)ds\right)$$

and

$$N_2(a) = \int_0^a e^{-\int_s^a[-i\omega_k+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\psi(s)ds.$$

Proof. Since

$$(\lambda I - \widehat{A}_{\nu_k}|_{\Pi_h^{\widehat{A}_{\nu_k}}(X)})^{-1}\begin{pmatrix}0\\ \varphi_1\end{pmatrix} = \frac{1}{(\lambda - i\omega_k)}\begin{pmatrix}0\\ \varphi_1\end{pmatrix}$$

and

$$(\lambda I - \widehat{A}_{\nu_k}|_{\Pi_h^{\widehat{A}_{\nu_k}}(X)})^{-1}\begin{pmatrix}0\\ \varphi_2\end{pmatrix} = \frac{1}{(\lambda + i\omega_k)}\begin{pmatrix}0\\ \varphi_2\end{pmatrix},$$

for each $\lambda \in i\mathbb{R} \setminus \{-i\omega_k, i\omega_k\}$,

$$(\lambda I - \widehat{A}_{\nu_k}|_{\Pi_h^{\widehat{A}_{\nu_k}}(X)})^{-1}\Pi_h^{\widehat{A}_{\nu_k}}\begin{pmatrix}0\\ \psi\end{pmatrix} = \begin{pmatrix}0\\ \phi_1\end{pmatrix}.$$

If $\lambda = i\omega_k$, we have

$$
\begin{aligned}
&(i\omega_k I - \widehat{A}_{\nu_k}|_{\Pi_h^{\widehat{A}_{\nu_k}}(X)})^{-1}\Pi_h^{\widehat{A}_{\nu_k}}\begin{pmatrix}0\\ \psi\end{pmatrix}\\
&\quad = \lim_{\substack{\lambda\to i\omega_k\\ \lambda\in\rho(\widehat{A}_{\nu_k})}} (\lambda I - \widehat{A}_{\nu_k}|_{\Pi_h^{\widehat{A}_{\nu_k}}(X)})^{-1}\Pi_h^{\widehat{A}_{\nu_k}}\begin{pmatrix}0\\ \psi\end{pmatrix} = \lim_{\substack{\lambda\to i\omega_k\\ \lambda\in\rho(\widehat{A}_{\nu_k})}} \begin{pmatrix}0\\ \phi_1\end{pmatrix}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&-\frac{\varphi_1(a)}{(\lambda - i\omega_k)} + \Delta(\nu_k,\lambda)^{-1}\begin{bmatrix}(-\nu_k\widehat{I}|_{\widehat{\varphi}=\psi})e^{-\int_0^a[\lambda+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\\ +\widehat{I}|_{\widehat{\varphi}=\psi}\int_0^a e^{-\int_s^a[\lambda+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\delta_{\nu_k}\chi(s)\bar{s}_{\nu_k}(s)ds\end{bmatrix}\\
&\quad = \frac{-\Delta(\nu_k,\lambda)\varphi_1(a) + (\lambda - i\omega_k)\begin{bmatrix}(-\nu_k\widehat{I}|_{\widehat{\varphi}=\psi})e^{-\int_0^a[\lambda+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\\ +\widehat{I}|_{\widehat{\varphi}=\psi}\int_0^a e^{-\int_s^a[\lambda+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\delta_{\nu_k}\chi(s)\bar{s}_{\nu_k}(s)ds\end{bmatrix}}{(\lambda - i\omega_k)\Delta(\nu_k,\lambda)}\\
&\quad = \frac{(\lambda - i\omega_k)^2}{(\lambda - i\omega_k)\Delta(\nu_k,\lambda)}\\
&\qquad \times \frac{\left\{-\Delta(\nu_k,\lambda)\varphi_1(a) + (\lambda - i\omega_k)\begin{bmatrix}(-\nu_k\widehat{I}|_{\widehat{\varphi}=\psi})e^{-\int_0^a[\lambda+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\\ +\widehat{I}|_{\widehat{\varphi}=\psi}\int_0^a e^{-\int_s^a[\lambda+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\delta_{\nu_k}\chi(s)\bar{s}_{\nu_k}(s)ds\end{bmatrix}\right\}}{(\lambda - i\omega_k)^2}
\end{aligned}
$$

and

$$
\lim_{\lambda\to i\omega_k}\frac{(\lambda - i\omega_k)^2}{(\lambda - i\omega_k)\Delta(\nu_k,\lambda)} = \left(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\right)^{-1}.
$$

We have

$$
\Delta(\nu_k,\lambda) = (\lambda - i\omega_k)\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda} + \frac{(\lambda - i\omega_k)^2}{2}\frac{d^2\Delta(\nu_k,i\omega_k)}{d\lambda^2} + (\lambda - i\omega_k)^3 g(\lambda - i\omega_k)
$$

with $g(0) = \frac{1}{3!}\frac{d^3\Delta(\nu_k,i\omega_k)}{d\lambda^3}$. Hence

$$
\begin{aligned}
&\lim_{\lambda\to i\omega_k}\frac{\left\{-\Delta(\nu_k,\lambda)\varphi_1(a) + (\lambda - i\omega_k)\begin{bmatrix}(-\nu_k\widehat{I}|_{\widehat{\varphi}=\psi})e^{-\int_0^a[\lambda+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\\ +\widehat{I}|_{\widehat{\varphi}=\psi}\int_0^a e^{-\int_s^a[\lambda+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\delta_{\nu_k}\chi(s)\bar{s}_{\nu_k}(s)ds\end{bmatrix}\right\}}{(\lambda - i\omega_k)^2}\\
&\quad = \lim_{\lambda\to i\omega_k}\frac{\left\{-(\lambda - i\omega_k)\begin{bmatrix}\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\varphi_1(a)\\ -\begin{bmatrix}(-\nu\widehat{I}|_{\widehat{\varphi}=\psi})e^{-\int_0^a[\lambda+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\\ +\widehat{I}|_{\widehat{\varphi}=\psi}\int_0^a e^{-\int_s^a[\lambda+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\delta_{\nu_k}\chi(s)\bar{s}_{\nu_k}(s)ds\end{bmatrix}\end{bmatrix}\right\}}{(\lambda - i\omega_k)^2}
\end{aligned}
$$

$$+\lim_{\lambda\to i\omega_k}\frac{-(\frac{(\lambda-i\omega_k)^2}{2}\frac{d^2\Delta(\nu_k,i\omega_k)}{d\lambda^2})\varphi_1(a)}{(\lambda-i\omega_k)^2}$$

$$=-M_1(a)-\frac{1}{2}\frac{d^2\Delta(\nu_k,i\omega_k)}{d\lambda^2}\varphi_1(a),$$

which yields (5.10). If $\lambda=-i\omega_k$, we can prove (5.11) similarly. □

Lemma 5.6. *We have*

$$\sigma(L)=\sigma(\widehat{A}_{\nu_k})\cup\{0\}.$$

Moreover, for $\lambda\in\rho(L)\cap\Omega=\Omega\setminus(\sigma(\widehat{A}_{\nu_k})\cup\{0\})$, we have

$$(\lambda I-L)^{-1}\begin{pmatrix} r\\ \begin{pmatrix}\varrho\\ \psi\end{pmatrix}\end{pmatrix}=\begin{pmatrix}\frac{r}{\lambda}\\ (\lambda I-\widehat{A}_{\nu_k})^{-1}\begin{pmatrix}\varrho\\ \psi\end{pmatrix}\end{pmatrix}$$

and the eigenvalues 0 *and* $\pm i\omega_k$ *of* L *are simple. The corresponding projectors* $\Pi_0,\Pi_{\pm i\omega_k}:Y\to Y$ *on the generalized eigenspace of* L *associated to* $0,\pm i\omega_k$*, respectively, are given by*

$$\Pi_0\begin{pmatrix} r\\ u\end{pmatrix}=\begin{pmatrix} r\\ 0\end{pmatrix},$$

$$\Pi_{\pm i\omega_k}\begin{pmatrix} r\\ u\end{pmatrix}=\begin{pmatrix} 0\\ \Pi^{\widehat{A}_{\nu_k}}_{\pm i\omega_k}u\end{pmatrix}.$$

In this context, the projectors $\Pi_c:Y\to Y$ and $\Pi_h:Y\to Y$ are defined by

$$\Pi_c(y)=(\Pi_0+\Pi_{i\omega_k}+\Pi_{-i\omega_k})(y),\quad \forall y\in Y,$$
$$\Pi_h(y)=(I-\Pi_c)(y),\quad \forall y\in Y.$$

Denote

$$Y_c:=\Pi_c(Y),\qquad Y_h:=\Pi_h(Y),$$

and

$$L_c:=L|_{Y_c},\qquad L_h:=L|_{Y_h}.$$

Then we have

$$\Pi_c\begin{pmatrix}0_{\mathbb{R}}\\ \begin{pmatrix}1\\ 0_{L^1}\end{pmatrix}\end{pmatrix}=\begin{pmatrix}0_{\mathbb{R}}\\ \Pi^{\widehat{A}_{\nu_k}}_{i\omega_k}\begin{pmatrix}1\\ 0_{L^1}\end{pmatrix}+\Pi^{\widehat{A}_{\nu_k}}_{-i\omega_k}\begin{pmatrix}1\\ 0_{L^1}\end{pmatrix}\end{pmatrix}=\begin{pmatrix}0_{\mathbb{R}}\\ \Pi^{\widehat{A}_{\nu_k}}_{c}\begin{pmatrix}1\\ 0_{L^1}\end{pmatrix}\end{pmatrix}$$

and

$$\Pi_c \left(\begin{pmatrix} 0 \\ \begin{pmatrix} 0 \\ \psi \end{pmatrix} \end{pmatrix} \right) = \begin{pmatrix} 0 \\ \Pi_{i\omega_k}^{\widehat{A}_{\nu_k}} \begin{pmatrix} 0 \\ \psi \end{pmatrix} + \Pi_{-i\omega_k}^{\widehat{A}_{\nu_k}} \begin{pmatrix} 0 \\ \psi \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ \Pi_c^{\widehat{A}_{\nu_k}} \begin{pmatrix} 0 \\ \psi \end{pmatrix} \end{pmatrix}.$$

Define the basis of $Y_c = \Pi_c(Y)$ by

$$e_1 = \begin{pmatrix} 1 \\ \begin{pmatrix} 0_{\mathbb{R}} \\ 0_{L^1} \end{pmatrix} \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0_{\mathbb{R}} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ b_1 \end{pmatrix} \end{pmatrix}, \qquad e_3 = \begin{pmatrix} 0_{\mathbb{R}} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ b_2 \end{pmatrix} \end{pmatrix}.$$

We have the following lemma.

Lemma 5.7. *For $\lambda \in i\mathbb{R}$ we have*

$$(\lambda I - L_h)^{-1} \Pi_h \left(\begin{pmatrix} 0 \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \right) = \begin{pmatrix} 0 \\ (\lambda I - \widehat{A}_{\nu_k}|_{\Pi_h^{\widehat{A}_{\nu_k}}(X)})^{-1} \Pi_h^{\widehat{A}_{\nu_k}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$

and

$$(\lambda I - L_h)^{-1} \Pi_h \left(\begin{pmatrix} 0 \\ \begin{pmatrix} 0 \\ \psi \end{pmatrix} \end{pmatrix} \right) = \begin{pmatrix} 0 \\ (\lambda I - \widehat{A}_{\nu_k}|_{\Pi_h^{\widehat{A}_{\nu_k}}(X)})^{-1} \Pi_h^{\widehat{A}_{\nu_k}} \begin{pmatrix} 0 \\ \psi \end{pmatrix} \end{pmatrix}.$$

In the following, we will compute the Taylor expansion of the reduced system of (5.3). We apply the procedure described at the end of Section 4 and apply the method with $k = 2$ in Theorem 4.2. For $j = 2$, we must find $L_2 \in \mathcal{L}_s(Y_c^2, Y_h \cap D(L))$ by solving the following equation for each $(w_1, w_2) \in Y_c^2$:

$$\frac{d}{dt}\left[L_2\left(e^{L_c t} w_1, e^{L_c t} w_2\right)\right](0) = L_h L_2(w_1, w_2) + \frac{1}{2!} \Pi_h D^2 H(0)(w_1, w_2). \tag{5.12}$$

Define $G_2 : Y \to Y_h \cap D(L)$ by

$$G_2(\Pi_c w) := L_2(\Pi_c w, \Pi_c w), \quad \forall w \in Y,$$

$\xi_2 : Y \to Y$ and $\xi_2^{-1} : Y \to Y$ by

$$\xi_2(w) := w - G_2(\Pi_c w) \quad \text{and} \quad \xi_2^{-1}(w) := w + G_2(\Pi_c w), \quad \forall w \in Y.$$

Then we define $H_2 : \overline{D(L)} \to Y$ by

$$\begin{aligned} H_2(w) = {} & H\left(\xi_2^{-1}(w)\right) + L G_2(\Pi_c w) - D G_2(\Pi_c w) L_c \Pi_c w \\ & - D G_2(\Pi_c w) \Pi_c H\left(\xi_2^{-1}(w)\right). \end{aligned}$$

By applying Theorem 4.2 to (5.3) with $k = 2$, we obtain the following theorem.

Theorem 5.8. *By using the change of variables*

$$w_2(t) = w(t) - G_2\big(\Pi_c w(t)\big) \quad \Leftrightarrow \quad w(t) = w_2(t) + G_2\big(\Pi_c w_2(t)\big),$$

the map $t \to w(t)$ is an integrated solution of the Cauchy problem (5.3) if and only if $t \to w_2(t)$ is an integrated solution of the Cauchy problem

$$\begin{cases} \dfrac{dw_2(t)}{dt} = Lw_2(t) + H_2\big(w_2(t)\big), & t \geq 0, \\ w_2(0) = w_2 \in \overline{D(L)}. \end{cases} \tag{5.13}$$

Moreover, the reduced system of the Cauchy problem (5.13) is given by the ordinary differential equations on $\mathbb{R} \times X_c$:

$$\begin{cases} \dfrac{d\widehat{\nu}(t)}{dt} = 0, \\ \dfrac{dx_c(t)}{dt} = (\widehat{A}_{\nu_k})_c x_c(t) + \Pi_c^{\widehat{A}_{\nu_k}} W(I + G_2)\begin{pmatrix} \widehat{\nu}(t) \\ x_c(t) \end{pmatrix} + \widehat{R}_c\begin{pmatrix} \widehat{\nu}(t) \\ x_c(t) \end{pmatrix}, \end{cases} \tag{5.14}$$

where $\widehat{R}_c \in C^4(\mathbb{R} \times X_c, X_c)$, and $\widehat{R}_c\begin{pmatrix} \widehat{\nu}(t) \\ x_c(t) \end{pmatrix}$ is a remainder of order 4, that is,

$$\widehat{R}_c\begin{pmatrix} \widehat{\nu} \\ x_c \end{pmatrix} = \big\|(\widehat{\nu}, x_c)\big\|^4 O(\widehat{\nu}, x_c),$$

where $O(\widehat{\nu}, x_c)$ is a function of $(\widehat{\nu}, x_c)$ which remains bounded when $(\widehat{\nu}, x_c)$ goes to 0, or equivalently,

$$D^j \widehat{R}_c(0) = 0 \quad \textit{for each } j = 1, \ldots, 3.$$

Furthermore,

$$\frac{\partial^j \widehat{R}_c(0)}{\partial^j \widehat{\nu}} = 0, \quad \forall j = 1, \ldots, 4,$$

which implies that

$$\widehat{R}_c\begin{pmatrix} \widehat{\nu} \\ x_c \end{pmatrix} = O\big(\widehat{\nu}^3 \|x_c\| + \widehat{\nu}^2 \|x_c\|^2 + \widehat{\nu}\|x_c\|^3 + \|x_c\|^4\big).$$

Proof. We apply Theorem 4.2 to system (5.3) and deduce that the reduced system of (5.13) consists of ordinary differential equations on X_c of the form

$$\frac{dw_c(t)}{dt} = L_c w_c(t) + \Pi_c^L H\big[w_c(t) + G_2\big(w_c(t)\big)\big] + R_c\big(w_c(t)\big),$$

where $R_c \in C^4(X_c, X_c)$ is the remainder term of order 4, which means that $D^j R_c(0) = 0$, $\forall j = 1, 2, 3$. Since

$$R_c(w_c) := \Pi_c^L \big\{ H\big[w_c + G_2(w_c) + \Psi_2(w_c)\big] - H\big[w_c + G_2(w_c)\big]\big\},$$

and the first component of H is 0, by using the formula for Π_c^L we deduce that $R_c(w_c)$ has the following form

$$R_c\begin{pmatrix} \widehat{\nu} \\ x_c \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}} \\ \widehat{R}_c\left(\begin{smallmatrix} \widehat{\nu} \\ x_c \end{smallmatrix}\right) \end{pmatrix}.$$

Moreover, since for each $\widehat{\nu} \in \mathbb{R}$ small enough, $\left(\begin{smallmatrix} \widehat{\nu} \\ 0_X \end{smallmatrix}\right)$ is an equilibrium solution of (5.13) and belongs to the center manifold X_c. It follows that

$$\begin{pmatrix} \widehat{\nu} \\ 0_X \end{pmatrix} = \Pi_c^L \begin{pmatrix} \widehat{\nu} \\ 0_X \end{pmatrix} + \Psi_2\left(\Pi_c^L \begin{pmatrix} \widehat{\nu} \\ 0_X \end{pmatrix}\right).$$

Thus

$$\Psi_2 \begin{pmatrix} \widehat{\nu} \\ 0_X \end{pmatrix} = 0, \quad \forall \widehat{\nu} \in \mathbb{R} \text{ small enough.}$$

So we must have

$$\widehat{R}_c \begin{pmatrix} \widehat{\nu} \\ 0_X \end{pmatrix} = 0, \quad \forall \widehat{\nu} \in \mathbb{R} \text{ small enough.}$$

We deduce that

$$\frac{\partial^j \widehat{R}_c(0)}{\partial^j \widehat{\nu}} = 0, \quad \forall j = 1, \dots, 4.$$

This completes the proof. □

In order to apply the above theorem and to compute the Taylor expansion of the reduced system it only remains to compute L_2.

