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Spreading speeds and traveling waves in competitive recursion systems

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Abstract This paper is concerned with the spreading speeds and traveling wave solutions of discrete time recursion systems, which describe the spatial propagation mode of two competitive invaders. We first establish the existence of traveling wave solutions when the wave speed is larger than a given threshold. Furthermore, we prove that the threshold is the spreading speed of one species while the spreading speed of the other species is distinctly slower compared to the case when the interspecific competition disappears. Our results also show that the interspecific competition does affect the spread of both species so that the eventual population densities at the coexistence domain are lower than the case when the competition vanishes.

Keywords Comparison principle · Upper and lower solutions · Traveling waves · Spreading speeds · Competitive invaders

Mathematics Subject Classification (2000) 45G15 · 45M05 · 92D25

1 Introduction

In the past three decades, traveling wave solutions and asymptotic speeds of spread (in short, spreading speeds) of spatio-temporal patterns have been widely studied, see Aronson (1977), Aronson and Weinberger (1975, 1978), Britton (1986), Fife (1979),

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Li et al. (2005), Ma (2007), Smoller (1994), van den Bosch et al. (1990), Volpert et al. (1994), Weinberger et al. (2002), and Ye and Li (1990) for reaction-diffusion systems and Cheng et al. (2008), Diekmann (1979), Radcliffe and Rass (1983, 1984, 1986), Thieme (1979a,b), Thieme and Zhao (2003), and Weng et al. (2003) for lattice differential systems and integral equations. It was shown in Weinberger (1982) that many of these results can be carried over to recursions of the form

$$u_{n+1} = Q[u_n], \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where $u_n = (u_n^1(x), \ldots, u_n^k(x)) \in \mathbb{R}^k$ is a vector-valued function on an Euclidean space or, more general, a habitat $\mathcal{H}$ in such a space (e.g., the integer lattice in $\mathbb{R}^n$), and Q is a translation invariant order-preserving operator with the properties that $Q[\mathbf{0}] = \mathbf{0}$ and $Q[\mathbf{E}] = \mathbf{E}$ for some positive constant vector $\mathbf{E} \in \mathbb{R}^k$. In population dynamics, $u_n(x)$ can be thought of as the population density of k species at time n at the point $x \in \mathcal{H}$. For more details about recursions, we refer to Allen et al. (1996), Carrillo and Fife (2005), Lewis (2000), Liang and Zhao (2007), Lin et al. (2010), Lui (1989a,b), Neubert and Caswell (2000), Weinberger (1982), and Weinberger et al. (2002).

For the recursion (1.1), much attention has been paid to the spatial propagation mode if Q is *order-preserving* or *cooperative* and the interesting steady-states are *comparable* in the sense of the same ordering, see Diekmann (1978, 1979), Kot (1992), Lewis (2000), Lewis et al. (2002), Li et al. (2005), Liang and Zhao (2007), Lui (1989a,b), Weinberger (1982, 2002), and Weinberger et al. (2002). In particular, Weinberger et al. (2002) considered (1.1) and extended Lui's results in Lui (1989a) so that they can treat the local invasion of an equilibrium of the cooperating species by a new species or mutant. Thus, their results can be applied to the invasion processes of certain models for cooperation (see Li et al. (2005)) or exclusion processes between two species (see Hardin (1960) for the biological sense of exclusion) in a Lotka–Volterra competition system (see Lewis et al. 2002) and the following recursion system (see Lewis et al. 2002; Li et al. 2005)

$$\begin{cases} p_{n+1}(x) = \int_{\mathbb{R}} \frac{(1+r_1)p_n(x-y)}{1+r_1(p_n(x-y)+a_1q_n(x-y))} s_1(y, dy), \\ q_{n+1}(x) = \int_{\mathbb{R}} \frac{(1+r_2)q_n(x-y)}{1+r_2(q_n(x-y)+a_2p_n(x-y))} s_2(y, dy), \end{cases} \tag{1.2}$$

where all the parameters are positive, $s_i(y, dy), i = 1, 2$, represent the probability measures for the dispersals of two species and $\int_{\mathbb{R}} s_i(y, dy) = 1, i = 1, 2$. More precisely, Lewis et al. (2002, Proposition 3.1) and Li et al. (2005, Example 3.1) formulated the spread of two competitive species when the species p described by p_n is the invader while the species q described by q_n is the resident (mathematically, this means that the equilibrium (0, 1) is involved). In fact, by introducing new variables $u = p, v = 1 - q$, (1.2) becomes a cooperative system and the new interesting equilibria in Lewis et al. (2002) are still ordered. Therefore, both the spreading speeds and traveling wave solutions concerned with the equilibrium (0, 1) can be investigated by the theory in Weinberger et al. (2002). Their results also imply that the spreading speed of system (1.2) can be linearly determinate (it should be noted that some systems are not linearly determinate, and we refer to Weinberger et al. (2007) for such an example). For further results of recursions, see also Hsu and Zhao (2008), Liang and Zhao (2007), and Weinberger (2002).

In agricultural industry, besides the competition-exclusion, the competition-coexistence (see Darlington 1972) is also very important. For example, several fishes (e.g., the carp, the grass carp, the chub) are often raised in the same pond in China to obtain maximal profit per cost (see Li 1992), all of them need oxygen, but the foods they need often have significant differences and the water levels they live in are also different, which relates to the competition-coexistence or competition-invasion problem. For the system (1.2), the trivial and positive equilibria will be involved if we try to formulate the competition-invasion process by traveling wave solutions and spreading speeds. Concretely, we shall study the following two questions of (1.2) in this paper: (i) the existence of traveling wave solutions connecting the trivial equilibrium $(0, 0)$ with a positive equilibrium (k_1, k_2) defined by

$$(k_1, k_2) = \left(\frac{1-a_1}{1-a_1a_2}, \frac{1-a_2}{1-a_1a_2}\right)$$

if $a_1, a_2 \in (0, 1)$; (ii) the spreading speeds of p, q when both species are invaders. From the view point of population dynamics, these problems are concerned with the spatio-temporal pattern of (1.2) when two invaders compete each other.

Formally, when p, q are positive, (1.2) is a competitive system and it would be a cooperative system with the standard partial ordering in $\mathbb{R}^2$ if we take the change of variables in Lewis et al. (2002). However, after the change, the interesting equilibria in this paper will become

$$(0, 1), \left(\frac{1-a_1}{1-a_1a_2}, \frac{a_2(1-a_1)}{1-a_1a_2}\right)$$

which are *not* ordered by the standard partial ordering in $\mathbb{R}^2$. Therefore, the results mentioned above cannot be applied, and we need some new techniques to study the spatio-temporal pattern of (1.2) when the trivial and positive equilibria are involved. To overcome the difficulty, for the traveling wave solution, we replace arguments based on monotone iteration by that based on invariant region and cross-iteration, and the spreading speeds can be investigated by the similar idea. In the study of partial differential equations, similar ideas have been used (see, for example, the monograph of Leung (1989), Pao (1992), Ye and Li (1990) and the references cited therein).

More precisely, to answer the first problem, we first consider the existence of traveling wave solutions of the following general recursion system

$$\begin{cases} u_{n+1}(x) = Q_1[u_n, v_n](x) = \int_{\mathbb{R}} f_1(u_n(x-y), v_n(x-y)) g_1(y, dy), \\ v_{n+1}(x) = Q_2[u_n, v_n](x) = \int_{\mathbb{R}} f_2(u_n(x-y), v_n(x-y)) g_2(y, dy), \end{cases} \tag{1.3}$$

in which $g_1 \in L^1, g_2 \in L^1$ are probability functions and $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$. We also require that (1.3) is a competitive system and has a trivial and a positive equilibria. By applying the comparison principle of the competitive system, we reduce the existence of traveling wave solutions to the existence of a pair of upper and lower solutions of (1.3). As an example, we then investigate the following recursion system

$$\begin{cases} p_{n+1}(x) = \int_{\mathbb{R}} \frac{e^{-\frac{y^2}{4d_1}}}{\sqrt{4\pi d_1}} \frac{(1+r_1)p_n(x-y)}{1+r_1(p_n(x-y)+a_1 q_n(x-y))} dy, \\ q_{n+1}(x) = \int_{\mathbb{R}} \frac{e^{-\frac{y^2}{4d_2}}}{\sqrt{4\pi d_2}} \frac{(1+r_2)q_n(x-y)}{1+r_2(q_n(x-y)+a_2 p_n(x-y))} dy, \end{cases} \tag{1.4}$$

hereafter d_1, d_2 are positive constants, $a_1, a_2 \in (0, 1)$ such that $\mathbf{K} = (k_1, k_2)$ is a positive equilibrium of (1.4). By constructing proper upper and lower solutions, we show the existence of traveling wave solutions connecting $\mathbf{0}$ with $\mathbf{K}$ if the wave speed is larger than

$$c^* =: \max \left\{ 2\sqrt{d_1 \ln(1+r_1)},\ 2\sqrt{d_2 \ln(1+r_2)} \right\},$$

and determine precisely the asymptotic behavior of traveling wave solutions near the equilibrium $\mathbf{0}$. It is well known that the existence of traveling wave solutions of evolutionary systems strongly depends on the stability of the equilibria, and we also refer to Cushing et al. (2004) for some conclusions established for the corresponding difference equations. Moreover, Li (2009) proved the existence of traveling wave solutions of (1.4), which starts from the equilibrium (0, 0) while the eventual steady state cannot be affirmed.

For the second question, we investigate the long time behavior of the corresponding initial value problem of (1.4), of which the discussion is based on the comparison principle appealing to the competitive system. These results imply that one species can also spread at speed c^* under proper assumptions, so it seems that the competition does not decrease its spreading speed. At the same time, we also prove that the spreading speed of the other species is significantly smaller than the case when the interspecific competition vanishes (namely, $a_1 = a_2 = 0$). To further formulate the effect of competition, we also consider the population densities at the coexistence domain, herein coexistence means that population densities of p and q are bounded from below by a positive constant. Our results imply that the eventual population densities depend on their interactions, which also indicates that the *interspecific competition* does play a negative role in the evolution process of multi-species competition communities (see Bengtsson (1989) and Bleasdale (1956) for some ecological example/effect of interspecific competition).

To illustrate the spreading speeds of such recursion systems, some numerical simulations are also presented in this paper. Here, we compare the spreading speeds of (p, q) with the uncoupled case (by letting $a_1 = a_2 = 0$). These numerical results are coincident with our theoretical conclusions on both the spreading speeds and population densities. Finally, we give a brief discussion on the spreading speeds and traveling wave solutions of competition recursion systems.

The rest of this paper is organized as follows. In Sect. 2, we consider the traveling wave solutions of an abstract competition system, these results are applied to (1.4) in Sect. 3. To investigate the spreading speeds of p and q, we give some results on the corresponding initial value of (1.4) in Sect. 4. In Sect. 5, the spreading speed of (1.4) is concerned. We then provide some numerical results in Sect. 6. The paper ends with some discussion on the traveling wave solutions and the spreading speeds of the competition systems.

2 Traveling wave solutions of discrete time recursions

In what follows, we shall use the standard partial ordering in $\mathbb{R}^2$ or $\mathbb{R}$. Denote

$$C(\mathbb{R},\mathbb{R}^2)=\left\{u(x)|u(x):\mathbb{R}\to\mathbb{R}^2 \text{ is uniformly continuous and bounded}\right\},$$

which is a Banach space equipped with the supremum norm $|\cdot|$. In particular, if $\mathbf{a},\mathbf{b}\in\mathbb{R}^2$ with $\mathbf{a}<\mathbf{b}$, then $C_{[\mathbf{a},\mathbf{b}]}$ is defined by

$$C_{[\mathbf{a},\mathbf{b}]}=\left\{u(x): u(x)\in C(\mathbb{R},\mathbb{R}^2) \text{ and } \mathbf{a}\le u(x)\le\mathbf{b},\ x\in\mathbb{R}\right\}.$$

Before investigating the traveling wave solutions of (1.4), we first consider the traveling wave solutions of (1.3). We make the following assumptions for (1.3) such that it at least contains (1.4) as a special example, and these will be imposed throughout this section.

(i) There exists a vector $\mathbf{E}=(e_1,e_2)>\mathbf{0}$ such that

$$Q_i[\widehat{\mathbf{0}}]=0 \quad\text{and}\quad Q_i[\widehat{\mathbf{E}}]=e_i,\quad i=1,2,$$

where $\widehat{\cdot}$ is the constant value function in the set $C(\mathbb{R},\mathbb{R}^2)$.

(ii) There exists a vector $\mathbf{M}=(M_1,M_2)\in\mathbb{R}^2$ with $\mathbf{M}\ge\mathbf{E}$ such that $C_{[\mathbf{0},\mathbf{M}]}$ is an invariant interval of (2.1) in the sense that for any $(x,y)\in C_{[\mathbf{0},\mathbf{M}]}$,

$$(Q_1[x,y],Q_2[x,y])\in C_{[\mathbf{0},\mathbf{M}]}.$$

(iii) Assume that $(x_1,y_1),(x_2,y_2)\in[\mathbf{0},\mathbf{M}]$. Then there exists $L>0$ such that

$$\|(f_1(x_1,y_1),f_2(x_1,y_1))-(f_1(x_2,y_2),f_2(x_2,y_2))\|\le L\|(x_1,y_1)-(x_2,y_2)\|,$$

hereafter $\|\cdot\|$ denotes the supremum norm in $\mathbb{R}^2$.

(iv) If $(u_2,v_2),(u_1,v_1)\in C_{[\mathbf{0},\mathbf{M}]}$ with $(u_2,v_2)\le(u_1,v_1)$, then

$$Q_1[u_1,v_2]\ge Q_1[u_2,v_1],\quad Q_2[u_2,v_1]\ge Q_2[u_1,v_2].$$

Namely, u and v in (1.3) compete each other.

Definition 2.1 A **traveling wave solution** of (1.3) is a special solution of the form $u_n(x)=\phi(x+cn)$, $v_n(x)=\psi(x+cn)$ with wave speed $c>0$ and wave profile $(\phi,\psi)\in C(\mathbb{R},\mathbb{R}^2)$.

