
Volume 251 Issue 3 August 1, 2011 ISSN 0022-0396
ELSEVIER
JOURNAL OF
DIFFERENTIAL
EQUATIONS
Editors-in-Chief
Alberto Bressan
Shui-Nee Chow
John Mallet-Paret
Konstantin Mischaikow
Founding Editors
Jack K. Hale
J. P. LaSalle
Editorial Board
Sigurd Angenent
John Ball
Courtney Coleman
Avner Friedman
Kening Lu
W.-M. Ni
Czesław Olech
James Serrin
Marshall Slemrod
Walter Strauss
Roger Temam
Sijue Wu
Yingfei Yi
Chongchun Zeng
Enrique Zuazua

J. Differential Equations 251 (2011) 728–757

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Attractors for non-autonomous parabolic problems with singular initial data

Xiaojun Li [a,*,1], Shigui Ruan [b]

[a] School of Science, Hohai University, Nanjing, Jiangsu 210098, China
[b] Department of Mathematics, University of Miami, Coral Gables, FL 33124-4250, USA

ARTICLE INFO

Article history:
Received 7 May 2010
Revised 3 May 2011
Available online 17 May 2011

Keywords:
Parabolic problems
Singular initial data
Existence
Non-autonomous equation
Uniform attractors

ABSTRACT

In this paper, we study the asymptotic behavior of solutions of non-autonomous parabolic problems with singular initial data. We first establish the well-posedness of the equation when the initial data belongs to $L^r(\Omega)$ $(1<r<\infty)$ and $W^{1,r}(\Omega)$ $(1<r<N)$, respectively. When the initial data belongs to $L^r(\Omega)$, we establish the existence of uniform attractors in $L^r(\Omega)$ for the family of processes with external forces being translation bounded but not translation compact in $L^p_{loc}(\mathbb{R}; L^r(\Omega))$. When we consider the existence of uniform attractors in $H^1_0(\Omega)$, the solution of equation lacks the higher regularity, so we introduce a new type of solution and prove the existence result. For the long time behavior of solutions of the equation in $W^{1,r}(\Omega)$, we only obtain the uniform attracting property in the weak topology.

1. Introduction

Let $\Omega\subset\mathbb{R}^N$ be a bounded smooth domain. Consider the following non-autonomous nonlinear reaction–diffusion equation

$$\begin{cases} u_t-\Delta u+f(x,u)=g(x,t) & \text{in } \Omega,\ t>\tau,\\ u=0 & \text{on } \partial\Omega,\\ u(\tau)=u_\tau, & \tau\in\mathbb{R}, \end{cases} \tag{1.1}$$

* Corresponding author.
E-mail addresses: lixjun05@mailsvr.hhu.edu.cn (X. Li), ruan@math.miami.edu (S. Ruan).

[1] The author was supported by the NNSF of China Grant 10871059, the Fundamental Research Funds for the Central Universities and NSF of Hohai University.

0022-0396/$ – see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.jde.2011.05.015

where $f(x,u) \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ satisfies

$$f(x,0) = 0, \tag{1.2}$$

$$\left|f(x,u) - f(x,v)\right| \leqslant C\left|a(x)\right||u-v|\left(|u|^{\rho-1} + |v|^{\rho-1} + 1\right) \tag{1.3}$$

with $\rho > 1$ and $a(x) \in L^\beta(\Omega)$, $\beta > 1$. Suppose that the external force $g(t) = g(\cdot,t)$ is translation bounded in $L^p_{loc}(\mathbb{R}; X)$, $p > 1$, i.e., $g \in L^p_b(\mathbb{R}; X)$,

$$\|g\|^p_{L^p_b(\mathbb{R};X)} = \sup_{t\in\mathbb{R}} \int_t^{t+1} \left\|g(s)\right\|^p_X ds < +\infty,$$

where the local p-power integral is the Bochner integral and X is a Banach space. We are interested in $X = L^r(\Omega)$ and $W^{1,r}(\Omega)$, respectively.

By a solution $u \in C([\tau,T]; X^1) \cap C((\tau,T]; X^{1+\epsilon}) \cap L^\infty_{loc}((\tau,T]; X^1)$ of (1.1), we mean that

$$\begin{cases} u(t) = e^{\Delta(t-\tau')}u(\tau') + \displaystyle\int_{\tau'}^t e^{\Delta(t-s)}\left[-f\left(x,u(s)\right) + g(x,s)\right] ds & \text{for } \tau < \tau' \leqslant t \leqslant T, \\ u(\tau') \to u(\tau) & \text{in } X^1 \text{ as } \tau' \to \tau, \end{cases} \tag{1.4}$$

where $\epsilon > 0$, X^α is the fractional power space associated to the operator Δ. This type of solution is also called an ϵ-regular mild solution in [6].

Elliptic and parabolic problems with the nonlinearity analogous with the one of (1.1) have drawn much attention. After the work [5], authors in [20] studied the existence, nonexistence and multiplicity of solutions for the problem

$$\begin{cases} -\Delta u = \lambda a(x)u^q + b(x)u^p & \text{in } \Omega, \\ u \geqslant 0, \quad u \not\equiv 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where Ω is a bounded domain in $\mathbb{R}^N$, λ is a parameter and the exponents p and q satisfy $0 \leqslant q < 1 < p$ with $p \leqslant 2^* - 1$ if $N \geqslant 3$, $p < \infty$ if $n = 1$ or 2. Here $2^* := 2N/(N-2)$. Let $\sigma_q = (\frac{2^*}{q+1})'$, $\sigma_p = (\frac{2^*}{p+1})'$. With some assumptions, they proved that if $a(x) \in L^{\tau_q}(\Omega)$ with $\tau_q > \sigma_q$ and $b(x) \in L^{\tau_p}(\Omega)$ with $\tau_p > \sigma_p$, then (1.5) has at least two solutions v and w; and if $a(x) \in L^{\sigma_q}(\Omega)$ and $b(x) \in L^{\sigma_p}(\Omega)$, then (1.5) has no solution. For the parabolic problem, authors in [1] investigated the dynamics of the semiflow φ induced on $H^1_0(\Omega)$ by the following Cauchy problem

$$\begin{cases} u_t - \Delta u = f(x,u) & \text{in } \Omega,\ t > 0, \\ u = 0 & \text{on } \partial\Omega,\ t > 0, \\ u(0) = u_0, & x \in \Omega, \end{cases} \tag{1.6}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. A model nonlinearity of (1.6) is

$$f(x,u) = a_0(x)u + \sum_{j=1}^k a_j(x)|u|^{p_j-1}u,$$

where $a_j(x)$ in $L^\infty(\Omega)$ for $j = 0, \dots, k$, $2 < p_1 < p_2 < \cdots < p_k < 2^*$, $2^* = \infty$ if $N = 1, 2$. When $f(x,u) = \lambda u + a(x)u^p$, $p > 1$, $u_0 \in L^\infty(\Omega)$ and $u_0 \geqslant 0$, the estimates of positive solutions of (1.6) with $a(x) \in C(\overline{\Omega})$ and the blow-up of solutions of (1.6) with $a(x) \in C^2(\overline{\Omega})$ have been studied respectively in [36]. Recently, if $f(x,u) = a(x)u^q + b(x)u^p$, $0 < q \leqslant 1 < p$, $a(x) \in L^\alpha(\Omega)$, $b(x) \in L^\beta(\Omega)$, $\alpha, \beta \geqslant 1$, it has been proved in [28] that there exists a unique positive solution

$$u \in C\big([0,T]; L^r(\Omega)\big) \cap L^\infty_{loc}\big((0,T); L^\infty(\Omega)\big) \tag{1.7}$$

of (1.6) with $u_0 \in L^r(\Omega)$, $1 \leqslant r < \infty$, and $u_0 \geqslant \gamma d_\Omega$, where γ is a positive constant, $d_\Omega = \operatorname{dist}(x, \partial\Omega)$. Authors in [38] analyze the dynamics of the following non-autonomous nonlinear parabolic model problem

$$\begin{cases} u_t - \Delta u = f(t,x,u) & \text{in } \Omega,\ t > s, \\ u = 0 & \text{on } \partial\Omega, \\ u(s) = u_s, \end{cases} \tag{1.8}$$

where Ω is a bounded domain in $\mathbb{R}^N$ and $f(t,x,u): \mathbb{R} \times \Omega \times \mathbb{R} \to \mathbb{R}$ is a suitable smooth function satisfying

$$f(t,x,u)u \leqslant C(t,x)|u|^2 + D(t,x)|u| \quad \text{for all } u \in \mathbb{R} \tag{1.9}$$

for some $C(t,x) \in C^\alpha(\mathbb{R}; L^p(\Omega))$ with $0 < \alpha \leqslant 1$ and $p > N/2$, and some function D with values in $L^r(\Omega)$, $1 \leqslant r \leqslant \infty$. Under some other assumptions, they prove that the solutions of (1.8) are global, and there exist two extremal complete trajectories φ_m and φ_M, one minimal and one maximal, that bound all complete trajectories corresponding to (1.8). Moreover, there exists a pullback attractor for the process corresponding to (1.8), which is bounded by φ_m and φ_M. See details in [38]. For other studies, see [4,37,42]. It is natural to consider problem (1.1) with general nonlinearity satisfying (1.2)–(1.3).

