SCIENCE CHINA
Mathematics

• ARTICLES •

May 2010 Vol. 53 No. 5: 1185–1194
doi: 10.1007/s11425-009-0123-6

Asymptotic stability of monostable wavefronts in discrete-time integral recursions

LIN Guo[1], LI WanTong[1] & RUAN ShiGui[2,*]

[1]*School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China;*
[2]*Department of Mathematics, University of Miami, P.O. Box 249085, Coral Gables, FL 33124-4250, USA*
Email: ling@lzu.edu.cn, wtli@lzu.edu.cn, ruan@math.miami.edu

Received July 28, 2008; accepted February 6, 2009; published online April 26, 2010

Abstract The aim of this work is to study the traveling wavefronts in a discrete-time integral recursion with a Gauss kernel in $\mathbb{R}^2$. We first establish the existence of traveling wavefronts as well as their precise asymptotic behavior. Then, by employing the comparison principle and upper and lower solutions technique, we prove the asymptotic stability and uniqueness of such monostable wavefronts in the sense of phase shift and circumnutation. We also obtain some similar results in $\mathbb{R}$.

Keywords discrete-time integral recursion, comparison principle, upper and lower solutions, monostable wave, stability

MSC(2000): 35B40, 45M05, 92D25

Citation: Lin G, Li W T, Ruan S G. Asymptotic stability of monostable wavefronts in discrete-time integral recursions. Sci China Math, 2010, 53(5): 1185-1194, doi: 10.1007/s11425-009-0123-6

1 Introduction

In 1982, Weinberger [27] proposed the following discrete-time recursion to model a class of biological processes

$$u_{n+1}(x) = Q[u_n](x), \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where the vector-valued function $u_n(x) \in \mathbb{R}^k$ represents the population density of k species at time n at the point $x \in \mathcal{H} \subseteq \mathbb{R}^m$, $\mathcal{H}$ is the habitat of the species, Q is an $\mathbb{R}^k$-valued mapping. In particular, Weinberger [27, p. 358] derived a concrete population genetics model of the form

$$u_{n+1}(\mathbf{x}) = Q[u_n](\mathbf{x}) = \int_{\mathbb{R}^2} m(\mathbf{x}-\mathbf{y})g(u_n(\mathbf{y}))d\mathbf{y}, \quad n = 0, 1, 2, \ldots, \tag{1.2}$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, $u_n \in \mathbb{R}$ denotes the gene fraction in the newly born individuals of the n-th generation, m is the probability function describing the migration of the individuals and $g(u_n)$ means the gene fraction before migration. If we assume that the probability function $m(\mathbf{x}-\mathbf{y})$ takes the form of a Gauss kernel function (also see Remark 5.6), which is the fundamental solution of a homogeneous reaction-diffusion equation in $\mathbb{R}^2$ (in fact, reaction-diffusion equations were also used to describe the spreading of gene

*Corresponding author

fraction in the pioneer work on traveling wavefronts of reaction-diffusion equations [7, 8]), then (1.2) reduces to the following discrete-time integral recursion

$$u_{n+1}(\mathbf{x}) = Q[u_n](\mathbf{x}) = \frac{1}{4\pi d}\int_{\mathbb{R}^2} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4d}} g(u_n(\mathbf{y}))d\mathbf{y}, \quad n = 0, 1, 2, \dots, \tag{1.3}$$

where $d > 0$ is a positive constant and $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^2$.

For model (1.2) and more general ones, the long term dynamical behavior has been investigated extensively in the past thirty years, especially on traveling wavefronts and the asymptotic speed of spread. We refer to [5, 6, 9–17, 22–24, 27–30], etc.

It is well-known that the stability of traveling wavefronts is very important in interpreting some phenomena in physics, biology and chemical reactions [1, 19, 25] and in understanding the long time behavior of evolutionary systems [20]. However, these results mentioned above only involve the existence of traveling wavefronts and the asymptotic speed of spread, and the *stability* of traveling wavefronts in the discrete-time recursions remains open. In this paper, we study the existence and stability of traveling frontwaves in discrete-time integral recursions.

By the studies on the stability of traveling wavefronts in reaction-diffusion equations [20, 25, 26] and lattice differential equations [2,3,18], it seems that the precise asymptotic behavior of traveling wavefronts of (1.3) should be established. Therefore, we shall first consider the existence and precise asymptotic behavior of traveling wavefronts in (1.3) before investigating the stability of monostable wavefronts. For this purpose, the characteristic equation of the linearized system near the unstable equilibrium state will be first investigated. Then we will construct proper upper and lower solutions depending on the constants that were established by the characteristic equation. This allows us to prove the existence and precise asymptotic behavior of traveling wavefronts in (1.3) by the monotone iteration technique [5, 24].

Since the stability of traveling wavefronts deals with the long term behavior of the corresponding Cauchy problem when the initial value is a spatial perturbation of the traveling wavefronts, it is necessary to study some properties of the initial value problem of model (1.3). In particular, we first construct a pair of upper and lower solutions by the traveling wavefronts, and then the squeezing technique [3, 18, 26] is applied to such discrete-time integral recursions to prove the asymptotic stability and uniqueness of traveling wavefronts in the sense of phase shift and circumnutation. In addition, the corresponding model defined on $\mathbb{R}$ is also considered.

The rest of this paper is organized as follows. In Section 2, we establish the existence and discuss the precise asymptotic behavior of traveling wavefronts of the discrete-time integral recursion (1.3). The corresponding Cauchy problem is investigated in Section 3. In Section 4, we study the asymptotic stability and uniqueness of the traveling wavefronts by the squeezing technique. Similar results on monostable wavefronts of discrete-time recursions in $\mathbb{R}$ are given in Section 5.