By Definition 2.1, let $t = x + cn \in \mathbb{R}$, then (ϕ, ψ) satisfies the following recursion system

$$\begin{cases} \phi(t+c) = Q_1[\phi, \psi](t) = \int_{\mathbb{R}} f_1(\phi(t-y), \psi(t-y))g_1(y, dy), \\ \psi(t+c) = Q_2[\phi, \psi](t) = \int_{\mathbb{R}} f_2(\phi(t-y), \psi(t-y))g_2(y, dy). \end{cases} \tag{2.1}$$

Recalling our main purpose is to formulate the competition-invasion of (1.2), so we also require the traveling wave solutions satisfy the following asymptotic boundary conditions

$$\lim_{t\to-\infty} \phi(t) = \lim_{t\to-\infty} \psi(t) = 0, \quad \lim_{t\to+\infty} \phi(t) = e_1, \quad \lim_{t\to+\infty} \psi(t) = e_2. \tag{2.2}$$

Clearly, to establish the existence of traveling wave solutions of (1.3) is equivalent to looking for fixed points of (2.1) and (2.2). Note that (2.1) is not a standard operator, it is difficult to apply the fixed point theorem directly. So we rewrite (2.1) as follows

$$\phi(t) = P_1^c[\phi, \psi](t), \quad \psi(t) = P_2^c[\phi, \psi](t), \quad t \in \mathbb{R}, \tag{2.3}$$

where P_i^c is an operator depending on the wave speed c and

$$P_i^c[\phi, \psi](t) =: Q_i[\phi, \psi](t-c), \quad (\phi, \psi) \in C(\mathbb{R}, \mathbb{R}^2), \quad t \in \mathbb{R}, \quad i = 1, 2.$$

Thus, the existence of a fixed point of (2.1) is equivalent to that of (2.3).

We now consider the existence of fixed points of (2.2)–(2.3). By the monotone condition (iv), the following comparison principle holds.

Lemma 2.2 *Assume that* $(\phi_1, \psi_1), (\phi_2, \psi_2) \in C(\mathbb{R}, \mathbb{R}^2)$. *Then*

$$\begin{cases} 0 \le P_1^c[\phi_2, \psi_1](t) \le P_1^c[\phi_1, \psi_2](t) \le M_1, \\ 0 \le P_2^c[\phi_1, \psi_2](t) \le P_2^c[\phi_2, \psi_1](t) \le M_2 \end{cases} \tag{2.4}$$

provided that $0 \le (\phi_2(t), \psi_2(t)) \le (\phi_1(t), \psi_1(t)) \le (M_1, M_2), t \in \mathbb{R}$.

In view of the above comparison principle, it is natural to introduce the following definition of upper and lower solutions, which depends on the monotone condition (iv). For similar definitions for reaction-diffusion systems, we refer to Leung (1989), Li et al. (2006), Lin et al. (2010), Pan (2009), Pao (1992, 2005), and Ye and Li (1990).

Definition 2.3 A pair of continuous vector functions $(\overline{\phi}(t), \overline{\psi}(t))$, $(\underline{\phi}(t), \underline{\psi}(t)) \in C_{\mathbf{[0,M]}}$ is called **upper and lower solutions** of (2.3), respectively, if for all $t \in \mathbb{R}$,

$$\begin{cases} (\overline{\phi}(t), \overline{\psi}(t)) \ge (\underline{\phi}(t), \underline{\psi}(t)), \\ \overline{\phi}(t) \ge P_1^c[\overline{\phi}, \underline{\psi}](t), \quad \overline{\psi}(t) \ge P_2^c[\underline{\phi}, \overline{\psi}](t), \\ \underline{\phi}(t) \le P_1^c[\underline{\phi}, \overline{\psi}](t), \quad \underline{\psi}(t) \le P_2^c[\overline{\phi}, \underline{\psi}](t). \end{cases}$$

If (2.3) has a pair of upper and lower solutions $(\overline{\phi}(t), \overline{\psi}(t)), (\underline{\phi}(t), \underline{\psi}(t)) \in C_{[0,M]}$, then we can define the set

$$\Gamma = \left\{(\phi, \psi) \in C(\mathbb{R}, \mathbb{R}^2) : (\underline{\phi}(t), \underline{\psi}(t)) \leq (\phi(t), \psi(t)) \leq (\overline{\phi}(t), \overline{\psi}(t)), t \in \mathbb{R}\right\}.$$

Clearly, Γ is nonempty. We shall prove the existence of traveling wave solutions in Γ. Furthermore, it is evident that the following conclusion holds.

Lemma 2.4 *Assume that* (2.3) *has a pair of upper and lower solutions. Then* Γ *is nonempty and convex. Moreover, it is bounded and closed with respect to the norm* $|\cdot|$.

Lemma 2.5 *Assume that* (2.3) *has a pair of upper and lower solutions. Then the mapping*

$$P^c := (P_1^c, P_2^c) : \Gamma \to \Gamma$$

is continuous with respect to the norm $|\cdot|$.

By Lemma 2.2, Definition 2.3 and the assumption (iii), Lemma 2.5 is clear and we omit the proof here.

Theorem 2.6 *Assume that* (2.3) *has a pair of upper and lower solutions* $(\overline{\phi}(t), \overline{\psi}(t))$ *and* $(\underline{\phi}(t), \underline{\psi}(t))$ *such that*

(a) $\lim_{t\to-\infty}(\overline{\phi}(t), \overline{\psi}(t)) = 0$;
(b) $\lim_{t\to\infty}(\overline{\phi}(t), \overline{\psi}(t)) = \lim_{t\to\infty}(\underline{\phi}(t), \underline{\psi}(t)) = (e_1, e_2)$.

If $P^c : \Gamma \to \Gamma$ *is compact, then there exists* $(\phi(t), \psi(t)) \in \Gamma$ *such that* (2.1) *and* (2.2) *hold, which is a desired traveling wave solution of* (1.2).

Proof It suffices to prove the existence of a fixed point of (2.3) with (2.2). In view of Lemmas 2.4 and 2.5 and the complete continuity of $P^c : \Gamma \to \Gamma$, we know that there exists $(\phi(t), \psi(t)) \in \Gamma$ satisfying (2.3) by Schauder's fixed point theorem.

Note that $P^c : \Gamma \to \Gamma$, then assumptions (a) and (b) imply that $(\phi(t), \psi(t)) \in \Gamma$ also satisfies the condition (2.2). The proof is complete. □

3 Traveling wave solutions of (1.4)

Let $(p_n(x), q_n(x)) = (\phi(x + cn), \psi(x + cn))$ be a traveling wave solution of (1.4). Then $(\phi(t), \psi(t))$ satisfies the recursion system

$$\begin{cases} \phi(t) = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} \frac{(1+r_1)\phi(t-y)}{1+r_1(\phi(t-y)+a_1\psi(t-y))} dy, \\ \psi(t) = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_2}} e^{-\frac{(y-c)^2}{4d_2}} \frac{(1+r_2)\psi(t-y)}{1+r_2(\psi(t-y)+a_2\phi(t-y))} dy, \quad t \in \mathbb{R}. \end{cases} \tag{3.1}$$

It is also required to satisfy the following asymptotic boundary conditions

$$\lim_{t\to-\infty}(\phi(t),\psi(t))=0,\quad \lim_{t\to\infty}(\phi(t),\psi(t))=(k_1,k_2). \tag{3.2}$$

Choose $\mathbf{M}=(1,1)=\mathbf{1}$ in this section. It is easy to verify that (3.1) satisfies the conditions (i)–(iv) with such an $\mathbf{M}$. In order to prove the existence of positive solutions of (3.1) with (3.2), it suffices to construct proper upper and lower solutions by Theorem 2.6. For this goal, we consider the characteristic equation of (3.1) as follows

$$\Lambda_i(\lambda,c)=\int_{\mathbb{R}}\frac{1+r_i}{\sqrt{4\pi d_i}}e^{-\frac{(y-c)^2}{4d_i}}e^{-\lambda y}dy=1,\quad i=1,2. \tag{3.3}$$

By some simple calculations, we obtain the following result.

Lemma 3.1 *Assume that $c>c^*=\max_{i=1,2}\{2\sqrt{d_i\ln(1+r_i)}\}$. Then $\Lambda_i(\lambda,c)=1$ has two positive roots, respectively. Let λ_i be the smaller root of $\Lambda_i(\lambda,c)=1$, then*

$$\lambda_i=\frac{c-\sqrt{c^2-4d_i\ln(1+r_i)}}{2d_i},\quad i=1,2.$$

Moreover, for any given $c>c^*$, there exists a constant $\eta\in(1,2)$ such that

$$\Lambda_i(\eta\lambda_i,c)<1,\quad \eta\lambda_i<\lambda_1+\lambda_2,\quad i=1,2. \tag{3.4}$$

Suppose that $c>c^*$ is given. By the constants in Lemma 3.1, we construct two continuous functions

$$\overline{\phi}(t)=\min\left\{e^{\lambda_1 t},1,k_1+\varepsilon_1k_1e^{-\gamma t}\right\},\quad \overline{\psi}(t)=\min\left\{e^{\lambda_2 t},1,k_2+\varepsilon_2k_2e^{-\gamma t}\right\}, \tag{3.5}$$

where $\varepsilon_1\ge 1,\varepsilon_2\ge 1$ and $\gamma>0$ will be explained later.

Next we define a constant $\rho>1$ by

$$\rho=1+\max\left\{\frac{r_1\Lambda_1(2\lambda_1,c)+r_1a_1\Lambda_1(\lambda_1+\lambda_2,c)}{1-\Lambda_1(\eta\lambda_1,c)},\frac{r_2\Lambda_2(2\lambda_2,c)+r_2a_2\Lambda_2(\lambda_1+\lambda_2,c)}{1-\Lambda_2(\eta\lambda_2,c)}\right\}.$$

Then for $i=1,2$, the function $e^{\lambda_i t}-\rho e^{\eta\lambda_i t}$ has a global maximum $m_i\in(0,1)$. Furthermore, let $N>3$ be large enough such that the following four items hold.

(i) $a_i+3/N<1$ and $\frac{3\frac{m_i}{N}}{1+r_i(3\frac{m_i}{N}+a_i)}>\frac{2\frac{m_i}{N}}{1+r_i(\frac{m_i}{N}+a_i)}$.

(ii) Let $t_i=\max\{t:e^{\lambda_i t}-\rho e^{\eta\lambda_i t}=\frac{m_i}{N}\}$ and define

$$\chi_i(t)=\begin{cases}e^{\lambda_i t}-\rho e^{\eta\lambda_i t}, & t<t_i,\\ \frac{m_i}{N}, & t>t_i.\end{cases}$$

If $L_i = \{t : \chi_i(t) > \frac{3m_i}{N}\}$, then $mes L_i > 1$, herein mes denotes the Lebesgue measure.

(iii) Define constants and intervals as follows

$$\begin{aligned} u_i &= \min\left\{t : e^{\lambda_i t} - \rho e^{\eta\lambda_i t} = \frac{m_i}{N}\right\}, \\ v_i &= \min\left\{t : e^{\lambda_i t} - \rho e^{\eta\lambda_i t} = \frac{3m_i}{N}\right\}, \\ I_i &= [u_i, t_i],\ J_i = (-\infty, t_i]. \end{aligned}$$

Then $t - v_i > t_i - v_i > c + 1$ if $t > t_i$. Moreover, it is obvious that

$$L_i \subset I_i \subset J_i, \quad i = 1, 2.$$

(iv) Since $\lim_{x\to\infty} \frac{\int_x^\infty e^{-\frac{y^2}{4d_i}} dy}{e^{-\frac{x^2}{4d_i}}} = 0$, then

$$2\int_x^\infty \frac{1}{\sqrt{4\pi d_i}} e^{-\frac{y^2}{4d_i}} dy < \frac{1}{\sqrt{4\pi d_i}} e^{-\frac{x^2}{4d_i}} \quad \text{if } x > t_i - v_i. \tag{3.6}$$

For the reader's convenience in understanding the geometric sense of these notations, we plot them in Fig. 1.

If $t > t_i$, then these facts imply that we can choose $N > 1$ large enough such that

$$\begin{aligned} &\int_{\mathbb{R}} \frac{(1+r_i)}{\sqrt{4\pi d_i}} e^{-\frac{(y-c)^2}{4d_i}} \frac{\chi_i(t-y)}{1 + r_i(\chi_i(t-y) + a_i)} dy \\ &= \left(\int_{t-y\in J_i} + \int_{t-y\in\mathbb{R}\setminus J_i} \right) \frac{(1+r_i)}{\sqrt{4\pi d_i}} e^{-\frac{(y-c)^2}{4d_i}} \frac{\chi_i(t-y)}{1 + r_i(\chi_i(t-y) + a_i)} dy \\ &= \left(\int_{t-y\in I_i\cup J_i} + \int_{t-y\in\mathbb{R}\setminus J_i} \right) \frac{(1+r_i)}{\sqrt{4\pi d_i}} e^{-\frac{(y-c)^2}{4d_i}} \frac{\chi_i(t-y)}{1 + r_i(\chi_i(t-y) + a_i)} dy \\ &> \left(\int_{t-y\in I_i} + \int_{t-y\in\mathbb{R}\setminus J_i} \right) \frac{(1+r_i)}{\sqrt{4\pi d_i}} e^{-\frac{(y-c)^2}{4d_i}} \frac{\chi_i(t-y)}{1 + r_i(\chi_i(t-y) + a_i)} dy \\ &= \left(\int_{t-y\in L_i} + \int_{t-y\in I_i\setminus L_i} + \int_{t-y\in\mathbb{R}\setminus J_i} \right) \frac{(1+r_i)}{\sqrt{4\pi d_i}} e^{-\frac{(y-c)^2}{4d_i}} \frac{\chi_i(t-y)}{1 + r_i(\chi_i(t-y) + a_i)} dy \end{aligned}$$