To investigate the behavior of solutions of (1.1) when time tends to infinity, the first task is to study the well-posedness of the problem. The autonomous case, study has been considered extensively. When the initial data $u(0) \in L^r(\Omega)$ and $-f(x,u) = |u|^{\rho-1}u$, after the work of [34,46,47], authors in [13] obtained the local existence and uniqueness of the solution of (1.6) in the sense of (1.7). In [6], authors studied the abstract parabolic problem

$$\begin{cases} u_t = Au + f(t,u), \quad t > t_0, \\ u(t_0) = u_0, \end{cases} \tag{1.10}$$

where the linear operator $A: D(A) \subset X^0 \to X^0$ satisfies that $-A$ is a sectorial operator in the Banach space X^0, $f(t,u)$ satisfies some local Lipschitz condition. They obtained that there exists a unique solution (ϵ-regular mild solution) of (1.10), and applied their abstract results to the heat equation and Navier–Stokes equation. For other related studies, we refer readers to [7,8,23,24,37,42].

To obtain the existence of uniform attractors in $L^r(\Omega)$ for the family of processes corresponding to (1.1), higher regularity of solutions of (1.1) than the results in [13,28] is needed. Since the nonlinearity $f(x,u)$ and external force $g(x,t)$ of (1.1) depend on x, and $g(x,t)$ belongs to $L^p_{loc}(\mathbb{R}; X)$, which is equipped with the local p-power mean convergence topology different from the topology of X^α associated to linear operator Δ, $-f(x,u) + g(x,t)$ is not an ϵ-regular map, and the abstract results for (1.10) in [6] cannot be applied directly to system (1.1). To overcome this, we decompose system (1.1) into a linear system and an autonomous nonlinear system. Using some estimates for the solution of the linear systems, we show that there exists a unique ϵ-regular solution for the nonlinear system, and get the local existence and regularity of solutions of (1.1) when the initial data belongs to $L^r(\Omega)$ and $W^{1,r}(\Omega)$, respectively.

There are papers both on the existence of attractors for autonomous evolution equations, e.g. [9–12,22,25,32,39–41,43,44,48], and on the existence of uniform attractors for non-autonomous evolution equations, e.g. [16–19,29–31,33,45]. We first establish the existence of uniform attractors in $L^r(\Omega)$ for (1.1) as $g(x,t)$ is translation bounded but not translation compact in $L^p_{loc}(\mathbb{R}; L^r(\Omega))$. It is shown that there are different dissipative conditions for $r = 2$ and the general case $1 < r < \infty$. Using the regularity of solutions of (1.1) obtained in Theorem 3.1, we establish the existence results in $L^r(\Omega)$ without any further assumption on $g(x,t)$. When we consider the existence of uniform attractors for (1.1) in $W^{1,r}(\Omega)$, the situation becomes more complicated. We know from Remark 3.1 that even if $g(x,t) \in L^p_b(\mathbb{R}, W^{1,r}(\Omega))$, solutions of (1.1) cannot enter in $W^{2,r}(\Omega)$. Therefore, this brings some difficulties in taking priori estimates of the solutions in obtaining compact uniformly absorbing set. For $r = 2$, we can overcome this. By the standard Fatou–Galerkin method, we show that there exists a unique weak solution $u \in C([\tau,T]; H^1_0(\Omega)) \cap L^2_{loc}((\tau,T); H^2(\Omega)) \cap L^\infty((\tau,T); H^1_0(\Omega))$, which also belongs to $C([\tau,T]; H^1_0(\Omega)) \cap C((\tau,T]; W^{1+2\epsilon,2}(\Omega))$ and satisfies (1.4). For the general case $1 < r < N$, we only get that there exists a bounded uniformly absorbing set in $W^{1,r}(\Omega)$ for the family of processes corresponding to (1.1), and the uniform attracting property in the weak topology for any bounded set $B \subset W^{1,r}(\Omega)$.

This paper is organized as follows: in next section, we give some definitions and recall some results which will be used in the following sections; in Section 3, we consider the well-posedness of (1.1) in $L^r(\Omega)$ and $W^{1,r}(\Omega)$, respectively; in Section 4, we prove the existence of uniform attractors in $L^r(\Omega)$; in Section 5, we first prove the existence of uniform attractors in $H^1_0(\Omega)$, and then show the uniform attracting property in the weak topology of the family of processes defined in $W^{1,r}(\Omega)$; in Section 6, we give the relationship between pullback, forward and uniform attractors corresponding to (1.1).

2. Preliminaries

Let Ω be a bounded smooth domain. Denote by $H^s_q(\Omega)$ the Bessel potential spaces and $H^{-s}_q(\Omega) := (H^s_{q'}(\Omega))'$, $1 < q < \infty$, $s \geqslant 0$, $\frac{1}{q} + \frac{1}{q'} = 1$. Notice that $H^s_q(\Omega) = W^{s,q}(\Omega)$, the standard Sobolev–Slobodeckii spaces, whenever $q = 2$ and $s \in \mathbb{R}$, or $q > 1$ and s is an integer. See details in [2,3]. We summarize some well-known embeddings as follows:

$$\begin{cases} H^{s_1}_{q_1}(\Omega) \hookrightarrow H^{s_2}_{q_2}(\Omega), & \text{if } s_1 - \dfrac{N}{q_1} \geqslant s_2 - \dfrac{N}{q_2},\ 1 > \dfrac{1}{q_1} \geqslant \dfrac{1}{q_2} > 0, \\ H^s_q(\Omega) \hookrightarrow C^\eta(\overline{\Omega}), & \text{if } s - \dfrac{N}{q} > \eta > 0. \end{cases} \tag{2.1}$$

Let $A : D(A) \subset X^0 \to X^0$ be a linear operator which satisfies that $-A$ is a sectorial operator in the Banach space X^0. Denote by X^α, $\alpha \geqslant 0$, the fractional power space associated to the operator A and by e^{At} the analytic semigroup generated by A. Without loss of generality we can assume that e^{At} is uniformly bounded, that is,

$$t^{\beta-\alpha} \left\| e^{At} x \right\|_{X^\beta} \leqslant M \|x\|_{X^\alpha}, \quad t > 0,\ 0 \leqslant \alpha \leqslant \beta. \tag{2.2}$$

See details in [26,35,41].

We recall the following compactness theorem (see [16, Theorem II.1.4], [27, Theorem I.5.1]).

Theorem 2.1. *Let E_0, E_1, E be three Banach spaces satisfying $E_1 \Subset E \subset E_0$. Assume that $p_1 \geqslant 1$ and $p_0 > 1$. Consider the space*

$$W_{p_1,p_0}(0,t; E_1, E_0) = \left\{ \psi(t), t \in [0,t] \,\middle|\, \psi(t) \in L^{p_1}(0,T; E_1),\ \psi'(t) \in L^{p_0}(0,T; E_0) \right\}$$

with the norm

$$\|\psi(t)\|_{W_{p_1,p_0}} = \left(\int_0^T \|\psi(s)\|_{E_1}^{p_1}\,ds\right)^{\frac{1}{p_1}} + \left(\int_0^T \|\psi'(s)\|_{E_0}^{p_0}\,ds\right)^{\frac{1}{p_0}}.$$

Then the following embedding is compact:

$$W_{p_1,p_0}(0,t;E_1,E_0) \Subset L^{p_1}(0,T;E).$$

Let $y(t) \in C^1([t_0,t_1])$, $y \geqslant 0$, and the following inequality

$$y'(t) + cy(t) \leqslant h(t) \tag{2.3}$$

holds with $c \geqslant 0$. We need the following result which comes from [16].