2 Existence of traveling wavefronts

In this section, we shall establish the existence of traveling wavefronts of the discrete-time integral recursion (1.3) by combining the monotone iteration technique with the upper and lower solutions method. In what follows, $\mathbf{x}$ will be denoted by (x_1, x_2) without further interpretation, and $d\mathbf{x}$ will be interpreted as dx_1dx_2, so for $\mathbf{y} = (y_1, y_2)$. For $n = 1, 2$, denote

$$C(\mathbb{R}^n, \mathbb{R}) = \{u(x) | u(x) : \mathbb{R}^n \to \mathbb{R} \text{ is uniformly continuous and bounded}\}.$$

It is well-known that $C(\mathbb{R}^n, \mathbb{R})$ is a Banach space equipped with sup norm $|\cdot|$. Assume that $a < b$ with $a, b \in \mathbb{R}$. Then $C_{[a,b]}$ will be interpreted as

$$C_{[a,b]} = \{u(x) | u(x) \in C(\mathbb{R}^n, \mathbb{R}) \text{ and } a \leqslant u(x) \leqslant b, x \in \mathbb{R}^n\}.$$

For convenience, we first give the assumptions on g, which will be imposed throughout Sections 2–5.

(g1) $g(0) = 0$, $g(1) = 1$ and $g(u) > u$ for all $u \in (0, 1)$;

(g2) $g(u)$ is a C^2 function if $u \in [0,1]$ and $0 \leqslant g'(u) \leqslant g'(0)$ if $u \in [0,1]$;
(g3) $0 \leqslant g'(u) < 1$ if $u \in [1-\rho, 1]$ with some $\rho \in (0,1)$;
(g4) $g((1+\delta)u) \leqslant (1+\delta)g(u)$ for any $u, \delta \in [0,1]$ such that $(1+\delta)u \in [0,1]$.

We should note that there are many functions satisfying the assumptions (g1)–(g4), such as the so-called Beverton-Holt stock recruitment curve, which takes the form

$$g(u) = \frac{\lambda u}{1+(\lambda-1)u}$$

with $\lambda > 1$. We refer to Kot [9] for more details.

Definition 2.1. *A traveling wave solution of* (1.3) *is a special solution with form* $u_n(\mathbf{x}) = \phi(x_1\cos\theta + x_2\sin\theta + cn)$, *in which* $(-c\cos\theta, -c\sin\theta)$ *with* $c>0$ *and* $\theta \in [0,2\pi]$ *is the wave speed that the wave profile* $\phi \in C(\mathbb{R},\mathbb{R})$ *spreads in* $\mathbb{R}^2$. *In particular, if* $\phi(\xi)$ *is monotone in* $\xi \in \mathbb{R}$, *then it is called a traveling wavefront.*

By Definition 2.1, the traveling wavefront $\phi(\xi)$ of (1.3) must satisfy the following integral equation

$$\phi(\xi+c) = \frac{1}{4\pi d}\int_{\mathbb{R}^2} e^{-\frac{|\mathbf{x}|^2}{4d}} g(\phi(\xi - x_1\cos\theta - x_2\sin\theta))d\mathbf{x}, \quad \xi \in \mathbb{R}. \tag{2.1}$$

Motivated by the background of traveling wavefronts [27], we also require that ϕ satisfies the following asymptotic boundary conditions

$$\lim_{\xi\to-\infty}\phi(\xi) = 0, \qquad \lim_{\xi\to\infty}\phi(\xi) = 1. \tag{2.2}$$

Thus, our intention is to prove the existence of a monotone solution of (2.1) and (2.2). For this purpose, we rewrite (2.1) as

$$\phi(\xi) = \frac{1}{4\pi d}\int_{\mathbb{R}^2} e^{-\frac{|\mathbf{x}|^2}{4d}} g(\phi(\xi - c - x_1\cos\theta - x_2\sin\theta))d\mathbf{x}, \quad \xi \in \mathbb{R}. \tag{2.3}$$

In order to apply the monotone iteration technique to prove the existence of a monotone solution of (2.3) and (2.2), we will construct a profile set by the upper and lower solutions which are defined as follows.

Definition 2.2. *A continuous function* $\phi(\xi) \in C_{[0,1]}(\mathbb{R},\mathbb{R})$ *is an upper solution* (*a lower solution*) *of* (2.3) *if it satisfies*

$$\phi(\xi) \geqslant (\leqslant) \frac{1}{4\pi d}\int_{\mathbb{R}^2} e^{-\frac{|\mathbf{x}|^2}{4d}} g(\phi(\xi - c - x_1\cos\theta - x_2\sin\theta))d\mathbf{x}, \quad \xi \in \mathbb{R}.$$

For $\lambda \geqslant 0, c > 0$, define

$$\Lambda(\lambda, c) = \frac{g'(0)}{4\pi d}\int_{\mathbb{R}^2} e^{-\frac{|\mathbf{x}|^2}{4d}} e^{-\lambda(x_1\cos\theta + x_2\sin\theta + c)}d\mathbf{x}.$$

Lemma 2.3. *Assume that* $c > c^* := 2\sqrt{d\ln(g'(0))}$. *Then* $\Lambda(\lambda, c) = 1$ *has two distinct positive roots* $\lambda_1(c)$ *and* $\lambda_2(c)$ *with* $\lambda_1(c) < \lambda_2(c)$ *if* $c > c^*$ *and no real roots if* $c < c^*$. *Moreover,* $\lambda_1(c)$ *and* $\lambda_2(c)$ *are given by*

$$\lambda_{1,2}(c) = \frac{c \pm \sqrt{c^2 - (c^*)^2}}{2d}$$

and there exists $\eta \in (1, \min\{2, \frac{\lambda_2(c)}{\lambda_1(c)}\})$ *such that* $\Lambda(\eta\lambda_1(c), c) < 1$.

The proof of Lemma 2.3 is clear and we omit it here. Using the constants in Lemma 2.3 with $q > 1$, we define continuous functions as follows:

$$\overline{\phi}(\xi) = \min\{e^{\lambda_1(c)\xi} + qe^{\eta\lambda_1(c)\xi}, 1\}, \qquad \underline{\phi}(\xi) = \max\{e^{\lambda_1(c)\xi} - qe^{\eta\lambda_1(c)\xi}, 0\}. \tag{2.4}$$

Lemma 2.4. $\overline{\phi}(\xi)$ *is an upper solution of* (2.3).