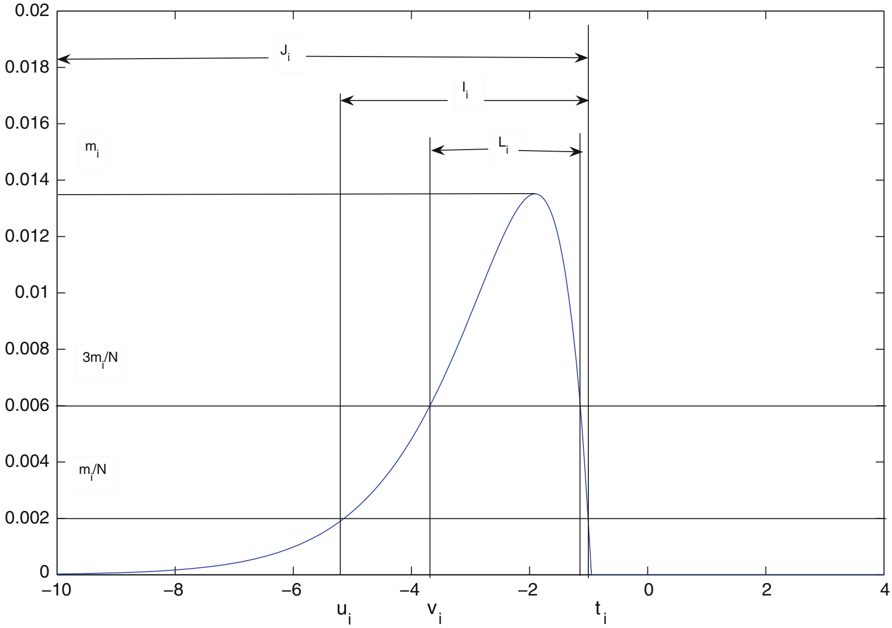


Fig. 1 The curve in the graph is defined by $e^t - 1.1e^{1.1t}$

$$
\begin{aligned}
&> \left(\int_{t-y\in I_i} + \int_{t-y\in L_i} + \int_{t-y\in \mathbb{R}\setminus J_i} \right) \frac{(1+r_i)}{\sqrt{4\pi d_i}} e^{-\frac{(y-c)^2}{4d_i}} \frac{\frac{m_i}{N}}{1+r_i(\frac{m_i}{N}+a_i)} dy \quad \text{(by (i))} \\
&> \left(\int_{t-y\in I_i} + \int_{t-y\in \mathbb{R}\setminus J_i} \right) \frac{(1+r_i)}{\sqrt{4\pi d_i}} e^{-\frac{(y-c)^2}{4d_i}} \frac{\frac{m_i}{N}}{1+r_i(\frac{m_i}{N}+a_i)} dy \\
&\quad + \int_{t-y\in [v_i, v_i+1]} \frac{(1+r_i)}{\sqrt{4\pi d_i}} e^{-\frac{(y-c)^2}{4d_i}} \frac{\frac{m_i}{N}}{1+r_i(\frac{m_i}{N}+a_i)} dy \quad \text{(by (ii) and (iii))} \\
&> \left(\int_{t-y\in I_i} + \int_{t-y\in \mathbb{R}\setminus J_i} \right) \frac{(1+r_i)}{\sqrt{4\pi d_i}} e^{-\frac{(y-c)^2}{4d_i}} \frac{\frac{m_i}{N}}{1+r_i(\frac{m_i}{N}+a_i)} dy \\
&\quad + \frac{(1+r_i)}{\sqrt{4\pi d_i}} e^{-\frac{(t-v_i-c)^2}{4d_i}} \frac{\frac{m_i}{N}}{1+r_i(\frac{m_i}{N}+a_i)} \quad \text{(by (iii))} \\
&> \left(\int_{t-y\in I_i} + \int_{t-y\in \mathbb{R}\setminus J_i} \right) \frac{(1+r_i)}{\sqrt{4\pi d_i}} e^{-\frac{(y-c)^2}{4d_i}} \frac{\frac{m_i}{N}}{1+r_i(\frac{m_i}{N}+a_i)} dy \\
&\quad + \frac{(1+r_i)}{\sqrt{4\pi d_i}} e^{-\frac{(t-u_i-c)^2}{4d_i}} \frac{\frac{m_i}{N}}{1+r_i(\frac{m_i}{N}+a_i)}
\end{aligned}
$$

$$
\begin{aligned}
&> \left(\int\limits_{t-y\in I_i} + \int\limits_{t-y\in\mathbb{R}\setminus J_i} \right) \frac{(1+r_i)}{\sqrt{4\pi d_i}} e^{-\frac{(y-c)^2}{4d_i}} \frac{\frac{m_i}{N}}{1+r_i(\frac{m_i}{N}+a_i)} dy \\
&\quad + \int\limits_{t-y\in J_i\setminus I_i} \frac{(1+r_i)}{\sqrt{4\pi d_i}} e^{-\frac{(y-c)^2}{4d_i}} \frac{\frac{m_i}{N}}{1+r_i(\frac{m_i}{N}+a_i)} dy \quad \text{(by (3.6))} \\
&= \left(\int\limits_{t-y\in J_i} + \int\limits_{t-y\in\mathbb{R}\setminus J_i} \right) \frac{(1+r_i)}{\sqrt{4\pi d_i}} e^{-\frac{(y-c)^2}{4d_i}} \frac{\frac{m_i}{N}}{1+r_i(\frac{m_i}{N}+a_i)} dy \\
&= \int\limits_{\mathbb{R}} \frac{(1+r_i)}{\sqrt{4\pi d_i}} e^{-\frac{(y-c)^2}{4d_i}} \frac{\frac{m_i}{N}}{1+r_i(\frac{m_i}{N}+a_i)} dy \\
&= \frac{m_i}{N} \times \frac{1+r_i}{1+r_i(\frac{m_i}{N}+a_i)} \\
&> \frac{m_i}{N}, \quad i=1,2.
\end{aligned} \tag{3.7}
$$

By t_1 and t_2, we further define two continuous functions

$$
\underline{\phi}(t) = \begin{cases} e^{\lambda_1 t} - \rho e^{\eta\lambda_1 t}, & t < t_1, \\ \frac{m_1}{N}, & t_1 < t < 0, \\ \max\left\{k_1 - \varepsilon_3 k_1 e^{-\gamma t}, \frac{m_1}{N}\right\}, & t > 0, \end{cases}
$$
$$
\underline{\psi}(t) = \begin{cases} e^{\lambda_2 t} - \rho e^{\eta\lambda_2 t}, & t < t_2, \\ \frac{m_2}{N}, & t_2 < t < 0, \\ \max\left\{k_2 - \varepsilon_4 k_2 e^{-\gamma t}, \frac{m_2}{N}\right\}, & t > 0, \end{cases}
$$

herein $\varepsilon_3 \geq 1, \varepsilon_4 \geq 1$ such that

$$
k_1\varepsilon_3 > a_1\varepsilon_2 k_2, \quad k_1\varepsilon_1 > a_1\varepsilon_4 k_2, \quad k_2\varepsilon_4 > a_2\varepsilon_1 k_1, \quad k_2\varepsilon_2 > a_2\varepsilon_3 k_1.
$$

Note that $a_1, a_2 \in (0, 1)$, then $\varepsilon_i, i = 1, 2, 3, 4$, are well defined.

Lemma 3.2 *Assume that* (3.7) *holds and $\gamma > 0$ is small enough. Then $(\overline{\phi}(t), \overline{\psi}(t))$ and $(\underline{\phi}(t), \underline{\psi}(t))$ are a pair of upper and lower solutions of* (3.1).

Proof Before verifying the inequalities in Definition 2.3, we point out that

$$
(\overline{\phi}(t), \overline{\psi}(t)) \geq (\underline{\phi}(t), \underline{\psi}(t)), \quad t \in \mathbb{R}
$$

because $\rho > 1, \varepsilon_i \geq 1$ and $N > 1$ hold. Moreover, since $\varepsilon_i > 1$, we can see that

$$
k_1 - \varepsilon_3 k_1 e^{-\gamma t} < 0, \quad k_2 - \varepsilon_4 k_2 e^{-\gamma t} < 0, \quad t < 0,
$$

which further imply that

$$k_1 - \varepsilon_3 k_1 e^{-\gamma t} \le \underline{\phi}(t), \quad k_2 - \varepsilon_4 k_2 e^{-\gamma t} \le \underline{\psi}(t), \quad t \in \mathbb{R}.$$

Now, it suffices to prove that these functions satisfy the corresponding inequalities in Definition 2.3. In particular, $\overline{\phi}(t)$ is an upper solution if

$$\overline{\phi}(t) \ge \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} \frac{(1+r_1)\overline{\phi}(t-y)}{1+r_1(\overline{\phi}(t-y)+a_1\underline{\psi}(t-y))} dy, \quad t \in \mathbb{R}. \tag{3.8}$$

(i) If $\overline{\phi}(t) = e^{\lambda_1 t}$, then $\overline{\phi}(y) > 0$, $\underline{\psi}(y) > 0$, $y \in \mathbb{R}$, imply that

$$\frac{(1+r_1)\overline{\phi}(t-y)}{1+r_1(\overline{\phi}(t-y)+a_1\underline{\psi}(t-y))} < (1+r_1)\overline{\phi}(t-y) \le (1+r_1)e^{\lambda_1(t-y)}.$$

Therefore, we need to prove that

$$e^{\lambda_1 t} \ge \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} (1+r_1)e^{\lambda_1(t-y)} dy,$$

which is clear by the definition of λ_1. Hence, (3.8) holds if $\overline{\phi}(t) = e^{\lambda_1 t}$.

(ii) If $\overline{\phi}(t) = 1$, then $1 \ge \overline{\phi}(y) > 0$, $\underline{\psi}(y) > 0$, $y \in \mathbb{R}$, indicate that

$$\frac{(1+r_1)\overline{\phi}(t-y)}{1+r_1(\overline{\phi}(t-y)+a_1\underline{\psi}(t-y))} < \frac{(1+r_1)\overline{\phi}(t-y)}{1+r_1\overline{\phi}(t-y)} \le 1, \quad t, y \in \mathbb{R}$$

by the definition of $\overline{\phi}(t)$. So (3.8) is clear if $\overline{\phi}(t) = 1$.

(iii) If $\overline{\phi}(t) = k_1 + \varepsilon_1 k_1 e^{-\gamma t}$, then $\varepsilon_1 k_1 - a_1 \varepsilon_4 k_2 > 0$ implies that

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} \frac{(1+r_1)\overline{\phi}(t-y)}{1+r_1(\overline{\phi}(t-y)+a_1\underline{\psi}(t-y))} dy \\
&< \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} \frac{(1+r_1)(k_1+\varepsilon_1 k_1 e^{-\gamma(t-y)})}{1+r_1((k_1+\varepsilon_1 k_1 e^{-\gamma(t-y)})+a_1(k_2-\varepsilon_4 k_2 e^{-\gamma(t-y)}))} dy \\
&= \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} \frac{k_1(1+\varepsilon_1 r_1)(1+e^{-\gamma(t-y)})}{1+r_1+r_1(\varepsilon_1 k_1 e^{-\gamma(t-y)}-a_1\varepsilon_4 k_2 e^{-\gamma(t-y)})} dy,
\end{aligned}$$

which is clear since $\underline{\psi}(t) \geq k_2 - \varepsilon_4 k_2 e^{-\gamma t}$, $t \in \mathbb{R}$. Thus it is sufficient to prove that

$$\begin{aligned}\varepsilon_1 e^{-\gamma t} &> \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} \frac{(1+r_1)(1+\varepsilon_1 e^{-\gamma(t-y)})}{1+r_1+r_1 e^{-\gamma(t-y)}(\varepsilon_1 k_1 - a_1\varepsilon_4 k_2)} dy - 1 \\ &= \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} \frac{\varepsilon_1(1+r_1)e^{-\gamma(t-y)} - r_1 e^{-\gamma(t-y)}(\varepsilon_1 k_1 - a_1\varepsilon_4 k_2)}{1+r_1+r_1 e^{-\gamma(t-y)}(\varepsilon_1 k_1 - a_1\varepsilon_4 k_2)} dy.\end{aligned} \tag{3.9}$$

It is obvious that (3.9) is equivalent to

$$\varepsilon_1 > \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} \frac{\varepsilon_1(1+r_1)e^{\gamma y} - r_1 e^{\gamma y}(\varepsilon_1 k_1 - a_1\varepsilon_4 k_2)}{1+r_1+r_1 e^{-\gamma(t-y)}(\varepsilon_1 k_1 - a_1\varepsilon_4 k_2)} dy,$$

which is true if

$$\varepsilon_1 > \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} \frac{(\varepsilon_1 + \varepsilon_1 r_1 - r_1\varepsilon_1 k_1 + r_1 a_1 \varepsilon_4 k_2)e^{\gamma y}}{1+r_1} dy \tag{3.10}$$

since $\varepsilon_1 k_1 > a_1\varepsilon_4 k_2$ and $\varepsilon_1 + \varepsilon_1 r_1 - r_1\varepsilon_1 k_1 + r_1 a_1\varepsilon_4 k_2 > 0$. In fact, there exists $\gamma_1 \in (0, 1)$ such that (3.10) holds for any $\gamma \in (0, \gamma_1)$ because of $\varepsilon_1 k_1 > a_1\varepsilon_4 k_2$. This completes the proof of (3.8).

In a similar way, we can show that there exists $\gamma_2 \in (0, \gamma_1)$ such that

$$\overline{\psi}(t) \geq \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_2}} e^{-\frac{(y-c)^2}{4d_2}} \frac{(1+r_2)\overline{\psi}(t-y)}{1+r_2(\overline{\psi}(t-y) + a_2\underline{\phi}(t-y))} dy$$

for any $t \in \mathbb{R}$ and $\gamma \in (0, \gamma_2)$.