Lemma 2.2. *Let $y(t)$ be uniformly continuous on $[t_0,\infty)$, $y \geqslant 0$, and satisfies* (2.3)*, where $c > 0$ and $h(t) \geqslant 0$ for all $t \geqslant t_0$. Suppose that*

$$\int_t^{t+1} h(s)\,ds \leqslant C_1, \quad \forall t \geqslant t_0.$$

Then

$$y(t) \leqslant y(t_0)e^{-c(t-t_0)} + C_1\left(1-e^{-c}\right)^{-1} \leqslant y(t_0)e^{-c(t-t_0)} + C_1\left(1+c^{-1}\right).$$

Consider a non-autonomous evolution equation of the type

$$\partial_t u = A_{\sigma(t)}(u), \quad t \in \mathbb{R}. \tag{2.4}$$

In system (1.1), $A_{\sigma(t)}(u) = \Delta u - f(x,u) + g(x,t)$, $\sigma(t) = g(x,t)$. For every $s \in \mathbb{R}$ we are given an operator $A_{\sigma(s)}(\cdot): E_1 \to E_0$, where E_1, E_0 are Banach spaces. The functional parameter $\sigma(s)$, $s \in \mathbb{R}$, in (2.4) reflects the dependence on time of the equation, and is called the *time symbol* (or the *symbol*) of Eq. (2.4). The values of the function $\sigma(s)$ belong to some metric or Banach space Ξ, i.e., $\sigma(s) \in \Xi$ for every (or almost every) $s \in \mathbb{R}$.

We supplement Eq. (2.4) with an initial data at $t = \tau$, $\tau \in \mathbb{R}$:

$$u|_{t=\tau} = u_\tau, \quad u_\tau \in E, \tag{2.5}$$

where E is a Banach space, $E_1 \subseteq E \subseteq E_0$. Assume that for any symbol $\sigma(s) \in \Sigma$, $\Sigma \subset \Xi$ is a parameter set, problem (2.4)–(2.5) is uniquely solvable for each $\tau \in \mathbb{R}$ and arbitrary $u_\tau \in E$. Let also $u(t) \in E$ for any $t \geqslant \tau$. Thus, $u(t)$ can be represented in the form

$$u(t) = U_\sigma(t,\tau)u_\tau, \quad u_\tau \in E,\ \tau \in \mathbb{R},\ t \geqslant \tau,\ \sigma = \sigma(s) \in \Sigma \subset \Xi.$$

Definition 2.1. The two-parameter family of mappings $\{U_\sigma(t,\tau),\ t \geqslant \tau,\ \tau \in \mathbb{R}\}$, $\sigma \in \Sigma$, acting in the Banach space E is said to be a family of processes with time symbol $\sigma \in \Sigma$ if for each $\sigma \in \Sigma$,

$$U_\sigma(t,\tau): E \to E, \quad t \geqslant \tau,\ \tau \in \mathbb{R}$$

and satisfies the following multiplicative properties:

$$U_\sigma(t,s)U_\sigma(s,\tau) = U_\sigma(t,\tau), \quad \forall t \geqslant s \geqslant \tau,\ \tau \in \mathbb{R},$$
$$U_\sigma(\tau,\tau) = Id \quad \text{is the identity operator},\ \tau \in \mathbb{R}.$$

Note that the following *translation identity* is valid for the family of processes $U_\sigma(t,\tau)$, $\sigma \in \Sigma$, generated by a problem, which is uniquely solvable, and for the translation semigroup $\{T(h) \mid h \geqslant 0\}$:

$$U_\sigma(t+h,\tau+h) = U_{T(h)\sigma}(t,\tau), \quad \forall \sigma \in \Sigma,\ t \geqslant \tau,\ \tau \in \mathbb{R},\ h \geqslant 0.$$

Definition 2.2. A family of processes $\{U_\sigma(t,\tau)\}$ is said to be $(E \times \Sigma, E)$ weakly continuous if for any $t \geqslant \tau$, $\tau \in \mathbb{R}$, the mapping $(u,\sigma) \to U_\sigma(t,\tau)u$ is weakly continuous from $E \times \Sigma$ to E.

Denote by $\mathcal{B}(E)$ the collection of the bounded sets of E. Let $B \in \mathcal{B}(E)$. Its Kuratowski measure of non-compactness $\kappa(B)$ is defined by

$$\kappa(B) = \inf\{\delta > 0 \mid B \text{ admits a finite cover by sets of diameter} \leqslant \delta\}.$$

For its properties, see details in [21]. Let $B_t \triangleq \bigcup_{\sigma \in \Sigma} \bigcup_{s \geqslant t} U_\sigma(s,\tau)B$.

Definition 2.3. A family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, is said to be uniformly (with respect to (w.r.t.) $\sigma \in \Sigma$) ω-limit compact if for any $\tau \in \mathbb{R}$ and $B \in \mathcal{B}(E)$, B_t is bounded for every t and $\lim_{t\to\infty} \kappa(B_t) = 0$.

A set B_0 belonging to E is said to be *uniformly* (*w.r.t.* $\sigma \in \Sigma$) *absorbing* for the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, if for any $\tau \in \mathbb{R}$ and every $B \in \mathcal{B}(E)$, there exists $t_0 = t_0(\tau, B) \geqslant \tau$ such that

$$\bigcup_{\sigma \in \Sigma} U_\sigma(t,\tau)B \subseteq B_0 \quad \text{for all } t \geqslant t_0.$$

A set P belonging to E is said to be *uniformly* (*w.r.t.* $\sigma \in \Sigma$) *attracting* for the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, if for an arbitrary fixed $\tau \in \mathbb{R}$ and any $B \in \mathcal{B}(E)$,

$$\lim_{t\to\infty} \Big(\sup_{\sigma \in \Sigma} \operatorname{dist}_E\big(U_\sigma(t,\tau)B, P\big)\Big) = 0.$$

Here $\operatorname{dist}_E(X,Y)$ denotes the Hausdorff distance from the set X to the set Y in the space E:

$$\operatorname{dist}_E(X,Y) = \sup_{x \in X} \inf_{y \in Y} \|y - x\|_E.$$

We now introduce the notion of the uniform attractor $\mathcal{A}_\Sigma$.

Definition 2.4. A closed set $\mathcal{A}_\Sigma$ is said to be the uniform (w.r.t. $\sigma \in \Sigma$) attractor of a family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, if it is uniformly (w.r.t. $\sigma \in \Sigma$) attracting (attracting property) and is contained in any closed uniformly (w.r.t. $\sigma \in \Sigma$) attracting set $\mathcal{A}'$ of the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$: $\mathcal{A}_\Sigma \subseteq \mathcal{A}'$ (minimality property).

To describe the general structure of the uniform attractor of a family of processes, we need the notion of the *kernel* of a process. A curve $u(s)$, $s \in \mathbb{R}$, is said to be a *complete trajectory* of the process $\{U_\sigma(t,\tau)\}$ if

$$U_\sigma(t,\tau)u(\tau) = u(t), \quad \forall t \geqslant \tau,\ \tau \in \mathbb{R}. \tag{2.6}$$

Definition 2.5. The kernel $\mathcal{K}_\sigma$ of the process $\{U_\sigma(t,\tau)\}$ consists of all bounded complete trajectories of the process $\{U_\sigma(t,\tau)\}$:

$$\mathcal{K}_\sigma = \{u(\cdot) \mid u(\cdot) \text{ satisfies (2.6) and } \|u(s)\|_E \leqslant M_u \text{ for all } s \in \mathbb{R}\}.$$

The set

$$\mathcal{K}_\sigma(t) = \{u(t) \mid u(\cdot) \in \mathcal{K}_\sigma\} \subset E, \quad t \in \mathbb{R},$$

is called the kernel section at time t.

Let $\omega_{\tau,\Sigma}(B) = \bigcap_{t \geqslant \tau} \overline{B_t}$. The following existence result for uniform attractors can be founded in [31].

Theorem 2.3. *Let Σ be a subset of some Banach space, and let $T(t)$ be a continuous invariant ($T(t)\Sigma = \Sigma$) semigroup on Σ satisfying the translation identity. A family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, possesses a compact uniform (w.r.t. $\sigma \in \Sigma$) attractor $\mathcal{A}_\Sigma$ satisfying*

$$\mathcal{A}_\Sigma = \omega_{0,\Sigma}(B_0) = \omega_{\tau,\Sigma}(B_0), \quad \forall \tau \in \mathbb{R}$$

if and only if it

(i) *has a bounded uniformly (w.r.t. $\sigma \in \Sigma$) absorbing set B_0; and*
(ii) *is uniformly (w.r.t. $\sigma \in \Sigma$) ω-limit compact.*

Moreover, if Σ is a weakly compact set, the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, with $\{T(h)\}_{h \geqslant 0}$ is $(E \times \Sigma, E)$ weakly continuous and satisfy (i)–(ii), then $\mathcal{A}_\Sigma$ satisfies

$$\mathcal{A}_\Sigma = \mathcal{A}_{\Sigma_0} = \omega_{0,\Sigma}(B_0) = \bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma(0).$$

Here, Σ_0 is the weak closure of Σ and $\mathcal{K}_\sigma(0)$ is the section at $t = 0$ of kernel $\mathcal{K}_\sigma$ of $\{U_\sigma(t,\tau)\}$ with symbol $\sigma \in \Sigma$.