Set

$$w_1 := \begin{pmatrix} \widehat{\nu}_1 \\ \widehat{u}_1 \end{pmatrix}, \; w_2 := \begin{pmatrix} \widehat{\nu}_2 \\ \widehat{u}_2 \end{pmatrix}, \; w_3 := \begin{pmatrix} \widehat{\nu}_3 \\ \widehat{u}_3 \end{pmatrix} \in \overline{D(L)},$$

with $\widehat{u}_i = \left(\begin{smallmatrix} 0_{\mathbb{R}} \\ \varphi_i \end{smallmatrix}\right)$, $i = 1, 2, 3$. We have

$$D^2 H(0)(w_1, w_2) = \begin{pmatrix} 0_{\mathbb{R}} \\ D^2 W(0)(w_1, w_2) \end{pmatrix}$$

and

$$D^3 H(0)(w_1, w_2, w_3) = \begin{pmatrix} 0_{\mathbb{R}} \\ D^3 W(0)(w_1, w_2, w_3) \end{pmatrix},$$

where

$$D^2W(0)(w_1, w_2) = \begin{pmatrix} C_1(w_1, w_2) \\ C_2(w_1, w_2) \end{pmatrix}$$

with

$$C_1(w_1, w_2) = -\widehat{\nu}_1 \int_0^{+\infty} \varphi_2(l)dl - \widehat{\nu}_2 \int_0^{+\infty} \varphi_1(l)dl,$$

$$C_2(w_1, w_2) = \delta_{\nu_k}\chi \begin{bmatrix} (\varphi_2 + \frac{d\bar{s}_{\nu_k}}{dv}\widehat{\nu}_2)\int_0^{+\infty}\varphi_1(l)dl \\ + (\varphi_2\widehat{\nu}_1 + \varphi_1\widehat{\nu}_2)\frac{d\overline{S}_{\nu_k}}{dv} \\ + (\varphi_1 + \frac{d\bar{s}_{\nu_k}}{dv}\widehat{\nu}_1)\int_0^{+\infty}\varphi_2(l)dl \end{bmatrix} + \frac{d\delta_{\nu_k}}{dv}\chi \begin{bmatrix} \widehat{\nu}_2(\bar{s}_{\nu_k}\int_0^{+\infty}\varphi_1(l)dl - \varphi_1(1-\overline{S}_{\nu_k})) \\ + \widehat{\nu}_1(\bar{s}_{\nu_k}\int_0^{+\infty}\varphi_2(l)dl - \varphi_2(1-\overline{S}_{\nu_k})) \end{bmatrix}$$

and

$$D^3W(0)(w_1, w_2, w_3) = \begin{pmatrix} 0 \\ C_3(w_1, w_2, w_3) \end{pmatrix}$$

with

$$\begin{aligned} C_3(w_1, w_2, w_3) = {} & \delta_{\nu_k}\chi \begin{bmatrix} (\varphi_2\widehat{\nu}_1\widehat{\nu}_3 + \varphi_3\widehat{\nu}_1\widehat{\nu}_2 + \varphi_1\widehat{\nu}_2\widehat{\nu}_3)\frac{d^2\overline{S}_{\nu_k}}{dv^2} \\ + \frac{d^2\bar{s}_{\nu_k}}{dv^2}\begin{pmatrix} \widehat{\nu}_2\widehat{\nu}_3\int_0^{+\infty}\varphi_1(l)dl + \widehat{\nu}_1\widehat{\nu}_2\int_0^{+\infty}\varphi_3(l)dl \\ + \widehat{\nu}_1\widehat{\nu}_3\int_0^{+\infty}\varphi_2(l)dl \end{pmatrix} \end{bmatrix} \\ & + \frac{d\delta_{\nu_k}}{dv}\widehat{\nu}_3\chi \begin{bmatrix} \varphi_1\int_0^{+\infty}\varphi_2 + \frac{d\bar{s}_{\nu_k}}{dv}\widehat{\nu}_2 dl + \varphi_2\int_0^{+\infty}\varphi_1 + \frac{d\bar{s}_{\nu_k}}{dv}\widehat{\nu}_1 dl \\ + \frac{d\bar{s}_{\nu_k}}{dv}(\widehat{\nu}_1\int_0^{+\infty}\varphi_2 dl + \widehat{\nu}_2\int_0^{+\infty}\varphi_1 dl) \end{bmatrix} \\ & + \frac{d\delta_{\nu_k}}{dv}\widehat{\nu}_2\chi \begin{bmatrix} \varphi_1\int_0^{+\infty}\varphi_3 + \frac{d\bar{s}_{\nu_k}}{dv}\widehat{\nu}_3 dl + \varphi_3\int_0^{+\infty}\varphi_1 + \frac{d\bar{s}_{\nu_k}}{dv}\widehat{\nu}_1 dl \\ + \frac{d\bar{s}_{\nu_k}}{dv}(\widehat{\nu}_1\int_0^{+\infty}\varphi_3 dl + \widehat{\nu}_3\int_0^{+\infty}\varphi_1 dl) \end{bmatrix} \\ & + \frac{d\delta_{\nu_k}}{dv}\widehat{\nu}_1\chi \begin{bmatrix} \varphi_2\int_0^{+\infty}\varphi_3 + \frac{d\bar{s}_{\nu_k}}{dv}\widehat{\nu}_3 dl + \varphi_3\int_0^{+\infty}\varphi_2 + \frac{d\bar{s}_{\nu_k}}{dv}\widehat{\nu}_2 dl \\ + \frac{d\bar{s}_{\nu_k}}{dv}(\widehat{\nu}_2\int_0^{+\infty}\varphi_3 dl + \widehat{\nu}_3\int_0^{+\infty}\varphi_2 dl) \end{bmatrix} \\ & + \frac{d^2\delta_{\nu_k}}{dv^2}\widehat{\nu}_3\widehat{\nu}_2\chi \left[-\varphi_1(1-\overline{S}_{\nu_k}) + \bar{s}_{\nu_k}\int_0^{+\infty}\varphi_1 dl \right] \\ & + \frac{d^2\delta_{\nu_k}}{dv^2}\widehat{\nu}_2\widehat{\nu}_1\chi \left[-\varphi_3(1-\overline{S}_{\nu_k}) + \bar{s}_{\nu_k}\int_0^{+\infty}\varphi_3 dl \right] \\ & + \frac{d^2\delta_{\nu_k}}{dv^2}\widehat{\nu}_3\widehat{\nu}_1\chi \left[-\varphi_2(1-\overline{S}_{\nu_k}) + \bar{s}_{\nu_k}\int_0^{+\infty}\varphi_2 dl \right]. \end{aligned}$$

Next recall that

$$\frac{d}{dt}\left[L_2\left(e^{L_c t}w_1, e^{L_c t}w_2\right)\right](0) = L_2(L_c w_1, w_2) + L_2(w_1, L_c w_2).$$

So system (5.12) can be rewritten as

$$L_2(L_c w_1, w_2) + L_2(w_1, L_c w_2) = L_h L_2(w_1, w_2) + \frac{1}{2!}\Pi_h D^2 H(0)(w_1, w_2). \tag{5.15}$$

(i) Computation of $L_2(e_1, e_1)$. We have

$$\Pi_h D^2 H(0)(e_1, e_1) = 0, \qquad L_c e_1 = 0.$$

So the equation

$$L_2(L_c e_1, e_1) + L_2(e_1, L_c e_1) = L_h L_2(e_1, e_1) + \frac{1}{2!}\Pi_h D^2 H(0)(e_1, e_1)$$

is equivalent to

$$0 = L_h L_2(e_1, e_1).$$

Since 0 belongs to the resolvent set of L_h, we obtain that

$$L_2(e_1, e_1) = 0. \tag{5.16}$$

(ii) Computation of $L_2(e_1, e_2)$. We have

$$D^2 H(0)(e_1, e_2) = \begin{pmatrix} 0 \\ \begin{pmatrix} -\int_0^{+\infty} b_1(l)dl \\ \Theta_{1,2} \end{pmatrix} \end{pmatrix}$$

with

$$\Theta_{1,2} = \delta_{\nu_k}\chi\left[b_1\frac{d\overline{S}_{\nu_k}}{dv} + \frac{d\overline{s}_{\nu_k}}{dv}\int_0^{+\infty} b_1(l)dl\right] + \frac{d\delta_{\nu_k}}{dv}\chi\left[\overline{s}_{\nu_k}\int_0^{+\infty} b_1(l)dl - b_1(1-\overline{S}_{\nu_k})\right]$$

and

$$L_c e_1 = 0, \qquad L_c e_2 = i\omega_k e_2.$$

So in this case system (5.15) becomes

$$i\omega_k L_2(e_1,e_2) = L_h L_2(e_1,e_2) + \frac{1}{2!}\Pi_h D^2 H(0)(e_1,e_2).$$

Thus

$$(i\omega_k I - L_h) L_2(e_1,e_2) = \frac{1}{2!}\Pi_h D^2 H(0)(e_1,e_2).$$

Now by using Lemma 5.7, we obtain

$$\begin{aligned}
L_2(e_1,e_2) &= \frac{1}{2}(i\omega_k I - L_h)^{-1}\Pi_h \left(\begin{pmatrix} 0 \\ \begin{pmatrix} -\int_0^{+\infty} b_1(l)dl \\ \Theta_{1,2} \end{pmatrix} \end{pmatrix}\right) \\
&= -\frac{\int_0^{+\infty} b_1(l)dl}{2}(i\omega_k I - L_h)^{-1}\Pi_h \left(\begin{pmatrix} 0 \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix}\end{pmatrix}\right) + \frac{1}{2}(i\omega_k I - L_h)^{-1}\Pi_h \left(\begin{pmatrix} 0 \\ \begin{pmatrix} 0 \\ \Theta_{1,2} \end{pmatrix}\end{pmatrix}\right) \\
&= -\frac{\int_0^{+\infty} b_1(l)dl}{2}\left(\begin{pmatrix} 0 \\ \begin{pmatrix} 0 \\ -(\frac{d\Delta(\nu_k,-i\omega_k)}{d\lambda})^{-1}\frac{b_2}{2i\omega_k} - (\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda})^{-2}(\frac{1}{2}\frac{d^2\Delta(\nu_k,i\omega_k)}{d\lambda^2}b_1 + M) \end{pmatrix}\end{pmatrix}\right) \\
&\quad + \frac{1}{2}\left(\begin{pmatrix} 0 \\ \begin{pmatrix} 0 \\ -\frac{\varphi_2}{2i\omega_k} + M_2 - (\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda})^{-1}(M_1 + \frac{1}{2}\frac{d^2\Delta(\nu_k,i\omega_k)}{d\lambda^2}\varphi_1) \end{pmatrix}\end{pmatrix}\right)
\end{aligned}$$

where M, M_1 and M_2 are defined in Lemmas 5.4 and 5.5 with $\psi = \Theta_{1,2}$. Hence

$$L_2(e_1,e_2) = L_2(e_2,e_1) = \left(\begin{pmatrix} 0 \\ \begin{pmatrix} 0 \\ \psi_{1,2} \end{pmatrix}\end{pmatrix}\right), \tag{5.17}$$

where

$$\begin{aligned}
\psi_{1,2} = &-\frac{\int_0^{+\infty} b_1(l)dl}{2}\left[\begin{array}{c} -(\frac{d\Delta(\nu_k,-i\omega_k)}{d\lambda})^{-1}\frac{b_2}{2i\omega_k} \\ -(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda})^{-2}(\frac{1}{2}\frac{d^2\Delta(\nu_k,i\omega_k)}{d\lambda^2}b_1 + M) \end{array}\right] \\
&- \frac{\varphi_2}{4i\omega_k} + \frac{1}{2}M_2 - \frac{1}{2}\left(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\right)^{-1}\left(M_1 + \frac{1}{2}\frac{d^2\Delta(\nu_k,i\omega_k)}{d\lambda^2}\varphi_1\right)
\end{aligned}$$

with

$$\psi = \Theta_{1,2}.$$

(iii) Computation of $L_2(e_1,e_3)$. We have

$$D^2 H(0)(e_1,e_3) = \left(\begin{pmatrix} 0 \\ \begin{pmatrix} -\int_0^{+\infty} b_2(l)dl \\ \Theta_{1,3} \end{pmatrix}\end{pmatrix}\right)$$

with

$$
\begin{aligned}
\Theta_{1,3} = {} & \delta_{\nu_k}\chi\left[b_2\frac{d\overline{S}_{\nu_k}}{dv} + \frac{d\overline{s}_{\nu_k}}{dv}\int_0^{+\infty} b_2(l)dl\right] \\
& + \frac{d\delta_{\nu_k}}{dv}\chi\left[\overline{s}_{\nu_k}\int_0^{+\infty} b_2(l)dl - b_2(1-\overline{S}_{\nu_k})\right]
\end{aligned}
$$

and

$$
L_c e_1 = 0, \qquad L_c e_3 = -i\omega_k e_3.
$$

In this case system (5.15) becomes

$$
-i\omega_k L_2(e_1,e_3) = L_h L_2(e_1,e_3) + \frac{1}{2!}\Pi_h D^2 H(0)(e_1,e_3).
$$

So

$$
(-i\omega_k I - L_h)L_2(e_1,e_3) = \frac{1}{2!}\Pi_h D^2 H(0)(e_1,e_3).
$$

Thus, by Lemma 5.7 we have

$$
\begin{aligned}
L_2(e_1,e_3) &= \frac{1}{2}(-i\omega_k I - L_h)^{-1}\Pi_h\begin{pmatrix}0\\ \begin{pmatrix}-\int_0^{+\infty} b_2(l)dl\\ \Theta_{1,3}\end{pmatrix}\end{pmatrix} \\
&= -\frac{\int_0^{+\infty} b_2(l)dl}{2}(-i\omega_k I - L_h)^{-1}\Pi_h\begin{pmatrix}0\\ \begin{pmatrix}1\\ 0\end{pmatrix}\end{pmatrix} + \frac{1}{2}(-i\omega_k I - L_h)^{-1}\Pi_h\begin{pmatrix}0\\ \begin{pmatrix}0\\ \Theta_{1,3}\end{pmatrix}\end{pmatrix} \\
&= -\frac{\int_0^{+\infty} b_2(l)dl}{2}\begin{pmatrix}0\\ \begin{pmatrix}0\\ \frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}^{-1}\frac{b_1}{2i\omega_k} - (\frac{d\Delta(\nu_k,-i\omega_k)}{d\lambda})^{-2}(\frac{1}{2}\frac{d^2\Delta(\nu_k,-i\omega_k)}{d\lambda^2}b_2 + N)\end{pmatrix}\end{pmatrix} \\
&\quad + \frac{1}{2}\begin{pmatrix}0\\ \begin{pmatrix}0\\ \frac{\varphi_1}{2i\omega_k} + N_2 - (\frac{d\Delta(\nu_k,-i\omega_k)}{d\lambda})^{-1}(N_1 + \frac{1}{2}\frac{d^2\Delta(\nu_k,-i\omega_k)}{d\lambda^2}\varphi_2)\end{pmatrix}\end{pmatrix}
\end{aligned}
$$

where N, N_1 and N_2 are defined in Lemmas 5.4 and 5.5 with $\psi = \Theta_{1,3}$. Then

$$
L_2(e_1,e_3) = L_2(e_3,e_1) = \begin{pmatrix}0\\ \begin{pmatrix}0\\ \psi_{1,3}\end{pmatrix}\end{pmatrix}, \tag{5.18}
$$

where

$$
\begin{aligned}
\psi_{1,3} = -\frac{\int_0^{+\infty} b_2(l)dl}{2}\Bigg[& \left(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\right)^{-1}\frac{b_1}{2i\omega_k} \\
& - \left(\frac{d\Delta(\nu_k,-i\omega_k)}{d\lambda}\right)^{-2}\left(\frac{1}{2}\frac{d^2\Delta(\nu_k,-i\omega_k)}{d\lambda^2}b_2 + N\right)\Bigg]
\end{aligned}
$$

$$+\frac{\varphi_1}{4i\omega_k}+\frac{1}{2}N_2-\frac{1}{2}\left(\frac{d\Delta(\nu_k,-i\omega_k)}{d\lambda}\right)^{-1}\left(N_1+\frac{1}{2}\frac{d^2\Delta(\nu_k,-i\omega_k)}{d\lambda^2}\varphi_2\right)$$

with

$$\psi=\Theta_{1,3}.$$

(iv) Computation of $L_2(e_2,e_2)$. We have

$$D^2H(0)(e_2,e_2)=\left(\begin{pmatrix}0\\ 0\\ 2\delta_{\nu_k}\chi b_1\int_0^{+\infty}b_1(l)dl\end{pmatrix}\right)$$

and

$$L_c e_2=i\omega_k e_2.$$

In this case system (5.15) becomes

$$2i\omega_k L_2(e_2,e_2)=L_hL_2(e_2,e_2)+\frac{1}{2!}\Pi_h D^2H(0)(e_2,e_2).$$

So

$$(2i\omega_k I-L_h)L_2(e_2,e_2)=\frac{1}{2!}\Pi_h D^2H(0)(e_2,e_2)$$

and Lemma 5.7 implies

$$\begin{aligned}L_2(e_2,e_2)&=\frac{1}{2}(2i\omega_k I-L_h)^{-1}\Pi_h\left(\begin{pmatrix}0\\ 0\\ 2\delta_{\nu_k}\chi b_1\int_0^{+\infty}b_1(l)dl\end{pmatrix}\right)\\ &=\left(\begin{pmatrix}0\\ 0\\ \psi_{2,2}\end{pmatrix}\right),\end{aligned}\tag{5.19}$$

where

$$\begin{aligned}\psi_{2,2}=&-\frac{\varphi_1}{2i\omega_k}-\frac{\varphi_2}{6i\omega_k}+\frac{1}{2}\int_0^a e^{-\int_s^a[2i\omega_k+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\psi(s)ds\\ &+\frac{1}{2}\Delta(\nu_k,2i\omega_k)^{-1}\left[\begin{array}{l}(-\nu_k\widehat{I}|_{\widehat{\varphi}=\psi,\lambda=2i\omega_k})e^{-\int_0^a[2i\omega_k+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\\ \quad+\widehat{I}|_{\widehat{\varphi}=\psi,\lambda=2i\omega_k}\int_0^a e^{-\int_s^a[2i\omega_k+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\delta_{\nu_k}\chi(s)\overline{s}_{\nu_k}(s)ds\end{array}\right],\end{aligned}$$

with

$$\psi=2\delta_{\nu_k}\chi b_1\int_0^{+\infty}b_1(l)dl.$$

(v) Computation of $L_2(e_2, e_3)$. We have

$$D^2H(0)(e_2, e_3) = \left(\begin{pmatrix} 0 \\ 0 \\ \delta_{\nu_k}\chi[b_2\int_0^{+\infty} b_1(l)dl + b_1\int_0^{+\infty} b_2(l)dl] \end{pmatrix}\right)$$

and

$$L_c e_2 = i\omega_k e_2, \qquad L_c e_3 = -i\omega_k e_3.$$

In this case system (5.15) reduces to

$$(i\omega_k - i\omega_k)L_2(e_2, e_3) = L_h L_2(e_2, e_3) + \frac{1}{2!}\Pi_h D^2H(0)(e_2, e_3).$$

Thus

$$-L_h L_2(e_2, e_3) = \frac{1}{2!}\Pi_h D^2H(0)(e_2, e_3),$$

and by Lemma 5.7 we have

$$\begin{aligned} L_2(e_2, e_3) &= L_2(e_3, e_2) \\ &= \frac{1}{2}(-L_h)^{-1}\Pi_h \left(\begin{pmatrix} 0 \\ 0 \\ \delta_{\nu_k}\chi[b_2\int_0^{+\infty} b_1(l)dl + b_1\int_0^{+\infty} b_2(l)dl] \end{pmatrix}\right) \\ &= \left(\begin{pmatrix} 0 \\ 0 \\ \psi_{2,3} \end{pmatrix}\right), \end{aligned} \tag{5.20}$$

where

$$\begin{aligned} \psi_{2,3} = {} & \frac{\varphi_1}{2i\omega_k} - \frac{\varphi_2}{2i\omega_k} + \frac{1}{2}\int_0^a e^{-\int_s^a [\delta_{\nu_k}(1-\bar{S}_{\nu_k})\chi(l)]dl}\psi(s)ds \\ & + \frac{1}{2}\Delta(\nu_k, 0)^{-1}\left[\begin{array}{l} (-v\widehat{I}|_{\widehat{\varphi}=\psi,\lambda=0})e^{-\int_0^a [\delta_{\nu_k}(1-\bar{S}_{\nu_k})\chi(l)]dl} \\ \quad + \widehat{I}|_{\widehat{\varphi}=\psi,\lambda=0}\int_0^a e^{-\int_s^a [\delta_{\nu_k}(1-\bar{S}_{\nu_k})\chi(l)]dl}\delta_{\nu_k}\chi(s)\bar{s}_{\nu_k}(s)ds \end{array}\right] \end{aligned}$$

with

$$\psi = \delta_{\nu_k}\chi\left[b_2\int_0^{+\infty} b_1(l)dl + b_1\int_0^{+\infty} b_2(l)dl\right].$$