Proof. It suffices to verify that $\overline{\phi}(\xi)$ satisfies the definition of an upper solution. If $\overline{\phi}(\xi)=1$, then $\overline{\phi}(y)\leqslant 1$ for $y\in\mathbb{R}$ by the definition of $\overline{\phi}(\xi)$, which further implies that $g(\overline{\phi}(\xi-c-x_1\cos\theta-x_2\sin\theta))\leqslant 1,\quad x_1,x_2\in\mathbb{R}$ by (g1) and (g2). Then the result holds when $\overline{\phi}(\xi)=1$.

If $\overline{\phi}(\xi)=e^{\lambda_1(c)\xi}+qe^{\eta\lambda_1(c)\xi}$, then $g(\overline{\phi}(y))\leqslant g'(0)[e^{\lambda_1(c)y}+qe^{\eta\lambda_1(c)y}]$ for $y\in\mathbb{R}$. Therefore, it is sufficient to prove that

$$\begin{aligned} e^{\lambda_1(c)\xi}+qe^{\eta\lambda_1(c)\xi} &\geqslant \frac{g'(0)}{4\pi d}\int_{\mathbb{R}^2} e^{-\frac{|\mathbf{x}|^2}{4d}}[e^{\lambda_1(c)(\xi-c-x_1\cos\theta-x_2\sin\theta)}+qe^{\eta\lambda_1(c)(\xi-c-x_1\cos\theta-x_2\sin\theta)}]d\mathbf{x}\\ &= e^{\lambda_1(c)\xi}\Lambda(\lambda_1(c),c)+qe^{\eta\lambda_1(c)\xi}\Lambda(\eta\lambda_1(c),c), \end{aligned}$$

which follows from the condition (g2) and Lemma 2.3. The proof is complete.

Lemma 2.5. *$\underline{\phi}(\xi)$ is a lower solution of* (2.3) *if $q>1$ is large enough.*

Proof. It suffices to verify that $\underline{\phi}(\xi)$ satisfies the definition of a lower solution. If $\underline{\phi}(\xi)=0$, then the result holds because $g(\underline{\phi}(y))\geqslant 0$ for all $y\in\mathbb{R}$.

By the condition (g2), there exists a constant $L>0$ such that $|g''(u)|<g'(0)L,\ u\in[0,1]$. If $\underline{\phi}(\xi)=e^{\lambda_1(c)\xi}-qe^{\eta\lambda_1(c)\xi}$, then we only need to prove that

$$\begin{aligned} \underline{\phi}(\xi)\leqslant{}& \frac{g'(0)}{4\pi d}\int_{\mathbb{R}^2} e^{-\frac{|\mathbf{x}|^2}{4d}}\underline{\phi}(\xi-c-x_1\cos\theta-x_2\sin\theta)d\mathbf{x}\\ &-\frac{g'(0)L}{4\pi d}\int_{\mathbb{R}^2} e^{-\frac{|\mathbf{x}|^2}{4d}}[\underline{\phi}(\xi-c-x_1\cos\theta-x_2\sin\theta)]^2d\mathbf{x}. \end{aligned}$$

Therefore, it is sufficient to show that

$$\begin{aligned} \underline{\phi}(\xi)\leqslant{}& \frac{g'(0)}{4\pi d}\int_{\mathbb{R}^2} e^{-\frac{|\mathbf{x}|^2}{4d}}e^{\lambda_1(c)(\xi-c-x_1\cos\theta-x_2\sin\theta)}d\mathbf{x}\\ &-\frac{qg'(0)}{4\pi d}\int_{\mathbb{R}^2} e^{-\frac{|\mathbf{x}|^2}{4d}}e^{\eta\lambda_1(c)(\xi-c-x_1\cos\theta-x_2\sin\theta)}d\mathbf{x}\\ &-\frac{Lg'(0)}{4\pi d}\int_{\mathbb{R}^2} e^{-\frac{|\mathbf{x}|^2}{4d}}e^{2\lambda_1(c)(\xi-c-x_1\cos\theta-x_2\sin\theta)}d\mathbf{x}\\ ={}& e^{\lambda_1(c)\xi}\Lambda(\lambda_1(c),c)-qe^{\eta\lambda_1(c)\xi}\Lambda(\eta\lambda_1(c),c)-Le^{2\lambda_1(c)\xi}\Lambda(2\lambda_1(c),c). \end{aligned}$$

By Lemma 2.3 and the definition of $\underline{\phi}(\xi)$, it is equivalent to prove that $q(\Lambda(\eta\lambda_1(c),c)-1)e^{\eta\lambda_1(c)\xi}\leqslant -L\Lambda(2\lambda_1(c),c)e^{2\lambda_1(c)\xi}$, which follows if $q>\frac{L\Lambda(2\lambda_1(c),c)}{1-\Lambda(\eta\lambda_1(c),c)}+1$. The proof is complete.

Repeating the monotone iteration processes in [5, Theorem 6.1] and [24, Theorem 3.3], we can prove the following result.

Theorem 2.6. *Assume that $c>c^*$ holds. Then* (2.1) *and* (2.2) *have a monotone solution $\phi(\xi)$ such that* $\lim_{\xi\to-\infty}\phi(\xi)e^{-\lambda_1(c)\xi}=1$.

Remark 2.7. Theorem 2.6 is similar to the result in [24]. However, the upper solution in [24] cannot be applied to the discussion in Lemma 4.2 of this paper, so we construct upper and lower solutions different from that in [24].

From the smoothness of the heat equation [21], it is evident that the following property of traveling wavefronts is true.

Theorem 2.8. *Assume that $\phi(\xi)$ is formulated by Theorem* 2.6. *Then $\phi(\xi)\in C^1(\mathbb{R},\mathbb{R})$ and* $\lim_{\xi\to-\infty}\phi'(\xi)e^{-\lambda_1(c)\xi}=\lambda_1(c)$.

3 Initial value problem

In this section, we first recall some results in [27] on the corresponding initial value problem of (1.3), which takes the form

$$\begin{cases} u_{n+1}(\mathbf{x})=Q[u_n](\mathbf{x}), & n=0,1,2,\dots,\\ u_0(\mathbf{x})=u(\mathbf{x}), \end{cases} \tag{3.1}$$

where $u(\mathbf{x}) \in C(\mathbb{R}^2, \mathbb{R})$ is a given function.