3. Well-posedness of (1.1)

Let $A = \Delta$. Define $\Phi(u)$ by

$$\Phi(u)(t) = e^{A(t-\tau)}u(\tau) + \int_\tau^t e^{A(t-s)}\big[-f\big(x,u(s)\big) + g(x,s)\big]\,ds. \tag{3.1}$$

The linear operator $A = \Delta$ with Dirichlet boundary conditions in a bounded and smooth domain Ω can be seen as an unbounded operator in $L^q(\Omega)$, $1 < q < \infty$, with domain $D(A) = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$. In this situation, $-A = -\Delta$ is a sectorial operator and generates an analytic semigroup e^{At} in $L^q(\Omega)$. Denote by $\{E_q^\alpha\}_{\alpha \in \mathbb{R}}$ the fractional power spaces associated to A with the norm $\|u\|_{E_q^\alpha} = \|A^\alpha u\|_{L^q(\Omega)}$, $u \in E_q^\alpha$. Notice that $E_q^0 = L^q(\Omega)$ and $E_q^1 = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$. We know from [3] that

$$\begin{cases} E_q^\alpha \hookrightarrow H_q^{2\alpha}, & \alpha \geqslant 0,\ 1 < q < \infty, \\ E_q^{-\alpha} = (E_{q'}^\alpha)', & \alpha \geqslant 0,\ 1 < q < \infty,\ \dfrac{1}{q} + \dfrac{1}{q'} = 1. \end{cases} \tag{3.2}$$

3.1. Local existence of solutions of (1.1) in $L^r(\Omega)$, $1 < r < \infty$

By the standard duality arguments, (2.1) and (3.2) imply that

$$\begin{cases} E_r^\alpha \hookrightarrow L^\nu(\Omega) \quad \text{for } \nu \leqslant \dfrac{Nr}{N-2\alpha r},\ 0 \leqslant \alpha < \dfrac{N}{2r}, \\ E_r^0 = L^r(\Omega), \\ E_r^\alpha \hookleftarrow L^s(\Omega) \quad \text{for } s \geqslant \dfrac{Nr}{N-2\alpha r},\ -\dfrac{N}{2r'} < \alpha \leqslant 0,\ r' = \dfrac{r}{r-1}. \end{cases} \tag{3.3}$$

Theorem 3.1. *Let $1 < r < \infty$ and $g(x,t) \in L_b^p(\mathbb{R}; L^r(\Omega))$, $p > 1$. Assume that $f(x,u)$ satisfies (1.2)–(1.3) with $a(x) \in L^\beta(\Omega)$, $\beta > 1$, and exponent $\rho > 1$ such that*

$$\frac{1}{\beta} + \frac{\rho - 1}{r} < \frac{2}{N} \quad \left(\textit{resp.},\ \frac{1}{\beta} + \frac{\rho-1}{r} = \frac{2}{N}\right). \tag{3.4}$$

Then for each $v \in L^r(\Omega)$, there exist $R = R(v) > 0$ and $T = T(v)$ such that for any $u_\tau \in L^r(\Omega)$ with $\|u_\tau - v\|_{L^r(\Omega)} \leqslant R$, there exists a continuous function $u(\cdot; u_\tau)$:

$$\begin{aligned} &u \in C\big([\tau,T]; L^r(\Omega)\big) \cap C\big((\tau,T]; E_r^\epsilon\big) \quad \textit{for some} \\ &0 < \epsilon \leqslant \epsilon_0 < \min\left\{\frac{N}{2r}, \frac{N}{2r} + \frac{N}{2\beta\rho} - \frac{N}{2r\rho}, \frac{1}{\rho}\right\}, \end{aligned} \tag{3.5}$$

which is the unique solution of (1.1) in the sense of (1.4). This solution is a classical solution and satisfies

$$u \in C\big((\tau,T]; E_r^\theta\big), \quad 0 < \theta \leqslant \theta_0 < \min\left\{\frac{p-1}{p}, \rho\epsilon + 1 + \frac{1-\rho}{2r}N - \frac{N}{2\beta}\right\}, \tag{3.6}$$

$$\lim_{t \to \tau^+} (t-\tau)^\theta \big\|u(\cdot, u_\tau)\big\|_{E_r^\theta} = 0, \quad 0 < \theta \leqslant \theta_0 < \min\left\{\frac{p-1}{p}, \rho\epsilon + 1 + \frac{1-\rho}{2r}N - \frac{N}{2\beta}\right\}. \tag{3.7}$$

If $u_{1\tau}, u_{2\tau} \in B(v,R)$, then

$$\begin{aligned} &(t-\tau)^\theta \big\|u_1(t,u_{1\tau}) - u_2(t,u_{2\tau})\big\|_{E_r^\theta} \leqslant M_1 \|u_{1\tau} - u_{2\tau}\|_{L^r(\Omega)}, \\ &\forall t \in [\tau,T],\ 0 < \theta \leqslant \theta_0 < \min\left\{\frac{p-1}{p}, \rho\epsilon + 1 + \frac{1-\rho}{2r}N - \frac{N}{2\beta}\right\}. \end{aligned} \tag{3.8}$$

Furthermore, the time of existence is uniform on any bounded set (resp. compact set) $\mathcal{S}$ of $L^r(\Omega)$.

Proof. Let $X^\alpha = E_r^{\alpha-1}$. From (3.3) we have

$$\begin{cases} X^\alpha \hookrightarrow L^\nu(\Omega) \quad \text{for } \nu \leqslant \dfrac{Nr}{N+2r-2\alpha r},\ 1 \leqslant \alpha < \dfrac{N}{2r}+1, \\ X^1 = L^r(\Omega), \\ X^\alpha \hookleftarrow L^s(\Omega) \quad \text{for } s \geqslant \dfrac{Nr}{N+2r-2\alpha r},\ 1 - \dfrac{N}{2r'} < \alpha \leqslant 1. \end{cases} \tag{3.9}$$

We first establish the following two claims.

Claim 1. *For some* $0 < \epsilon \leqslant \epsilon_0 < \min\{\frac{N}{2r}, \frac{N}{2r} + \frac{N}{2\beta\rho} - \frac{N}{2r\rho}, \frac{1}{\rho}\}$*, there exists* $\gamma(\epsilon)$ *with*

$$\rho\epsilon \leqslant \gamma(\epsilon) = \rho\epsilon + 1 + \frac{1-\rho}{2r}N - \frac{N}{2\beta} < 1 \tag{3.10}$$

such that for any $u, \varphi \in C((\tau, T]; X^{1+\epsilon})$,

$$\|f(x,u)\|_{X^{\gamma(\epsilon)}} \leqslant C_2 \|a(x)\|_{L^\beta(\Omega)} \big(\|u\|^\rho_{X^{1+\epsilon}} + 1\big), \tag{3.11}$$

$$\|f(x,u) - f(x,\varphi)\|_{X^{\gamma(\epsilon)}} \leqslant C_2 \|a(x)\|_{L^\beta(\Omega)} \|u-\varphi\|_{X^{1+\epsilon}} \big(\|u\|^{\rho-1}_{X^{1+\epsilon}} + \|\varphi\|^{\rho-1}_{X^{1+\epsilon}} + 1\big). \tag{3.12}$$

Proof of Claim 1. Since $\frac{1}{\beta} + \frac{\rho-1}{r} \leqslant \frac{2}{N}$, we know that there exists $\gamma(\epsilon)$ such that (3.10) holds and $\gamma(\epsilon) > 1 + \frac{N}{2r} - \frac{N}{2}$ for some $\epsilon \in (0, \epsilon_0]$. Furthermore, (3.10) implies that

$$\frac{1}{\beta} + \frac{[N - 2\epsilon r]}{Nr}\rho \leqslant \frac{N + 2r - 2\gamma(\epsilon) r}{Nr}.$$

Choosing $m > 1$ such that

$$\frac{1}{\beta} < \frac{1}{\beta} + \frac{[N - 2\epsilon r]}{Nr}\rho \leqslant \frac{1}{m} \leqslant \frac{N + 2r - 2\gamma(\epsilon) r}{Nr},$$

together with (1.3) we have

$$\begin{aligned} \|f(x,u)\|_{L^m(\Omega)} &\leqslant C_3\bigg[\bigg(\int |a(x)|^m |u|^{\rho m}\,dx\bigg)^{\frac{1}{m}} + 1\bigg] \\ &\leqslant C_4 \|a(x)\|_{L^\beta(\Omega)} \big(\|u\|^\rho_{L^{\frac{m\beta}{\beta-m}\rho}(\Omega)} + 1\big). \end{aligned} \tag{3.13}$$

From (3.9) and (3.13) we have

$$f : X^{1+\epsilon} \hookrightarrow L^{\frac{m\beta}{\beta-m}\rho}(\Omega) \hookrightarrow L^m(\Omega) \hookrightarrow X^{\gamma(\epsilon)}, \tag{3.14}$$

which implies that (3.11) holds.