(vi) Computation of $L_2(e_3, e_3)$. We have

$$D^2H(0)(e_3, e_3) = \left(\begin{pmatrix} 0 \\ 0 \\ 2\delta_{\nu_k}\chi b_2\int_0^{+\infty} b_2(l)dl \end{pmatrix}\right)$$

and

$$L_c e_3 = -i\omega_k e_3.$$

In this case system (5.15) becomes

$$-2i\omega_k L_2(e_3, e_3) = L_h L_2(e_3, e_3) + \frac{1}{2!}\Pi_h D^2 H(0)(e_3, e_3).$$

Hence

$$(-2i\omega_k I - L_h)L_2(e_3, e_3) = \frac{1}{2!}\Pi_h D^2 H(0)(e_3, e_3).$$

It follows from Lemma 5.7 that

$$\begin{aligned} L_2(e_3, e_3) &= \frac{1}{2}(-2i\omega_k I - L_h)^{-1}\Pi_h \left(\begin{pmatrix} 0 \\ \begin{pmatrix} 0 \\ 2\delta_{\nu_k}\chi b_2 \int_0^{+\infty} b_2(l)dl \end{pmatrix} \end{pmatrix} \right) \\ &= \frac{1}{2}\begin{pmatrix} 0 \\ (-2i\omega_k I - \widehat{A}_{\nu_k}|_{\Pi_h^{\widehat{A}_{\nu_k}}(X)})^{-1}\Pi_h^{\widehat{A}_{\nu_k}} \begin{pmatrix} 0 \\ 2\delta_{\nu_k}\chi b_2 \int_0^{+\infty} b_2(l)dl \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \begin{pmatrix} 0 \\ \psi_{3,3} \end{pmatrix} \end{pmatrix}, \end{aligned} \tag{5.21}$$

where

$$\begin{aligned} \psi_{3,3} = {} & \frac{\varphi_1}{6i\omega_k} + \frac{\varphi_2}{2i\omega_k} + \frac{1}{2}\int_0^a e^{-\int_s^a [-2i\omega_k + \delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\psi(s)ds \\ & + \frac{1}{2}\Delta(\nu_k, -2i\omega_k)^{-1} \\ & \times \begin{bmatrix} (-\nu_k \widehat{I}|_{\widehat{\varphi}=\psi,\lambda=-2i\omega_k})e^{-\int_0^a [-2i\omega_k + \delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl} \\ + \widehat{I}|_{\widehat{\varphi}=\psi,\lambda=-2i\omega_k} \int_0^a e^{-\int_s^a [-2i\omega_k + \delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\delta_{\nu_k}\chi(s)\overline{s}_{\nu_k}(s)ds \end{bmatrix} \end{aligned}$$

with

$$\psi = 2\delta_{\nu_k}\chi b_2 \int_0^{+\infty} b_2(l)dl.$$

Next, we can express the above computations in terms of the following basis of $Y_c = \Pi_c(Y)$:

$$\widetilde{e}_1 = \begin{pmatrix} 1 \\ \begin{pmatrix} 0_{\mathbb{R}} \\ 0_{L^1} \end{pmatrix} \end{pmatrix}, \qquad \widetilde{e}_2 = \begin{pmatrix} 0_{\mathbb{R}} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ c_1 \end{pmatrix} \end{pmatrix}, \qquad \widetilde{e}_3 = \begin{pmatrix} 0_{\mathbb{R}} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ c_2 \end{pmatrix} \end{pmatrix}$$

with

$$c_1 = b_1 + b_2, \qquad c_2 = \frac{b_1 - b_2}{i}. \tag{5.22}$$

Lemma 5.9. *The symmetric and bilinear map $L_2 : Y_c^2 \to Y_h \cap D(L)$ is defined by*

(a) $L_2(\widetilde{e}_1, \widetilde{e}_1) = 0$;
(b) $L_2(\widetilde{e}_1, \widetilde{e}_2)$ *and* $L_2(\widetilde{e}_2, \widetilde{e}_1)$ *are defined by*

$$L_2(\widetilde{e}_1, \widetilde{e}_2) = L_2(\widetilde{e}_2, \widetilde{e}_1) = \begin{pmatrix} 0_{\mathbb{R}} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ 2\operatorname{Re}\psi_{1,2} \end{pmatrix} \end{pmatrix};$$

(c) $L_2(\widetilde{e}_1, \widetilde{e}_3)$ *and* $L_2(\widetilde{e}_3, \widetilde{e}_1)$ *are defined by*

$$L_2(\widetilde{e}_1, \widetilde{e}_3) = L_2(\widetilde{e}_3, \widetilde{e}_1) = \begin{pmatrix} 0_{\mathbb{R}} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ 2\operatorname{Im}\psi_{1,2} \end{pmatrix} \end{pmatrix};$$

(d) $L_2(\widetilde{e}_2, \widetilde{e}_2)$ *is defined by*

$$L_2(\widetilde{e}_2, \widetilde{e}_2) = \begin{pmatrix} 0_{\mathbb{R}} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ 2\operatorname{Re}\psi_{2,2} + 2\psi_{2,3} \end{pmatrix} \end{pmatrix};$$

(e) $L_2(\widetilde{e}_2, \widetilde{e}_3)$ *and* $L_2(\widetilde{e}_3, \widetilde{e}_2)$ *are defined by*

$$L_2(\widetilde{e}_2, \widetilde{e}_3) = L_2(\widetilde{e}_3, \widetilde{e}_2) = \begin{pmatrix} 0_{\mathbb{R}} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ 2\operatorname{Im}\psi_{2,2} \end{pmatrix} \end{pmatrix};$$

(f) $L_2(\widetilde{e}_3, \widetilde{e}_3) = \begin{pmatrix} 0_{\mathbb{R}} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ -2\operatorname{Re}\psi_{2,2} + 2\psi_{2,3} \end{pmatrix} \end{pmatrix}$,

where $\psi_{1,2}$, $\psi_{2,2}$, and $\psi_{2,3}$ are defined in (5.17), (5.19), and (5.20), respectively.

Proof. We use the fact that

$$\widetilde{e}_1 = e_1, \qquad \widetilde{e}_2 = e_2 + e_3, \quad \text{and} \quad \widetilde{e}_3 = \frac{e_2 - e_3}{i}.$$

(a) By using (5.16), (a) follows.
(b) We have

$$L_2(\widetilde{e}_1, \widetilde{e}_2) = L_2(e_1, e_2) + L_2(e_1, e_3).$$

So by using (5.17) and (5.18), we obtain

$$L_2(\widetilde{e}_1, \widetilde{e}_2) = \begin{pmatrix} 0_{\mathbb{R}} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ \psi_{1,2}+\psi_{1,3} \end{pmatrix} \end{pmatrix}.$$

Since $\psi_{1,2}(a) = \overline{\psi_{1,3}(a)}$, $\forall a > 0$, (b) follows.

(c) We have

$$L_2(\widetilde{e}_1, \widetilde{e}_3) = L_2\left(e_1, \frac{1}{i}(e_2 - e_3)\right) = \frac{1}{i}\left[L_2(e_1, e_2) - L_2(e_1, e_3)\right].$$

It follows from (5.17) and (5.18) that

$$L_2(\widetilde{e}_1, \widetilde{e}_3) = \frac{1}{i}\begin{pmatrix} 0_{\mathbb{R}} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ \psi_{1,2}-\psi_{1,3} \end{pmatrix} \end{pmatrix}$$

and (c) follows.

(d) By using (5.19), (5.20) and (5.21), we have

$$\begin{aligned} L_2(\widetilde{e}_2, \widetilde{e}_2) &= L_2(e_2 + e_3, e_2 + e_3) \\ &= L_2(e_2, e_2) + 2L_2(e_2, e_3) + L_2(e_3, e_3) \\ &= \begin{pmatrix} 0_{\mathbb{R}} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ \psi_{2,2}+2\psi_{2,3}+\psi_{3,3} \end{pmatrix} \end{pmatrix}. \end{aligned}$$

Note that $\psi_{2,2}(a) = \overline{\psi_{3,3}(a)}$, $\forall a > 0$, (d) follows.

(e) We have

$$\begin{aligned} L_2(\widetilde{e}_2, \widetilde{e}_3) &= L_2\left(e_2 + e_3, \frac{1}{i}(e_2 - e_3)\right) \\ &= \frac{1}{i}\left[L_2(e_2, e_2) - L_2(e_3, e_3)\right] \\ &= \frac{1}{i}\begin{pmatrix} 0_{\mathbb{R}} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ \psi_{2,2}-\psi_{3,3} \end{pmatrix} \end{pmatrix} \end{aligned}$$

and (e) follows.

(f) We have

$$\begin{aligned} L_2(\widetilde{e}_3, \widetilde{e}_3) &= L_2\left(\frac{1}{i}(e_2 - e_3), \frac{1}{i}(e_2 - e_3)\right) \\ &= -\left[L_2(e_2, e_2) - 2L_2(e_2, e_3) + L_2(e_3, e_3)\right] \\ &= -\begin{pmatrix} 0_{\mathbb{R}} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ \psi_{2,2}-2\psi_{2,3}+\psi_{3,3} \end{pmatrix} \end{pmatrix}. \end{aligned}$$

This completes the proof. $\square$

In the following lemma we compute the Taylor expansion of the reduced system (5.14) by using the formula obtained for L_2 in Lemma 5.9.

Lemma 5.10. *The reduced system (5.14) expressed in terms of the basis $\{\widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3\}$ has the following form*

$$\begin{cases} \dfrac{d\widehat{\nu}(t)}{dt} = 0, \\ \dfrac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = M_c \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + (\widehat{H}_2 + \widehat{H}_3 + \widehat{R}_{1c})\begin{pmatrix} \widehat{\nu}(t) \\ \binom{x(t)}{y(t)} \end{pmatrix}, \end{cases} \tag{5.23}$$

where the matrix M_c is given by

$$M_c = \begin{bmatrix} 0 & \omega_k \\ -\omega_k & 0 \end{bmatrix},$$

the map $\widehat{H}_2 : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by

$$\begin{aligned} \widehat{H}_2\begin{pmatrix} \widehat{\nu} \\ \binom{x}{y} \end{pmatrix} &= \left(-\widehat{\nu}\int_0^{+\infty} \xi(l)dl\right)\left|\frac{d\Delta(\nu_k, i\omega_k)}{d\lambda}\right|^{-2}\begin{pmatrix} \mathrm{Re}(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}) \\ \mathrm{Im}(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}) \end{pmatrix} \\ &\quad + \left|\frac{d\Delta(\nu_k, i\omega_k)}{d\lambda}\right|^{-2}\begin{pmatrix} \mathrm{Re}(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\frac{\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta}}{C_2|_{\lambda=-i\omega_k}}) \\ \mathrm{Im}(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\frac{\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta}}{C_2|_{\lambda=-i\omega_k}}) \end{pmatrix}, \end{aligned} \tag{5.24}$$

where

$$\begin{aligned} \xi &= xc_1 + yc_2 + 2(x^2 - y^2)\,\mathrm{Re}(\psi_{2,2}) + 2(x^2 + y^2)\psi_{2,3} \\ &\quad + 4\widehat{\nu}x\,\mathrm{Re}(\psi_{1,2}) + 4\widehat{\nu}y\,\mathrm{Im}(\psi_{1,2}) + 4xy\,\mathrm{Im}(\psi_{2,2}) \end{aligned}$$

and

$$\begin{aligned} \theta &= \delta_{\nu_k}\chi\left[\left(\xi + \frac{d\overline{s}_{\nu_k}}{d\nu}\widehat{\nu}\right)\int_0^{+\infty}\xi(l)dl + \xi\widehat{\nu}\frac{d\overline{S}_{\nu_k}}{d\nu}\right] \\ &\quad + \frac{d\delta_{\nu_k}}{d\nu}\chi\widehat{\nu}\left(\overline{s}_{\nu_k}\int_0^{+\infty}\xi(l)dl - \xi(1 - \overline{S}_{\nu_k})\right) \end{aligned}$$

where $\psi_{1,2}(a)$, $\psi_{2,2}(a)$ and $\psi_{2,3}(a)$ are obtained in (5.17), (5.19), and (5.20), respectively. The map $\widehat{H}_3 : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by

$$\widehat{H}_3\begin{pmatrix} \widehat{\nu} \\ \binom{x}{y} \end{pmatrix} = \frac{1}{6}\left|\frac{d\Delta(\nu_k, i\omega_k)}{d\lambda}\right|^{-2}\begin{pmatrix} \mathrm{Re}(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\frac{\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\varsigma}}{C_2|_{\lambda=-i\omega_k}}) \\ \mathrm{Im}(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\frac{\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\varsigma}}{C_2|_{\lambda=-i\omega_k}}) \end{pmatrix}, \tag{5.25}$$

with

$$\begin{aligned}
\varsigma = {} & 3\delta_{\nu_k}\chi\begin{bmatrix} (xc_1+yc_2)\widehat{\nu}^2\frac{d^2\overline{S}_{\nu_k}}{d\nu^2} \\ +\frac{d^2\overline{s}_{\nu_k}}{d\nu^2}\widehat{\nu}^2\int_0^{+\infty}(xc_1+yc_2)(l)dl \end{bmatrix} \\
& +6\frac{d\delta_{\nu_k}}{d\nu}\widehat{\nu}\chi\begin{bmatrix} (xc_1+yc_2)\int_0^{+\infty}xc_1+yc_2+\frac{d\overline{s}_{\nu_k}}{d\nu}\widehat{\nu}dl \\ +\frac{d\overline{s}_{\nu_k}}{d\nu}\widehat{\nu}\int_0^{+\infty}xc_1+yc_2dl \end{bmatrix} \\
& +3\frac{d^2\delta_{\nu_k}}{d\nu^2}\widehat{\nu}^2\chi\left[-(xc_1+yc_2)(1-\overline{S}_{\nu_k})+\overline{s}_{\nu_k}\int_0^{+\infty}(xc_1+yc_2)dl\right],
\end{aligned} \tag{5.26}$$

and the remainder term $\widehat{R}_{1c}\in C^4(\mathbb{R}^3,\mathbb{R}^2)$ *is given by*

$$\widehat{R}_{1c}\left(\begin{pmatrix}\widehat{\nu}\\ \begin{pmatrix}x\\y\end{pmatrix}\end{pmatrix}\right)=O\left(\widehat{\nu}^3\left\|\begin{pmatrix}x\\y\end{pmatrix}\right\|+\widehat{\nu}^2\left\|\begin{pmatrix}x\\y\end{pmatrix}\right\|^2+\widehat{\nu}\left\|\begin{pmatrix}x\\y\end{pmatrix}\right\|^3+\left\|\begin{pmatrix}x\\y\end{pmatrix}\right\|^4\right). \tag{5.27}$$

Proof. By using the Taylor expansion of W around 0, the reduced system (5.14) can be rewritten as follows:

$$\begin{aligned}
\frac{d\widehat{\nu}(t)}{dt} &= 0, \\
\frac{dx_c(t)}{dt} &= (\widehat{A}_{\nu_k})_c x_c(t)+\frac{1}{2!}\Pi_c^{\widehat{A}_{\nu_k}}D^2W(0)\left((I+G_2)\begin{pmatrix}\widehat{\nu}(t)\\x_c(t)\end{pmatrix}\right)^2 \\
&\quad +\frac{1}{3!}\Pi_c^{\widehat{A}_{\nu_k}}D^3W(0)\left((I+G_2)\begin{pmatrix}\widehat{\nu}(t)\\x_c(t)\end{pmatrix}\right)^3+\widetilde{R}_{1c}\begin{pmatrix}\widehat{\nu}(t)\\x_c(t)\end{pmatrix}.
\end{aligned}$$

Set

$$x_c=\begin{pmatrix}0\\xc_1+yc_2\end{pmatrix}=\begin{pmatrix}0\\x(b_1+b_2)+y(\frac{b_1-b_2}{i})\end{pmatrix}.$$

Since $\{\widetilde{e}_1,\widetilde{e}_2,\widetilde{e}_3\}$ is used as the basis for $Y_c=\Pi_c(Y)$, i.e., $\{\binom{0}{c_1},\binom{0}{c_2}\}$ is a basis of $X_c:=\Pi_c^{\widehat{A}_{\nu_k}}(X)$, we obtain that

$$M_c=\begin{bmatrix}0 & \omega_k\\ -\omega_k & 0\end{bmatrix}.$$

(vii) Computation of $\widehat{H}_2(Y)$. We have

$$\begin{aligned}
(I+G_2)\begin{pmatrix}\widehat{\nu}\\x_c\end{pmatrix} &= \widehat{\nu}\widetilde{e}_1+x\widetilde{e}_2+y\widetilde{e}_3+L_2(\widehat{\nu}\widetilde{e}_1+x\widetilde{e}_2+y\widetilde{e}_3,\widehat{\nu}\widetilde{e}_1+x\widetilde{e}_2+y\widetilde{e}_3) \\
&= \widehat{\nu}\widetilde{e}_1+x\widetilde{e}_2+y\widetilde{e}_3+\widehat{\nu}^2L_2(\widetilde{e}_1,\widetilde{e}_1)+x^2L_2(\widetilde{e}_2,\widetilde{e}_2)+y^2L_2(\widetilde{e}_3,\widetilde{e}_3) \\
&\quad +2\widehat{\nu}xL_2(\widetilde{e}_1,\widetilde{e}_2)+2\widehat{\nu}yL_2(\widetilde{e}_1,\widetilde{e}_3)+2xyL_2(\widetilde{e}_2,\widetilde{e}_3).
\end{aligned}$$

By Lemma 5.9, it follows that

$$(I+G_2)\begin{pmatrix}\widehat{\nu}\\ x_c\end{pmatrix}=\begin{pmatrix}\widehat{\nu}\\ \begin{pmatrix}0\\ \xi\end{pmatrix}\end{pmatrix}, \tag{5.28}$$

where

$$\xi(a)=xc_1(a)+yc_2(a)+2\left(x^2-y^2\right)\operatorname{Re}\left(\psi_{2,2}(a)\right)+2\left(x^2+y^2\right)\psi_{2,3}(a) \\ +4\widehat{\nu}x\operatorname{Re}\left(\psi_{1,2}(a)\right)+4\widehat{\nu}y\operatorname{Im}\left(\psi_{1,2}(a)\right)+4xy\operatorname{Im}\left(\psi_{2,2}(a)\right).$$

Then we deduce that

$$\frac{1}{2!}D^2W(0)\left(\begin{pmatrix}\widehat{\nu}\\ \begin{pmatrix}0\\ \xi\end{pmatrix}\end{pmatrix}\right)^2=\begin{pmatrix}-\widehat{\nu}\int_0^{+\infty}\xi(l)dl\\ \theta\end{pmatrix}.$$

By projecting on X_c, using (5.5), (5.8), (5.9), and the same identification as above, we obtain that

$$\begin{aligned}
&\frac{1}{2!}\Pi_c^{\widehat{A}_{\nu_k}}D^2W(0)\left((I+G_2)\begin{pmatrix}\widehat{\nu}(t)\\ x_c(t)\end{pmatrix}\right)^2\\
&=\Pi_c^{\widehat{A}_{\nu_k}}\begin{pmatrix}-\widehat{\nu}\int_0^{+\infty}\xi(l)dl\\ \theta\end{pmatrix}\\
&=\left(-\widehat{\nu}\int_0^{+\infty}\xi(l)dl\right)\Pi_c^{\widehat{A}_{\nu_k}}\begin{pmatrix}1\\ 0\end{pmatrix}+\Pi_c^{\widehat{A}_{\nu_k}}\begin{pmatrix}0\\ \theta\end{pmatrix}\\
&=\left(-\widehat{\nu}\int_0^{+\infty}\xi(l)dl\right)\left|\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\right|^{-2}\begin{pmatrix}0\\ \operatorname{Re}(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda})c_1+\operatorname{Im}(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda})c_2\end{pmatrix}\\
&+\begin{pmatrix}0\\ \varphi_1|_{\psi=\theta}+\varphi_2|_{\psi=\theta}\end{pmatrix},
\end{aligned}$$

where c_1 and c_2 are defined in (5.22), φ_1 and φ_2 are defined in (5.8) and (5.9). Note that

$$\begin{aligned}
&\begin{pmatrix}0\\ \varphi_1|_{\psi=\theta}+\varphi_2|_{\psi=\theta}\end{pmatrix}\\
&=\begin{pmatrix}0\\ (\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda})^{-1}(\frac{\widehat{I}|_{\lambda=i\omega_k,\widehat{\varphi}=\theta}}{C_2|_{\lambda=i\omega_k}})b_1+(\frac{d\Delta(\nu_k,-i\omega_k)}{d\lambda})^{-1}(\frac{\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta}}{C_2|_{\lambda=-i\omega_k}})b_2\end{pmatrix}\\
&=\left|\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\right|^{-2}\begin{pmatrix}0\\ \operatorname{Re}(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\frac{\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta}}{C_2|_{\lambda=-i\omega_k}})c_1+\operatorname{Im}(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\frac{\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta}}{C_2|_{\lambda=-i\omega_k}})c_2\end{pmatrix}.
\end{aligned}$$

Then (5.24) follows.