We now prove that $(\underline{\phi}(t), \underline{\psi}(t))$ is a lower solution. For $\underline{\phi}(t)$, it suffices to prove that

$$\underline{\phi}(t) \leq \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} \frac{(1+r_1)\underline{\phi}(t-y)}{1+r_1(\underline{\phi}(t-y) + a_1\overline{\psi}(t-y))} dy, \quad t \in \mathbb{R}. \tag{3.11}$$

(a) If $\underline{\phi}(t) = e^{\lambda_1 t} - \rho e^{\eta\lambda_1 t}$, then we only need to verify that

$$e^{\lambda_1 t} - \rho e^{\eta\lambda_1 t} \leq \int_{\mathbb{R}} \frac{(1+r_1)}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} \frac{(e^{\lambda_1(t-y)} - \rho e^{\eta\lambda_1(t-y)})}{1+r_1(e^{\lambda_1(t-y)} + a_1 e^{\lambda_2(t-y)})} dy. \tag{3.12}$$

Since $\frac{1}{1+u} > 1 - u, u > 0$, then (3.12) holds provided

$$e^{\lambda_1 t} - \rho e^{\eta\lambda_1 t} \le \int_{\mathbb{R}} \frac{(1+r_1)}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} \left(e^{\lambda_1(t-y)} - \rho e^{\eta\lambda_1(t-y)}\right) \times \left[1 - r_1(e^{\lambda_1(t-y)} + a_1 e^{\lambda_2(t-y)})\right] dy,$$

which is equivalent to prove that

$$\begin{aligned}
&\rho(\Lambda_1(\eta\lambda_1, c) - 1)e^{\eta\lambda_1 t} \\
&\le -\int_{\mathbb{R}} \frac{r_1(1+r_1)}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} \left(e^{\lambda_1(t-y)} - \rho e^{\eta\lambda_1(t-y)}\right)\left(e^{\lambda_1(t-y)} + a_1 e^{\lambda_2(t-y)}\right) dy \\
&= -\int_{\mathbb{R}} \frac{r_1(1+r_1)}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} e^{\lambda_1(t-y)}\left(e^{\lambda_1(t-y)} + a_1 e^{\lambda_2(t-y)}\right) dy \\
&\quad +\rho \int_{\mathbb{R}} \frac{r_1(1+r_1)}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} e^{\eta\lambda_1(t-y)}\left(e^{\lambda_1(t-y)} + a_1 e^{\lambda_2(t-y)}\right) dy. \qquad (3.13)
\end{aligned}$$

Note that (3.13) holds if

$$\begin{aligned}
\rho(\Lambda_1(\eta\lambda_1, c) - 1)e^{\eta\lambda_1 t} &\le -\int_{\mathbb{R}} \frac{r_1(1+r_1)}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} \left[e^{\lambda_1(t-y)}\left(e^{\lambda_1(t-y)} + a_1 e^{\lambda_2(t-y)}\right)\right] dy \\
&= -r_1\Lambda_1(2\lambda_1, c)e^{2\lambda_1 t} - r_1 a_1 \Lambda_1(\lambda_1 + \lambda_2, c)e^{(\lambda_1+\lambda_2)t}.
\end{aligned}$$

Then (3.12) is true by the definition of ρ.

(b) If $\underline{\phi}(t) = \frac{m_1}{N}$, since $\overline{\psi}(t) \le 1$ and $\underline{\phi}(t) > 0$, it follows that

$$\frac{\underline{\phi}(t-y)}{1 + r_1(\underline{\phi}(t-y) + a_1\overline{\psi}(t-y))} \ge \frac{\underline{\phi}(t-y)}{1 + r_1(\underline{\phi}(t-y) + a_1)} \quad \text{for any } t, y \in \mathbb{R},$$

and (3.11) is evident by (3.7) and $\underline{\phi}(t) \ge \frac{m_1}{N}$ for $t > 0$.

(c) We now consider (3.11) with $t > t_3$, herein $\underline{\phi}(t_3) = k_1 - \varepsilon_3 k_1 e^{-\gamma t_3} = \frac{m_1}{N}$. If $\underline{\phi}(t-y) \ge k_1 - \varepsilon_3 k_1 e^{-\gamma(t-y)} \ge \frac{m_1}{2N}$, then $\overline{\psi}(t-y) \le k_2 + \varepsilon_2 k_2 e^{-\gamma(t-y)}$ and

$$\begin{aligned}
\frac{(1+r_1)\underline{\phi}(t-y)}{1 + r_1(\underline{\phi}(t-y) + a_1\overline{\psi}(t-y))} &\ge \frac{(1+r_1)\left(k_1 - \varepsilon_3 k_1 e^{-\gamma(t-y)}\right)}{1 + r_1 + r_1\left(a_1\varepsilon_2 k_2 - \varepsilon_3 k_1\right) e^{-\gamma(t-y)}} \\
&= \frac{k_1\left(1 - \varepsilon_3 e^{-\gamma(t-y)}\right)}{1 + \frac{r_1(a_1\varepsilon_2 k_2 - \varepsilon_3 k_1)}{1+r_1} e^{-\gamma(t-y)}}.
\end{aligned}$$

Assume that $1 > nx > mx > 0$. Then it is clear that

$$\frac{1-nx}{1-mx} > 1-(n-m)x - mnx^2.$$

Because of $\varepsilon_3 > \varepsilon_3 k_1 - a_1\varepsilon_2 k_2 > 0$, we have

$$\frac{k_1\left(1-\varepsilon_3 e^{-\gamma(t-y)}\right)}{1+\frac{(a_1\varepsilon_2k_2-\varepsilon_3k_1)}{1+r_1}e^{-\gamma(t-y)}} \geq k_1\left(1-\left(\varepsilon_3+\frac{(a_1\varepsilon_2k_2-\varepsilon_3k_1)}{1+r_1}\right)e^{-\gamma(t-y)}\right) + \frac{k_1\varepsilon_3\left(a_1\varepsilon_2k_2-\varepsilon_3k_1\right)}{1+r_1}e^{-2\gamma(t-y)}.$$

Note that $k_1\varepsilon_3 e^{-\gamma(t-y)} \leq k_1 - \frac{m_1}{2N}$ and $\varepsilon_3 k_1 - a_1\varepsilon_2 k_2 > 0$, we obtain

$$\begin{aligned}\frac{k_1\left(1-\varepsilon_3 e^{-\gamma(t-y)}\right)}{1+\frac{(a_1\varepsilon_2k_2-\varepsilon_3k_1)}{1+r_1}e^{-\gamma(t-y)}} &\geq k_1\left(1-\left(\varepsilon_3+\frac{(a_1\varepsilon_2k_2-\varepsilon_3k_1)}{1+r_1}\right)e^{-\gamma(t-y)}\right)\\ &\quad+\frac{\left(k_1-\frac{m_1}{2N}\right)\left(a_1\varepsilon_2k_2-\varepsilon_3k_1\right)}{1+r_1}e^{-\gamma(t-y)}\\ &= k_1\left(1-\left(\varepsilon_3+\frac{m_1\left(a_1\varepsilon_2k_2-\varepsilon_3k_1\right)}{2k_1N\left(1+r_1\right)}\right)e^{-\gamma(t-y)}\right).\end{aligned}$$

If $\underline{\phi}(t-y) = \frac{m_1}{N}$, then $\frac{(1+r_1)\frac{m_1}{N}}{1+r_1(\frac{m_1}{N}+a_1)} > \frac{m_1}{N}$ implies that

$$\frac{(1+r_1)\underline{\phi}(t-y)}{1+r_1(\underline{\phi}(t-y)+a_1\overline{\psi}(t-y))} \geq k_1 - \varepsilon_3 k_1 e^{-\gamma(t-y)}.$$

For any given $\varepsilon_3 > 0$ and $\gamma \in (0,\gamma_2)$, there exists $\overline{N}(\varepsilon_3) > 0$ such that

$$t_3 - t_4 = \overline{N}\left(\gamma\right),$$

in which $t_4 \in \mathbb{R}$ such that $k_1 - \varepsilon_3 k_1 e^{-\gamma t_4} = \frac{m_1}{2N}$. Then it is clear that

$$\overline{N}\left(\gamma\right) \to \infty \quad \text{if } \gamma \to 0+. \tag{3.14}$$

We have the following estimate

$$\begin{aligned}&\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1}}e^{-\frac{(y-c)^2}{4d_1}}\frac{(1+r_1)\underline{\phi}(t-y)}{1+r_1(\underline{\phi}(t-y)+a_1\overline{\psi}(t-y))}dy\\ &= \int_{-\infty}^{t-t_4} \frac{1}{\sqrt{4\pi d_1}}e^{-\frac{(y-c)^2}{4d_1}}\frac{(1+r_1)\underline{\phi}(t-y)}{1+r_1(\underline{\phi}(t-y)+a_1\overline{\psi}(t-y))}dy\end{aligned}$$

$$
\begin{aligned}
&+\int_{t-t_4}^{\infty}\frac{1}{\sqrt{4\pi d_1}}e^{-\frac{(y-c)^2}{4d_1}}\frac{(1+r_1)\underline{\phi}(t-y)}{1+r_1(\underline{\phi}(t-y)+a_1\overline{\psi}(t-y))}dy\\
&\geq\int_{-\infty}^{t-t_4}\frac{1}{\sqrt{4\pi d_1}}e^{-\frac{(y-c)^2}{4d_1}}k_1\left(1-\left(\varepsilon_3+\frac{m_1\left(a_1\varepsilon_2k_2-\varepsilon_3k_1\right)}{2k_1N\left(1+r_1\right)}\right)e^{-\gamma(t-y)}\right)dy\\
&+\int_{t-t_4}^{\infty}\frac{1}{\sqrt{4\pi d_1}}e^{-\frac{(y-c)^2}{4d_1}}\left(k_1-\varepsilon_3k_1e^{-\gamma(t-y)}\right)dy\\
&=k_1-k_1\int_{-\infty}^{t-t_4}\frac{1}{\sqrt{4\pi d_1}}e^{-\frac{(y-c)^2}{4d_1}}\left(\varepsilon_3+\frac{m_1\left(a_1\varepsilon_2k_2-\varepsilon_3k_1\right)}{2k_1N\left(1+r_1\right)}\right)e^{-\gamma(t-y)}dy\\
&-\varepsilon_3k_1\int_{t-t_4}^{\infty}\frac{1}{\sqrt{4\pi d_1}}e^{-\frac{(y-c)^2}{4d_1}}e^{-\gamma(t-y)}dy\\
&\geq k_1-k_1\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi d_1}}e^{-\frac{(y-c)^2}{4d_1}}\left(\varepsilon_3+\frac{m_1\left(a_1\varepsilon_2k_2-\varepsilon_3k_1\right)}{2k_1N\left(1+r_1\right)}\right)e^{-\gamma(t-y)}dy\\
&-\varepsilon_3k_1\int_{t-t_4}^{\infty}\frac{1}{\sqrt{4\pi d_1}}e^{-\frac{(y-c)^2}{4d_1}}e^{-\gamma(t-y)}dy.
\end{aligned}
$$

By simple calculation, it is clear that

$$
\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi d_1}}e^{-\frac{(y-c)^2}{4d_1}}e^{\gamma y}dy=e^{\gamma c+d_1\gamma^2}.
$$

Moreover, $t-t_4\geq t_3-t_4=\overline{N}(\gamma)$ implies that

$$
\begin{aligned}
-\varepsilon_3k_1\int_{t-t_4}^{\infty}\frac{1}{\sqrt{4\pi d_1}}e^{-\frac{(y-c)^2}{4d_1}}e^{-\gamma(t-y)}dy&\geq-\varepsilon_3k_1\int_{\overline{N}(\gamma)}^{\infty}\frac{1}{\sqrt{4\pi d_1}}e^{-\frac{(y-c)^2}{4d_1}}e^{-\gamma(t-y)}dy\\
&=-\varepsilon_3k_1e^{-\gamma t}\int_{\overline{N}(\gamma)}^{\infty}\frac{1}{\sqrt{4\pi d_1}}e^{-\frac{(y-c)^2}{4d_1}}e^{\gamma y}dy.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&k_1 - k_1 \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} \left(\varepsilon_3 + \frac{m_1 \left(a_1 \varepsilon_2 k_2 - \varepsilon_3 k_1 \right)}{2k_1 N \left(1 + r_1\right)} \right) e^{-\gamma(t-y)} dy \\
&\quad - \varepsilon_3 k_1 \int_{t-t_4}^{\infty} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} e^{-\gamma(t-y)} dy \\
&\quad \geq k_1 - k_1 e^{-\gamma t} \left[\left(\varepsilon_3 + \frac{m_1 \left(a_1 \varepsilon_2 k_2 - \varepsilon_3 \right)}{2k_1 N \left(1 + r_1\right)} \right) e^{\gamma c + d_1 \gamma^2} \right. \\
&\quad \left. + \varepsilon_3 k_1 \int_{\overline{N}(\gamma)}^{\infty} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} e^{\gamma y} dy \right].
\end{aligned}
$$

Thus, it suffices to prove that

$$
\left(\varepsilon_3 + \frac{m_1 \left(a_1 \varepsilon_2 k_2 - \varepsilon_3 k_1 \right)}{2k_1 N \left(1 + r_1\right)} \right) e^{\gamma c + d_1 \gamma^2} + \varepsilon_3 \int_{\overline{N}(\gamma)}^{\infty} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} e^{\gamma y} dy \leq \varepsilon_3. \tag{3.15}
$$

Note that (3.14) is true and $\frac{m_1(a_1\varepsilon_2 k_2 - \varepsilon_3 k_1)}{2k_1 N(1+r_1)} < 0$ is independent of γ. Hence, there exists $\gamma_3 \in (0, \gamma_2)$ such that (3.15) holds for all $\gamma \in (0, \gamma_3)$. This completes the proof of (3.11).

In a similar way, we can prove that there exists $\gamma_4 \in (0, \gamma_3]$ such that

$$
\underline{\psi}(t) \leq \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_2}} e^{-\frac{(y-c)^2}{4d_2}} \frac{(1 + r_2)\underline{\psi}(t - y)}{1 + r_2(\underline{\psi}(t - y) + a_2 \overline{\phi}(t - y))} dy, \quad t \in \mathbb{R}
$$

for any $\gamma \in (0, \gamma_4)$. The proof is complete. □

Define a set

$$
\Gamma^* = \left\{ (\phi, \psi) \in C(\mathbb{R}, \mathbb{R}^2) : (\underline{\phi}, \underline{\psi}) \leq (\phi, \psi) \leq (\overline{\phi}, \overline{\psi}) \right\}.
$$

For $(\phi, \psi) \in \Gamma^*$, we denote (T_1^c, T_2^c) as follows

$$
\begin{cases}
T_1^c(\phi, \psi)(t) = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} \frac{(1+r_1)\phi(t-y)}{1+r_1(\phi(t-y)+a_1\psi(t-y))} dy, \\
T_2^c(\phi, \psi)(t) = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_2}} e^{-\frac{(y-c)^2}{4d_2}} \frac{(1+r_2)\psi(t-y)}{1+r_2(\psi(t-y)+a_2\phi(t-y))} dy.
\end{cases}
$$

Lemma 3.3 *Assume that Lemma 3.2 holds. Then*

$$T^c := (T_1^c, T_2^c) : \Gamma^* \to \Gamma^*$$

and the mapping is completely continuous with respect to the supremum norm $|\cdot|$.

Proof $T^c : \Gamma^* \to \Gamma^*$ is obvious by the comparison principle and the definition of upper and lower solutions. It is also clear that $T^c : \Gamma^* \to \Gamma^*$ is continuous with respect to the supremum norm. We now prove the compactness of the mapping T^c.