By similar arguments, we get (3.12). $\square$

Claim 2. *For any* $t_1 < t_2$, $0 < \theta \leqslant \theta_0 < \frac{1}{q}$,

$$\bigg\| \int_{t_1}^{t_2} e^{A(t_2-s)} g(x,s)\,ds \bigg\|_{X^{1+\theta}} \leqslant M_2 e^{t_2-t_1}(t_2-t_1)^{-\theta+\frac{1}{q}},$$

where

$$M_2 = M\bigg(\frac{e^p}{e^p-1}\bigg)^{\frac{1}{p}} \frac{1}{(1-q\theta_0)^{\frac{1}{q}}} \|g(x,t)\|_{L^p_b(\mathbb{R}; L^r(\Omega))}, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

Proof of Claim 2. From (2.2) we get that

$$
\begin{aligned}
&\left\| \int_{t_1}^{t_2} e^{A(t_2-s)} g(x,s)\,ds \right\|_{X^{1+\theta}} \\
&\leqslant M \int_{t_1}^{t_2} (t_2-s)^{-\theta} \left\| g(x,s) \right\|_{L^r(\Omega)} ds \\
&\leqslant M \left(\int_{t_1}^{t_2} (t_2-s)^{-q\theta} e^{qs}\, ds \right)^{\frac{1}{q}} \left(\int_{t_1}^{t_2} e^{-ps} \left\| g(x,s) \right\|_{L^r(\Omega)}^p ds \right)^{\frac{1}{p}} \\
&\leqslant M e^{t_2} \frac{1}{(1-q\theta_0)^{\frac{1}{q}}} (t_2-t_1)^{-\theta+\frac{1}{q}} \Biggl(e^{-pt_1} \int_{t_1}^{t_1+1} \left\| g(x,s) \right\|_{L^r(\Omega)}^p ds \\
&\quad + e^{-p(t_1+1)} \int_{t_1+1}^{t_1+2} \left\| g(x,s) \right\|_{L^r(\Omega)}^p ds + \cdots + e^{-p(t_1+n)} \int_{t_1+n}^{t_1+n+1} \left\| g(x,s) \right\|_{L^r(\Omega)}^p ds + \cdots \Biggr)^{\frac{1}{p}} \\
&\leqslant M e^{t_2-t_1} \frac{1}{(1-q\theta_0)^{\frac{1}{q}}} (t_2-t_1)^{-\theta+\frac{1}{q}} \left(1 + e^{-p} + e^{-2p} + \cdots\right)^{\frac{1}{p}} \left\| g(x,t) \right\|_{L_b^p(\mathbb{R};L^r(\Omega))} \\
&= M \left(\frac{e^p}{e^p-1} \right)^{\frac{1}{p}} \frac{1}{(1-q\theta_0)^{\frac{1}{q}}} \left\| g(x,t) \right\|_{L_b^p(\mathbb{R};L^r(\Omega))} e^{t_2-t_1} (t_2-t_1)^{-\theta+\frac{1}{q}}. \qquad \square
\end{aligned}
$$

Note that the solution $u(t)$ of (1.1) can be decomposed into the sum

$$u(t) = v(t) + w(t),$$

where $v(t)$ and $w(t)$ solve the problems

$$
\begin{cases}
v_t - \Delta v = g(x,t) & \text{in } \Omega,\ t > \tau, \\
v = 0 & \text{on } \partial\Omega, \\
v(\tau) = 0, & \tau \in \mathbb{R},
\end{cases}
\tag{3.15}
$$

and

$$
\begin{cases}
w_t - \Delta w + \widetilde{f}(x,w) = 0 & \text{in } \Omega,\ t > \tau, \\
w = 0 & \text{on } \partial\Omega, \\
w(\tau) = u_\tau, & \tau \in \mathbb{R},
\end{cases}
\tag{3.16}
$$

respectively, where $\widetilde{f}(x,w) = f(x, w+v)$.

By Claim 2, as in the proof of Theorem 1 of [6], the linear equation (3.15) has a unique solution $v(t)$ in the sense of (1.4) such that

$$v(t) \in C\bigl([\tau,T]; X^1\bigr) \cap C\bigl((\tau,T]; X^{1+\theta}\bigr)$$

with $0 < \theta < \frac{1}{q}$ and satisfies

$$v(t) = \int_{\tau}^{t} e^{A(t-s)} g(x,s)\, ds. \tag{3.17}$$

For the nonlinear function $\widetilde{f}(x,w)$ of (3.16), choosing ρ such that $\frac{1}{\rho} \leqslant \frac{1}{q}$, by Claims 1–2 and (3.17) we obtain

$$\begin{aligned}
\|\widetilde{f}(x,w)\|_{X^{\gamma(\epsilon)}} &= \|f(x,u)\|_{X^{\gamma(\epsilon)}} \\
&\leqslant C_2 \|a(x)\|_{L^{\beta}(\Omega)} \big(\|u\|^{\rho}_{X^{1+\epsilon}} + 1\big) \\
&= C_2 \|a(x)\|_{L^{\beta}(\Omega)} \big(\|w+v\|^{\rho}_{X^{1+\epsilon}} + 1\big) \\
&\leqslant C_5 \|a(x)\|_{L^{\beta}(\Omega)} \big(\|w\|^{\rho}_{X^{1+\epsilon}} + 1\big)
\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}
&\|\widetilde{f}(x,w_1) - \widetilde{f}(x,w_2)\|_{X^{\gamma(\epsilon)}} \\
&\quad = \|f(x,w_1+v) - f(x,w_2+v)\|_{X^{\gamma(\epsilon)}} \\
&\quad \leqslant C_2 \|a(x)\|_{L^{\beta}(\Omega)} \|w_1 - w_2\|_{X^{1+\epsilon}} \big(\|w_1+v\|^{\rho-1}_{X^{1+\epsilon}} + \|w_2+v\|^{\rho-1}_{X^{1+\epsilon}} + 1\big) \\
&\quad \leqslant C_5 \|a(x)\|_{L^{\beta}(\Omega)} \|w_1 - w_2\|_{X^{1+\epsilon}} \big(\|w_1\|^{\rho-1}_{X^{1+\epsilon}} + \|w_2\|^{\rho-1}_{X^{1+\epsilon}} + 1\big).
\end{aligned} \tag{3.19}$$

By (3.18)–(3.19), applying Theorem 1 of [6], we know that Eq. (3.16) has a unique ϵ-regular solution

$$w(t) \in C\big([\tau,T]; X^1\big) \cap C\big((\tau,T]; X^{1+\epsilon}\big)$$

for some ϵ satisfying (3.5). Therefore,

$$\begin{aligned}
\|u(t) - u_\tau\|_{X^1} &= \|v(t) + w(t) - u_\tau\|_{X^1} \\
&\leqslant \|w(t) - u_\tau\|_{X^1} + \|v(t)\|_{X^1} \to 0 \quad \text{as } t \to \tau^+
\end{aligned}$$

and $u(t)$ is the unique solution of (1.1) in the sense of (1.4).

By Claim 2 and Theorem 1 of [6], we obtain (3.6)–(3.8). This completes the proof. □

3.2. Local existence of solutions of (1.1) in $W^{1,r}(\Omega)$, $1 < r < N$

Theorem 3.2. *Let $1 < r < N$ and $g(x,t) \in L^p_b(\mathbb{R}; L^r(\Omega))$, $p > 2$. Assume that $f(x,u)$ satisfies (1.2)–(1.3) with $a(x) \in L^{\beta}(\Omega)$, $\beta > 1$ and exponent $\rho > 1$ such that*

$$\frac{N-r}{N+r}\rho + \frac{Nr}{\beta(N+r)} < 1 \quad \left(\textit{resp.}, \ \frac{N-r}{N+r}\rho + \frac{Nr}{\beta(N+r)} = 1\right). \tag{3.20}$$