Set

$$\widehat{R}_{1c}\begin{pmatrix}\widehat{\nu}\\ x_c\end{pmatrix}=\widetilde{R}_{1c}\begin{pmatrix}\widehat{\nu}\\ x_c\end{pmatrix}+\frac{1}{3!}\Pi_c^{\widehat{A}_{\nu_k}}\left\{D^3W(0)\left((I+G_2)\begin{pmatrix}\widehat{\nu}\\ x_c\end{pmatrix}\right)^3-D^3W(0)\begin{pmatrix}\widehat{\nu}\\ x_c\end{pmatrix}^3\right\}.$$

Then by (5.28), we deduce that the remainder term satisfies the order condition (5.27). Thus it only remains to compute $\frac{1}{3!}D^3W(0)\begin{pmatrix}\widehat{\nu}\\ x_c\end{pmatrix}^3$.

(viii) Computation of $\widehat{H}_3(Y)$. We have

$$D^3W(0)\begin{pmatrix}\widehat{\nu}\\ x_c\end{pmatrix}^3=\begin{pmatrix}0\\ \varsigma\end{pmatrix}.$$

Then we obtain

$$\begin{aligned}\frac{1}{3!}\Pi_c^{\widehat{A}_{\nu_k}}D^3W(0)\begin{pmatrix}\widehat{\nu}\\ x_c\end{pmatrix}^3&=\frac{1}{6}\Pi_c^{\widehat{A}_{\nu_k}}\begin{pmatrix}0\\ \varsigma\end{pmatrix}\\ &=\frac{1}{6}\begin{pmatrix}0\\ \varphi_1|_{\psi=\varsigma}+\varphi_2|_{\psi=\varsigma}\end{pmatrix}\\ &=\frac{1}{6}\left|\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\right|^{-2}\begin{pmatrix}0\\ \operatorname{Re}\big(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\frac{\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\varsigma}}{C_2|_{\lambda=-i\omega_k}}\big)c_1+\operatorname{Im}\big(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\frac{\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\varsigma}}{C_2|_{\lambda=-i\omega_k}}\big)c_2\end{pmatrix}\end{aligned}$$

and (5.25) follows. □

The main result of this section is the following theorem in which we summarize the above results.

Theorem 5.11. *The reduced system (5.14) expressed in terms of the basis $\{\widetilde{e}_1,\widetilde{e}_2,\widetilde{e}_3\}$ has the following form*

$$\begin{cases}\dfrac{d\widehat{\nu}(t)}{dt}=0,\\ \dfrac{d}{dt}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}=M_c\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}+(\widetilde{H}_2+\widetilde{H}_3+\widehat{R}_{2c})\begin{pmatrix}\widehat{\nu}(t)\\ \begin{pmatrix}x(t)\\ y(t)\end{pmatrix}\end{pmatrix},\end{cases}\tag{5.29}$$

where

$$M_c=\begin{bmatrix}0&\omega_k\\ -\omega_k&0\end{bmatrix}.$$

The map $\widetilde{H}_2:\mathbb{R}^3\to\mathbb{R}^2$ is defined by

$$\widetilde{H}_2\left(\begin{pmatrix}\widehat{\nu}\\ \begin{pmatrix}x\\ y\end{pmatrix}\end{pmatrix}\right)=\varkappa_2(\widehat{\nu},x,y)\left|\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\right|^{-2}\begin{pmatrix}\mathrm{Re}(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda})\\ \mathrm{Im}(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda})\end{pmatrix}$$
$$+\left|\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\right|^{-2}\begin{pmatrix}\mathrm{Re}(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\frac{\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\bar{\theta}_1}}{C_2|_{\lambda=-i\omega_k}})\\ \mathrm{Im}(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\frac{\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\bar{\theta}_1}}{C_2|_{\lambda=-i\omega_k}})\end{pmatrix},$$

where

$$\varkappa_2(\widehat{\nu},x,y)=-\widehat{\nu}\int_0^{+\infty}(xc_1+yc_2)(l)dl,$$

$$\bar{\theta}_1=\delta_{\nu_k}\chi\begin{bmatrix}(xc_1+yc_2+\frac{d\bar{s}_{\nu_k}}{d\nu}\widehat{\nu})\int_0^{+\infty}(xc_1+yc_2)(l)dl\\ +(xc_1+yc_2)\widehat{\nu}\frac{d\overline{S}_{\nu_k}}{d\nu}\end{bmatrix}$$
$$+\frac{d\delta_{\nu_k}}{d\nu}\chi\widehat{\nu}\left(\bar{s}_{\nu_k}\int_0^{+\infty}(xc_1+yc_2)dl-(xc_1+yc_2)(1-\overline{S}_{\nu_k})\right),$$

and the map $\widetilde{H}_3:\mathbb{R}^3\to\mathbb{R}^2$ *is defined by*

$$\widetilde{H}_3\left(\begin{pmatrix}\widehat{\nu}\\ \begin{pmatrix}x\\ y\end{pmatrix}\end{pmatrix}\right)=\varkappa_3(\widehat{\nu},x,y)\left|\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\right|^{-2}\begin{pmatrix}\mathrm{Re}(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda})\\ \mathrm{Im}(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda})\end{pmatrix}$$
$$+\left|\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\right|^{-2}\begin{pmatrix}\mathrm{Re}(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\frac{\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\bar{\theta}_2}}{C_2|_{\lambda=-i\omega_k}})\\ \mathrm{Im}(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\frac{\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\bar{\theta}_2}}{C_2|_{\lambda=-i\omega_k}})\end{pmatrix}$$
$$+\frac{1}{6}\left|\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\right|^{-2}\begin{pmatrix}\mathrm{Re}(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\frac{\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\varsigma}}{C_2|_{\lambda=-i\omega_k}})\\ \mathrm{Im}(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\frac{\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\varsigma}}{C_2|_{\lambda=-i\omega_k}})\end{pmatrix},$$

where

$$\varkappa_3(\widehat{\nu},x,y)=-\widehat{\nu}\int_0^{+\infty}\bar{\xi}_1(l)dl$$

$$\bar{\theta}_2=\delta_{\nu_k}\chi\begin{bmatrix}(xc_1+yc_2+\frac{d\bar{s}_{\nu_k}}{d\nu}\widehat{\nu})\int_0^{+\infty}\bar{\xi}_1(l)dl\\ +\bar{\xi}_1\int_0^{+\infty}(xc_1+yc_2)(l)dl\\ +\bar{\xi}_1\widehat{\nu}\frac{d\overline{S}_{\nu_k}}{d\nu}\end{bmatrix}$$
$$+\frac{d\delta_{\nu_k}}{d\nu}\chi\widehat{\nu}\left(\bar{s}_{\nu_k}\int_0^{+\infty}\bar{\xi}_1(l)dl-\bar{\xi}_1(1-\overline{S}_{\nu_k})\right)$$

with

$$\bar{\xi}_1 = 2\left(x^2 - y^2\right)\mathrm{Re}(\psi_{2,2}) + 2\left(x^2 + y^2\right)\psi_{2,3} \\ + 4\widehat{\nu}x\,\mathrm{Re}(\psi_{1,2}) + 4\widehat{\nu}y\,\mathrm{Im}(\psi_{1,2}) + 4xy\,\mathrm{Im}(\psi_{2,2})$$

where $\psi_{1,2}(a)$, $\psi_{2,2}(a)$, $\psi_{2,3}(a)$ *and* ς *are obtained in (5.17), (5.19), (5.20), and (5.26), respectively. The remainder term* $\widehat{R}_{2c} \in C^4(\mathbb{R}^3, \mathbb{R}^2)$ *is given by*

$$\widehat{R}_{2c}\left(\begin{pmatrix} \widehat{\nu} \\ \begin{pmatrix} x \\ y \end{pmatrix} \end{pmatrix}\right) = O\left(\widehat{\nu}^3\left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\| + \widehat{\nu}^2\left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|^2 + \widehat{\nu}\left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|^3 + \left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|^4\right).$$

From Theorem 5.11, dropping the auxiliary equation introduced for handling the parameter, we get the following equation

$$\begin{aligned}\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= M_c\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \varkappa_2(\widehat{\nu}, x, y)\left|\frac{d\Delta(\nu_k, i\omega_k)}{d\lambda}\right|^{-2}\begin{pmatrix} \mathrm{Re}(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}) \\ \mathrm{Im}(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}) \end{pmatrix} \\ &+ \left|\frac{d\Delta(\nu_k, i\omega_k)}{d\lambda}\right|^{-2}\begin{pmatrix} \mathrm{Re}(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\frac{\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\bar{\theta}_1}}{C_2|_{\lambda=-i\omega_k}}) \\ \mathrm{Im}(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\frac{\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\bar{\theta}_1}}{C_2|_{\lambda=-i\omega_k}}) \end{pmatrix} \\ &+ \varkappa_3(\widehat{\nu}, x, y)\left|\frac{d\Delta(\nu_k, i\omega_k)}{d\lambda}\right|^{-2}\begin{pmatrix} \mathrm{Re}(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}) \\ \mathrm{Im}(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}) \end{pmatrix} \\ &+ \left|\frac{d\Delta(\nu_k, i\omega_k)}{d\lambda}\right|^{-2}\begin{pmatrix} \mathrm{Re}(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\frac{\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\bar{\theta}_2}}{C_2|_{\lambda=-i\omega_k}}) \\ \mathrm{Im}(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\frac{\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\bar{\theta}_2}}{C_2|_{\lambda=-i\omega_k}}) \end{pmatrix} \\ &+ \frac{1}{6}\left|\frac{d\Delta(\nu_k, i\omega_k)}{d\lambda}\right|^{-2}\begin{pmatrix} \mathrm{Re}(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\frac{\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\varsigma}}{C_2|_{\lambda=-i\omega_k}}) \\ \mathrm{Im}(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\frac{\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\varsigma}}{C_2|_{\lambda=-i\omega_k}}) \end{pmatrix} + \widehat{R}_{2c}.\end{aligned} \tag{5.30}$$

In the following we will study the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions. We first make some preliminary remarks. Rewrite the system of ordinary differential equations (5.30) as the following form

$$\frac{dX}{dt} = F(X, \widehat{\nu}), \tag{5.31}$$

where the stationary point is $X = 0 \in \mathbb{R}^2$ and the critical value of the bifurcation parameter $\widehat{\nu}$ is 0. Since the equilibrium solutions belong to the center manifold, we have for each $|\widehat{\nu}|$ small enough that

$$F(0, \widehat{\nu}) = 0.$$

One may observe that $\partial_x F(0,\widehat{\nu})$ is unknown whenever $\widehat{\nu} \neq 0$. System (5.29) only provides an approximation at the order 2 of $\partial_x F(0,\widehat{\nu})$ with respect to $\widehat{\nu}$. Nevertheless by using Proposition 4.22 in Magal and Ruan [43], we know that the eigenvalues $\lambda(\widehat{\nu})$ of $\partial_x F(0,\widehat{\nu})$ are the roots of the original characteristic equation

$$
\begin{aligned}
&0 = \Delta(\nu,\lambda) = 1 - e^{-\lambda} + \frac{\lambda^2}{\nu h(\nu)} + \frac{\lambda}{\nu} \\
&\quad \Leftrightarrow \quad 0 = \Delta_1(\nu,\lambda) := \lambda\big[\lambda + \nu\kappa(c)\big] + \nu^2\kappa(c)\big(1 - e^{-\lambda}\big) \\
&\quad \Leftrightarrow \quad 0 = \Delta_1(\widehat{\nu} + \nu_k,\lambda) := \lambda\big[\lambda + (\widehat{\nu} + \nu_k)\kappa(c)\big] + \big(\widehat{\nu} + \nu_k\big)^2\kappa(c)\big(1 - e^{-\lambda}\big), \\
&\quad \lambda \in \Omega, \qquad \kappa(c) = \frac{(\frac{1}{c})^2}{1 - \sqrt{1 - \frac{1}{c^2}}}
\end{aligned}
$$

and

$$\lambda(0) = \pm i\omega_k.$$

From Proposition 5.1, we know that for $c \geq 1$ the characteristic equation has a unique pair of complex conjugate roots $\widetilde{\lambda}(\widehat{\nu}), \overline{\widetilde{\lambda}(\widehat{\nu})}$ close to $i\omega_k, -i\omega_k$ for $\widehat{\nu}$ in a neighborhood of 0. Here $\widetilde{\lambda}(\widehat{\nu}) = \alpha(\widehat{\nu}) + i\omega(\widehat{\nu})$, $\alpha(0) = 0$ and $i\omega(0) = i\omega_k$ and

$$\alpha'(0) > 0.$$

The spectrum of $\partial_x F(0,\widehat{\nu})$ is

$$\sigma\big(\partial_x F(0,\widehat{\nu})\big) = \big\{\widetilde{\lambda}_n(\widehat{\nu}), \overline{\widetilde{\lambda}_n(\widehat{\nu})}\big\}.$$

By using a standard procedure (see for example the proof of Lemma 3.3 on page 92 in Kuznetsov [34]) and by introducing a complex variable z, system (5.31) can be written for sufficiently small $|\widehat{\nu}|$ as a single equation:

$$\dot{z} = \widetilde{\lambda}(\widehat{\nu})z + g(z,\bar{z};\widehat{\nu}), \tag{5.32}$$

where

$$\widetilde{\lambda}(\widehat{\nu}) = \alpha(\widehat{\nu}) + i\omega(\widehat{\nu}), \qquad g(z,\bar{z},\widehat{\nu}) = \sum_{i+j=2}^{3} \frac{1}{i!j!} g_{ij}(\widehat{\nu}) z^i \bar{z}^j + O\big(|z|^3\big).$$

It is easy to check that (5.31) satisfies

1) $F(0,\widehat{\nu}) = 0$ for $\widehat{\nu}$ in an open interval containing 0, and $0 \in \mathbb{R}^2$ is an isolated stationary point of F;
2) $F(X,\widehat{\nu})$ is jointly $C^{L+2}(L \geq 2)$ in X and $\widehat{\nu}$ in a neighborhood of $(0,0) \in \mathbb{R}^2 \times \mathbb{R}$;

3) $D_X F(0, \widehat{\nu})$ has a pair of complex conjugate eigenvalues λ and $\bar{\lambda}$ such that

$$\lambda(\widehat{\nu}) = \alpha(\widehat{\nu}) + i\omega(\widehat{\nu}),$$

where $\omega(0) = \omega_k > 0$, $\alpha(0) = 0$, $\alpha'(0) \neq 0$,

By Hassard et al. [28, Theorem II, page 16], for (5.31) we have: there exist an $\varepsilon_p > 0$ and a C^{L+1}-function $\widehat{\nu}(\varepsilon)$,

$$\widehat{\nu}(\varepsilon) = \sum_{1}^{[\frac{L}{2}]} \widehat{\nu}_{2i}\varepsilon^{2i} + O\big(\varepsilon^{L+1}\big) \quad (0 < \varepsilon < \varepsilon_p) \tag{5.33}$$

such that for each $\varepsilon \in (0, \varepsilon_p)$ there exists a periodic solution $P_\varepsilon(t)$ with period $T(\varepsilon)$, occurring for $\widehat{\nu} = \widehat{\nu}(\varepsilon)$. The period $T(\varepsilon)$ of $P_\varepsilon(t)$ is a C^{L+1}-function

$$T(\varepsilon) = \frac{2\pi}{\omega_k}\left[1 + \sum_{1}^{[\frac{L}{2}]} \tau_{2i}\varepsilon^{2i}\right] + O\big(\varepsilon^{L+1}\big) \quad (0 < \varepsilon < \varepsilon_p). \tag{5.34}$$

Exactly two of the Floquet exponents of $P_\varepsilon(t)$ approach 0 as $\varepsilon \downarrow 0$. One is 0 for $\varepsilon \in (0, \varepsilon_p)$, the other is a C^{L+1}-function

$$\kappa(\varepsilon) = \sum_{1}^{[\frac{L}{2}]} \kappa_{2i}\varepsilon^{2i} + O\big(\varepsilon^{L+1}\big) \quad (0 < \varepsilon < \varepsilon_p). \tag{5.35}$$

$P_\varepsilon(t)$ is orbitally asymptotically stable with asymptotic phase if $\kappa(\varepsilon) < 0$, and is unstable if $\kappa(\varepsilon) > 0$.