For any $(\phi, \psi) \in \Gamma^*$ and $t \in \mathbb{R}$, it is clear that $(T_1^c(\phi, \psi)(t), T_2^c(\phi, \psi)(t))$ is equicontinuous. In fact, for any $s \in \mathbb{R}$,

$$\begin{aligned}
&|T_1^c(\phi, \psi)(t) - T_1^c(\phi, \psi)(t+s)| \\
&= \left| \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} \frac{(1+r_1)\phi(t-y)}{1+r_1(\phi(t-y)+a_1\psi(t-y))} dy \right. \\
&\quad \left. - \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} \frac{(1+r_1)\phi(t+s-y)}{1+r_1(\phi(t+s-y)+a_1\psi(t+s-y))} dy \right| \\
&\le \int_{\mathbb{R}} \left| \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} - \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y+s-c)^2}{4d_1}} \right| \frac{(1+r_1)\phi(t-y)}{1+r_1(\phi(t-y)+a_1\psi(t-y))} dy \\
&\le \int_{\mathbb{R}} \left| \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} - \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y+s-c)^2}{4d_1}} \right| dy,
\end{aligned}$$

the equicontinuity of $T_1^c(\phi, \psi)(t)$ follows, so for $T_2^c(\phi, \psi)(t)$.

For any given $\epsilon > 0$, there exists $T > 0$ such that

$$\sup_{t<-T} \left\{ \overline{\phi}(t) + \overline{\psi}(t) \right\} + \sup_{t>T} \left\{ \overline{\phi}(t) - \underline{\phi}(t) + \overline{\psi}(t) - \underline{\psi}(t) \right\} < \epsilon, \tag{3.16}$$

which is clear by the asymptotic behavior of the upper and lower solutions.

The equicontinuity of the mapping also implies that there exists sequences $\{\phi_n(t), \psi_n(t)\}_{n=1}^{N_\epsilon} \in \Gamma^*$ with finite N_ϵ such that $\{\phi_n(t), \psi_n(t)\}_{n=1}^{N_\epsilon}$ with $|t| \le T$ is a finite ϵ-net of the set

$$\left\{ (T_1^c(\phi, \psi)(t), T_2^c(\phi, \psi)(t)) : |t| \le T \text{ and } (\phi, \psi) \in \Gamma^* \right\},$$

which is based on the Ascoli–Arzela lemma. Furthermore, (3.16) means that $\{\phi_n(t), \psi_n(t)\}_{n=1}^{N_\epsilon}$ is also a finite ϵ-net of the following set

$$\left\{ (T_1^c(\phi, \psi)(t), T_2^c(\phi, \psi)(t)) : (\phi, \psi) \in \Gamma^* \right\}.$$

Thus the above set is precompact and the mapping is compact. The proof is complete. □

Summarizing the above discussions, we have the following result.

Theorem 3.4 *Assume that $c > c^*$. Then* (3.1) *and* (3.2) *has a solution* $(\phi(t), \psi(t))$ *such that*

$$\lim_{t\to-\infty} \phi(t)e^{-\lambda_1 t} = 1, \quad \lim_{t\to-\infty} \psi(t)e^{-\lambda_2 t} = 1,$$

where the constants λ_1 and λ_2 are defined in Lemma 3.1.

4 Initial value problem

Before investigating the spreading speed of (1.2), we consider the initial value problem

$$\begin{cases} p_{n+1}(x) = \int_{\mathbb{R}} \frac{e^{-\frac{y^2}{4d_1}}}{\sqrt{4\pi d_1}} \frac{(1+r_1)p_n(x-y)}{1+r_1(p_n(x-y)+a_1q_n(x-y))} dy, \quad n = 0, 1, 2, \ldots, \\ q_{n+1}(x) = \int_{\mathbb{R}} \frac{e^{-\frac{y^2}{4d_2}}}{\sqrt{4\pi d_2}} \frac{(1+r_2)q_n(x-y)}{1+r_2(q_n(x-y)+a_2p_n(x-y))} dy, \quad n = 0, 1, 2, \ldots, \\ p_0(x) = p(x), \quad q_0(x) = q(x), \quad x \in \mathbb{R}, \end{cases} \tag{4.1}$$

where $(p(x), q(x)) \in C_{[\mathbf{0},\mathbf{1}]}(\mathbb{R}, \mathbb{R}^2)$. In particular, the following result is clear.

Theorem 4.1 *Assume that $(p(x), q(x)) \in C_{[\mathbf{0},\mathbf{1}]}(\mathbb{R}, \mathbb{R}^2)$. Then $(p_n(x), q_n(x)) \in C_{[\mathbf{0},\mathbf{1}]}(\mathbb{R}, \mathbb{R}^2)$ for all $n = 1, 2, \ldots$.*

In order to establish the comparison principle of (4.1), we introduce the definition of upper and lower solutions of (4.1) as follows.

Definition 4.2 Assume that $(\overline{p}_n(x), \overline{q}_n(x)), (\underline{p}_n(x), \underline{q}_n(x)) \in C_{[\mathbf{0},\mathbf{1}]}$ for all $n = 0, 1, \cdots$. If they satisfy the following inequalities

$$\begin{cases} \overline{p}_{n+1}(x) \geq \int_{\mathbb{R}} \frac{e^{-\frac{y^2}{4d_1}}}{\sqrt{4\pi d_1}} \frac{(1+r_1)\overline{p}_n(x-y)}{1+r_1(\overline{p}_n(x-y)+a_1\underline{q}_n(x-y))} dy, \quad n = 0, 1, 2, \ldots, \\ \overline{q}_{n+1}(x) \geq \int_{\mathbb{R}} \frac{e^{-\frac{y^2}{4d_2}}}{\sqrt{4\pi d_2}} \frac{(1+r_2)\overline{q}_n(x-y)}{1+r_2(\overline{q}_n(x-y)+a_2\underline{p}_n(x-y))} dy, \quad n = 0, 1, 2, \ldots, \\ \underline{p}_{n+1}(x) \leq \int_{\mathbb{R}} \frac{e^{-\frac{y^2}{4d_1}}}{\sqrt{4\pi d_1}} \frac{(1+r_1)\underline{p}_n(x-y)}{1+r_1(\underline{p}_n(x-y)+a_1\overline{q}_n(x-y))} dy, \quad n = 0, 1, 2, \ldots, \\ \underline{q}_{n+1}(x) \leq \int_{\mathbb{R}} \frac{e^{-\frac{y^2}{4d_2}}}{\sqrt{4\pi d_2}} \frac{(1+r_2)\underline{q}_n(x-y)}{1+r_2(\underline{q}_n(x-y)+a_2\overline{p}_n(x-y))} dy, \quad n = 0, 1, 2, \ldots, \\ (\underline{p}_0(x), \underline{q}_0(x)) \leq (p(x), q(x)) \leq (\overline{p}_0(x), \overline{q}_0(x)), \quad x \in \mathbb{R}, \end{cases} \tag{4.2}$$

then $(\overline{p}_n(x), \overline{q}_n(x))$ and $(\underline{p}_n(x), \underline{q}_n(x))$ are called a pair of **upper** and **lower solutions** of the system (4.1), respectively.

Theorem 4.3 *Assume that $(\overline{p}_n(x), \overline{q}_n(x))$ and $(\underline{p}_n(x), \underline{q}_n(x))$ are a pair of upper and lower solutions of* (4.1)*, respectively. Then the following statements hold.*

(i) *For all* $n = 0, 1, 2, \ldots,$

$$(\overline{p}_n(x), \overline{q}_n(x)) \geq (\underline{p}_n(x), \underline{q}_n(x)), \quad x \in \mathbb{R}.$$

(ii) *The unique solution* $(p_n(x), q_n(x))$ *of* (4.1) *satisfies*

$$(\overline{p}_n(x), \overline{q}_n(x)) \geq (p_n(x), q_n(x)) \geq (\underline{p}_n(x), \underline{q}_n(x)), \quad x \in \mathbb{R}, \ n = 0, 1, 2, \ldots.$$

The above theorem is clear and we omit the proof here. Moreover, it is obvious that (4.1) admits the following property.

Proposition 4.4 *Assume that both* $p(x)$ *and* $q(x)$ *have nonempty supports. Then*

$$p_n(x) > 0, \quad q_n(x) > 0$$

for all $x \in \mathbb{R}$ *and* $n = 1, 2, \ldots.$

Let $N > 0$ such that $(1+r_1)\int_{-N}^{N} \frac{e^{-\frac{y^2}{4d_1}}}{\sqrt{4\pi d_1}} dy > 1+r_1 a_1$. Define an auxiliary recursion $(p_n'(x), q_n'(x))$ as follows

$$\begin{cases} p_{n+1}'(x) = \int_{-N}^{N} \frac{e^{-\frac{y^2}{4d_1}}}{\sqrt{4\pi d_1}} \frac{(1+r_1)p_n'(x-y)}{1+r_1(p_n'(x-y)+a_1 q_n'(x-y))} dy, & n = 0, 1, 2, \ldots, \\ q_{n+1}'(x) = \int_{\mathbb{R}} \frac{e^{-\frac{y^2}{4d_2}}}{\sqrt{4\pi d_2}} \frac{(1+r_2)q_n'(x-y)}{1+r_2(q_n'(x-y)+a_2 p_n'(x-y))} dy, & n = 0, 1, 2, \ldots, \\ p_0'(x) = p(x), \quad q_0'(x) = q(x), \quad x \in \mathbb{R}. \end{cases} \tag{4.3}$$

Then we can define upper and lower solutions and establish the comparison principle for system (4.3) similar to that for system (4.1), and we omit it here.

Lemma 4.5 *For any* $n = 0, 1, \ldots, x \in \mathbb{R}$, *we have*

$$0 \leq p_n'(x) \leq p_n(x) \leq 1, \quad 0 \leq q_n(x) \leq q_n'(x) \leq 1.$$

5 The spreading speed of (4.1)

We now give the definition of spreading speed.

Definition 5.1 Assume that $u_n(x)$ is a nonnegative function. Then c_* is called the **spreading speed** of $u_n(x)$ if

(a) $\lim_{n\to\infty} \sup_{|x|>(c_*+\epsilon)n} u_n(x) = 0$ for any $\epsilon > 0$;
(b) $\lim_{n\to\infty} \inf_{|x|<(c_*-\epsilon)n} u_n(x) > 0$ for any $\epsilon \in (0, c_*)$.

By Definition 5.1, the spreading speed states the observed phenomena if an observer were to move to the right or left at a fixed speed (Weinberger et al. 2002). This is very important since it describes the speed at which the geographic range of the new population expands in population dynamics (Hsu and Zhao 2008). Thus, we shall investigate the spreading speed of (1.4) by considering the corresponding initial value problem.

Theorem 5.2 *Assume that* $p_0(x)$ *and* $q_0(x)$ *have nonempty compact supports and satisfy* $0 \le p_0(x), q_0(x) \le 1, x \in \mathbb{R}$. *Then*

$$\lim_{n\to\infty} \sup_{|x|>(c^*+\epsilon)n} \{p_n(x) + q_n(x)\} = 0$$

for any $\epsilon > 0$, *where* c^* *is defined in Lemma* 3.1.

Proof Let $c = c^* + \frac{\epsilon}{2}$. Define constants β_1 and β_2 as follows

$$\beta_i = \frac{c - \sqrt{c^2 - 4d_i \ln(1 + r_i)}}{2d_i}, \quad i = 1, 2.$$

For $T_1 > 0$, define a continuous function sequence pair $(\overline{p}_n(x), \overline{q}_n(x))$ by

$$\overline{p}_n(x) = \min\left\{e^{\beta_1(x+cn+T_1)}, e^{\beta_1(-x+cn+T_1)}, 1\right\}$$

and

$$\overline{q}_n(x) = \min\left\{e^{\beta_2(x+cn+T_1)}, e^{\beta_2(-x+cn+T_1)}, 1\right\}.$$

Let $T_1 > 0$ be large enough such that $(\overline{p}_0(x), \overline{q}_0(x)) \ge (p(x), q(x))$. Then it is easy to prove that $(\overline{p}_n(x), \overline{q}_n(x))$ is an upper solution if we take $(0, 0)$ as a lower solution. The proof is similar to that of Lemma 3.2 and we omit it here.

By the comparison principle (Theorem 4.3) and the fact that

$$\lim_{n\to\infty} \sup_{|x|>(c^*+\epsilon)n} \{\overline{p}_n(x) + \overline{q}_n(x)\} = 0,$$

we obtain the result. The proof is complete. □

Theorem 5.3 *Assume that* $p_0(x)$ *and* $q_0(x)$ *have nonempty compact supports and satisfy* $0 \le p_0(x), q_0(x) \le 1, x \in \mathbb{R}$. *Also assume that*

$$d_1 \ln\left(\frac{1 + r_1}{1 + r_1 a_1}\right) > d_2 \ln(1 + r_2). \tag{5.1}$$

Then for any $\epsilon \in (0, c^*)$,

$$\lim_{n\to\infty} \inf_{|x|<(c^*-\epsilon)n} p_n(x) > 0.$$

Remark 5.4 Theorems 5.2 and 5.3 imply that c^* is the spreading speed of the unknown function $p_n(x)$ if (5.1) holds, which equals to the case when $a_1 = 0$ (namely, the inter-specific competition vanishes).

Before proving Theorem 5.3, we first give two lemmas in which the conditions of Theorem 5.3 will be imposed.

Lemma 5.5 *Define* $c_1 = 2\sqrt{d_1 \ln(\frac{1+r_1}{1+r_1 a_1})}$ *and* $c_2 = 2\sqrt{d_2 \ln(1+r_2)}$. *Then*

$$\lim_{n\to\infty} \inf_{|x|<(c_1-\epsilon)n} p_n(x) > 0, \quad \lim_{n\to\infty} \sup_{|x|>(c_2+\epsilon)n} q_n(x) = 0$$

if $\epsilon > 0$ *such that* $c_1 - \epsilon > 0$.

Proof By Theorem 4.1, it is obvious that

$$p_{n+1}(x) \geq \int_{\mathbb{R}} \frac{e^{-\frac{y^2}{4d_1}}}{\sqrt{4\pi d_1}} \frac{(1+r_1)p_n(x-y)}{1+r_1(p_n(x-y)+a_1)} dy, \quad n = 0, 1, 2, \ldots.$$

Then $\lim_{n\to\infty} \inf_{|x|<(c_1-\epsilon)n} p_n(x) > 0$ since c_1 is the spreading speed of the recursion defined by

$$u_{n+1}(x) = \int_{\mathbb{R}} \frac{e^{-\frac{y^2}{4d_1}}}{\sqrt{4\pi d_1}} \frac{(1+r_1)u_n(x-y)}{1+r_1(u_n(x-y)+a_1)} dy, \quad n = 0, 1, 2, \ldots,$$

see Liang and Zhao (2007) and Weinberger et al. (2002).