Then for each $v \in W^{1,r}(\Omega)$, there exist $R = R(v) > 0$ and $T = T(v)$ such that for any $u_\tau \in W^{1,r}(\Omega)$ with $\|u_\tau - v\|_{W^{1,r}(\Omega)} \leqslant R$, there exists a continuous function $u(\cdot; u_\tau)$:

$$u \in C\big([\tau, T]; W^{1,r}(\Omega)\big) \cap C\big((\tau, T]; E_r^{\frac{1}{2}+\epsilon}\big) \quad \textit{for some}$$
$$0 < \epsilon \leqslant \epsilon_0 < \min\left\{\frac{N}{2r} - \frac{1}{2}, \frac{1}{2\rho} + \frac{N-r}{2r} + \frac{N}{2\beta\rho} - \frac{N+r}{2r\rho}, \frac{1}{2\rho}\right\}, \tag{3.21}$$

which is the unique solution of (1.1) *in the sense of* (1.4)*. This solution is a classical solution and satisfies*

$$u \in C\big((\tau, T]; E_r^{\frac{1}{2}+\theta}\big), \quad 0 < \theta \leqslant \theta_0 < \min\left\{\frac{1}{2} - \frac{1}{p}, \rho\epsilon + \frac{N+r}{2r} - \frac{N-r}{2r}\rho - \frac{N}{2\beta}\right\}, \tag{3.22}$$

$$\lim_{t\to\tau^+} (t-\tau)^\theta \big\|u(\cdot, u_\tau)\big\|_{E_r^{\frac{1}{2}+\theta}} = 0,$$
$$0 < \theta \leqslant \theta_0 < \min\left\{\frac{1}{2} - \frac{1}{p}, \rho\epsilon + \frac{N+r}{2r} - \frac{N-r}{2r}\rho - \frac{N}{2\beta}\right\}.$$

If $u_{1\tau}, u_{2\tau} \in B(v, R)$*, then*

$$(t-\tau)^\theta \big\|u_1(t, u_{1\tau}) - u_2(t, u_{2\tau})\big\|_{E_r^{\frac{1}{2}+\theta}} \leqslant M_3 \|u_{1\tau} - u_{2\tau}\|_{W^{1,r}(\Omega)},$$
$$\forall t \in [\tau, T],\ 0 < \theta \leqslant \theta_0 < \min\left\{\frac{1}{2} - \frac{1}{p}, \rho\epsilon + \frac{N+r}{2r} - \frac{N-r}{2r}\rho - \frac{N}{2\beta}\right\}.$$

Furthermore, the time of existence is uniform on any bounded set (*resp. compact set*) $\mathcal{S}$ *of* $W^{1,r}(\Omega)$*.*

Proof. Let $X^\alpha = E_r^{\alpha-\frac{1}{2}}$. From (3.3) we have that

$$\begin{cases} X^\alpha \hookrightarrow L^\nu(\Omega) & \text{for } \nu \leqslant \dfrac{Nr}{N+r-2\alpha r},\ \dfrac{1}{2} \leqslant \alpha < \dfrac{1}{2} + \dfrac{N}{2r}, \\ X^{\frac{1}{2}} = L^r(\Omega), \\ X^\alpha \hookleftarrow L^s(\Omega) & \text{for } s \geqslant \dfrac{Nr}{N+r-2\alpha r},\ \dfrac{1}{2} - \dfrac{N}{2r'} < \alpha \leqslant \dfrac{1}{2}. \end{cases} \tag{3.23}$$

Similar to the proof of Claim 1 in the proof of Theorem 3.1, for any $u, \varphi \in C((\tau, T]; X^{1+\epsilon})$ and some $0 < \epsilon < \min\{\frac{N}{2r} - \frac{1}{2}, \frac{1}{2\rho} + \frac{N-r}{2r} + \frac{N}{2\beta\rho} - \frac{N+r}{2r\rho}\}$, there exists $\gamma(\epsilon)$ with

$$\rho\epsilon \leqslant \gamma(\epsilon) = \frac{N+r}{2r} - \frac{N-r}{2r}\rho - \frac{N}{2\beta} + \rho\epsilon \leqslant \frac{1}{2} \tag{3.24}$$

such that (3.11)–(3.12) hold with the constant C_2'. In fact, since

$$\frac{N-r}{N+r}\rho + \frac{Nr}{\beta(N+r)} \leqslant 1,$$

there exists $\gamma(\epsilon)$ such that (3.24) holds for some ϵ in (3.21). From (3.24) we get that

$$\frac{1}{\beta} < \frac{1}{\beta} + \frac{N+r-2(1+\epsilon)r}{Nr}\rho \leqslant \frac{N+r-2\gamma(\epsilon)r}{Nr}.$$

Choosing $1 < m < \beta$ such that

$$\frac{1}{\beta} + \frac{N + r - 2(1+\epsilon)r}{Nr}\rho \leqslant \frac{1}{m} \leqslant \frac{N + r - 2\gamma(\epsilon)r}{Nr},$$

using (3.13)–(3.14) and (3.23), we obtain (3.11)–(3.12).

For $t_1 < t_2$, $0 < \theta \leqslant \theta_0 < \frac{1}{q} - \frac{1}{2}$, as the proof of Claim 2 in the proof of Theorem 3.1, we have

$$\begin{aligned}
&\left\| \int_{t_1}^{t_2} e^{A(t_2-s)} g(x,s)\,ds \right\|_{X^{1+\theta}} \\
&\quad \leqslant M \int_{t_1}^{t_2} (t_2 - s)^{\frac{1}{2}-(1+\theta)} \big\| g(x,s) \big\|_{L^r(\Omega)}\,ds \\
&\quad \leqslant M \left(\frac{e^p}{e^p - 1} \right)^{\frac{1}{p}} \big\| g(x,t) \big\|_{L_b^p(\mathbb{R};L^r(\Omega))} \left(\frac{1}{1 - q(\theta_0 + \frac{1}{2})} \right)^{\frac{1}{q}} e^{t_2 - t_1} (t_2 - t_1)^{(-\frac{1}{2}-\theta)+\frac{1}{q}}.
\end{aligned}$$

The rest of the proof is similar to that of Theorem 3.1. The proof is completed. □

Remark 3.1. If $g(x,t) \in L_b^p(\mathbb{R}; W^{1,r}(\Omega))$, the solution of (1.1) with $u(\tau) = u_\tau \in W^{1,r}(\Omega)$ has higher regularity, that is,

$$u \in C\big((\tau, T]; E_r^{\frac{1}{2}+\theta}\big), \quad 0 < \theta \leqslant \theta_0 < \min\left\{ \frac{p-1}{p}, \rho\epsilon + \frac{N+r}{2r} - \frac{N-r}{2r}\rho - \frac{N}{2\beta} \right\},$$

and for $u_{1\tau}, u_{2\tau} \in B(u_\tau, R)$,

$$\begin{gathered}
(t-\tau)^\theta \big\| u_1(t, u_{1\tau}) - u_2(t, u_{2\tau}) \big\|_{E_r^{\frac{1}{2}+\theta}} \leqslant M_3 \| u_{1\tau} - u_{2\tau} \|_{W^{1,r}(\Omega)}, \\
\forall t \in [\tau, T],\ 0 < \theta \leqslant \theta_0 < \min\left\{ \frac{p-1}{p}, \rho\epsilon + \frac{N+r}{2r} - \frac{N-r}{2r}\rho - \frac{N}{2\beta} \right\}.
\end{gathered}$$

4. Existence of attractors in $L^r(\Omega)$

4.1. $r = 2$, $\frac{1}{\beta} + \frac{\rho-1}{2} < \frac{2}{N}$

Let $H = L^2(\Omega)$, $V = H_0^1(\Omega)$, and $V' = (H_0^1(\Omega))'$ the dual of V. Denote by $\|\cdot\|$ the norm of H. In order to establish the global existence of solution of (1.1) with the initial data belonging to H, we need the following dissipative condition: Suppose that there exist constants c_0 and c_0' with $0 \leqslant c_0 < (1-\alpha_0)\lambda_1$, where α_0 is a constant satisfying $0 < \alpha_0 < 1$ and λ_1 is the first eigenvalue of $-\Delta$ in V, such that

$$f(x,u)u \geqslant -c_0|u|^2 - c_0'. \tag{4.1}$$

Similar dissipative conditions are also introduced in [7,22,44,48], specially in [7].

For a fixed external force $g_0(t) := g_0(x,t) \in L_b^p(\mathbb{R}; X)$, consider the following translation:

$$\Sigma_0 = \big\{ T(h)g_0(t) \,\big|\, h \in \mathbb{R} \big\} = \big\{ g_0(h+t) \,\big|\, h \in \mathbb{R} \big\},$$

where $T(h)$, $h \in \mathbb{R}$, is the translation operator. Denote by

$$\mathcal{H}_X(g_0) = \big[\big\{T(h)g_0(t) \;\big|\; h \in \mathbb{R}\big\}\big]_{L^{p,w}_{loc}(\mathbb{R};X)}$$

the closure of Σ_0 in $L^{p,w}_{loc}(\mathbb{R};X)$, which is the subspace of $L^p_{loc}(\mathbb{R};X)$ endowed with the local weak convergence topology. By Proposition V.4.2 of [16] we know that

$$\big\|g(t)\big\|_{L^p_b(\mathbb{R};X)} \leqslant \big\|g_0(t)\big\|_{L^p_b(\mathbb{R};X)}, \quad \forall g(t) \in \mathcal{H}_X(g_0). \tag{4.2}$$

Theorem 4.1. *Assume that $f(x,u)$ satisfies (1.2)–(1.3) with $a(x) \in L^\beta$, $\beta > 1$ and exponent $\rho > 1$ such that*

$$\frac{1}{\beta} + \frac{\rho - 1}{2} < \frac{2}{N},$$

and (4.1) holds. Let $g_0(t) \in L^p_b(\mathbb{R};H)$, $p > 2$. Then the family of processes $\{U_g(t,\tau)\}$, $g \in \mathcal{H}_H(g_0)$, corresponding to problem (1.1) possesses a compact uniform (w.r.t. $g \in \mathcal{H}_H(g_0)$) attractor $\mathcal{A}_{\mathcal{H}_H(g_0)}$ in H satisfying:

$$\mathcal{A}_{\mathcal{H}_H(g_0)} = \omega_{0,\mathcal{H}_H(g_0)}(B_0) = \bigcup_{g \in \mathcal{H}_H(g_0)} \mathcal{K}_g(0), \tag{4.3}$$

where B_0 is the uniformly (w.r.t. $g \in \mathcal{H}_H(g_0)$) absorbing set in H.