From the above results we know that the stability of the bifurcating periodic solution and the direction of the Hopf bifurcation are determined by the sign of $\kappa(\varepsilon)$ and $\widehat{\nu}(\varepsilon)$. Now the problem becomes to compute the coefficients κ_{2i} and $\widehat{\nu}_{2i}$ in (5.35) and (5.33). Applying the results in Hassard et al. [28, pages 45–51] and using a transformation of the following form

$$\begin{aligned} z &= \xi + \chi(\xi, \bar{\xi}; \widehat{\nu}) \\ &= \xi + \sum_{i+j=2}^{L+1} \frac{1}{i!j!}\chi_{ij}(\widehat{\nu})\xi^i\bar{\xi}^j, \qquad \chi_{ij} \equiv 0 \text{ for } i = j+1, \end{aligned}$$

we can change Eq. (5.32) into the Poincaré normal form as follows:

$$\dot{\xi} = \lambda(\widehat{\nu})\xi + \sum_{j=1}^{[L/2]} c_j(\widehat{\nu})\xi|\xi|^{2j} + O\big(|\xi|\big|(\xi, \widehat{\nu})\big|^{L+1}\big) \equiv C(\xi, \bar{\xi}, \widehat{\nu}), \tag{5.36}$$

where $C(\xi, \bar{\xi}, \widehat{\nu})$ is C^{L+2} jointly in $\xi, \bar{\xi}, \widehat{\nu}$ in a neighborhood of $0 \in \mathbb{C} \times \mathbb{C} \times \mathbb{R}$, then the periodic solution of period $T(\varepsilon)$ such that $\xi(0, \widehat{\nu}) = \varepsilon$ of (5.36) has the form

$$\xi = \varepsilon \exp\bigl[2\pi i t/T(\varepsilon)\bigr] + O\bigl(\varepsilon^{L+2}\bigr),$$

where

$$T(\varepsilon) = \frac{2\pi}{\omega_k}\left[1 + \sum_1^L \tau_i \varepsilon^i\right] + O\bigl(\varepsilon^{L+1}\bigr) \tag{5.37}$$

and

$$\widehat{\nu}(\varepsilon) = \sum_1^L \widehat{\nu}_i \varepsilon^i + O\bigl(\varepsilon^{L+1}\bigr). \tag{5.38}$$

Furthermore $\tau_1, \ldots, \tau_4$ and $\widehat{\nu}_1, \ldots, \widehat{\nu}_4$ are given by the following formulae:

$$\begin{aligned}
&\widehat{\nu}_1 = 0, \qquad \widehat{\nu}_2 = -\frac{\operatorname{Re} c_1(0)}{\alpha'(0)}, \qquad \widehat{\nu}_3 = 0,\\
&\widehat{\nu}_4 = -\frac{1}{\alpha'(0)}\left[\operatorname{Re} c_2(0) + \widehat{\nu}_2 \operatorname{Re} c_1'(0) + \frac{\alpha''(0)}{2}\widehat{\nu}_2^2\right],\\
&\tau_1 = 0, \qquad \tau_2 = \frac{-1}{\omega_k}\bigl[\operatorname{Im} c_1(0) + \widehat{\nu}_2 \omega'(0)\bigr], \qquad \tau_3 = 0,\\
&\tau_4 = -\frac{1}{\omega_k}\left[\omega'(0)\widehat{\nu}_4 + \frac{\omega''(0)}{2}\widehat{\nu}_2^2 + \operatorname{Im} c_1'(0)\widehat{\nu}_2 + \operatorname{Im} c_2(0) - \omega_k \tau_2^2\right].
\end{aligned} \tag{5.39}$$

The Floquet exponents of the periodic solution are given by

$$\kappa(\varepsilon) = \sum_1^{[\frac{L}{2}]} \kappa_{2i} \varepsilon^{2i} + O\bigl(\varepsilon^{L+1}\bigr), \tag{5.40}$$

where

$$\kappa_2 = 2 \operatorname{Re} c_1(0), \qquad \kappa_4 = 4 \operatorname{Re} c_2(0) + 2 \operatorname{Re} c_1'(0)\widehat{\nu}_2, \tag{5.41}$$

$$c_1(0) = \frac{i}{2\omega_k}\left(g_{20}(0) g_{11}(0) - 2\bigl|g_{11}(0)\bigr|^2 - \frac{1}{3}\bigl|g_{02}(0)\bigr|^2\right) + \frac{g_{21}(0)}{2}. \tag{5.42}$$

To use the bifurcation formulae for $\kappa(\varepsilon)$, $\widehat{\nu}(\varepsilon)$ and $T(\varepsilon)$, we need only compute $c_1(0)$, $c_1'(0)$, and $c_2(0)$. For sufficiently small ε, if $\kappa_2 \neq 0$, $\widehat{\nu}_2 \neq 0$, the stability of the bifurcating periodic solutions and the direction of the Hopf bifurcation are determined by the sign of κ_2 and $\widehat{\nu}_2$. In applications, usually computation of κ_2 and $\widehat{\nu}_2$ is sufficient.

In the following, for system (5.31) we shall obtain explicit expressions for $\widehat{\nu}_2$ and κ_2 only. From (5.39) and (5.31) we know that to compute κ_2 and $\widehat{\nu}_2$ we only need $c_1(0)$. By (5.42), in order to obtain $c_1(0)$ we only need to compute $g_{20}(0)$, $g_{11}(0)$, $g_{02}(0)$ and $g_{21}(0)$. Thus we only need to compute at $\widehat{\nu} = 0$. Setting $\widehat{\nu} = 0$ in (5.30), we obtain the following system

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = M_c \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \left|\frac{d\Delta(\nu_k, i\omega_k)}{d\lambda}\right|^{-2} \begin{pmatrix} \operatorname{Re}(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda} \frac{\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_1}}{C_2|_{\lambda=-i\omega_k}}) \\ \operatorname{Im}(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda} \frac{\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_1}}{C_2|_{\lambda=-i\omega_k}}) \end{pmatrix}$$
$$+ \left|\frac{d\Delta(\nu_k, i\omega_k)}{d\lambda}\right|^{-2} \begin{pmatrix} \operatorname{Re}(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda} \frac{\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_2}}{C_2|_{\lambda=-i\omega_k}}) \\ \operatorname{Im}(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda} \frac{\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_2}}{C_2|_{\lambda=-i\omega_k}}) \end{pmatrix} + \widehat{R}_{2c}, \tag{5.43}$$

where

$$\theta_1 = \delta_{\nu_k}\chi\left[(xc_1 + yc_2)\int_0^{+\infty}(xc_1 + yc_2)(l)dl\right],$$
$$\theta_2 = \delta_{\nu_k}\chi\begin{bmatrix} (xc_1 + yc_2)\int_0^{+\infty}\xi_1(l)dl \\ +\xi_1\int_0^{+\infty}(xc_1 + yc_2)(l)dl \end{bmatrix}$$

with

$$\xi_1 = 2\left(x^2 - y^2\right)\operatorname{Re}(\psi_{2,2}) + 2\left(x^2 + y^2\right)\psi_{2,3} + 4xy\operatorname{Im}(\psi_{2,2}).$$

By introducing a complex variable $z = y + ix$, system (5.43) can be written as a single equation

$$\dot{z} = i\omega_k z(t) + g(z, \bar{z}),$$

where $g(z, \bar{z})$ is an expansion in powers of z and $\bar{z}$:

$$g(z, \bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots$$

in which

$$g_{11} = \varrho_1\left\{\begin{array}{l} \varrho_2\begin{bmatrix} (\operatorname{Im}\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_1})_{02} + (\operatorname{Im}\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_1})_{20} \\ +i(\operatorname{Re}\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_1})_{02} + i(\operatorname{Re}\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_1})_{20} \end{bmatrix} \\ +\varrho_3\begin{bmatrix} (\operatorname{Re}\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_1})_{02} + (\operatorname{Re}\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_1})_{20} \\ -i(\operatorname{Im}\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_1})_{02} - i(\operatorname{Im}\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_1})_{20} \end{bmatrix} \end{array}\right\},$$
$$g_{20} = g_{11} + \varrho, \qquad g_{02} = g_{11} - \varrho$$

with

$$(\operatorname{Re}\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_1})_{11} = \int_0^{+\infty}\int_0^a \left\{\begin{array}{l} \cos\omega_k(a-s)e^{-\int_s^a \delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)dl}\delta_{\nu_k}\chi(s) \\ {[c_1(s)\int_0^{+\infty}c_2(l)dl + c_2(s)\int_0^{+\infty}c_1(l)dl]} \end{array}\right\} dsda,$$
$$(\operatorname{Im}\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_1})_{11} = \int_0^{+\infty}\int_0^a \left\{\begin{array}{l} \sin\omega_k(a-s)e^{-\int_s^a \delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)dl}\delta_{\nu_k}\chi(s) \\ {[c_1(s)\int_0^{+\infty}c_2(l)dl + c_2(s)\int_0^{+\infty}c_1(l)dl]} \end{array}\right\} dsda,$$

$$(\operatorname{Re}\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_1})_{02}$$
$$=\int_0^{+\infty}\int_0^a\left\{\cos\omega_k(a-s)e^{-\int_s^a\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)dl}\delta_{\nu_k}\chi(s)c_2(s)\int_0^{+\infty}c_2(l)dl\right\}dsda,$$
$$(\operatorname{Im}\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_1})_{02}$$
$$=\int_0^{+\infty}\int_0^a\left\{\sin\omega_k(a-s)e^{-\int_s^a\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)dl}\delta_{\nu_k}\chi(s)c_2(s)\int_0^{+\infty}c_2(l)dl\right\}dsda,$$
$$(\operatorname{Re}\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_1})_{20}$$
$$=\int_0^{+\infty}\int_0^a\left\{\cos\omega_k(a-s)e^{-\int_s^a\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)dl}\delta_{\nu_k}\chi(s)c_1(s)\int_0^{+\infty}c_1(l)dl\right\}dsda,$$
$$(\operatorname{Im}\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_1})_{20}$$
$$=\int_0^{+\infty}\int_0^a\left\{\sin\omega_k(a-s)e^{-\int_s^a\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)dl}\delta_{\nu_k}\chi(s)c_1(s)\int_0^{+\infty}c_1(l)dl\right\}dsda,$$
$$\varrho_1=\frac{1}{2}\left|\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\right|^{-2}|C_2|_{\lambda=-i\omega_k}|^{-2},$$
$$\varrho_2=\operatorname{Re}C_2\operatorname{Re}\left(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\right)+\operatorname{Im}C_2\operatorname{Im}\left(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\right),$$
$$\varrho_3=\operatorname{Re}C_2\operatorname{Im}\left(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\right)-\operatorname{Im}C_2\operatorname{Re}\left(\frac{d\Delta(\nu_k,i\omega_k)}{d\lambda}\right),$$
$$\varrho=\varrho_1\left\{\begin{array}{l}\varrho_2[(\operatorname{Re}\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_2})_{11}-i(\operatorname{Im}\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_2})_{11}]\\ \quad-\varrho_3[(\operatorname{Im}\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_2})_{11}+i(\operatorname{Re}\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_2})_{11}]\end{array}\right\},$$

and

$$g_{21}=\frac{1}{2}\varrho_1(\varrho_2-i\varrho_3)\left[\begin{array}{l}\overline{(\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_2})_{12}}+3\overline{(\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_2})_{30}}\\ \quad+i\overline{(\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_2})_{21}}+i3\overline{(\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_2})_{03}}\end{array}\right]$$

with

$$(\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_2})_{12}$$
$$=\int_0^{+\infty}\int_0^a e^{-\int_s^a[-i\omega_k+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\delta_{\nu_k}\chi(s)\left[\begin{array}{l}2c_1(s)\int_0^{+\infty}[-\operatorname{Re}(\psi_{2,2})+\psi_{2,3}](l)dl\\ \quad+4c_2(s)\int_0^{+\infty}\operatorname{Im}(\psi_{2,2})(l)dl\\ 2[-\operatorname{Re}(\psi_{2,2})+\psi_{2,3}](s)\int_0^{+\infty}c_1(l)dl\\ \quad+4\operatorname{Im}(\psi_{2,2})(s)\int_0^{+\infty}c_2(l)dl\end{array}\right]dsda,$$

$$
\begin{aligned}
&(\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_2})_{30} \\
&= \int_0^{+\infty}\int_0^{a} e^{-\int_s^a[-i\omega_k+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\delta_{\nu_k}\chi(s)\begin{bmatrix} 2c_1(s)\int_0^{+\infty}[\operatorname{Re}(\psi_{2,2})+\psi_{2,3}](l)dl \\ 2[\operatorname{Re}(\psi_{2,2})+\psi_{2,3}](s)\int_0^{+\infty}c_1(l)dl \end{bmatrix} dsda, \\
&(\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_2})_{21} \\
&= \int_0^{+\infty}\int_0^{a} e^{-\int_s^a[-i\omega_k+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\delta_{\nu_k}\chi(s)\begin{bmatrix} 2c_2(s)\int_0^{+\infty}[\operatorname{Re}(\psi_{2,2})+\psi_{2,3}](l)dl \\ +4c_1(s)\int_0^{+\infty}\operatorname{Im}(\psi_{2,2})(l)dl \\ 2[\operatorname{Re}(\psi_{2,2})+\psi_{2,3}](s)\int_0^{+\infty}c_2(l)dl \\ +4\operatorname{Im}(\psi_{2,2})(s)\int_0^{+\infty}c_1(l)dl \end{bmatrix} dsda, \\
&(\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_2})_{03} \\
&= \int_0^{+\infty}\int_0^{a} e^{-\int_s^a[-i\omega_k+\delta_{\nu_k}(1-\overline{S}_{\nu_k})\chi(l)]dl}\delta_{\nu_k}\chi(s)\begin{bmatrix} 2c_2(s)\int_0^{+\infty}[-\operatorname{Re}(\psi_{2,2})+\psi_{2,3}](l)dl \\ 2[-\operatorname{Re}(\psi_{2,2})+\psi_{2,3}](s)\int_0^{+\infty}c_2(l)dl \end{bmatrix} dsda.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\operatorname{Re} c_1(0) &= \operatorname{Re}\left[\frac{i}{2\omega_k}(g_{20}g_{11})+\frac{g_{21}}{2}\right] \\
&= -\frac{2\operatorname{Re} g_{11}\operatorname{Im} g_{11}+\operatorname{Im}\varrho\operatorname{Re} g_{11}+\operatorname{Re}\varrho\operatorname{Im} g_{11}}{2\omega_k} \\
&\quad + \frac{1}{4}\varrho_1\left\{\begin{matrix} \varrho_2\begin{bmatrix}\operatorname{Re}(\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_2})_{12}+3\operatorname{Re}(\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_2})_{30} \\ +\operatorname{Im}(\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_2})_{21}+3\operatorname{Im}(\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_2})_{03}\end{bmatrix} \\ +\varrho_3\begin{bmatrix}\operatorname{Im}(\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_2})_{12}-3\operatorname{Im}(\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_2})_{30} \\ +\operatorname{Re}(\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_2})_{21}+3\operatorname{Re}(\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_2})_{03}\end{bmatrix}\end{matrix}\right\}, \\
\operatorname{Re} g_{11} &= \varrho_1\left\{\begin{matrix}\varrho_2[(\operatorname{Im}\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_1})_{02}+(\operatorname{Im}\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_1})_{20}] \\ +\varrho_3[(\operatorname{Re}\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_1})_{02}+(\operatorname{Re}\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_1})_{20}]\end{matrix}\right\}, \\
\operatorname{Im} g_{11} &= \varrho_1\left\{\begin{matrix}\varrho_2[(\operatorname{Re}\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_1})_{02}+(\operatorname{Re}\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_1})_{20}] \\ -\varrho_3[(\operatorname{Im}\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_1})_{02}+(\operatorname{Im}\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_1})_{20}]\end{matrix}\right\}, \\
\operatorname{Re}\varrho &= \varrho_1\left\{\begin{matrix}\varrho_2(\operatorname{Re}\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_1})_{11} \\ -\varrho_3(\operatorname{Im}\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_1})_{11}\end{matrix}\right\}, \\
\operatorname{Im}\varrho &= -\varrho_1\left\{\begin{matrix}\varrho_2(\operatorname{Im}\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_1})_{11} \\ +\varrho_3(\operatorname{Re}\widehat{I}|_{\lambda=-i\omega_k,\widehat{\varphi}=\theta_1})_{11}\end{matrix}\right\},
\end{aligned}
$$

and

$$
\widehat{\nu}_2 = -\frac{\operatorname{Re} c_1(0)}{\alpha'(0)}, \qquad \kappa_2 = 2\operatorname{Re} c_1(0).
$$

For the symmetric case (ii) in Proposition 5.1, we only need to replace ν_k and ω_k by $\widehat{\nu}_k$ and $\widehat{\omega}_k$, respectively in the expression of $\operatorname{Re} c_1(0)$. From the above discussion we can state the following result.

Theorem 5.12. *The direction of the Hopf bifurcation is determined by the sign of $\widehat{\nu}_2$: if $\widehat{\nu}_2 > 0$ (< 0), then the bifurcating periodic solutions exist for $\nu > \nu_k$ ($\nu < \nu_k$). The periodic solutions are stable (unstable) if $\kappa_2 < 0$ (> 0).*

The existence of Hopf bifurcation in the structured evolutionary epidemiological model of influenza A drift (1.2) obtained in Magal and Ruan [44] and the stability of the bifurcated periodic solutions given in Theorem 5.12 indicate that influenza A has an intrinsic tendency to oscillate due to the evolutionary and/or immunological changes of the influenza viruses. This will be very helpful in understanding the seasonal occurrence of influenza A, in predicting the epidemics of specific influenza A strains, and in designing effective vaccine programs.

5.2. An age structured population model

Now we apply the normal form theory developed in the previous sections to the age structured population model (1.6). At first, we make the following assumptions.

Assumption 5.13. Assume that $\mu > 0$, $\alpha > 0$, $\gamma \in L^\infty(0, +\infty)$ is a map defined by

$$\gamma(a) = (a-\tau)^n e^{-\varsigma(a-\tau)} 1_{[\tau,+\infty)}(a) = \begin{cases} (a-\tau)^n e^{-\varsigma(a-\tau)}, & \text{if } a \geq \tau, \\ 0, \text{otherwise}, \end{cases}$$

where $\tau \geq 0$, $\varsigma \geq 0, n \in \mathbb{N}$, and assume that $\varsigma > 0$ whenever $n \geq 1$. The map $h : \mathbb{R} \to \mathbb{R}$ is defined by

$$h(x) = x \exp(-\beta x), \quad \forall x \in \mathbb{R},$$

where $\beta > 0$.