We now prove that $\lim_{n\to\infty} \sup_{|x|>(c_2+\epsilon)n} q_n(x) = 0$. Let $c = c_2 + \frac{\epsilon}{2}$, define

$$\beta_2 = \frac{c - \sqrt{c^2 - 4d_2 \ln(1+r_2)}}{2d_2}.$$

Construct continuous functions as follows

$$\overline{p}_n(x) = 1, \quad \overline{q}_n(x) = \min\left\{e^{\beta_2(x+cn+T_1)}, e^{\beta_2(-x+cn+T_1)}, 1\right\}, \quad \underline{p}_n(x) = \underline{q}_n(x) = 0.$$

Let $T_1 > 0$ be large enough such that $\overline{q}_0(x) \geq q(x)$. Then we can easily prove that $(\overline{p}_n(x), \overline{q}_n(x))$ and $(\underline{p}_n(x), \underline{q}_n(x))$ are a pair of upper and lower solutions of (5.1). By the comparison principle and the asymptotic behavior of $\overline{q}_n(x)$, $\lim_{n\to\infty} \sup_{|x|>(c_2+\epsilon)n} q_n(x) = 0$. The proof is complete. □

Similarly, the following result holds.

Lemma 5.6 *Assume that* $\epsilon \in (0, c_1)$ *is given and* $N > 0$ *is large enough. Then*

$$\lim_{n\to\infty} \inf_{|x|<(c_1-\epsilon)n} p_n'(x) > 0, \quad \lim_{n\to\infty} \inf_{|x|>(c_2+\epsilon)n} q_n'(x) = 0.$$

Proof of Theorem 5.3 We prove the results by the comparison principle and the idea in Diekmann (1979). Assume that $0 < \epsilon_1 < \epsilon_2$. Then the result of Theorem 5.3 holds for ϵ_2 if the result is true for ϵ_1. Thus we only consider the case

$$4\epsilon \in (0, \min\{c^* - c_1, c_1 - c_2\}).$$

Define function sequences as follows

$$\overline{p}'_n(x) = 1, \quad \overline{q}'_n(x) = \min\left\{e^{\beta_2(x+(c_2+\frac{\epsilon}{2})n+T_1)}, e^{\beta_2(-x+(c_2+\frac{\epsilon}{2})n+T_1)}, 1\right\}, \tag{5.2}$$

where β_2, T_1 are the same as in Lemma 5.5. Moreover, for $\alpha \in \mathbb{R}, \beta > 0$, let

$$q(y; \alpha, \beta) := \begin{cases} e^{-\alpha y} \sin \beta y & \text{for } y \in [0, \frac{\pi}{\beta}], \\ 0 & \text{for } y \in \mathbb{R} \backslash [0, \frac{\pi}{\beta}] \end{cases}$$

and

$$r(y; \alpha, \beta, \gamma) := \max_{\eta \geq -\gamma} q(y + \eta; \alpha, \beta),$$

where $\gamma > 0$. For $D > 0$, we further define

$$\omega(n, x) := r(|x|; \alpha, \beta, D + cn).$$

Claim Assume that $\sigma > 0$ is small enough and $N > 0$ is large enough, then there exist constants α, β, D and $T^* > 0$ such that $(\sigma\omega(n, x), 0)$ is a lower solution of (4.3) if $c = c^* - \frac{3\epsilon}{4}$ and $n \geq T^*$, herein the corresponding upper solution is defined by (5.2).

By the comparison principle on (4.3) and the above claim, we have

$$\lim_{n\to\infty} \inf_{|x|<(c^*-\epsilon)n} p'_n(x) > 0.$$

By Lemma 4.5, the proof is complete once we prove the above claim.

Proof of the Claim Let $N > 0$ be large enough such that

$$\int_{-N}^{N} \frac{1 + r_1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} e^{-\lambda y} dy = 1$$

has no positive root λ if $c < c^* - \frac{\epsilon}{4}$, which is true from Lemma 3.1.

Let $T > 0$ be large enough such that $\frac{\epsilon}{4}T > N$. Then Lemma 5.6 and (5.2) imply that there exist $0 < \delta' \ll \delta < 1$ such that

$$\sup_{|x|>\left(c_2+\frac{3}{4}\epsilon\right)n} \overline{q}'_n(x) < \delta', \qquad \inf_{(c_2+\epsilon)n \leq |x| \leq \left(c_1-\frac{1}{4}\epsilon\right)n} p'_n(x) > \delta$$

for all $n > T$. These further indicate that

$$p'_{n+1}(x) \geq \int_{-N}^{N} \frac{e^{-\frac{y^2}{4d_1}}}{\sqrt{4\pi d_1}} \frac{(1 + r_1)p'_n(x - y)}{1 + r_1(p'_n(x - y) + a_1\delta')} dy$$

if $|x| > (c_2 + \epsilon)n > (c_2 + \epsilon)T$.

Let $N > 0$ be large enough such that

$$\frac{1+r_1}{1+r_1 a_1 \delta'} \int_{-N}^{N} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} e^{-\lambda y} dy = 1$$

has no positive root if $c < c^* - \frac{\epsilon}{2}$ and δ' is small enough (if it is necessary, we can choose $T > 0$ again such that $\epsilon T > 4N$).

Let $\sigma > 0$ be small enough and $D > 0$ be large enough. Then

$$0 \leq \sigma \omega(n, x) \leq p_n'(x), \quad x \in \mathbb{R}$$

if $n = T + 1$. Namely, $\sigma \omega(n, x)$ satisfies the condition of the initial value for the definition of a lower solution. Regarding $(p_{T+1}'(x), q_{T+1}'(x))$ as the new initial value of (4.3), then it suffices to prove that

$$\underline{p}_{n+1}'(x) \leq \int_{-N}^{N} \frac{e^{-\frac{y^2}{4d_1}}}{\sqrt{4\pi d_1}} \frac{(1+r_1)\underline{p}_n'(x-y)}{1+r_1(\underline{p}_n'(x-y)+a_1\overline{q}_n'(x))} dy \tag{5.3}$$

with $\underline{p}_n'(x) = \sigma \omega(n, x)$ for $n > T$ and $c = c^* - \frac{3\epsilon}{4}$. □

Let $\tau = \sup_{n \in \mathbb{Z}, x \in \mathbb{R}} \sigma \omega(n, x)$ (which is small if $\sigma > 0$ is small enough). Then the result holds if $x \in \mathbb{R}$ such that $\underline{p}_n'(|x| + N) = \tau$ because $1 + r_1 > 1 + r_1(a_1 + \delta')$. Otherwise, $\overline{q}_n'(y) \leq \delta'$ with $|x - y| \leq N$. Then the proof is similar to that in Diekmann (1979, Lemma 4) for a single equation since

$$\frac{1+r_1}{1+r_1 a_1 \delta'} \int_{-N}^{N} \frac{1}{\sqrt{4\pi d_1}} e^{-\frac{(y-c)^2}{4d_1}} e^{-\lambda y} dy = 1$$

has no positive root if $c < c^* - \frac{\epsilon}{2}$ and δ' is small enough. So we omit the remaining details of the proof of (5.3) here. The proof is complete. □

The above result (Theorem 5.3) implies that if one species has stronger spreading ability formulated by (5.1) than the other one, then the competition will not affect its spreading speed. We further formulate the spreading effect of p_n as follows.

Theorem 5.7 *Assume that the conditions of Theorem 5.3 hold. Then for any fixed constant $\epsilon > 0$ with $2\epsilon \in (0, c^* - c_2)$,*

$$\lim_{n\to\infty} \inf_{(c_2+\epsilon)n<|x|<(c^*-\epsilon)n} p_n(x) = \lim_{n\to\infty} \sup_{(c_2+\epsilon)n<|x|<(c^*-\epsilon)n} p_n(x) = 1. \tag{5.4}$$

Proof Let $\{\epsilon_k\}_{k=1}^{\infty}$ be an increasing sequence with

$$\frac{\epsilon}{2} = \epsilon_1 < \epsilon_2 < \cdots < \epsilon, \quad \lim_{k\to\infty} \epsilon_k = \epsilon.$$

Define a sequence $\{p_k^*\}_{k=1}^{\infty}$ as

$$p_k^* = \lim_{n\to\infty} \inf_{(c_2+\epsilon_k)n<|x|<(c_1-\epsilon_k)n} p_n(x).$$

Then Theorem 5.3 implies that there exist $\delta_* > 0$ and $K_* > 0$ such that

$$1 \geq p_k^* \geq \delta_* \quad \text{if } k \geq K_*.$$

By the definition of lim inf, it is easy to see that

$$p_{k+1}^* \geq p_k^*, \quad k = 1, 2, \ldots.$$

Thus, there exists $p^* > 0$ such that

$$\lim_{k\to\infty} p_k^* = p^* > 0.$$

For any given $k > 1$ and $0 < \varepsilon \ll \delta_*$, there exists $N_1 > 0$ such that

$$2(1+r_1)\int_{N_1}^{\infty} \frac{e^{-\frac{x^2}{4d_1}}}{\sqrt{4\pi d_1}} dx < \frac{\varepsilon}{4}.$$

Moreover, we can choose $N_2 > N_1$ such that

$$(\epsilon_{k+1} - \epsilon_k)N_2 > 2N_1.$$

By Definition 5.1, there exists $N_3 > N_2$ such that

$$\inf_{(c_2+\epsilon_k)n<|x|<(c_1-\epsilon_k)n} p_n(x) > p_k^* - \frac{\varepsilon}{4},$$
$$\inf_{(c_2+\epsilon_{k+1})n<|x|<(c_1-\epsilon_{k+1})n} p_n(x) < p_{k+1}^* + \frac{\varepsilon}{4},$$
$$\sup_{|x|>(c_2+\frac{\epsilon}{2})n} q_n(x) < \frac{\varepsilon}{4} \quad \text{for all } n > N_3.$$

By the definition of inf, it is clear that

$$p_{k+1}^* + \frac{\varepsilon}{4} \geq \frac{(1+r_1)\left(p_k^* - \frac{\varepsilon}{4}\right)}{1 + r_1\left(p_k^* - \frac{\varepsilon}{4} + \frac{a_1\varepsilon}{4}\right)} \int_{-N_1}^{N_1} \frac{e^{-\frac{x^2}{4d_1}}}{\sqrt{4\pi d_1}} dx.$$

Let $\varepsilon \to 0$. Then $\delta_* > 0$ implies that

$$p_{k+1}^* \geq \frac{(1+r_1)p_k^*}{1+r_1 p_k^*}.$$

So $p^* \geq 1$ is clear by letting $k \to \infty$. Note that $p_n(x) \leq 1$ for all $n \geq 0$, $x \in \mathbb{R}$, then (5.4) holds. The proof is complete. □

Remark 5.8 Let $p_* = \lim_{n\to\infty} \inf_{(c_2+\epsilon)n<|x|<(c^*-\epsilon)n} p_n(x)$, we cannot directly obtain

$$p_* \geq \frac{(1+r_1)p_*}{1+r_1 p_*}$$

since the definition of $p_{n+1}(x)$ depends on $p_n(y), q_n(y)$, $y \in \mathbb{R}$.

The above theorem implies that the "frontier" of $p_n(x)$ spreads as if there is not competition between different species. However, the interspecific competition does decrease the spreading speed of the other species. We formulate this as follows.

Theorem 5.9 *Let $c_3 = 2\sqrt{d_2 \ln \frac{1+r_2}{1+r_2a_2(1-a_1)}}$. Assume that the conditions of Theorem 5.3 hold and $c_1 > c_2 + c_3$. Then for any $\epsilon > 0$,*

$$\lim_{n\to\infty} \sup_{|x|>(c_3+\epsilon)n} q_n(x) = 0. \tag{5.5}$$

Proof If $a_2 = 0$, then the result is clear. So we only consider the case $a_2 > 0$. Let $\delta \in (0, 1-a_1)$, $2\epsilon \in (0, c_2 - c_3)$ be small constants such that

$$d_2\lambda^2 - \left(c_3 + \frac{\epsilon}{2}\right)\lambda + \ln \frac{1+r_2}{1+r_2a_2(1-a_1-\delta)} = 0$$

has two equivalent zeros. It is clear that $\delta \to 0$ implies $\epsilon \to 0$. In particular, let $\delta > 0$ be small enough such that

$$c_3 + c_2 < c_1 - \epsilon.$$

Define a recursion $r_n(x)$ as follows

$$r_{n+1}(x) = \int_{\mathbb{R}} \frac{e^{-\frac{y^2}{4d_1}}}{\sqrt{4\pi d_1}} \frac{(1+r_1)r_n(x-y)}{1+r_1(r_n(x-y)+a_1)} dy, \quad n = 1, 2, \ldots, \quad r_0(x) = p(x).$$

Then there exists an integer $T' > 0$ such that

$$\inf_{|x|<(c_1-\frac{\epsilon}{2})n} r_n(x) \geq 1 - a_1 - \frac{\delta}{2} \quad \text{for all } n \geq T'.$$

Define constants

$$\beta_2 = \frac{c_2}{2d_2}, \quad \beta_3 = \frac{c_3 + \frac{\epsilon}{2}}{2d_2}$$

such that

$$\Lambda' =: d_2\beta_3^2 - \beta_3\left(c_3 + \frac{\epsilon}{2}\right) + \ln\left(\frac{1+r_2}{1+r_2a_2\left(1-a_1-\frac{\delta}{2}\right)}\right) < 0.$$

In particular, let $T' > 0$ be large enough such that

$$(c_1 - \epsilon - c_2 - c_3)T' > N^*$$

with

$$e^{\Lambda'} + (1+r_2)\left(\int_{-\infty}^{-N^*} + \int_{N^*}^{\infty}\right) e^{-\frac{(y-c)^2}{4d_2}} e^{-\beta_3 y}\, dy < 1. \tag{5.6}$$

Construct continuous functions as follows

$$\begin{aligned}
&\overline{p}_n(x) = 1, \quad \underline{p}_n(x) = r_n(x), \quad \underline{q}_n(x) = 0,\\
&\overline{q}_n(x) = \min\left\{e^{\beta_2(x+c_2n+T_1)}, e^{\beta_2(-x+c_2n+T_1)}, e^{\beta_3\left(x+\left(c_3+\frac{\epsilon}{2}\right)n+T_2\right)},\right.\\
&\qquad\qquad\left. e^{\beta_3\left(-x+\left(c_3+\frac{\epsilon}{2}\right)n+T_2\right)}, 1\right\},
\end{aligned}$$

in which $T_1, T_2 > 0$ and $\beta_2 T_1 = \beta_3 T_2$. For any given $n \geq 0$, $|x| \to \infty$ implies that

$$\overline{q}_n(x) = \min\left\{e^{\beta_2(x+c_2n+T_1)}, e^{\beta_2(-x+c_2n+T_1)}\right\},$$

then there exist T_1, T_2 such that $q_{T'}(x) \leq \overline{q}_{T'}(x)$. We now prove that $(\overline{p}_n, \overline{q}_n)$ and $(\underline{p}_n, \underline{q}_n)$ are a pair of upper and lower solutions if $n \geq T'$.