Theorem 3.1. *Assume that $u(\mathbf{x}) \in C_{[0,1]}$. Then $u_n(\mathbf{x}) \in C_{[0,1]}$ for all $n = 0, 1, 2, \dots$*

The result in Theorem 3.1 is clear by the assumptions (g1) and (g2), so we omit the proof here. In order to establish the comparison principle for (3.1), we need to introduce the upper and lower solutions.

Definition 3.2. *Assume that $v_n(\mathbf{x}) \in C_{[0,1]}$ for all $n = 0, 1, 2, \dots$ Then $v_n(\mathbf{x})$ is called an upper solution (a lower solution) of (3.1) if*

$$v_{n+1}(\mathbf{x}) \geqslant (\leqslant) Q[v_n](\mathbf{x}), \quad n = 0, 1, 2, \dots, \quad \text{and} \quad v_0(\mathbf{x}) \geqslant (\leqslant) u(\mathbf{x}). \tag{3.2}$$

Theorem 3.3. *Assume that $\overline{u}_n(\mathbf{x})$ and $\underline{u}_n(\mathbf{x})$ are the upper and lower solutions of (3.1), respectively.*

(i) *If $\overline{u}_0(\mathbf{x}) \geqslant \underline{u}_0(\mathbf{x})$, then $\overline{u}_n(\mathbf{x}) \geqslant \underline{u}_n(\mathbf{x})$ for all $n \geqslant 1$;*

(ii) *If $\overline{u}_0(\mathbf{x}) \geqslant u(\mathbf{x}) \geqslant \underline{u}_0(\mathbf{x})$, then $\overline{u}_n(\mathbf{x}) \geqslant u(\mathbf{x}) \geqslant \underline{u}_n(\mathbf{x})$ for all $n \geqslant 1$;*

(iii) *If $\overline{u}_0(\mathbf{x}) \geqslant \underline{u}_0(\mathbf{x})$, then*

$$\overline{u}_1(\mathbf{x}) - \underline{u}_1(\mathbf{x}) \geqslant \frac{1}{4\pi d} \int_{\mathbb{R}^2} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4d}} [g(\overline{u}_0(\mathbf{y})) - g(\underline{u}_0(\mathbf{y}))] d\mathbf{y} \geqslant 0 \quad \text{for all } \mathbf{x} \in \mathbb{R}^2.$$

Theorem 3.3 is clear and we refer to [27].

Remark 3.4. Since the traveling wavefront is monotone, the above item (iii) implies that the traveling wavefront of (1.3) is strictly monotone such that $\phi'(\xi) > 0$ for $\xi \in \mathbb{R}$.

Lemma 3.5. *Assume that $\phi(x_1 \cos\theta + x_2 \sin\theta + cn)$ is the traveling wavefront of (1.3) formulated by Theorem 2.6. Then for any $\theta_1 \in [0, 2\pi]$, there exist $\delta_0 \in (0, 1)$, $\beta > 0$ and $\sigma > 0$ such that for each $\delta \in (0, \delta_0]$ and $\xi^+ \in \mathbb{R}$, the continuous function*

$$\overline{u}_n(\mathbf{x}) = \min\{(1 + \delta e^{-\beta n}) \phi(x_1 \cos\theta_1 + x_2 \sin\theta_1 + cn + \xi^+ - \sigma\delta e^{-\beta n}), 1\} \tag{3.3}$$

is an upper solution of (3.1) if $\overline{u}_0(\mathbf{x}) \geqslant u(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^2$.

Proof. Without loss of generality, we only consider the case $\xi^+ = 0$. If $\overline{u}_n(\mathbf{x}) = 1$, then the result holds, so we only consider the case $\overline{u}_n(\mathbf{x}) \neq 1$. Let $\xi = x_1 \cos\theta_1 + x_2 \sin\theta_1 + cn$ and $\tau = y_1 \cos\theta_1 + y_2 \sin\theta_1$. We shall prove the following inequality

$$(1 + \delta e^{-\beta(n+1)}) \phi(\xi + c - \delta\sigma e^{-\beta(n+1)}) \geqslant \frac{1}{4\pi d} \int_{\mathbb{R}^2} e^{-\frac{|\mathbf{y}|^2}{4d}} g((1 + \delta e^{-\beta n}) \phi(\xi - \delta\sigma e^{-\beta n} - \tau)) d\mathbf{y} \tag{3.4}$$

for all $\xi \in \mathbb{R}$ and $n \geqslant 0$. Let $M > 0$ be large enough but finite. We shall consider the above equation in three cases.

By the definition of traveling wavefronts, (3.4) is equivalent to

$$\begin{aligned} \delta e^{-\beta(n+1)} \phi(\xi + c - \delta\sigma e^{-\beta(n+1)}) \geqslant{}& \frac{1}{4\pi d} \int_{\mathbb{R}^2} e^{-\frac{|\mathbf{y}|^2}{4d}} g((1 + \delta e^{-\beta n}) \phi(\xi - \delta\sigma e^{-\beta n} - \tau)) d\mathbf{y} \\ & - \frac{1}{4\pi d} \int_{\mathbb{R}^2} e^{-\frac{|\mathbf{y}|^2}{4d}} g(\phi(\xi - \delta\sigma e^{-\beta(n+1)} - \tau)) d\mathbf{y}. \end{aligned} \tag{3.5}$$

According to (g4), (3.5) holds provided that

$$\begin{aligned} & \delta e^{-\beta(n+1)} \phi(\xi + c - \delta\sigma e^{-\beta(n+1)}) \\ & \quad \geqslant \frac{1 + \delta e^{-\beta n}}{4\pi d} \int_{\mathbb{R}^2} e^{-\frac{|\mathbf{y}|^2}{4d}} g(\phi(\xi - \delta\sigma e^{-\beta n} - \tau)) d\mathbf{y} - \frac{1}{4\pi d} \int_{\mathbb{R}^2} e^{-\frac{|\mathbf{y}|^2}{4d}} g(\phi(\xi - \delta\sigma e^{-\beta(n+1)} - \tau)) d\mathbf{y} \\ & \quad = \delta e^{-\beta n} \phi(\xi + c - \delta\sigma e^{-\beta n}) + \phi(\xi + c - \delta\sigma e^{-\beta n}) - \phi(\xi + c - \delta\sigma e^{-\beta(n+1)}). \end{aligned}$$