Proof. We first show that the family of processes $\{U_g(t,\tau)\}$, $g \in \mathcal{H}_H(g_0)$, corresponding to problem (1.1) is well defined.

Taking the scalar product in H of (1.1) with u, we have

$$\frac{1}{2}\frac{d}{dt}\big\|u(t)\big\|^2 + \big\|\nabla u(t)\big\|^2 + \big\langle f\big(x,u(t)\big), u(t)\big\rangle = \big\langle g(x,t), u(t)\big\rangle. \tag{4.4}$$

By (4.1), we have that

$$\big\langle f\big(x,u(t)\big), u(t)\big\rangle \geqslant -c_0\|u\|^2 - c_0'|\Omega|. \tag{4.5}$$

Using the fact that $p > 2$ and Young's inequality we obtain

$$\begin{aligned}
\big|\big\langle g(x,t), u(t)\big\rangle\big| &\leqslant \big\|g(x,t)\big\|\big\|u(t)\big\| \\
&\leqslant \frac{1}{q}\big\|u(t)\big\|^q + \frac{1}{p}\big\|g(x,t)\big\|^p \\
&\leqslant \alpha_0\lambda_1\big\|u(t)\big\|^2 + \left[\frac{1}{q}(\alpha_0\lambda_1)^{-\frac{q}{2}}\right]^{\frac{2}{2-q}} + \frac{1}{p}\big\|g(x,t)\big\|^p.
\end{aligned} \tag{4.6}$$

Let $M_4 = c_0'|\Omega| + [\frac{1}{q(\alpha_0\lambda_1)^{\frac{q}{2}}}]^{\frac{2}{2-q}}$. By Poincaré inequality, it follows from (4.4)–(4.6) that

$$\frac{d}{dt}\big\|u(t)\big\|^2 + 2\big((1-\alpha_0)\lambda_1 - c_0\big)\big\|u(t)\big\|^2 \leqslant 2M_4 + \frac{2}{p}\big\|g(x,t)\big\|^p.$$

Let $\Lambda = 2((1-\alpha_0)\lambda_1 - c_0)$. Using Lemma 2.2 and (4.2), from the above inequality we have

$$\begin{aligned}\|u(t)\|^2 &\leqslant \|u(\tau)\|^2 e^{-\Lambda(t-\tau)} + \int_\tau^t e^{-\Lambda(t-s)}\left(2M_4 + \frac{2}{p}\|g(x,s)\|^p\right)ds\\ &\leqslant \|u(\tau)\|^2 e^{-\Lambda(t-\tau)} + \left(2M_4 + \frac{2}{p}\|g_0(x,t)\|^p_{L_b^p(\mathbb{R};H)}\right)\left(1+\frac{1}{\Lambda}\right).\end{aligned} \tag{4.7}$$

Thus, the family of process $\{U_g(t,\tau)\}$: $U_g(t,\tau)u_\tau = u(t)$, $U_g(t,\tau)H \to H$, $g \in \mathcal{H}_H(g_0)$, $t \geqslant \tau$, $\tau \in \mathbb{R}$, is well defined, where $u(t)$ is the solution of (1.1).

Let

$$R_0^2 = 2\left(2M_4 + \frac{2}{p}\|g_0(x,t)\|^p_{L_b^p(\mathbb{R};H)}\right)\left(1+\frac{1}{\Lambda}\right).$$

(4.7) also implies that the family of processes possesses a uniformly absorbing set

$$B_0 = \{u \in H \mid \|u\| \leqslant R_0\},$$

that is, for any bounded set $B \subset H$, there exists $T_0 = T_0(\tau, B)$ such that

$$\bigcup_{g\in\mathcal{H}_H(g_0)} U_g(t,\tau)B \subset B_0, \quad \forall t > T_0.$$

Since $p > 2$, by the regularity of solutions of (1.1) obtained in Theorem 3.1, we can choose $\theta = \frac{1}{2}$ such that

$$B_1 = \bigcup_{g\in\mathcal{H}_H(g_0)} \bigcup_{\tau\in\mathbb{R}} U_g(\tau+1,\tau)B_0$$

is also a uniformly absorbing set and bounded in V. By standard Sobolev compact embedding, we know that the family of processes $\{U_g(t,\tau)\}$, $g \in \mathcal{H}_H(g_0)$, is ω-limit compact in H and possesses a compact uniform (w.r.t. $g \in \mathcal{H}_H(g_0)$) attractor $\mathcal{A}_{\mathcal{H}_H(g_0)}$ in H.

Next, we show that (4.3) holds. We claim that if $u_{\tau_n} \to u_\tau$ in H, $g_n \rightharpoonup g_0$ weakly in $L^p_{loc}(\mathbb{R}; H)$, then for any fixed $t \geqslant \tau$, $t \in \mathbb{R}$,

$$U_{g_n}(t,\tau)u_{\tau_n} \rightharpoonup U_{g_0}(t,\tau)u_\tau \quad \text{weakly in } H. \tag{4.8}$$

For fixed $\tau \in \mathbb{R}$, let

$$u_n(t) = U_{g_n}(t,\tau)u_{\tau_n}, \qquad u(t) = U_{g_0}(t,\tau)u_\tau, \tag{4.9}$$

be the solutions of the equation

$$\frac{d}{dt}u_n(t) - \Delta u_n(t) + f(x, u_n(t)) = g_n(x,t) \tag{4.10}$$

and (1.1), respectively. The regularity of solutions of (4.10) implies that

$$\int_\tau^t \left\| \nabla u_n(s) \right\| ds \leqslant C_6(\tau). \tag{4.11}$$

As in Claim 1 in the proof of Theorem 3.1, we can choose $0 < \epsilon \leqslant \frac{1}{2}$ and $\gamma(\epsilon) \geqslant \frac{1}{2}$ such that

$$\begin{aligned} \left\| f\big(x, u_n(t)\big) \right\|_{V'} &\leqslant C_7 \left\| f\big(x, u_n(t)\big) \right\|_{E_2^{\gamma(\epsilon)-1}} \\ &\leqslant C_8 \left\| a(x) \right\|_{L^\beta(\Omega)} \big(\left\| u_n(t) \right\|_{E_2^\epsilon}^\rho + 1 \big) \\ &\leqslant C_9 \left\| a(x) \right\|_{L^\beta(\Omega)} \big(\left\| \nabla u_n(t) \right\|^\rho + 1 \big). \end{aligned} \tag{4.12}$$

Since the operator A is an isometry between V and V', from (4.11) and (4.12) we have

$$\begin{aligned} \int_\tau^t \left\| u_n'(s) \right\|_{V'}^2 ds &\leqslant \int_\tau^t \left\| A u_n(s) \right\|_{V'}^2 ds + \int_\tau^t \left\| f\big(x, u_n(s)\big) \right\|_{V'}^2 ds + \int_\tau^t \left\| g_n(x,s) \right\|_{V'}^2 ds \\ &\leqslant C_{10}\big(\tau, \left\| a(x) \right\|_{L^\beta(\Omega)}, \left\| g_0(x,t) \right\|_{L_b^p(\mathbb{R};H)}, R_0\big). \end{aligned} \tag{4.13}$$

Similar to the derivation of (4.7), together with (4.12)–(4.13), we obtain that

$$\begin{aligned} \big\{ u_n(t) \big\} \quad &\text{is bounded in } L^\infty(\mathbb{R}_\tau; H), \\ &\text{bounded in } L^2_{loc}(\mathbb{R}_\tau; V), \\ \big\{ u_n'(t) \big\} \quad &\text{is bounded in } L^2_{loc}\big(\mathbb{R}_\tau; V'\big), \end{aligned} \tag{4.14}$$

where $\mathbb{R}_\tau = [\tau, \infty)$. Using Theorem 2.1, we know that

$$\big\{ u_n(t) \big\} \quad \text{is precompact in } L^2_{loc}(\mathbb{R}_\tau; H). \tag{4.15}$$