For the operator A defined by (1.7), we have

$$\rho(A) = \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) > -\mu\},$$

and for each $\lambda \in \rho(A)$,

$$(\lambda I - A)^{-1}\begin{pmatrix} \chi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi \end{pmatrix}$$

$$\Leftrightarrow \quad \varphi(a) = e^{-(\lambda+\mu)a}\chi + \int_0^a e^{-(\lambda+\mu)(a-s)}\psi(s)ds.$$

It is readily checked that

$$\left\|(\lambda I - A)^{-1}\right\| \le \frac{1}{\lambda+\mu}, \quad \forall\lambda > -\mu.$$

So A is a Hille–Yosida operator. Now we consider A_0, the part of A in X_0, which is defined by

$$A_0\begin{pmatrix}0\\ \varphi\end{pmatrix} = A\begin{pmatrix}0\\ \varphi\end{pmatrix} = \begin{pmatrix}0\\ -\varphi' - \mu\varphi\end{pmatrix}, \quad \forall\begin{pmatrix}0\\ \varphi\end{pmatrix} \in D(A_0),$$

and

$$D(A_0) = \left\{\begin{pmatrix}0\\ \varphi\end{pmatrix} \in \{0_{\mathbb{R}}\} \times W^{1,1}\big((0,+\infty),\mathbb{R}\big) : \varphi(0) = 0\right\}.$$

The linear operator A_0 is the infinitesimal generator of a strongly continuous semigroup $\{T_{A_0}(t)\}_{t\ge 0}$ of bounded linear operators on X_0, which is defined by

$$T_{A_0}(t)\begin{pmatrix}0\\ \varphi\end{pmatrix} = \begin{pmatrix}0\\ \widehat{T}_{A_0}(t)\varphi\end{pmatrix}$$

with

$$\widehat{T}_{A_0}(t)(\varphi)(a) = \begin{cases} e^{-\mu t}\varphi(a-t), & \text{if } a-t \ge 0,\\ 0, & \text{otherwise.}\end{cases}$$

Then we consider $\{S_A(t)\}_{t\ge 0} \subset \mathcal{L}(X)$ the integrated semigroup generated by A. That is the family of bounded linear operators on X, such that for each $x = \begin{pmatrix}\chi\\ \psi\end{pmatrix} \in X$, the map $t \to S_A(t)x$ is an integrated solution of the Cauchy problem

$$\frac{dS_A(t)x}{dt} = AS_A(t)x + x, \quad \text{for } t \ge 0, \quad \text{and} \quad S_A(0)x = 0.$$

Thus, we deduce that

$$S_A(t)\begin{pmatrix}\chi\\ \psi\end{pmatrix} = \begin{pmatrix}0\\ L(t)\chi\end{pmatrix} + \int_0^t T_{A_0}(l)\begin{pmatrix}0\\ \psi\end{pmatrix} dl$$

with

$$L(t)(\chi)(a) = \begin{cases} 0, & \text{if } a-t \ge 0,\\ e^{-\mu a}\chi, & \text{if } a-t \le 0.\end{cases}$$

Finally define a convolution

$$(S_A * f)(t) = \int_0^t S_A(t-s)f(s)ds$$

for $f \in L^1(0,\tau;X)$. Then for each $f \in L^1(0,\tau;X)$, the map $t \to (S_A * f)(t)$ belongs to $C^1([0,\tau],X_0) \cap C([0,\tau],D(A))$, and

$$(S_A \diamond f)(t) := \frac{d}{dt}(S_A * f)(t)$$

satisfies

$$(S_A \diamond f)(t) = A\int_0^t (S_A \diamond f)(l)dl + \int_0^t f(l)dl, \quad \forall t \in [0,\tau].$$

Then the integrated solution of system (1.9) is unique and is given by

$$v(t) = T_{A_0}(t)\begin{pmatrix} 0 \\ \varphi \end{pmatrix} + \big(S_A \diamond \alpha H\big(v(.)\big)\big)(t), \quad \forall t \geq 0.$$

Set

$$X_{0+} := \{0\} \times L^1_+(0,+\infty).$$

The positive equilibrium solution of (1.9) is given for each $\alpha > \alpha_0$ by

$$\bar{v}_\alpha = \begin{pmatrix} 0 \\ \bar{u}_\alpha \end{pmatrix} \quad \text{with } \bar{u}_\alpha(a) = \bar{C}\exp(-\mu a),$$

where

$$\alpha_0 = \frac{1}{\int_0^{+\infty}\gamma(a)e^{-\mu a}da}, \quad \text{and} \quad \bar{C} := \frac{\ln(\alpha\int_0^{+\infty}\gamma(a)e^{-\mu a}da)}{\beta\int_0^{+\infty}\gamma(a)e^{-\mu a}da}.$$

The linearized system of (1.9) around $\bar{v}_\alpha$ is

$$\frac{dw(t)}{dt} = Aw(t) + \alpha DH(\bar{v}_\alpha)w(t) \quad \text{for } t \geq 0, \qquad v(t) \in X_0,$$

where

$$\alpha DH(\bar{v}_\alpha)\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} \eta(\alpha)\int_0^{+\infty}\gamma(a)\varphi(a)da \\ 0 \end{pmatrix}$$

and

$$\eta(\alpha) = \alpha h'\left(\int_0^{+\infty}\gamma(a)\bar{u}_\alpha(a)da\right) = \frac{1 - \ln(\alpha\int_0^{+\infty}\gamma(a)e^{-\mu a}da)}{\int_0^{+\infty}\gamma(a)e^{-\mu a}da}.$$

To simplify the notation, we set

$$B_\alpha x = Ax + \alpha DH(\bar{v}_\alpha)x \quad \text{with } D(B_\alpha) = D(A).$$

In the following we summarize some results obtained in Magal and Ruan [43].

Lemma 5.14. *Let Assumption 5.13 be satisfied. Then the linear operator $B_\alpha : D(A) \subset X \to X$ is a Hille–Yosida operator and*

$$\omega_{\mathrm{ess}}\big((B_\alpha)_0\big) \leq -\mu.$$

Set

$$\Omega := \big\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > -\mu\big\}.$$

By using the above lemma, we deduce that for each $\lambda \in \Omega$,

$$\lambda \in \sigma(B_\alpha) \quad \Leftrightarrow \quad \Delta(\alpha, \lambda) = 0,$$

where the characteristic function is

$$\Delta(\alpha, \lambda) := 1 - \eta(\alpha) \int_0^{+\infty} \gamma(a) e^{-(\lambda+\mu)a} da \quad \text{for each } \lambda \in \Omega.$$

Moreover, by using the fact that $\gamma(a) = (a-\tau)^n e^{-\varsigma(a-\tau)} 1_{[\tau,+\infty)}(a)$, for each $\lambda \in \Omega$, the characteristic equation

$$\Delta(\alpha, \lambda) = 0$$

is equivalent to

$$1 = n!\eta(\alpha) \frac{e^{-(\lambda+\mu)\tau}}{(\varsigma + \lambda + \mu)^{n+1}}. \tag{5.44}$$

In the following, we regard α as the bifurcation parameter and have the following result on Hopf bifurcation in model (1.6) (Magal and Ruan [43]).

Proposition 5.15. *Let Assumption 5.13 be satisfied and assume that $\tau > 0$. Then the characteristic equation (5.44) with $\alpha = \alpha_k$, $k \in \mathbb{N} \setminus \{0\}$, has a unique pair of purely imaginary roots $\pm i\omega_k$, where*

$$1 = n!\eta(\alpha_k) \frac{e^{-\mu\tau}}{(\sqrt{(\mu+\varsigma)^2 + \omega_k^2})}$$

and $\omega_k > 0$ is the unique solution of

$$-\left(\omega\tau + (n+1)\arctan\frac{\omega}{\varsigma+\mu}\right) = \pi - 2k\pi,$$

so that the age structured model (1.6) undergoes a Hopf bifurcation at the equilibrium $u = \overline{u}_{\alpha_k}$. In particular, a non-trivial periodic solution bifurcates from the equilibrium $u = \overline{u}_{\alpha_k}$.

In the following we study the direction and stability of the Hopf bifurcation by applying the normal form theory developed in previous sections to the Cauchy problem (1.9). We first include the parameter α into the state variable. Consider the system

$$\begin{cases} \dfrac{d\alpha(t)}{dt} = 0, \\ \dfrac{dv(t)}{dt} = Av(t) + \alpha(t)H\big(v(t)\big), \\ \alpha(0) = \alpha_0 \in \mathbb{R}, \quad v(0) = v_0 \in X_0. \end{cases}$$

Making a change of variables

$$v(t) = \widehat{v}(t) + \overline{v}_\alpha,$$

we obtain the system

$$\begin{cases} \dfrac{d\alpha(t)}{dt} = 0, \\ \dfrac{d\widehat{v}(t)}{dt} = A\widehat{v}(t) + \alpha(t)H\big(\widehat{v}(t) + \overline{v}_\alpha\big) - \alpha(t)H(\overline{v}_\alpha). \end{cases}$$

Now setting

$$\alpha = \widehat{\alpha} + \alpha_k,$$

we obtain

$$\begin{cases} \dfrac{d\widehat{\alpha}(t)}{dt} = 0, \\ \dfrac{d\widehat{v}(t)}{dt} = A\widehat{v}(t) + \widehat{H}(\widehat{\alpha}, \widehat{v}), \end{cases} \tag{5.45}$$

where

$$\widehat{H}(\widehat{\alpha}, \widehat{v}) := (\widehat{\alpha} + \alpha_k)\big[H\big(\widehat{v}(t) + \overline{v}_{(\widehat{\alpha}+\alpha_k)}\big) - H(\overline{v}_{(\widehat{\alpha}+\alpha_k)})\big].$$

We have

$$\partial_{\widehat{v}}\widehat{H}(\widehat{\alpha}, \widehat{v})(w) = (\widehat{\alpha} + \alpha_k)DH(\widehat{v} + \overline{v}_{(\widehat{\alpha}+\alpha_k)})(w)$$

and

$$\begin{aligned}\partial_{\widehat{\alpha}}\widehat{H}(\widehat{\alpha},\widehat{v})(\widetilde{\alpha}) = \widetilde{\alpha}\bigg\{ & H(\widehat{v}+\overline{v}_{(\widehat{\alpha}+\alpha_k)}) - H(\overline{v}_{(\widehat{\alpha}+\alpha_k)}) \\ & + (\widehat{\alpha}+\alpha_k)\bigg[DH(\widehat{v}+\overline{v}_{(\widehat{\alpha}+\alpha_k)})\left(\frac{d\overline{v}_{(\widehat{\alpha}+\alpha_k)}}{d\widehat{\alpha}}\right) \\ & - DH(\overline{v}_{(\widehat{\alpha}+\alpha_k)})\left(\frac{d\overline{v}_{(\widehat{\alpha}+\alpha_k)}}{d\widehat{\alpha}}\right)\bigg]\bigg\}.\end{aligned}$$

So

$$\partial_{\widehat{v}}\widehat{H}(0,0) = \alpha_k DH(\overline{v}_{\alpha_k}) \quad \text{and} \quad \partial_{\widehat{\alpha}}\widehat{H}(0,0) = 0.$$

Set

$$\mathcal{X} = \mathbb{R}\times X, \qquad \mathcal{X}_0 = \mathbb{R}\times\overline{D(A)}.$$

Consider the linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$ defined by

$$\mathcal{A}\begin{pmatrix}\widehat{\alpha}\\ \widehat{v}\end{pmatrix} = \begin{pmatrix}0\\ (A+\alpha_k DH(\overline{v}_{\alpha_k}))\widehat{v}\end{pmatrix} = \begin{pmatrix}0\\ B_{\alpha_k}\widehat{v}\end{pmatrix}$$

with

$$D(\mathcal{A}) = \mathbb{R}\times D(A),$$

and the map $F : \overline{D(\mathcal{A})} \to \mathcal{X}$ defined by

$$F\begin{pmatrix}\widehat{\alpha}\\ \widehat{v}\end{pmatrix} = \begin{pmatrix}0\\ W\begin{pmatrix}\widehat{\alpha}\\ \widehat{v}\end{pmatrix}\end{pmatrix},$$

where $W : \overline{D(\mathcal{A})} \to X$ is defined by

$$W\begin{pmatrix}\widehat{\alpha}\\ \widehat{v}\end{pmatrix} := (\widehat{\alpha}+\alpha_k)\big[H(\widehat{v}+\overline{v}_{(\widehat{\alpha}+\alpha_k)}) - H(\overline{v}_{(\widehat{\alpha}+\alpha_k)})\big] - \alpha_k DH(\overline{v}_{\alpha_k})(\widehat{v}).$$

Then we have

$$F\begin{pmatrix}0\\0\end{pmatrix} = 0 \quad \text{and} \quad DF\begin{pmatrix}0\\0\end{pmatrix} = 0.$$

Now we can reformulate system (5.45) as the following system

$$\frac{dw(t)}{dt} = \mathcal{A}w(t) + F\big(w(t)\big), \qquad w(0) = w_0 \in \overline{D(\mathcal{A})}. \tag{5.46}$$

The following lemmas were obtained in Magal and Ruan [43].

Lemma 5.16. *Let Assumption 5.13 be satisfied and assume that $\tau > 0$. Then*

$$\sigma(B_{\alpha_k}|_{\widehat{\Pi}_c(X)}) = \{i\omega_k, -i\omega_k\}, \qquad \sigma(B_{\alpha_k}|_{(I-\widehat{\Pi}_c)(X)}) = \sigma(B_{\alpha_k}) \setminus \{i\omega_k, -i\omega_k\}$$

with

$$\widehat{\Pi}_c = \widehat{\Pi}_{i\omega_k} + \widehat{\Pi}_{-i\omega_k},$$

$$\widehat{\Pi}_{\pm i\omega_k}\begin{pmatrix}\delta\\ \psi\end{pmatrix} = \begin{pmatrix} 0 \\ \frac{d\Delta(\alpha,\pm i\omega_k)}{d\lambda}^{-1}[\delta + \int_0^{+\infty}\int_s^{+\infty}\gamma(l)e^{-(\pm i\omega_k+\mu)(l-s)}dl\psi(s)ds]e^{-(\pm i\omega_k+\mu)\cdot}\end{pmatrix}$$

and

$$\widehat{\Pi}_c\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix} 0 \\ \frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}^{-1}e^{-(i\omega_k+\mu)\cdot} + \frac{d\Delta(\alpha_k,-i\omega_k)}{d\lambda}^{-1}e^{-(-i\omega_k+\mu)\cdot}\end{pmatrix}.$$

Observe that by construction we have

$$B_{\alpha_k}\begin{pmatrix}0\\ e^{-(\pm i\omega_k+\mu)\cdot}\end{pmatrix} = \begin{pmatrix}0\\ -(\frac{d}{da}+\mu I)e^{-(\pm i\omega_k+\mu)\cdot}\end{pmatrix} = \pm i\omega_k\begin{pmatrix}0\\ e^{-(\pm i\omega_k+\mu)\cdot}\end{pmatrix}.$$

Set

$$\widehat{\Pi}_h := (I-\widehat{\Pi}_c), \qquad X_c := \widehat{\Pi}_c(X), \quad \text{and} \quad X_h := \widehat{\Pi}_h(X).$$

Lemma 5.17. *Let Assumption 5.13 be satisfied and assume that $\tau > 0$. Then*

$$\sigma(\mathcal{A}) = \sigma(B_{\alpha_k}) \cup \{0\}.$$

Moreover, we have for $\lambda \in \rho(\mathcal{A}) \cap \Omega = \Omega \setminus (\sigma(B_{\alpha_k}) \cup \{0\})$ that

$$(\lambda-\mathcal{A})^{-1}\begin{pmatrix} r \\ \begin{pmatrix}\delta\\ \psi\end{pmatrix}\end{pmatrix} = \begin{pmatrix}\frac{r}{\lambda} \\ (\lambda - B_{\alpha_k})^{-1}\begin{pmatrix}\delta\\ \psi\end{pmatrix}\end{pmatrix}$$

and the eigenvalues 0 and $\pm i\omega_k$ of A are simple. The corresponding projectors $\Pi_0, \Pi_{\pm i\omega_k} : \mathcal{X} + i\mathcal{X} \to \mathcal{X} + i\mathcal{X}$ are defined by

$$\Pi_0\begin{pmatrix}r\\ v\end{pmatrix} = \begin{pmatrix}r\\ 0\end{pmatrix}, \qquad \Pi_{\pm i\omega_k}\begin{pmatrix}r\\ v\end{pmatrix} = \begin{pmatrix}0\\ \widehat{\Pi}_{\pm i\omega_k}v\end{pmatrix}, \quad \forall\begin{pmatrix}r\\ v\end{pmatrix} \in \mathcal{X} + i\mathcal{X}.$$

Note that we have

$$\overline{\Pi_{i\omega_k}(x)} = \Pi_{-i\omega_k}(\overline{x}), \quad \forall x \in \mathcal{X} + i\mathcal{X}.$$

In this context, the projectors $\Pi_c : \mathcal{X} \to \mathcal{X}$ and $\Pi_h : \mathcal{X} \to \mathcal{X}$ are defined by

$$\Pi_c(x) := (\Pi_0 + \Pi_{i\omega_k} + \Pi_{-i\omega_k})(x), \quad \forall x \in \mathcal{X},$$
$$\Pi_h(x) := (I - \Pi_c)(x), \quad \forall x \in \mathcal{X}.$$

We denote

$$\mathcal{X}_c := \Pi_c(\mathcal{X}), \qquad \mathcal{X}_h := \Pi_h(\mathcal{X}), \qquad \mathcal{A}_c := \mathcal{A}|_{\mathcal{X}_c}, \qquad \mathcal{A}_h := \mathcal{A}|_{\mathcal{X}_h}.$$

Now we have the decomposition

$$\mathcal{X} = \mathcal{X}_c \oplus \mathcal{X}_h.$$

Define the basis of $\mathcal{X}_c$ by

$$\widehat{e_1} := \begin{pmatrix} 1 \\ \begin{pmatrix} 0_{\mathbb{R}} \\ 0_C \end{pmatrix} \end{pmatrix}, \qquad \widehat{e_2} := \begin{pmatrix} 0_{\mathbb{R}} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ c_1 \end{pmatrix} \end{pmatrix}, \qquad \widehat{e_3} =: \begin{pmatrix} 0_{\mathbb{R}} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ c_2 \end{pmatrix} \end{pmatrix}$$

with

$$c_1 = e^{-(\mu + i\omega_k)\cdot} \quad \text{and} \quad c_2 = e^{-(\mu - i\omega_k)\cdot}.$$

We have

$$\mathcal{A}\widehat{e_1} = 0, \qquad \mathcal{A}\widehat{e_2} = i\omega_k \widehat{e_2}, \quad \text{and} \quad \mathcal{A}\widehat{e_3} = -i\omega_k \widehat{e_3}.$$

Set

$$w := \begin{pmatrix} \widehat{\alpha} \\ \widehat{v} \end{pmatrix} = \begin{pmatrix} \widehat{\alpha} \\ \begin{pmatrix} 0 \\ \widehat{u} \end{pmatrix} \end{pmatrix} \in \overline{D(\mathcal{A})},$$
$$\widehat{\Pi}_c \widehat{v} := \widehat{v}_c, \qquad \widehat{\Pi}_h \widehat{v} := \widehat{v}_h \qquad w_c := \Pi_c w = \begin{pmatrix} \widehat{\alpha} \\ \widehat{\Pi}_c \widehat{v} \end{pmatrix} = \begin{pmatrix} \widehat{\alpha} \\ \widehat{v}_c \end{pmatrix}$$

and

$$w_h := \Pi_h w = (I - \Pi_c) w = \begin{pmatrix} 0 \\ \widehat{\Pi}_h \widehat{v} \end{pmatrix} = \begin{pmatrix} 0 \\ \widehat{v}_h \end{pmatrix}.$$

Notice that $\left\{ \begin{pmatrix} 0 \\ c_1 \end{pmatrix}, \begin{pmatrix} 0 \\ c_2 \end{pmatrix} \right\}$ is the basis of X_c. Set

$$\widehat{v}_c = \begin{pmatrix} 0 \\ x_1 c_1 + x_2 c_2 \end{pmatrix}$$

and

$$\chi := \int_0^{+\infty} \gamma(a) e^{-\mu a} da = \frac{n! \exp(-\mu\tau)}{(\mu + \varsigma)^{n+1}}.$$