Note that $\phi(\xi + c - \delta\sigma e^{-\beta(n+1)}) > \phi(\xi + c - \delta\sigma e^{-\beta n})$. Then it is sufficient to prove that

$$\phi(\xi + c - \delta\sigma e^{-\beta(n+1)}) - \phi(\xi + c - \delta\sigma e^{-\beta n}) > \delta e^{-\beta n} [1 - e^{-\beta}] \phi(\xi + c - \delta\sigma e^{-\beta n}),$$

which is true once

$$\phi(\xi + c - \delta\sigma e^{-\beta n}) \leqslant \sigma\phi'(\mu) \tag{3.6}$$

for all $\mu \in [\xi + c - \delta\sigma e^{-\beta n}, \xi + c - \delta\sigma e^{-\beta(n+1)}]$.

If $\xi + c - \delta\sigma e^{-\beta n} < -M$, then (3.6) is clear if $\sigma > 0$ is large enough, which depends on the precise asymptotic behavior in Theorems 2.6 and 2.8.

If $|\xi + c - \delta\sigma e^{-\beta n}| \leqslant M$, then (3.6) holds if $\beta > 0$ is small enough and $\sigma > 0$ is large enough, which depends on the *strict monotonicity* of $\phi(\xi)$ (see Remark 3.4).

We now consider the case $\xi + c - \delta\sigma e^{-\beta n} > M$. Choose $\kappa \in (0, 1)$ such that $g'(x) < \kappa < 1$ if $x > \phi(M - \delta\sigma) > \kappa$, which is well-defined by (g3) if $M > 0$ is large enough. Then it is evident that

$$\begin{aligned}
&\frac{1}{4\pi d}\int_{\tau\leqslant 0} e^{-\frac{|\mathbf{y}|^2}{4d}} g((1+\delta e^{-\beta n})\phi(\xi-\delta\sigma e^{-\beta n}-\tau))d\mathbf{y}\\
&\quad\leqslant \frac{1+\delta e^{-\beta n}}{4\pi d}\int_{\tau\leqslant 0} e^{-\frac{|\mathbf{y}|^2}{4d}} g(\phi(\xi-\delta\sigma e^{-\beta n}-\tau))d\mathbf{y} + \frac{\kappa\delta e^{-\beta n}}{2} - \frac{\delta e^{-\beta n}\phi(\xi-\delta\sigma e^{-\beta n})}{2}
\end{aligned}$$

by the monotonicity of $\phi(\xi)$, which further implies that

$$\begin{aligned}
&\frac{1}{4\pi d}\int_{\mathbb{R}^2} e^{-\frac{|\mathbf{y}|^2}{4d}} g((1+\delta e^{-\beta n})\phi(\xi-\delta\sigma e^{-\beta n}-\tau))d\mathbf{y}\\
&\quad\leqslant \frac{1+\delta e^{-\beta n}}{4\pi d}\int_{\mathbb{R}^2} e^{-\frac{|\mathbf{y}|^2}{4d}} g(\phi(\xi-\delta\sigma e^{-\beta n}-\tau))d\mathbf{y} + \frac{\kappa\delta e^{-\beta n}}{2} - \frac{\delta e^{-\beta n}\phi(\xi-\delta\sigma e^{-\beta n})}{2}\\
&\quad= (1+\delta e^{-\beta n})\phi(\xi+c-\delta\sigma e^{-\beta n}) + \frac{\kappa\delta e^{-\beta n}}{2} - \frac{\delta e^{-\beta n}\phi(\xi-\delta\sigma e^{-\beta n})}{2}.
\end{aligned}$$

Therefore, (3.4) holds for $\xi > M$ provided that

$$\begin{aligned}
&(1+\delta e^{-\beta(n+1)})\phi(\xi+c-\delta\sigma e^{-\beta(n+1)}) - (1+\delta e^{-\beta n})\phi(\xi+c-\delta\sigma e^{-\beta n})\\
&\quad\geqslant (1+\delta e^{-\beta(n+1)})\phi(\xi+c-\delta\sigma e^{-\beta n}) - (1+\delta e^{-\beta n})\phi(\xi+c-\delta\sigma e^{-\beta n})\\
&\quad= \delta e^{-\beta n}(e^{-\beta}-1)\phi(\xi+c-\delta\sigma e^{-\beta n}) \geqslant \frac{\kappa\delta e^{-\beta n}}{2} - \frac{\delta e^{-\beta n}\phi(\xi-\delta\sigma e^{-\beta n})}{2},
\end{aligned}$$

which is true if $\beta > 0$ is small enough. The proof is complete.

Lemma 3.6. *Assume that $\phi(x_1\cos\theta + x_2\sin\theta + cn)$ is the traveling wavefront of* (1.3) *formulated by Theorem* 2.6. *Then for any $\theta_1 \in [0, 2\pi]$, there exist $\delta_0 \in (0, 1)$, $\beta > 0$ and $\sigma > 0$ such that for each $\delta \in (0, \delta_0]$ and $\xi^- \in \mathbb{R}$, the continuous function*

$$\underline{u}_n(\mathbf{x}) = (1-\delta e^{-\beta n})\phi(x\cos\theta_1 + y\sin\theta_1 + cn + \xi^- + \delta\sigma e^{-\beta n}) \tag{3.7}$$

is a lower solution of (3.1) *if $\underline{u}_0(\mathbf{x}) \leqslant u(\mathbf{x})$ holds.*

The proof of Lemma 3.6 is similar to that of Lemma 3.5, so we omit it here.

Remark 3.7. In Lemmas 3.5 and 3.6, the choices of σ and β are uniform for $\delta \in (0, \delta_0]$, which will be very important in what follows.