Note that $\overline{q}_n(x) \leq 1$, then it is clear that $\overline{p}_n(x), \underline{p}_n(x), \underline{q}_n(x)$ satisfy the definitions of upper and lower solutions of (4.1).

For $\overline{q}_n(x)$, if

$$\overline{q}_n(x) = \min\left\{e^{\beta_2(x+c_2n+T_1)}, e^{\beta_2(-x+c_2n+T_1)}\right\},$$

then the result is clear (see the definition of β_2). If

$$\overline{q}_n(x) = \min\left\{e^{\beta_3\left(x+\left(c_3+\frac{\epsilon}{2}\right)n+T_2\right)}, e^{\beta_3\left(-x+\left(c_3+\frac{\epsilon}{2}\right)n+T_2\right)}, 1\right\},$$

then $c_3 + c_2 < c_1 - \epsilon$ implies that $|x| < (c_2 + c_4 + \frac{\epsilon}{2})n < (c_1 - \frac{\epsilon}{2})n$. In particular, $\inf_{|x|<(c_1-\frac{\epsilon}{2})n} r_n(x) \geq 1 - a_1 - \frac{\delta}{2}$ for $n \geq T'$ indicates that

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{e^{-\frac{y^2}{4d_2}}}{\sqrt{4\pi d_2}} \frac{(1+r_2)\overline{q}_n(x-y)}{1+r_2(\overline{q}_n(x-y)+a_2\underline{p}_n(x-y))} dy \\
&\quad \leq \int_{\mathbb{R}} \frac{e^{-\frac{y^2}{4d_2}}}{\sqrt{4\pi d_2}} \frac{(1+r_2)\overline{q}_n(x-y)}{1+r_2(\overline{q}_n(x-y)+a_2(1-a_1-\frac{\delta}{2}))} dy \\
&\quad + (1+r_2)\left(\int_{-\infty}^{-N^*} + \int_{N^*}^{\infty}\right) e^{-\frac{(y-c)^2}{4d_2}} \overline{q}_n(x-y)dy, \quad n \geq T'.
\end{aligned}$$

Then (5.6) implies that

$$\int_{\mathbb{R}} \frac{e^{-\frac{y^2}{4d_2}}}{\sqrt{4\pi d_2}} \frac{(1+r_2)\overline{q}_n(x-y)}{1+r_2(\overline{q}_n(x-y)+a_2\underline{p}_n(x-y))} dy \leq \overline{q}_{n+1}(x).$$

Thus, $(\overline{p}_n(x), \overline{q}_n(x))$ and $(\underline{p}_n(x), \underline{q}_n(x))$ are a pair of upper and lower solutions if $n \geq T'$.

Applying the comparison principle, we obtain the result. The proof is complete. $\square$

For any given $2\epsilon \in (0, c_1 - c_2)$, it is clear that

$$\lim_{n\to\infty} \inf_{|x|<(c_1-\epsilon)n} p_n(x) \geq 1 - a_1. \tag{5.7}$$

Then (5.7) implies that

$$\lim_{n\to\infty} \sup_{|x|<(c_2+\epsilon)n} q_n(x) \leq 1 - a_2(1 - a_1). \tag{5.8}$$

(5.8) further indicates that

$$\lim_{n\to\infty} \inf_{|x|<(c_1-\epsilon)n} p_n(x) \geq 1 - a_1(1 - a_2(1 - a_1)).$$

Repeating the processes, we have

$$\lim_{n\to\infty} \inf_{|x|<(c_1-\epsilon)n} p_n(x) \geq k_1, \quad \lim_{n\to\infty} \sup_{|x|<(c_2+\epsilon)n} q_n(x) \leq k_2. \tag{5.9}$$

Combining (5.9) with the proof of Theorem 5.9, we can prove the following result.

Theorem 5.10 *Let* $c_4 = 2\sqrt{d_2 \ln(\frac{1+r_2}{1+r_2 a_2 k_1})}$. *Assume that the conditions of Theorem 5.9 hold. Then for any* $\epsilon > 0$,

$$\lim_{n\to\infty} \sup_{|x|>(c_4+\epsilon)n} q_n(x) = 0. \tag{5.10}$$

Remark 5.11 In Theorem 5.10, we only estimated the spreading speed of q, precise results need further investigation.

Moreover, Theorems 5.7 and 5.10 imply the following result.

Theorem 5.12 *Theorem 5.7 remains true if* c_2 *is replaced by* c_4.

We now formulate the eventual spreading effects as follows.

Theorem 5.13 *Define a constant by*

$$c_5 = \min\left\{2\sqrt{d_1 \ln\left(\frac{1+r_1}{1+r_1 a_1}\right)}, 2\sqrt{d_2 \ln\left(\frac{1+r_2}{1+r_2 a_2}\right)}\right\}.$$

Then $\lim_{n\to\infty,|x|<(c_5-\epsilon)n} p_n(x) = k_1$, $\lim_{n\to\infty,|x|<(c_5-\epsilon)n} q_n(x) = k_2$ *for any* $\epsilon \in (0, c_5)$.

Proof Similar to that of Lemma 5.5, we can prove that there exist constants

$$\phi_*, \psi_*, \phi^*, \psi^* \in (0, 1]$$

such that

$$\lim_{n\to\infty} \inf_{|x|<(c_5-\epsilon)n} p_n(x) = \phi_*, \quad \lim_{n\to\infty} \inf_{|x|<(c_5-\epsilon)n} q_n(x) = \psi_*,$$
$$\lim_{n\to\infty} \sup_{|x|<(c_5-\epsilon)n} p_n(x) = \phi^*, \quad \lim_{n\to\infty} \sup_{|x|<(c_5-\epsilon)n} q_n(x) = \psi^*.$$

Combining the standard definitions of lim inf and lim sup with the recipe in the proof of Theorem 5.7, we can prove that

$$\phi_* + a_1\psi^* \geq 1, \quad \psi_* + a_2\phi^* \geq 1,$$
$$\phi^* + a_1\psi_* \leq 1, \quad \psi^* + a_2\phi_* \leq 1.$$

Note that $(\phi_*, \psi_*) \leq (\phi^*, \psi^*)$. Then

$$(\phi_*, \psi_*) = (\phi^*, \psi^*) = (k_1, k_2).$$

The proof is complete. □

Similar to the proof of Theorem 5.13, we may obtain the following conclusion.

Theorem 5.14 *Assume that $c_6 \geq c_5$ such that*

$$\lim_{n\to\infty} \inf_{|x|<(c_6-\epsilon)n} u_n(x)v_n(x) > 0$$

for any $\epsilon \in (0, c_6)$. *Then* $\lim_{n\to\infty,|x|<(c_6-\epsilon)n}(p_n(x), q_n(x)) = (k_1, k_2)$ *for any* $\epsilon \in (0, c_6)$.

6 Numerical simulations

In this section, we give some numerical simulations to illustrate our main results in Sect. 5. We consider the recursion system as follows:

$$\begin{cases} p_{n+1}(x) = \frac{25}{\sqrt{5\pi}} \int_{\mathbb{R}} e^{-125y^2} \frac{2p_n(x-y)}{1+p_n(x-y)+0.5q_n(x-y)} dy, & n = 0, 1, 2, \ldots, \\ q_{n+1}(x) = \frac{100}{\sqrt{5\pi}} \int_{\mathbb{R}} e^{-2000y^2} \frac{2q_n(x-y)}{1+q_n(x-y)+0.5p_n(x-y)} dy, & n = 0, 1, 2, \ldots. \end{cases} \tag{6.1}$$

It is easy to see that $(\frac{2}{3}, \frac{2}{3})$ is the coexistence equilibrium of (6.1). We begin with the delta function. For $n = 0$, we have the distribution of (p_1, q_1) given in Fig. 2.

Figures 3 and 4 indicate that both p and q begin spreading into new habitats.

Figures 5, 6, 7 and 8 are obtained by replacing 0.5 by 0 in (6.1). Figures 3, 4, 5, 6, 7 and 8 demonstrate that competition has little effect on the spreading speed at the beginning, but the population densities are decreasing at the coexistence habitat (Fig. 9).

Figures 10, 11 and 12 imply that the interspecific competition significantly decreases the spreading speed of q, while the "frontier" of p seems to out-compete that of q, see Theorems 5.3, 5.7 and 5.12 (Fig. 13).

To draw Fig. 14, we replace 0.5 by 0 in (6.1). These figures indicate that the interspecific competition decreases the spreading speed of q, see Theorems 5.9 and 5.10.

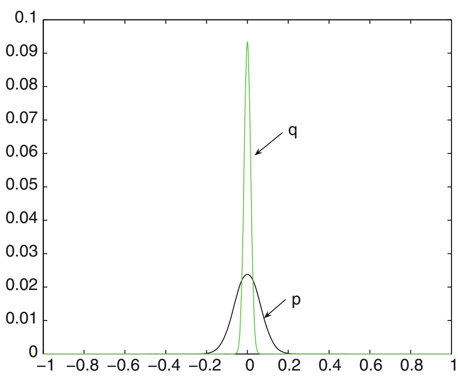


Fig. 2 The distribution of $(p_1(x), q_1(x))$ defined by (6.1)

Fig. 3 The distribution of $(p_5(x), q_5(x))$

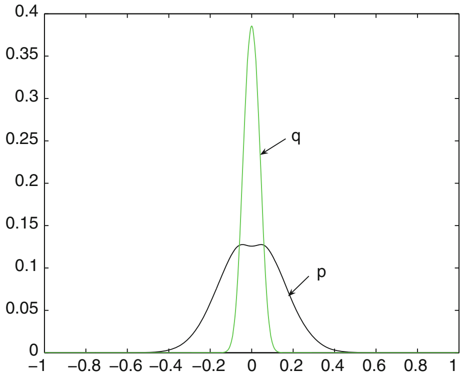


Fig. 4 The distribution of $(p_{10}(x), q_{10}(x))$

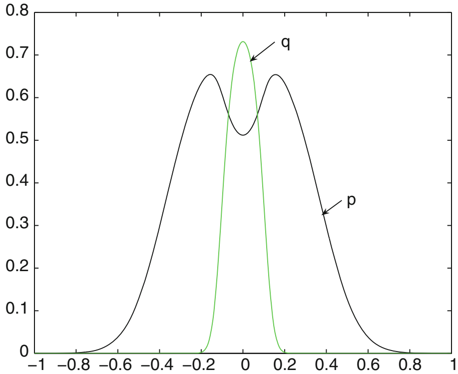
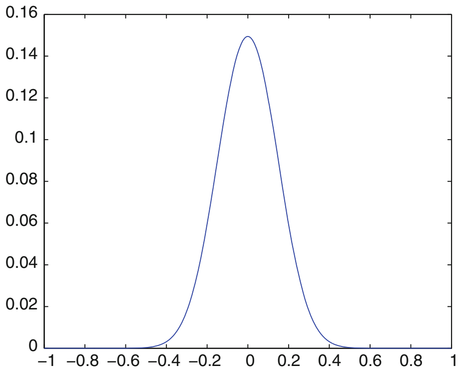


Fig. 5 The distribution of $p_5(x)$



Figures 15 and 16 show that the population densities decrease significantly and tend to the coexistence equilibrium ($(\frac{2}{3}, \frac{2}{3})$ is the positive equilibrium of (6.1)), see Theorem 5.13. It should be noted that the boundaries of the figures are not accurate since we must cut off the boundaries when we do numerical simulations.

Fig. 6 The distribution of $q_5(x)$

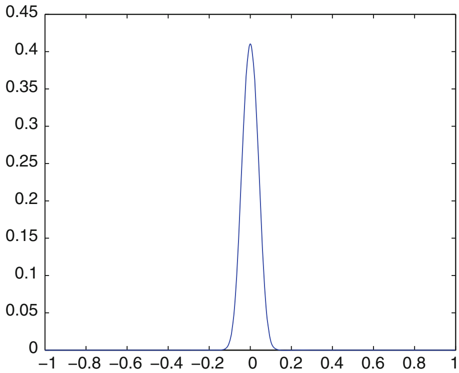


Fig. 7 The distribution of $p_{10}(x)$

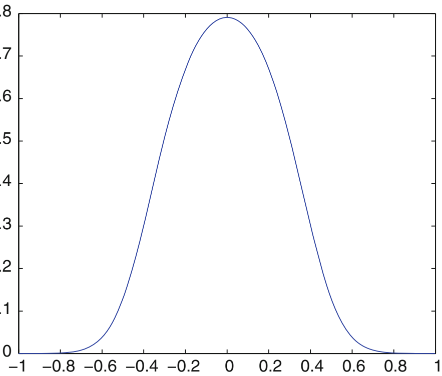
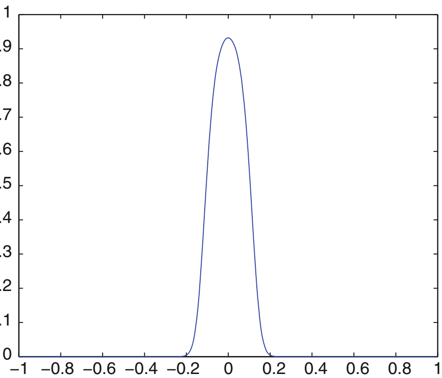


Fig. 8 The distribution of $q_{10}(x)$



7 Discussion

In this paper, we have established the existence of traveling wave solutions and the spreading speeds of the recursion system (1.4), in which we regarded both of the two competitive species as invaders. By constructing upper and lower solutions, the existence of traveling wave solutions was proved if the wave speed is larger than c^*. Such an existence result implies that there is a transition zone moving from the trivial steady state with no species to the coexistence steady state with both species,