We observe that for each

$$w_1 := \begin{pmatrix} \widehat{\alpha}_1 \\ v_1 \end{pmatrix}, \ w_2 := \begin{pmatrix} \widehat{\alpha}_2 \\ v_2 \end{pmatrix} \in \overline{D(\mathcal{A})}$$

with $v_i = \begin{pmatrix} 0_{\mathbb{R}} \\ \varphi_i \end{pmatrix}$, $i = 1, 2$,

$$\begin{aligned} D^2 W(0)(w_1, w_2) &= D^2 W(0)\left(\begin{pmatrix} \widehat{\alpha}_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} \widehat{\alpha}_2 \\ v_2 \end{pmatrix}\right) \\ &= \alpha_k D^2 H(\overline{v}_{\alpha_k})(v_1, v_2) + \widehat{\alpha}_2 DH(\overline{v}_{\alpha_k})(v_1) + \widehat{\alpha}_1 DH(\overline{v}_{\alpha_k})(v_2) \\ &\quad + \widehat{\alpha}_2 \alpha_k D^2 H(\overline{v}_{\alpha_k})\left(v_1, \left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}\right) \\ &\quad + \widehat{\alpha}_1 \alpha_k D^2 H(\overline{v}_{\alpha_k})\left(v_2, \left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}\right) \end{aligned}$$

with

$$D^2 H(\overline{v}_{\alpha_k})\left(\begin{pmatrix} 0 \\ \varphi_1 \end{pmatrix}, \begin{pmatrix} 0 \\ \varphi_2 \end{pmatrix}\right) = \begin{pmatrix} \frac{\beta[\ln(\alpha_k \chi)-2]}{\alpha_k \chi} \prod_{i=1}^2 \int_0^{+\infty} \gamma(a)\varphi_i(a)da \\ 0 \end{pmatrix}.$$

Then

$$\frac{1}{2!} D^2 W(0)(w)^2 = \frac{1}{2!} D^2 W(0)\begin{pmatrix} \widehat{\alpha} \\ \begin{pmatrix} 0 \\ \widehat{u} \end{pmatrix} \end{pmatrix}^2 = \begin{pmatrix} \widetilde{\widetilde{\psi}} \\ 0 \end{pmatrix},$$

where

$$\widetilde{\widetilde{\psi}} = -\frac{\widehat{\alpha}}{\alpha_k \chi} \int_0^{+\infty} \gamma(a)\widehat{u}(a)da + \frac{\beta(\ln(\alpha_k \chi) - 2)}{2\chi}\left(\int_0^{+\infty} \gamma(a)\widehat{u}(a)da\right)^2.$$

By projecting on X_c and using Lemma 5.16, we obtain

$$\begin{aligned} \frac{1}{2!} \widehat{\Pi}_c D^2 W(0)\begin{pmatrix} \widehat{\alpha} \\ \begin{pmatrix} 0 \\ \widehat{u} \end{pmatrix} \end{pmatrix}^2 &= \widetilde{\widetilde{\psi}} \widehat{\Pi}_c \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \widetilde{\widetilde{\psi}} \begin{bmatrix} 0 \\ \frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}^{-1} c_1 + \frac{d\Delta(\alpha_k, -i\omega_k)}{d\lambda}^{-1} c_2 \end{bmatrix}. \end{aligned}$$

Now we compute $\frac{1}{2!} D^2 W(0)(w_c)^2$, $\frac{1}{2!} \Pi_c D^2 F(0)(w_c)^2$, $\frac{1}{2!} \Pi_h D^2 F(0)(w_c)^2$ and $\frac{1}{3!} D^3 W(0)(w_c)^3$ expressed in terms of the basis $\{\widehat{e}_1, \widehat{e}_2, \widehat{e}_3\}$. We first obtain that

$$\frac{1}{2!} D^2 W(0)(w_c)^2 = \frac{1}{2!} D^2 W(0)\begin{pmatrix} \widehat{\alpha} \\ \begin{pmatrix} 0 \\ x_1 c_1 + x_2 c_2 \end{pmatrix} \end{pmatrix}^2 = \begin{pmatrix} \widetilde{\psi} \\ 0 \end{pmatrix},$$

where

$$\widetilde{\psi} = -\frac{\widehat{\alpha}}{\alpha_k \chi} \int_0^{+\infty} \gamma(a)(x_1 c_1 + x_2 c_2)(a)da + \frac{\beta(\ln(\alpha_k \chi) - 2)}{2\chi} \left(\int_0^{+\infty} \gamma(a)(x_1 c_1 + x_2 c_2)(a)da \right)^2.$$

Thus

$$\frac{1}{2!}\Pi_c D^2 F(0)(w_c)^2 = \begin{pmatrix} 0 \\ \frac{1}{2!}\widehat{\Pi}_c D^2 W(0)\left(\begin{pmatrix} \widehat{\alpha} \\ \widehat{v}_c \end{pmatrix}\right)^2 \end{pmatrix} = \begin{pmatrix} 0 \\ \widetilde{\psi}\widehat{\Pi}_c\binom{1}{0} \end{pmatrix} = \begin{pmatrix} 0 \\ \widetilde{\psi}\left[\begin{smallmatrix} 0 \\ \frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}^{-1} c_1 + \frac{d\Delta(\alpha_k, -i\omega_k)}{d\lambda}^{-1} c_2 \end{smallmatrix}\right] \end{pmatrix} \tag{5.47}$$

and

$$\frac{1}{2!}\Pi_h D^2 F(0)(w_c)^2 = \frac{1}{2!}(I - \Pi_c) D^2 F(0)\left(\begin{pmatrix} \widehat{\alpha} \\ \widehat{v}_c \end{pmatrix}\right)^2 = \begin{pmatrix} 0 \\ \widetilde{\psi}\left[\begin{smallmatrix} 1 \\ -\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}^{-1} c_1 - \frac{d\Delta(\alpha_k, -i\omega_k)}{d\lambda}^{-1} c_2 \end{smallmatrix}\right] \end{pmatrix}. \tag{5.48}$$

Next we obtain

$$\frac{1}{3!} D^3 W(0)(w_c)^3 = \frac{1}{3!} D^3 W(0) \begin{pmatrix} \widehat{\alpha} \\ \widehat{v}_c \end{pmatrix}^3 = \begin{pmatrix} \widehat{\psi} \\ 0 \end{pmatrix}$$

with

$$\widehat{\psi} = \frac{1}{(\alpha_k)^2(1 - \ln(\alpha_k \chi))}\widehat{\alpha}^2\left(\frac{x_1 + x_2}{2}\right) + \frac{2\beta\chi}{\alpha_k(1 - \ln(\alpha_k \chi))^2}\widehat{\alpha}\left(\frac{x_1 + x_2}{2}\right)^2 + \frac{4\beta^2(-\ln(\alpha_k \chi) + 3)\chi^2}{3(1 - \ln(\alpha_k \chi))^3}\left(\frac{x_1 + x_2}{2}\right)^3.$$

By Lemma 5.16, we obtain

$$\frac{1}{3!}\Pi_c D^3 F(0)(w_c)^3 = \begin{pmatrix} 0 \\ \frac{1}{3!}\widehat{\Pi}_c D^3 W(0)(w_c)^3 \end{pmatrix} = \begin{pmatrix} 0 \\ \widehat{\psi}\widehat{\Pi}_c\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ \widehat{\psi}\left[\begin{smallmatrix} 0 \\ \frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}^{-1} c_1 + \frac{d\Delta(\alpha_k, -i\omega_k)}{d\lambda}^{-1} c_2 \end{smallmatrix}\right] \end{pmatrix}. \tag{5.49}$$

In the following we will compute the normal form of system (5.46). Define $\Theta_m^c : V^m(\mathcal{X}_c, \mathcal{X}_c) \to V^m(\mathcal{X}_c, \mathcal{X}_c)$ by

$$\Theta_m^c(G_c) := [\mathcal{A}_c, G_c], \quad \forall G_c \in V^m(\mathcal{X}_c, \mathcal{X}_c), \tag{5.50}$$

and $\Theta_m^h : V^m(\mathcal{X}_c, \mathcal{X}_h \cap D(\mathcal{A})) \to V^m(\mathcal{X}_c, \mathcal{X}_h)$ by

$$\Theta_m^h(G_h) := [\mathcal{A}, G_h], \quad \forall G_h \in V^m\big(\mathcal{X}_c, \mathcal{X}_h \cap D(\mathcal{A})\big). \tag{5.51}$$

We decompose $V^m(\mathcal{X}_c, \mathcal{X}_c)$ into the direct sum

$$V^m(\mathcal{X}_c, \mathcal{X}_c) = \mathcal{R}_m^c \oplus \mathcal{C}_m^c,$$

where

$$\mathcal{R}_m^c := R\big(\Theta_m^c\big),$$

is the range of Θ_m^c, and $\mathcal{C}_m^c$ is some complementary space of $\mathcal{R}_m^c$ into $V^m(\mathcal{X}_c, \mathcal{X}_c)$. Define $\mathcal{P}_m : V^m(\mathcal{X}_c, \mathcal{X}) \to V^m(\mathcal{X}_c, \mathcal{X})$ the bounded linear projector satisfying

$$\mathcal{P}_m\big(V^m(\mathcal{X}_c, \mathcal{X})\big) = \mathcal{R}_m^c \oplus V^m(\mathcal{X}_c, \mathcal{X}_h), \quad \text{and} \quad (I - \mathcal{P}_m)\big(V^m(\mathcal{X}_c, \mathcal{X})\big) = \mathcal{C}_m^c.$$

Now we apply the method described in Theorem 4.4 for $k = 3$ to system (5.46). The main point is to compute $G_2 \in V^2(\mathcal{X}_c, D(\mathcal{A}))$ defined such that

$$[\mathcal{A}, G_2](w_c) = \mathcal{P}_2\left[\frac{1}{2!}D^2F(0)(w_c, w_c)\right] \quad \text{for each } w_c \in \mathcal{X}_c \tag{5.52}$$

in order to obtain the normal form because the reduced system is the following

$$\begin{aligned} \frac{dw_c(t)}{dt} &= \mathcal{A}_c w_c(t) + \frac{1}{2!}\Pi_c D^2F_3(0)\big(w_c(t), w_c(t)\big) \\ &\quad + \frac{1}{3!}\Pi_c D^3F_3(0)\big(w_c(t), w_c(t), w_c(t)\big) + R_c\big(w_c(t)\big) \end{aligned} \tag{5.53}$$

where

$$\begin{aligned} \frac{1}{2!}\Pi_c D^2F_3(0)(w_c, w_c) &= \frac{1}{2!}\Pi_c D^2F_2(0)(w_c, w_c) \\ &= \frac{1}{2!}\Pi_c D^2F(0)(w_c, w_c) - [\mathcal{A}_c, \Pi_c G_2](w_c) \end{aligned} \tag{5.54}$$

and

$$\begin{aligned} \frac{1}{3!}\Pi_c D^3F_3(0)(w_c, w_c, w_c) &= \frac{1}{3!}\Pi_c D^3F_2(0)(w_c, w_c, w_c) - \Pi_c[\mathcal{A}, G_3](w_c) \\ &= \frac{1}{3!}\Pi_c D^3F_2(0)(w_c, w_c, w_c) - [\mathcal{A}_c, \Pi_c G_3](w_c). \end{aligned} \tag{5.55}$$

Set

$$G_{m,k} := \Pi_k G_m, \quad \forall k = c, h, \ m \geq 2.$$

Recall that (5.52) is equivalent to finding $G_{2,c} \in V^2(\mathcal{X}_c, \mathcal{X}_c)$ and $G_{2,h} \in V^2(\mathcal{X}_c, \mathcal{X}_h \cap D(\mathcal{A}))$ satisfying

$$[\mathcal{A}_c, G_{2,c}] = \Pi_c \mathcal{P}_2 \left[\frac{1}{2!} D^2 F(0)(w_c, w_c) \right] \tag{5.56}$$

and

$$[\mathcal{A}, G_{2,h}] = \Pi_h \mathcal{P}_2 \left[\frac{1}{2!} D^2 F(0)(w_c, w_c) \right]. \tag{5.57}$$

From (5.55), we know that the third order term $\frac{1}{3!} \Pi_c D^3 F_2(0)(w_c, w_c, w_c)$ in the equation is needed after computing the normal form up to the second order. In the following lemma we find the expression of $\frac{1}{3!} \Pi_c D^3 F_2(0)(w_c, w_c, w_c)$.

Lemma 5.18. *Let $G_2 \in V^2(\mathcal{X}_c, D(\mathcal{A}))$ be defined in (5.52). Then after the change of variables*

$$w = \overline{w} + G_2(\Pi_c \overline{w}), \tag{5.58}$$

system (5.46) becomes (after dropping the bars)

$$\frac{dw(t)}{dt} = \mathcal{A} w(t) + F_2\big(w(t)\big), \qquad w(0) = w_0 \in \overline{D(\mathcal{A})},$$

where

$$F_2\big(w(t)\big) = F\big(w(t)\big) - [\mathcal{A}, G_2]\big(w_c(t)\big) + O\big(\big\|w(t)\big\|^3\big).$$

In particular,

$$\begin{aligned}
&\frac{1}{3!} \Pi_c D^3 F_2(0)(w_c, w_c, w_c) \\
&\quad = \Pi_c D^2 F(0)\big(w_c, G_2(w_c)\big) + \frac{1}{3!} \Pi_c D^3 F(0)(w_c, w_c, w_c) \\
&\qquad - DG_{2,c}(w_c) \left[\frac{1}{2!} \Pi_c D^2 F(0)(w_c, w_c) - [\mathcal{A}_c, G_{2,c}](w_c) \right].
\end{aligned} \tag{5.59}$$

Proof. From Proposition 3.8, the first part is obvious. We only need to prove the formula (5.59). Set

$$w_k(t) := \Pi_k w(t), \qquad \overline{w}_k(t) := \Pi_k \overline{w}(t), \quad \forall k = c, h.$$

We can split the system (5.46) as

$$\frac{dw_c(t)}{dt} = \mathcal{A}_c w_c(t) + \Pi_c F\big(w_c(t) + w_h(t)\big),$$
$$\frac{dw_h(t)}{dt} = \mathcal{A}_h w_h(t) + \Pi_h F\big(w_c(t) + w_h(t)\big).$$

Note that (5.58) is equivalent to

$$w_c = \overline{w}_c + G_{2,c}(\overline{w}_c), \qquad w_h = \overline{w}_h + G_{2,h}(\overline{w}_c).$$

Since $\dim(\mathcal{X}_c) < +\infty$, we have

$$\begin{aligned}
\dot{\overline{w}}_c(t) &= \big[I + DG_{2,c}\big(\overline{w}_c(t)\big)\big]^{-1}\big[\mathcal{A}_c\big(\overline{w}_c(t) + G_{2,c}\big(\overline{w}_c(t)\big)\big) + \Pi_c F\big(\overline{w}(t) + G_2\big(\overline{w}_c(t)\big)\big)\big] \\
&= \mathcal{A}_c\big(\overline{w}_c(t) + G_{2,c}\big(\overline{w}_c(t)\big)\big) + \Pi_c F\big(\overline{w}(t) + G_2\big(\overline{w}_c(t)\big)\big) \\
&\quad - DG_{2,c}\big(\overline{w}_c(t)\big)\big[\mathcal{A}_c\big(\overline{w}_c(t) + G_{2,c}\big(\overline{w}_c(t)\big)\big) + \Pi_c F\big(\overline{w}(t) + G_2\big(\overline{w}_c(t)\big)\big)\big] \\
&\quad + DG_{2,c}\big(\overline{w}_c(t)\big)DG_{2,c}\big(\overline{w}_c(t)\big)\begin{bmatrix} \mathcal{A}_c(\overline{w}_c(t) + G_{2,c}(\overline{w}_c(t))) \\ + \Pi_c F(\overline{w}(t) + G_2(\overline{w}_c(t))) \end{bmatrix} + O\big(\big\|\overline{w}(t)\big\|^4\big).
\end{aligned}$$

Hence

$$\begin{aligned}
&\Pi_c F_2\big(\overline{w}(t)\big) \\
&\quad = \Pi_c F\big(\overline{w}(t) + G_2\big(\overline{w}_c(t)\big)\big) - [\mathcal{A}_c, G_{2,c}]\big(\overline{w}_c(t)\big) \\
&\qquad - DG_{2,c}\big(\overline{w}_c(t)\big)\big[\Pi_c F\big(\overline{w}(t) + G_2\big(\overline{w}_c(t)\big)\big) - [\mathcal{A}_c, G_{2,c}]\big(\overline{w}_c(t)\big)\big] + O\big(\big\|\overline{w}(t)\big\|^4\big).
\end{aligned}$$

Let $w_c \in \mathcal{X}_c$. It follows that

$$\begin{aligned}
\Pi_c F_2(w_c) &= \Pi_c F\big(w_c + G_2(w_c)\big) - [\mathcal{A}_c, G_{2,c}](w_c) \\
&\quad - DG_{2,c}(w_c)\big[\Pi_c F\big(w_c + G_2(w_c)\big) - [\mathcal{A}_c, G_{2,c}](w_c)\big] + O\big(\|w_c\|^4\big).
\end{aligned}$$

Thus we have

$$\begin{aligned}
\Pi_c F_2(w_c) &= \frac{1}{2!}\Pi_c D^2 F(0)(w_c, w_c) - [\mathcal{A}_c, G_{2,c}](w_c) \\
&\quad + \Pi_c D^2 F(0)\big(w_c, G_2(w_c)\big) + \frac{1}{3!}\Pi_c D^3 F(0)(w_c, w_c, w_c) \\
&\quad - DG_{2,c}(w_c)\left[\frac{1}{2!}\Pi_c D^2 F(0)(w_c, w_c) - [\mathcal{A}_c, G_{2,c}](w_c)\right] + O\big(\|w_c\|^4\big).
\end{aligned}$$

Then (5.59) follows and the proof is complete. □

Set

$$w_c = \widehat{\alpha}\widehat{e}_1 + x_1\widehat{e}_2 + x_2\widehat{e}_3.$$

We shall compute the normal form expressed in terms of the basis $\{\widehat{e}_1, \widehat{e}_2, \widehat{e}_3\}$. Consider $V^m(\mathbb{C}^3, \mathbb{C}^3)$ and $V^m(\mathbb{C}^3, \mathcal{X}_h \cap D(\mathcal{A}))$, which denote the linear space of the homogeneous polynomials of degree m in 3 real variables, $\widehat{\alpha}$, $x = (x_1, x_2)$ with coefficients in $\mathbb{C}^3$ and $\mathcal{X}_h \cap D(\mathcal{A})$, respectively. The operators Θ_m^c and Θ_m^h considered in (5.50) and (5.51) now act in the spaces $V^m(\mathbb{C}^3, \mathbb{C}^3)$ and $V^m(\mathbb{C}^3, \mathcal{X}_h \cap D(\mathcal{A}))$, respectively, and satisfy

$$
\begin{aligned}
\Theta_m^c(G_{m,c})\begin{pmatrix}\widehat{\alpha}\\ x_1\\ x_2\end{pmatrix} &= [\mathcal{A}_c, G_{m,c}]\begin{pmatrix}\widehat{\alpha}\\ x_1\\ x_2\end{pmatrix} = DG_{m,c}\mathcal{A}_c\begin{pmatrix}\widehat{\alpha}\\ x_1\\ x_2\end{pmatrix} - \mathcal{A}_c G_{m,c}\begin{pmatrix}\widehat{\alpha}\\ x_1\\ x_2\end{pmatrix}\\
&= \begin{pmatrix} D_x G_{m,c}^1\begin{pmatrix}\widehat{\alpha}\\ x_1\\ x_2\end{pmatrix} M_c x\\ \left(D_x\begin{pmatrix}G_{m,c}^2\\ G_{m,c}^3\end{pmatrix}\begin{pmatrix}\widehat{\alpha}\\ x_1\\ x_2\end{pmatrix} M_c\begin{pmatrix}x_1\\ x_2\end{pmatrix} - M_c\begin{pmatrix}G_{m,c}^2\\ G_{m,c}^3\end{pmatrix}\begin{pmatrix}\widehat{\alpha}\\ x_1\\ x_2\end{pmatrix}\right)\end{pmatrix},\\
\Theta_m^h(G_{m,h}) &= [\mathcal{A}, G_{m,h}] = DG_{m,h}\mathcal{A}_c - \mathcal{A}_h G_{m,h},\\
\forall G_{m,c}\begin{pmatrix}\widehat{\alpha}\\ x_1\\ x_2\end{pmatrix} &= \begin{pmatrix}G_{m,c}^1\\ G_{m,c}^2\\ G_{m,c}^3\end{pmatrix}\begin{pmatrix}\widehat{\alpha}\\ x_1\\ x_2\end{pmatrix} \in V^m\left(\mathbb{C}^3, \mathbb{C}^3\right), \quad \forall G_{m,h} \in V^m\left(\mathbb{C}^3, \mathcal{X}_h \cap D(\mathcal{A})\right)
\end{aligned}
\tag{5.60}
$$

with

$$
\mathcal{A}_c = \begin{bmatrix} 0 & 0 & 0\\ 0 & i\omega_k & 0\\ 0 & 0 & -i\omega_k\end{bmatrix} \quad \text{and} \quad M_c = \begin{bmatrix} i\omega_k & 0\\ 0 & -i\omega_k\end{bmatrix}.
$$