Lemma 3.8. *Assume that $u_n(\mathbf{x}), v_n(\mathbf{x})$ are defined by* (3.1) *with the initial values $u(\mathbf{x}), v(\mathbf{x})$, respectively. If $u(\mathbf{x}) \geqslant v(\mathbf{x})$ and $u(\mathbf{x}), v(\mathbf{x}) \in C_{[0,1]}$, then $0 \leqslant u_1(\mathbf{x}) - v_1(\mathbf{x}) \leqslant g'(0)|u(\cdot) - v(\cdot)|$, $\mathbf{x} \in \mathbb{R}^2$.*

Lemma 3.8 follows from the assumption (g2), so the proof is omitted.

4 Stability of traveling wavefronts

In this section, we shall prove that the traveling wavefronts established in Theorem 2.6 are stable in the sense of phase shift and circumnutation. The main result is first listed as follows.

Theorem 4.1. *Let $\phi(x_1\cos\theta + x_2\sin\theta + cn)$ be a traveling wavefront of* (1.3) *formulated by Theorem* 2.6. *Assume that there exists $\theta_1 \in [0, 2\pi]$ such that $u(\mathbf{x})$ satisfies*

$$\lim_{\xi\to-\infty} u(\mathbf{x})e^{-\lambda_1(c)\xi} = \rho, \qquad \liminf_{\xi\to\infty} u(\mathbf{x}) > 0$$

uniformly in $\xi = x_1\cos\theta_1 + x_2\sin\theta_1$. *Let* $\xi_0 = \frac{1}{\lambda_1(c)}\ln\rho$ *and* $u_n(\mathbf{x})$ *be defined by* (3.1) *with* $u_0(\mathbf{x}) = u(\mathbf{x})$. *Then*

$$\lim_{n\to\infty}\sup_{\mathbf{x}\in\mathbb{R}^2}\left|\frac{u_n(\mathbf{x})}{\phi(x_1\cos\theta_1 + x_2\sin\theta_1 + cn + \xi_0)} - 1\right| = 0.$$

Before proving Theorem 4.1, we first establish several lemmas, throughout which the conditions of Theorem 4.1 are imposed.

Lemma 4.2. *For any* $p > 0$ *with* $\mathbf{p} = (p\cos\theta_1, p\sin\theta_1)$, *there exists* $\xi(p)$ *such that* $u_n(\mathbf{x} - 2\mathbf{p}) < \phi(\xi + \xi_0) < u_n(\mathbf{x} + 2\mathbf{p})$ *for all* $x_1\cos\theta_1 + x_2\sin\theta_2 + cn = \xi \leqslant \xi(p)$ *and* $n = 0, 1, 2, \ldots$

Proof. Let $q > 1$ be large enough. Then

$$\overline{\phi}(x_1\cos\theta_1 + x_2\sin\theta_1 + \xi_0) \geqslant u(\mathbf{x} - \mathbf{p}) \quad \text{and} \quad \underline{\phi}(x_1\cos\theta_1 + x_2\sin\theta_1 + \xi_0) \leqslant u(\mathbf{x} + \mathbf{p})$$

hold for all $\mathbf{x} \in \mathbb{R}^2$, where $\overline{\phi}$ and $\underline{\phi}$ are defined by (2.4). Similar to those in Lemmas 2.4 and 2.5, we can verify that $\overline{\phi}(\xi + \xi_0)$ and $\underline{\phi}(\xi + \xi_0)$ are the upper and lower solutions of (3.1), respectively. Then the result is clear by the asymptotic behavior of traveling wavefronts in Theorem 2.6. The proof is complete.

Lemma 4.3. *There exist constants* $\delta \in (0, 1), \beta > 0, \sigma > 0$ *and* $z_0 > 0$ *such that*

$$(1 - \delta e^{-\beta n})\phi(\xi + \xi_0 - z_0 + \delta\sigma e^{-\beta n}) \leqslant u_n(\mathbf{x}) \leqslant \min\{(1 + \delta e^{-\beta n})\phi(\xi + \xi_0 + z_0 - \delta\sigma e^{-\beta n}), 1\}$$

for any $\xi = x_1\cos\theta_1 + x_2\sin\theta_2 + cn \in \mathbb{R}$ *and* $n \geqslant 1$.

Proof. In Lemma 4.2, let $p = 1$. Then the result is true if $z_0 - \sigma > 2$ and $\xi < \xi(1)$ hold. If $\xi > \xi(1)$ with $n = 1$, it is clear that there exists $\delta' > 0$ such that $u_1(\mathbf{x}) > \delta'$ holds uniformly. Therefore, there exist $\delta \in (0, 1), \beta > 0, \sigma > 0$ such that for $\xi = x_1\cos\theta_1 + x_2\sin\theta_2 + c$,

$$(1 - \delta e^{-\beta})\phi(\xi + \xi_0 - z_0 + \delta\sigma e^{-\beta}) \leqslant u_1(\mathbf{x}) \leqslant \min\{(1 + \delta e^{-\beta})\phi(\xi + \xi_0 + z_0 - \delta\sigma e^{-\beta}), 1\}$$

if $z_0 > 0$ is large enough. In particular, let $z_0 > 0$ be large enough. Then $\delta \in (0, 1), \beta > 0, \sigma > 0$ also satisfy the conditions in Lemmas 3.5 and 3.6. By Theorem 3.3, the result holds and this completes the proof.

Lemma 4.4. *There exists a constant* $M_0 > 0$ *such that* $(1 - \varepsilon)\phi(\xi + 3\varepsilon\sigma) \leqslant \phi(\xi) \leqslant (1 + \varepsilon)\phi(\xi - 3\varepsilon\sigma)$ *for any* $\varepsilon \in (0, \delta)$ *and* $\xi \geqslant M_0 + \xi_0$.

Proof. Note that $\lim_{\xi\to\infty}\phi(\xi) = 1$. Then the result is obvious if $M_0 > 0$ is large enough.