Fig. 9 The distribution of $(p_{15}(x), q_{15}(x))$

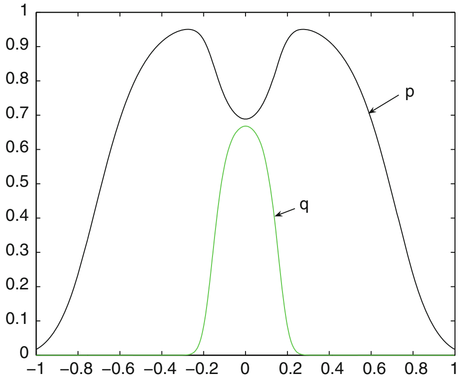


Fig. 10 The distribution of $(p_{20}(x), q_{20}(x))$

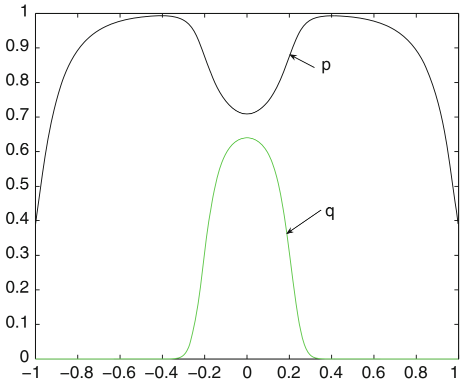
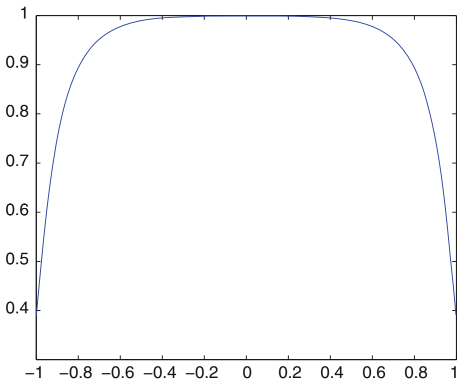


Fig. 11 The distributions of $p_{20}(x)$



which means that the invasion of both species can be successful even if they compete each other. Furthermore, we showed that c^* is the spreading speed of one species under proper conditions, which indicates that the interspecific competition cannot decrease the spreading speed of some species. At the same time, we considered the spreading speed of the other species, which is smaller than the case when the interspecific

Fig. 12 The distributions of $q_{20}(x)$

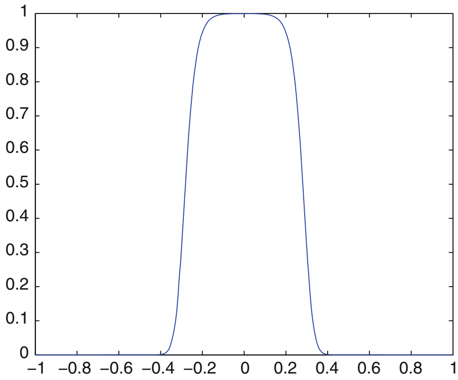


Fig. 13 The distribution of $(p_{40}(x), q_{40}(x))$

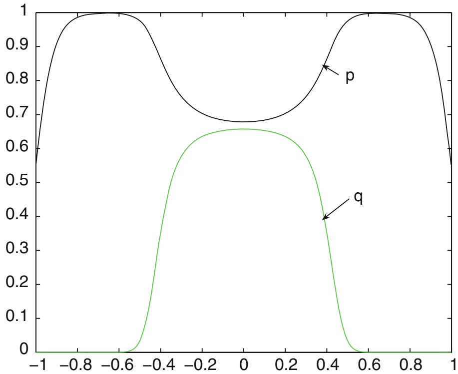
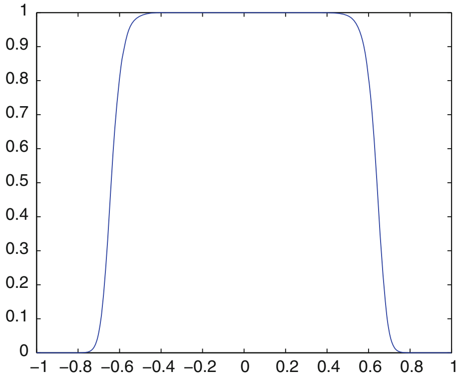


Fig. 14 The distribution of $q_{40}(x)$



competition vanishes. Moreover, it was proved that the invasion of both species is eventually successful and the interspecific competition does affect the propagation effect such that the population densities in the coexistence habitat are distinctly lower than the case when the competition vanishes. Therefore, the interspecific competition may play a negative role in the evolution process of multi-species competition communities. Recalling the condition that ensures the coexistence of both competitive species

Fig. 15 The distribution of $(p_{60}(x), q_{60}(x))$

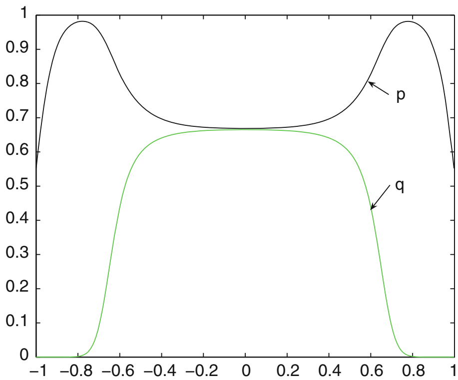
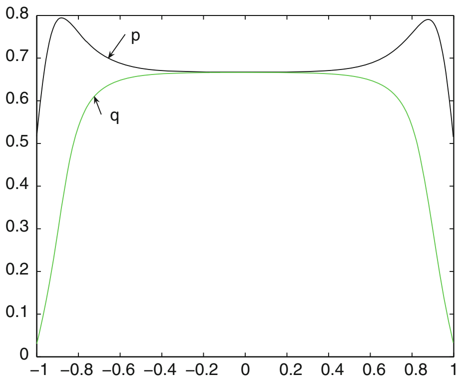


Fig. 16 The distribution of $(p_{80}(x), q_{80}(x))$



(see Cushing et al. (2004) for some conclusions established for the corresponding difference equations), we see that the interspecific competition in this paper is weak since $a_1, a_2 \in (0, 1)$ (if $a_1 \in (0, 1)$ and $a_2 \in (0, 1)$ do not hold, see Lewis et al. (2002)). Note that $k_1 + k_2 > 1$ holds and the invasion of both species is successful, our results also demonstrate that the weak interspecific competition may be useful in obtaining the maximal profit per cost. At least such a conclusion in the literature is analogous to the custom of cultivating freshwater fish in China (see Li 1992).

It should be noted that we started to describe the propagation mode that two species were introduced simultaneously into a new environment and they compete each other (one also refers to Li 2009). Hence the problem is concerned with the competition-coexistence (see Darlington 1972) and competition-invasion so that it is different from that in Lewis et al. (2002), in which the population exclusion process (see Hardin 1960) was formulated by the spreading speeds and the traveling wave solutions. The background of the problem also implies that the previous results established for the cooperative/monotone systems cannot be applied here, even after the linear transformation of unknown functions. It is easy to understand that the method used in this paper can be generalized to more general systems, for example, the probability functions can be more general and different from the Gaussian in model (1.2). We shall consider several other models in our forthcoming papers to display more properties of the spreading speeds in competition systems.

We proved that the interspecific competition cannot change the spreading speed of $p_n(x)$ while it may decrease the spreading speed of $q_n(x)$ under the condition (5.1), and we conjecture that the spreading speed of $q_n(x)$ is smaller than $2\sqrt{d_2 \ln \frac{1+r_2}{1+r_2 a_2 k_1}}$. Another important problem is the monotonicity of traveling wave solutions of system (1.2). Motivated by the spreading speeds of p_n and q_n formulated by system (1.2), we also conjecture that (1.2) has non-monotone traveling wave solutions and shall consider these two problems in our future work.

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References

Allen EJ, Allen LJS, Gilliam X (1996) Dispersal and competition models for plants. J Math Biol 34:455–481
Aronson DG (1977) The asymptotic speed of propagation of a simple epidemic. In: Fitzgibbon WE, Walker HF (eds) Nonlinear diffusion. Pitman, London, pp 1–23
Aronson DG, Weinberger HF (1975) Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation. In: Goldstein JA (ed) Partial differential equations and related topics. Lecture notes in mathematics, vol 446. Springer, Berlin, pp 5–49
Aronson DG, Weinberger HF (1978) Multidimensional nonlinear diffusion arising in population dynamics. Adv Math 30:33–76
Bengtsson J (1989) Interspecific competition increases local extinction rate in a metapopulation system. Nature 340:713–715
Bleasdale JKA (1956) Interspecific competition in higher plants. Nature 178:150–151
Britton N (1986) Reaction-diffusion equations and their applications to biology. Academic Press, San Diego
Carrillo C, Fife P (2005) Spatial effects in discrete generation population models. J Math Biol 50:161–188
Cheng CP, Li W-T, Wang ZC (2008) Spreading speeds and traveling waves in a delayed population model with stage structure on a two-dimensional spatial lattice. IMA J Appl Math 73:592–618
Cushing JM, Levarge S, Chitnis N, Henson SM (2004) Some discrete competition models and the competitive exclusion principle. J Differ Equ Appl 10:1139–1151
Darlington PJ (1972) Competition, competitive repulsion, and coexistence. Proc Natl Acad Sci USA 69:3151–3155
Diekmann O (1978) Thresholds and traveling waves for the geographical spread of infection. J Math Biol 6:109–130
Diekmann O (1979) Run for your life. A note on the asymptotic speed of propagation of an epidemic. J Differ Equ 33:58–73
Fife P (1979) Mathematical aspects of reacting and diffusing systems. Lecture notes in biomathematics, vol 28. Springer, Berlin
Hardin G (1960) The competitive exclusion principle. Science 131:1292–1297
Hsu SB, Zhao XQ (2008) Spreading speeds and traveling waves for nonmonotone integrodifference equations. SIAM J Math Anal 40:776–789
Kot M (1992) Discrete-time travelling waves: ecological examples. J Math Biol 30:413–436
Leung AW (1989) Systems of nonlinear partial differential equations: applications to biology and engineering. Kluwer Academic Publishers, Dordrecht
Lewis MA (2000) Spread rate for a nonlinear stochastic invasion. J Math Biol 41:430–454

Lewis MA, Li B, Weinberger HF (2002) Spreading speed and linear determinacy for two-species competiotion models. J Math Biol 45:219–233
Li S (1992) Population ecology of freshwater fishes. Agricultural Press of China, Beijing
Li B (2009) Some remarks on traveling wave solutions in competition models. Discrete Contin Dyn Syst Ser B 12:389–399
Li B, Weinberger HF, Lewis MA (2005) Spreading speeds as slowest wave speeds for cooperative systems. Math Biosci 196:82–98
Li W-T, Lin G, Ruan S (2006) Existence of traveling wave solutions in delayed reaction diffusion systems with applications to diffusion-competition systems. Nonlinearity 19:1253–1273
Liang X, Zhao XQ (2007) Asymptotic speeds of spread and traveling waves for monotone semiflows with applications. Comm Pure Appl Math 60:1–40
Lin G, Li W-T, Ma M (2010) Travelling wave solutions in delayed reaction diffusion systems with applications to multi-species models. Discrete Contin Dyn Syst Ser B 19:393–414
Lin G, Li W-T, Ruan S Asymptotic stability of monostable wavefronts in discrete-time integral recursions. Sci China Ser A (2010). doi:10.1007/s11425-009-0123-6
Lui R (1989a) Biological growth and spread modeled by systems of recursions. I. Mathematical theory. Math Biosci 93:269–295
Lui R (1989b) Biological growth and spread modeled by systems of recursions. II. Biological theory. Math Biosci 93:297–312
Ma S (2007) Traveling waves for non-local delayed diffusion equations via auxiliary equations. J Differ Equ 237:259–277
Neubert MG, Caswell H (2000) Demography and dispersal: calculation and sensitivity analysis of invasion speed for structured populations. Ecology 81:1613–1628
Pan S (2009) Traveling wave solutions in delayed diffusion systems via a cross iteration scheme. Nonlinear Anal RWA 10:2807–2818
Pao CV (1992) Nonlinear parabolic and elliptic equations. Plenum, New York
Pao CV (2005) Strongly coupled elliptic systems and applications to Lotka–Volterra models with cross-diffusion. Nonlinear Anal TMA 60:1197–1217
Radcliffe J, Rass L (1983) Wave solutions for the deterministic nonreducible n-type epidemic. J Math Biol 17:45–66
Radcliffe J, Rass L (1984) The uniqueness of wave solutions for the deterministic nonreducible n-type epidemic. J Math Biol 19:303–308
Radcliffe J, Rass L (1986) The asymptotic spread of propagation of the deterministic non-reducible n-type epidemic. J Math Biol 23:341–359
Smoller J (1994) Shock waves and reaction diffusion equations. Springer, New York
Thieme HR (1979a) Asymptotic estimates of the solutions of nonlinear integral equations and asymptotic speeds for the spread of populations. J Reine Angew Math 306:94–121
Thieme HR (1979b) Density-dependent regulation of spatially distributed populations and their asymptotic speed of spread. J Math Biol 8:173–187
Thieme HR, Zhao XQ (2003) Asymptotic speeds of spread and traveling waves for integral equations and delayed reaction diffusion models. J Differ Equ 195:430–470
van den Bosch F, Metz JAJ, Diekmann O (1990) The velocity of spatial population expansion. J Math Biol 28:529–565
Volpert AI, Volpert VA, Volpert VA (1994) Traveling wave solutions of parabolic systems. Translations of mathematical monographs, vol 140. American Mathematical Society, Providence
Weinberger HF (1982) Long-time behavior of a class of biological model. SIAM J Math Anal 13:353–396
Weinberger HF (2002) On spreading speeds and traveling waves for growth and migration models in a periodic habitat. J Math Biol 45:511–548
Weinberger HF, Lewis MA, Li B (2002) Analysis of linear determinacy for spread in cooperative models. J Math Biol 45:183–218
Weinberger HF, Lewis MA, Li B (2007) Anomalous spreading speeds of cooperative recursion systems. J Math Biol 55:207–222
Weng P, Huang H, Wu J (2003) Asymptotic speed of propagation of wave fronts in a lattice delay differential equation with global interaction. IMA J Appl Math 68:409–439
Ye Q, Li Z (1990) Introduction to reaction diffusion equations. Science Press, Beijing