We define $\overline{\Theta}_m^c : V^m(\mathbb{C}^3, \mathbb{C}^2) \to V^m(\mathbb{C}^3, \mathbb{C}^2)$ by

$$
\begin{aligned}
&\overline{\Theta}_m^c\begin{pmatrix}G_{m,c}^2\\ G_{m,c}^3\end{pmatrix} = D_x\begin{pmatrix}G_{m,c}^2\\ G_{m,c}^3\end{pmatrix} M_c\begin{pmatrix}x_1\\ x_2\end{pmatrix} - M_c\begin{pmatrix}G_{m,c}^2\\ G_{m,c}^3\end{pmatrix},\\
&\forall\begin{pmatrix}G_{m,c}^2\\ G_{m,c}^3\end{pmatrix} \in V^m\left(\mathbb{C}^3, \mathbb{C}^2\right).
\end{aligned}
\tag{5.61}
$$

Lemma 5.19. *For $m \in \mathbb{N}\backslash\{0, 1\}$, we have the decomposition*

$$
V^m\left(\mathbb{C}^3, \mathbb{C}^2\right) = R\left(\overline{\Theta}_m^c\right) \oplus N\left(\overline{\Theta}_m^c\right)
\tag{5.62}
$$

and

$$
N\left(\overline{\Theta}_m^c\right) = \operatorname{span}\left\{\begin{pmatrix}0\\ x_1^{q_1}x_2^{q_2}\alpha^{q_3}\end{pmatrix}, \begin{pmatrix}x_1^{q_1'}x_2^{q_2'}\alpha^{q_3'}\\ 0\end{pmatrix} \,\middle|\, \begin{array}{l} q_1 - q_2 = -1,\\ q_1' - q_2' = 1,\ q_i, q_i' \in \mathbb{N},\ i = 1, 2, 3\end{array}\right\}.
\tag{5.63}
$$

Proof. The canonical basis of $V^m(\mathbb{C}^3,\mathbb{C}^2)$ is

$$\Phi=\left\{\begin{pmatrix} x_1^{q_1}x_2^{q_2}\alpha^{q_3}\\ 0\end{pmatrix},\begin{pmatrix} 0\\ x_1^{q_1}x_2^{q_2}\alpha^{q_3}\end{pmatrix}\,\middle|\,q_1+q_2+q_3=m\right\}.$$

Since $M_c=\begin{bmatrix} i\omega_k & 0\\ 0 & -i\omega_k\end{bmatrix}$, for each $\begin{pmatrix} x_1^{q_1}x_2^{q_2}\alpha^{q_3}\\ 0\end{pmatrix},\begin{pmatrix} 0\\ x_1^{q_1}x_2^{q_2}\alpha^{q_3}\end{pmatrix}\in\Phi$, we have

$$\begin{aligned}\overline{\Theta}_m^c\begin{pmatrix} x_1^{q_1}x_2^{q_2}\alpha^{q_3}\\ 0\end{pmatrix}&=D_x\begin{pmatrix} x_1^{q_1}x_2^{q_2}\alpha^{q_3}\\ 0\end{pmatrix}M_c\begin{pmatrix} x_1\\ x_2\end{pmatrix}-M_c\begin{pmatrix} x_1^{q_1}x_2^{q_2}\alpha^{q_3}\\ 0\end{pmatrix}\\ &=i\omega_k(q_1-q_2-1)\begin{pmatrix} x_1^{q_1}x_2^{q_2}\alpha^{q_3}\\ 0\end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\overline{\Theta}_m^c\begin{pmatrix} 0\\ x_1^{q_1}x_2^{q_2}\alpha^{q_3}\end{pmatrix}&=D_x\begin{pmatrix} 0\\ x_1^{q_1}x_2^{q_2}\alpha^{q_3}\end{pmatrix}M_c\begin{pmatrix} x_1\\ x_2\end{pmatrix}-M_c\begin{pmatrix} 0\\ x_1^{q_1}x_2^{q_2}\alpha^{q_3}\end{pmatrix}\\ &=i\omega_k(q_1-q_2+1)\begin{pmatrix} 0\\ x_1^{q_1}x_2^{q_2}\alpha^{q_3}\end{pmatrix}.\end{aligned}$$

Hence the operators $\overline{\Theta}_m^c$ defined in (5.61) have diagonal matrix representation in the canonical basis of $V^m(\mathbb{C}^3,\mathbb{C}^2)$. Thus, (5.62) and (5.63) hold. $\square$

From (5.63), we obtain

$$\begin{aligned}N\big(\overline{\Theta}_2^c\big)&=\operatorname{span}\left\{\begin{pmatrix} x_1\widehat{\alpha}\\ 0\end{pmatrix},\begin{pmatrix} 0\\ x_2\widehat{\alpha}\end{pmatrix}\right\},\\ N\big(\overline{\Theta}_3^c\big)&=\operatorname{span}\left\{\begin{pmatrix} x_1^2x_2\\ 0\end{pmatrix},\begin{pmatrix} x_1\widehat{\alpha}^2\\ 0\end{pmatrix},\begin{pmatrix} 0\\ x_1x_2^2\end{pmatrix},\begin{pmatrix} 0\\ x_2\widehat{\alpha}^2\end{pmatrix}\right\}.\end{aligned}\tag{5.64}$$

Define P_m^R and $P_m^N: V^m(\mathbb{C}^3,\mathbb{C}^2)\to V^m(\mathbb{C}^3,\mathbb{C}^2)$ the bounded linear projectors satisfying

$$P_m^R\big(V^m\big(\mathbb{C}^3,\mathbb{C}^2\big)\big)=R\big(\overline{\Theta}_m^c\big)$$

and

$$P_m^N\big(V^m\big(\mathbb{C}^3,\mathbb{C}^2\big)\big)=N\big(\overline{\Theta}_m^c\big).$$

We are now ready to compute the normal form of the reduced system expressed in terms of the basis $\{\widehat{e}_1,\widehat{e}_2,\widehat{e}_3\}$ of $\mathcal{X}_c$. From (5.47), (5.48), (5.56), (5.57), (5.60)–(5.62) and (5.64), we know that to find $G_2\in V^2(\mathcal{X}_c,D(\mathcal{A}))$ defined in (5.52) is equivalent to finding $G_{2,c}=\begin{pmatrix} G_{2,c}^1\\ G_{2,c}^2\\ G_{2,c}^3\end{pmatrix}:=\Pi_cG_2\in V^2(\mathbb{C}^3,\mathbb{C}^3)$ and $G_{2,h}:=\Pi_hG_2\in V^2(\mathbb{C}^3,\mathcal{X}_h\cap D(\mathcal{A}))$ such that

$$[\mathcal{A}_c, G_{2,c}]\left(\begin{pmatrix} \widehat{\alpha} \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{pmatrix}\right) = \begin{pmatrix} D_x G_{2,c}^1 M_c x \\ D_x \begin{pmatrix} G_{2,c}^2 \\ G_{2,c}^3 \end{pmatrix} M_c \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - M_c \begin{pmatrix} G_{2,c}^2 \\ G_{2,c}^3 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ P_2^R(\widehat{H}_2^1) \end{pmatrix}$$

and

$$\begin{aligned} [\mathcal{A}, G_{2,h}] &= DG_{2,h}\mathcal{A}_c - \mathcal{A}_h G_{2,h} \\ &= \begin{pmatrix} 0 \\ \widetilde{\psi}\begin{bmatrix} 1 \\ -\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}^{-1} c_1 - \frac{d\Delta(\alpha_k, -i\omega_k)}{d\lambda}^{-1} c_2 \end{bmatrix} \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} \widehat{H}_2^1\left(\begin{pmatrix} \widehat{\alpha} \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{pmatrix}\right) &= \widetilde{\psi}\begin{bmatrix} \frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}^{-1} \\ \frac{d\Delta(\alpha_k, -i\omega_k)}{d\lambda}^{-1} \end{bmatrix} \\ &= \begin{pmatrix} A_1 x_1 \widehat{\alpha} + A_2 \widehat{\alpha} x_2 + \frac{1}{2} a_{20} x_1^2 + a_{11} x_1 x_2 + \frac{1}{2} a_{02} x_2^2 \\ \bar{A}_1 x_2 \widehat{\alpha} + \overline{A}_2 \widehat{\alpha} x_1 + \frac{1}{2} \bar{a}_{02} x_1^2 + \bar{a}_{11} x_1 x_2 + \frac{1}{2} \bar{a}_{20} x_2^2 \end{pmatrix} \end{aligned}$$

with

$$\begin{aligned} A_1 = A_2 &= -\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}^{-1} \frac{1}{\alpha_k(1 - \ln(\alpha_k \chi))}, \\ a_{20} = a_{11} = a_{02} &= \frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}^{-1} \frac{\chi\beta(\ln(\alpha_k \chi) - 2)}{(1 - \ln(\alpha_k \chi))^2}. \end{aligned}$$

From (5.54), it is easy to obtain the second order terms of the normal form expressed in terms of the basis $\{\widehat{e}_1, \widehat{e}_2, \widehat{e}_3\}$:

$$\begin{aligned} \frac{1}{2!}\Pi_c D^2 F_3(0)(w_c, w_c) &= (\widehat{e}_1, \widehat{e}_2, \widehat{e}_3)\begin{pmatrix} 0 \\ P_2^N(\widehat{H}_2^1) \end{pmatrix} \\ &= (\widehat{e}_1, \widehat{e}_2, \widehat{e}_3)\begin{pmatrix} 0 \\ \begin{pmatrix} A_1 x_1 \widehat{\alpha} \\ \overline{A_1} x_2 \widehat{\alpha} \end{pmatrix} \end{pmatrix} \\ &= A_1 x_1 \widehat{\alpha}\widehat{e}_2 + \overline{A_1} x_2 \widehat{\alpha}\widehat{e}_3. \end{aligned}$$

Notice that the terms $O(|x|\alpha^2)$ are irrelevant to determine the generic Hopf bifurcation. Hence, it is only needed to compute the coefficients of

$$\begin{pmatrix} 0 \\ \begin{pmatrix} x_1^2 x_2 \\ 0 \end{pmatrix} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ \begin{pmatrix} 0 \\ x_1 x_2^2 \end{pmatrix} \end{pmatrix}$$

in the third order terms of the normal form. Firstly from Chu et al. [12, pages 22–24] and by computing we obtain

$$G_{2,h}\left(\begin{pmatrix} 0 \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{pmatrix}\right) = \begin{pmatrix} 0 \\ \begin{pmatrix} 0 \\ \overline{\Psi} \end{pmatrix} \end{pmatrix}$$

with

$$\overline{\Psi} = x_1^2 \psi_{2,2} + x_2^2 \psi_{3,3} + 2x_1 x_2 \psi_{2,3}$$

and

$$G_{2,c}\left(\begin{pmatrix} 0 \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{pmatrix}\right) = \begin{pmatrix} 0 \\ \begin{pmatrix} \frac{1}{i2\omega_k}(a_{20}x_1^2 - 2a_{11}x_1x_2 - \frac{1}{3}a_{02}x_2^2) \\ \frac{1}{i2\omega_k}(\frac{1}{3}\bar{a}_{02}x_1^2 + 2\bar{a}_{11}x_1x_2 - \bar{a}_{20}x_2^2) \end{pmatrix} \end{pmatrix}.$$

Therefore we have

$$\left[D^2 W(0)\big(w_c, G_2(w_c)\big)\right]_{\widehat{\alpha}=0} = \begin{pmatrix} \frac{\beta(\ln(\alpha_k \chi)-2)}{\chi} S_1 S_2 \\ 0 \end{pmatrix}$$

with

$$S_1 = \int_0^{+\infty} \gamma(a)(x_1 c_1 + x_2 c_2)(a)\,da,$$

$$S_2 = \int_0^{+\infty} \gamma(a) \begin{pmatrix} \frac{1}{i2\omega_k}(a_{20}x_1^2 - 2a_{11}x_1x_2 - \frac{1}{3}a_{02}x_2^2)c_1 \\ + \frac{1}{i2\omega_k}(\frac{1}{3}\bar{a}_{02}x_1^2 + 2\bar{a}_{11}x_1x_2 - \bar{a}_{20}x_2^2)c_2 + \overline{\Psi} \end{pmatrix}(a)\,da.$$

Hence

$$\begin{aligned} &\left[\widehat{\Pi}_c D^2 W(0)\big(w_c, G_2(w_c)\big)\right]_{\widehat{\alpha}=0} \\ &= \frac{\beta(\ln(\alpha_k \chi) - 2)}{\chi} S_1 S_2 \left(\widehat{\Pi}_c \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \\ &= \frac{\beta(\ln(\alpha_k \chi) - 2)}{\chi} S_1 S_2 \begin{bmatrix} 0 \\ \frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}^{-1} c_1 + \frac{d\Delta(\alpha_k, -i\omega_k)}{d\lambda}^{-1} c_2 \end{bmatrix}. \end{aligned} \tag{5.65}$$

From (5.49), (5.59) and (5.65), we have

$$\begin{aligned} &\left[\frac{1}{3!}\Pi_c D^3 F_2(0)(w_c, w_c, w_c)\right]_{\widehat{\alpha}=0} \\ &= \begin{pmatrix} 0 \\ \begin{pmatrix} \widehat{\psi}\begin{bmatrix} 0 \\ \frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}^{-1} c_1 + \frac{d\Delta(\alpha_k, -i\omega_k)}{d\lambda}^{-1} c_2 \end{bmatrix} \\ + \frac{\beta(\ln(\alpha_k \chi)-2)}{\chi} S_1 S_2 \begin{bmatrix} 0 \\ \frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}^{-1} c_1 + \frac{d\Delta(\alpha_k, -i\omega_k)}{d\lambda}^{-1} c_2 \end{bmatrix} \end{pmatrix} \end{pmatrix}. \end{aligned}$$

Now we obtain the third order terms of the normal form expressed in terms of the basis $\{\widehat{e}_1, \widehat{e}_2, \widehat{e}_3\}$

$$\frac{1}{3!}\Pi_c D^3 F_3(0)(w_c, w_c, w_c)$$

$$= (\widehat{e}_1, \widehat{e}_2, \widehat{e}_3)\begin{pmatrix} 0 \\ P_3^N \begin{pmatrix} \frac{4\beta^2(-\ln(\alpha_k\chi)+3)\chi^2}{3(1-\ln(\alpha_k\chi))^3}(\frac{x_1+x_2}{2})^3 \begin{bmatrix} \frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}^{-1} \\ \frac{d\Delta(\alpha_k, -i\omega_k)}{d\lambda}^{-1} \end{bmatrix} \\ + \begin{pmatrix} \frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}^{-1} \frac{\beta(\ln(\alpha_k\chi)-2)}{\chi} S_1 S_2 \\ \frac{d\Delta(\alpha_k, -i\omega_k)}{d\lambda}^{-1} \frac{\beta(\ln(\alpha_k\chi)-2)}{\chi} S_1 S_2 \end{pmatrix} \end{pmatrix} \end{pmatrix} + O(|x|\widehat{\alpha}^2)$$

$$= (\widehat{e}_1, \widehat{e}_2, \widehat{e}_3)\begin{pmatrix} 0 \\ \begin{pmatrix} C_1 x_1^2 x_2 \\ \overline{C_1} x_1 x_2^2 \end{pmatrix} \end{pmatrix} + O(|x|\widehat{\alpha}^2)$$

with

$$C_1 = \begin{bmatrix} \frac{i}{2\omega_k}(a_{11}^2 - \frac{7}{3}|a_{11}|^2) + \frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}^{-1} \frac{\beta^2(-\ln(\alpha_k\chi)+3)\chi^2}{2(1-\ln(\alpha_k\chi))^3} \\ + \frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}^{-1} \frac{\beta(\ln(\alpha_k\chi)-2)}{1-\ln(\alpha_k\chi)} \begin{bmatrix} \int_0^{+\infty} \gamma(a)\psi_{2,2}(a)da \\ +\int_0^{+\infty} \gamma(a)2\psi_{2,3}(a)da \end{bmatrix} \end{bmatrix}.$$

Therefore we obtain the following normal form of the reduced system

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = M_c \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} A_1 x_1 \widehat{\alpha} \\ \overline{A_1} x_2 \widehat{\alpha} \end{pmatrix} + \begin{pmatrix} C_1 x_1^2 x_2 \\ \overline{C_1} x_1 x_2^2 \end{pmatrix} + O(|x|\widehat{\alpha}^2 + |(\widehat{\alpha}, x)|^4).$$

The normal form above can be written in real coordinates (w_1, w_2) through the change of variables $x_1 = w_1 - iw_2$, $x_2 = w_1 + iw_2$. Setting $w_1 = \rho\cos\xi$, $w_2 = \rho\sin\xi$, this normal form becomes

$$\begin{cases} \dot{\rho} = \iota_1\widehat{\alpha}\rho + \iota_2\rho^3 + O(\widehat{\alpha}^2\rho + |(\rho, \widehat{\alpha})|^4), \\ \dot{\xi} = -\sigma_k + O(|(\rho, \widehat{\alpha})|), \end{cases} \tag{5.66}$$

where

$$\iota_1 = \mathrm{Re}(A_1), \qquad \iota_2 = \mathrm{Re}(C_1).$$

Following Chow and Hale [8] we know that the sign of $\iota_1\iota_2$ determines the direction of the bifurcation and that the sign of ι_2 determines the stability of the nontrivial periodic orbits. In summary we have the following theorem.

Theorem 5.20. *The flow of (1.6) on the center manifold of the origin at $\alpha = \alpha_k$, $k \in \mathbb{N}^+$, is given by (5.66). Furthermore, we have the following*

(i) *Hopf bifurcation is supercritical if $\iota_1\iota_2 < 0$, and subcritical if $\iota_1\iota_2 > 0$;*
(ii) *the nontrivial periodic solution is stable if $\iota_2 < 0$, and unstable if $\iota_2 > 0$.*

Remark 5.21. We would like to mention that since the normal form theory and the computation procedure developed in Sections 3 and 4 is for general semilinear Cauchy problems, the theory and technique of computing the reduced system and normal form could be applied to other types of equations such as transport equations (Perthame [53]), reaction–diffusion equations (Kokubu [33], Eckmann et al. [18]), and partial differential equations with delay (Faria [22, 23]).

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