Lemma 4.5. *Let* z *and* M *be any given positive constants and* $u_n^+(\mathbf{x}), u_n^-(\mathbf{x})$ *be solutions of* (3.1) *with initial values*

$$u^+(\mathbf{x}) = \phi(\varsigma + \xi_0 + z)\chi(\varsigma + M) + \phi(\varsigma + \xi_0 + 2z)\left[1 - \chi(\varsigma + M)\right],$$
$$u^-(\mathbf{x}) = \phi(\varsigma + \xi_0 - z)\chi(\varsigma + M) + \phi(\varsigma + \xi_0 - 2z)\left[1 - \chi(\varsigma + M)\right],$$

respectively, where $\varsigma = x_1\cos\theta_1 + x_2\sin\theta_1, \chi(y) = \min\{\max\{0, -y\}, 1\}$ *for all* $y \in \mathbb{R}$. *Then there exists an* $\epsilon \in (0, \min\{\delta/2, z/(3\delta)\})$ *such that for any* $\varsigma \in [-M, \infty)$,

$$u_1^+(\mathbf{x}) \leqslant (1 + \epsilon)\phi(\varsigma + \xi_0 + 2z - 3\epsilon\sigma), \quad u_1^-(\mathbf{x}) \geqslant (1 - \epsilon)\phi(x + \xi_0 - 2z - 3\epsilon\sigma),$$

in which $\varsigma = x_1\cos\theta_1 + x_2\sin\theta_1 + c$.

Proof. By the definition of $\chi(y)$, it is easy to see that $u^+(\mathbf{x}) \leqslant \phi(\varsigma + \xi_0 + 2z)$, $\varsigma = x_1\cos\theta_1 + x_2\sin\theta_1$. Thus $\phi(\xi + \xi_0 + 2z)$ is an upper solution of (3.1) with the initial value $u^+(\mathbf{x})$. Theorem 3.3 and the definition of $\chi(y)$ imply that

$$u_n^+(\mathbf{x}) < \phi(\xi + \xi_0 + 2z), \quad \xi = x_1\cos\theta_1 + x_2\sin\theta_2 + cn \in \mathbb{R}, \quad n = 1, 2, \ldots$$

Let M_0 be defined by Lemma 4.3. Then the result is clear for $\xi > M_0$. For the case $\xi \in [-M, M_0]$, we should note that the definition of $u_1^+(\mathbf{x})$ depends *only* on $\phi(\xi)$. Then the uniform continuity of $u_1^+(\mathbf{x})$ and $\phi(\xi)$ implies that the result is true if $\epsilon > 0$ is small enough.

Similarly, we can prove the result of $u_1^-(\mathbf{x})$.

Proof of Theorem 4.1. Define constants z^+ and z^- as follows:

$$z^+ \triangleq \inf\{z|z \in A^+\}, \quad A^+ = \left\{ z \geqslant 0 \,\middle|\, \limsup_{n\to\infty} \sup_{\xi\in\mathbb{R}} \frac{u_n(\mathbf{x})}{\phi(\xi+\xi_0+2z)} \leqslant 1 \right\},$$

$$z^- \triangleq \sup\{z|z \in A^-\}, \quad A^- = \left\{ z \geqslant 0 \,\middle|\, \liminf_{n\to\infty} \inf_{\xi\in\mathbb{R}} \frac{u_n(\mathbf{x})}{\phi(\xi+\xi_0-2z)} \geqslant 1 \right\},$$

in which $\xi = x_1\cos\theta + x_2\sin\theta + cn$. Then Lemma 4.3 implies that z^+ and z^- are well-defined. We now prove that $z^+ = z^- = 0$. Were the statements false, then there would exist $N > 0$ such that

$$\sup_{\xi\in\mathbb{R}} \frac{u_n(\mathbf{x})}{\phi\left(\xi+\xi_0+2z^+\right)} \leqslant 1+\overline{\epsilon}, \quad n > N, \quad \xi = x_1\cos\theta + x_2\sin\theta + cn,$$

where $g'(0)\overline{\epsilon} = \epsilon\phi(-M+\xi_0-3\epsilon\sigma)$ and ϵ is given in Lemma 4.5 with $z = z^+ > 0$. In particular, in Lemma 4.5, let

$$u^+(\mathbf{x}) = \phi(\xi+\xi_0+2z^+), \quad \xi = x_1\cos\theta_1 + x_2\sin\theta_1 + cN \in [-M, +\infty)$$

with $M > 0$ large enough. Then $u_N(\mathbf{x}) \leqslant \phi(\xi+\xi_0+2z^+)+\overline{\epsilon} = u^+(\mathbf{x})+\overline{\epsilon}$ if $\xi = x_1\cos\theta_1+x_2\sin\theta_1+cN \in [-M, +\infty)$. Lemma 3.8 implies that

$$u_{N+1}(\mathbf{x}) \leqslant u_1^+(\mathbf{x}) + \epsilon\phi(-M+\xi_0-3\epsilon\sigma) < (1+2\epsilon)\phi(\xi+\xi_0+2z^+-3\epsilon\sigma)$$

for all $\xi = x_1\cos\theta_1 + x_2\sin\theta_1 + cN + c \in [-M, \infty)$. By Lemma 4.2, $u_{N+1}(\mathbf{x}) \leqslant \phi(\xi+\xi_0+z^+)$, where $\xi = x_1\cos\theta_1 + x_2\sin\theta_1 + cN + c \in (-\infty, -M]$ since $M > 0$ is large enough.

Let $\beta > 0$ be small enough. Then

$$u_{N+1}(\mathbf{x}) \leqslant \min\{(1+2\epsilon e^{-\beta})\phi(\xi+\xi_0+2z^+-\epsilon\sigma-2\epsilon\sigma e^{-\beta}), 1\}$$

with $\xi = x_1\cos\theta_1 + x_2\sin\theta_1 + cN + c \in \mathbb{R}$. By the comparison principle and Lemmas 3.5 and 3.6 (also see Remark 3.7), we can prove that

$$\limsup_{n\to\infty} \sup_{\xi\in\mathbb{R}} \frac{u_n\left(\mathbf{x}\right)}{\phi\left(\xi+\xi_0+2z^+-\epsilon\sigma\right)} \leqslant 1, \quad \xi = x_1\cos\theta + x_2\sin\theta + cn,$$

which is a contradiction to the definition of z^+. Thus $z^+ = 0$.

In a similar way, we can prove that $z^- = 0$. The proof is complete.

Theorem 4.6. *Assume that* $\phi(x_1\cos\theta + x_2\sin\theta + cn)$ *and* $\phi_1(x_1\cos\theta_1 + x_2\sin\theta_1 + cn)$ *are traveling wavefronts of* (2.2). *If*

$$\lim_{\xi\to-\infty} \phi(\xi)e^{-\lambda_1(c)\xi} = 1, \qquad \lim_{\xi\to-\infty} \phi_1(\xi)e^{-\lambda_1(c)\xi} = \rho,$$

then $\phi(\xi) = \phi_1(\xi-\xi_0)$ *with* $\xi_0 = \frac{1}{\lambda_1(c)}\ln\rho$ *and* $\xi \in \mathbb{R}$.

Theorem 4.6 is a direct consequence of Theorem 4.1, so the proof is omitted.

Remark 4.7. Theorems 4.1 and 4.6 imply that the traveling wavefronts are asymptotically stable and unique in the sense of phase shift and circumnutation, which is different from that in a discrete media, see [4] and [27, p. 358].

5 The case $\mathcal{H} = \mathbb{R}$

In this section, we will give the corresponding results in the one-dimensional spatial case, which can be regarded as the degenerate case of that in Sections 2–5. First, we consider the discrete-time recursions on $\mathbb{R}$ [10],

$$v_{n+1}(x) = Q[v_n](x) = \frac{1}{\sqrt{4\pi d}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4d}} g(v_n(y))dy, \quad n = 0, 1, 2, \ldots, \tag{5.1}$$

where $d > 0, x, y \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ satisfies (g1)–(g4).

Definition 5.1. *A traveling wavefront of* (5.1) *is a special solution of the form* $v_n(x) = \psi(x + cn)$ *with* $c > 0$ *and* $\psi(t)$ *being nondecreasing in* $t \in \mathbb{R}$.

Substituting $v_n(x) = \psi(x + cn)$ into (5.1), then ψ satisfies

$$\psi(t + c) = \frac{1}{\sqrt{4\pi d}} \int_{\mathbb{R}} e^{-\frac{x^2}{4d}} g(\psi(t - x))dx, \quad t \in \mathbb{R}. \tag{5.2}$$

We are also interested in the following asymptotic boundary conditions

$$\lim_{t\to-\infty} \psi(t) = 0, \qquad \lim_{t\to\infty} \psi(t) = 1. \tag{5.3}$$

Similar to that in Section 2, we can establish the following result.

Theorem 5.2. *Assume that* $c > c^*$ *holds. Then* (5.2) *and* (5.3) *have a monotone solution* $\psi(t)$ *such that* $\lim_{t\to-\infty} \psi(t)e^{-\lambda_1(c)t} = 1$ *and* $\lim_{t\to-\infty} \psi'(t)e^{-\lambda_1(c)t} = \lambda_1(c)$, *where* c^* *and* $\lambda_1(c)$ *are defined by Lemma* 2.3.

Consider the following initial value problem

$$\begin{cases} v_{n+1}(x) = Q[v_n](x), \quad n = 0, 1, 2, \ldots, \\ v_0(x) = v(x), \end{cases} \tag{5.4}$$

in which $v(x) \in C_{[0,1]}$ and the operator Q is defined by (5.1).

Theorem 5.3. *Let* $\psi(x + cn)$ *be a traveling wavefront of* (5.1) *formulated by Theorem* 5.2. *Assume that* $v(x)$ *satisfies*

$$\lim_{x\to-\infty} v(x)e^{-\lambda_1(c)x} = \rho \qquad \Big(\lim_{x\to\infty} v(x)e^{\lambda_1(c)x} = \rho\Big)$$

and

$$\liminf_{x\to\infty} v(x) > 0 \qquad \Big(\liminf_{x\to-\infty} v(x) > 0\Big).$$

Let $t_0 = \frac{1}{\lambda_1(c)} \ln \rho$ *and* $v_n(x)$ *be defined by* (5.4) *with* $v_0(x) = v(x)$. *Then*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \frac{v_n(x)}{\psi(x + cn + t_0)} - 1 \right| = 0 \qquad \left(\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \frac{v_n(x)}{\psi(-x + cn + t_0)} - 1 \right| = 0 \right).$$

The proof of Theorem 5.3 is similar to that of Theorem 4.1, so we omit it here. Moreover, the following result is evident by Theorem 5.3.

Theorem 5.4. *Assume that* $\psi(x+cn)$ *is a traveling wavefront of* (5.1) *formulated by Theorem* 5.2, *and* $\psi_1(x + cn) \in C_{[0,1]}$ *and* $\psi_2(-x + cn) \in C_{[0,1]}$ *satisfy* (5.2) *and* (5.3). *If*

$$\lim_{t\to-\infty} \psi_1(t)e^{-\lambda_1(c)t} = \rho_1, \quad \lim_{t\to-\infty} \psi_2(t)e^{-\lambda_1(c)t} = \rho_2,$$

then $\psi(t) = \psi_1(t - t_1) = \psi_2(t - t_2)$ *with* $t_1 = \frac{1}{\lambda_1(c)} \ln \rho_1, t_2 = \frac{1}{\lambda_1(c)} \ln \rho_2$.

Remark 5.5. Theorems 5.3 and 5.4 imply that even for the one-dimensional case, the traveling wavefront of (5.1) is stable and unique in the sense of phase shift and circumnutation, since there are only two directions in the case of $\mathbb{R}$.

The paper ends with the following two remarks.

Remark 5.6. It is easy to see that the method used in this paper can be generalized to the case that the probability functions are more general than the Gaussian kernels (see, e.g., [17]).

Remark 5.7. We can also consider the corresponding problems for the reaction-diffusion equations and obtain similar results. We shall investigate these in our forthcoming papers.

Acknowledgements This work was partially supported by National Natural Science Foundation of China (Grant No. 10871085) and US National Science Foundation (Grant Nos. DMS-0412047, DMS-0715772).

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