Therefore, by taking, if necessary, a subsequence (which we still denote by $u_n(t)$), there exists $\widetilde{u}(t) \in L^\infty(\mathbb{R}_\tau; H) \cap L^2_{loc}(\mathbb{R}_\tau; V)$ such that

$$\begin{aligned} u_n(t) &\rightharpoonup \widetilde{u}(t) \quad *\text{-weakly in } L^\infty(\mathbb{R}_\tau; H), \\ &\rightharpoonup \widetilde{u}(t) \quad \text{weakly in } L^2_{loc}(\mathbb{R}_\tau; V), \\ &\to \widetilde{u}(t) \quad \text{strongly in } L^2_{loc}(\mathbb{R}_\tau; H). \end{aligned} \tag{4.16}$$

In particularly, as $n \to \infty$,

$$\begin{aligned} u_n'(t) &\rightharpoonup \widetilde{u}'(t) \quad \text{weakly in } L^2_{loc}\big(\mathbb{R}_\tau; V'\big), \\ \Delta u_n(t) &\rightharpoonup \Delta \widetilde{u}(t) \quad \text{weakly in } L^2_{loc}\big(\mathbb{R}_\tau; V'\big), \\ f\big(x, u_n(t)\big) &\rightharpoonup w(t) \quad \text{weakly in } L^2_{loc}\big(\mathbb{R}_\tau; V'\big), \\ g_n(x,t) &\rightharpoonup g_0(x,t) \quad \text{weakly in } L^p_{loc}(\mathbb{R}_\tau; H). \end{aligned} \tag{4.17}$$

Taking $n \to \infty$, we obtain the equality

$$\widetilde{u}'(t) - \Delta\widetilde{u}(t) + w(t) = g_0(x,t)$$

in the distribution sense of the space $\mathcal{D}'(\mathbb{R}_\tau; V')$. Thanks to (4.15), using Theorem II.1.8 of [16], we get that $\widetilde{u}(t) \in C([\tau, T_0']; H)$ for any $T_0' > 0$, which implies that $\widetilde{u}(\tau) = u_\tau$, since $u_{\tau_n} \to u_\tau$ strongly in H.

Now, we show that $w(t) = f(x, \widetilde{u}(t))$, which implies that $\widetilde{u}(t)$ is the solution of (1.1), and by uniqueness, $\widetilde{u}(t) = u(t)$. Due to the strong convergence in (4.16), we can extract a subsequence of $\{u_n(t)\}$ (which we still denote by $u_n(t)$) such that $u_n(x,t) \to \widetilde{u}(x,t)$ $(n \to \infty)$ for almost every $(x,t) \in \Omega \times [\tau, T_0']$. By the continuity of $f(x,u)$, $f(x, u_n(t)) \to f(x, \widetilde{u}(t))$ $(n \to \infty)$ for almost every $(x,t) \in \Omega \times [\tau, T_0']$. From (4.11)–(4.12), we get that

$$\int_\tau^t \big\| f\big(x, u_n(s)\big)\big\|_{V'}\, ds \leqslant C_{11},$$

where C_{11} is independent on n. Similar to the proof of Lemma I.1.3 in [27], we obtain that $f(x, u_n(t)) \rightharpoonup f(x, \widetilde{u}(t))$ weakly in $L^2_{loc}(\mathbb{R}_\tau; V')$. Thus, $w(t) = f(x, \widetilde{u}(t))$.

Finally, by the strong convergence of (4.15), we get that for almost every $t \geqslant \tau$, $u_n(t)$ converges strongly to $u(t)$ in H. Therefore,

$$\big\langle u_n(t), \upsilon\big\rangle \to \big\langle u(t), \upsilon\big\rangle \quad \text{for } a.e.\ t \geqslant \tau,\ \upsilon \in C_0^\infty(\Omega).$$

It follows from (4.15) that $\{\langle u_n(t), \upsilon\rangle\}$ is uniformly bounded. For all $\upsilon \in C_0^\infty(\Omega)$ and $t_1 \geqslant 0$, by (4.17) we have

$$\big\langle u_n(t+t_1) - u_n(t), \upsilon\big\rangle = \int_t^{t+t_1} \big\langle \partial_t u_n(s), \upsilon\big\rangle ds \leqslant C_{12} t_1 \|\upsilon\|_V \big\|u_n'(t)\big\|_{L^2_{loc}(\mathbb{R}_\tau; V')} \leqslant C_{13} t_1 \|\upsilon\|_V,$$

which implies that $\{\langle u_n(t), \upsilon\rangle\}$ is locally equicontinuous. Thus, by (4.15) again,

$$\big\langle u_n(t), \upsilon\big\rangle \to \big\langle u(t), \upsilon\big\rangle, \quad \forall t \geqslant \tau,\ \upsilon \in C_0^\infty(\Omega),$$

which implies that (4.8) holds, since $C_0^\infty(\Omega)$ is dense in H.

Using Theorem 2.3, we know that (4.3) holds. This completes the proof. □

Remark 4.1. For $1 < p \leqslant 2$, from Theorem 3.1 we notice that the solution of (1.1) with $g(x,t) \in L^p_b(\mathbb{R}; L^r(\Omega))$ does not belong to $W^{1,r}(\Omega)$. If we improve the regularity of $g(x,t)$, that is, $g(x,t) \in L^p_b(\mathbb{R}; W^{1,r}(\Omega))$, choosing $\theta = \frac{1}{2}$ and $0 < \theta' < \frac{1}{2}$ such that $\theta - \theta' < \frac{1}{q}$, similar to the proof of Claim 2 in the proof of Theorem 3.1, for $t_1 < t_2$ we can obtain

$$\begin{aligned}
&\left\| \int_{t_1}^{t_2} e^{A(t_2 - s)} g(x,s)\, ds \right\|_{X^{1+\theta}} \\
&\quad \leqslant C_1' M \int_{t_1}^{t_2} (t_2 - s)^{-(\theta - \theta')} \big\|g(x,s)\big\|_{X^{1+\theta'}}\, ds \\
&\quad = C_2' M \left(\frac{e^p}{e^p - 1}\right)^{\frac{1}{p}} \frac{1}{[1 - q(\theta_0 - \theta')]^{\frac{1}{q}}} \big\|g(x,t)\big\|_{L^p_b(\mathbb{R}; W^{1,r}(\Omega))} e^{t_2 - t_1} (t_2 - t_1)^{-(\theta - \theta') + \frac{1}{q}},
\end{aligned}$$

where C_1' and C_2' are two constants. Therefore, as the proof of Theorem 3.1, we get that the solution of (1.1) with $g(x,t) \in L_b^p(\mathbb{R}; W^{1,r}(\Omega))$ and $u(\tau) \in L^r(\Omega)$ can enter in $W^{1,r}(\Omega)$.

For $p = 2$, substituting the assumption on $g_0(x,t)$ in Theorem 4.1 with $g_0(x,t) \in L_b^p(\mathbb{R}; V)$, we obtain the same result as in Theorem 4.1 with $\mathcal{H}_H(g_0)$ being replaced by $\mathcal{H}_V(g_0)$.

For $1 < p < 2$, substituting the dissipative condition (4.1) and $g_0(x,t) \in L_b^p(\mathbb{R}; H)$ in Theorem 4.1 with (5.21) and $g_0(x,t) \in L_b^p(\mathbb{R}; V)$, respectively, proceeding as in the derivation of (5.22) and (5.23) (see Section 5.2), we get that there exists a uniformly (w.r.t. $g \in \mathcal{H}_V(g_0)$) absorbing set for the family of processes $\{U_g(t,\tau)\}$, $g \in \mathcal{H}_V(g_0)$ in H. In this situation, as the proof of Theorem 4.1, we can obtain that there exists a compact uniformly absorbing set in H for the family of processes and the family of processes is weak continuous in H. Therefore, the result of Theorem 4.1 still holds with $\mathcal{H}_H(g_0)$ being replaced by $\mathcal{H}_V(g_0)$. $\square$

4.2. $\frac{1}{\beta} + \frac{\rho-1}{r} < \frac{2}{N}, r > 1$

In this case, the theory of Hilbert spaces cannot be used. We need other dissipative conditions instead of (4.1). Assume that

$$\lim_{s\to\infty} \inf \frac{f(x,s)s}{|s|^2} > 0 \tag{4.18}$$

and

$$\begin{cases} p \geqslant r, & \text{if } r > 2, \\ p > 2, & \text{if } 1 < r \leqslant 2. \end{cases} \tag{4.19}$$

Multiplying (1.1) by $|u|^{r-2}u$ and integrating by parts, using the boundary condition, we get

$$\begin{aligned} &\frac{1}{r}\frac{d}{dt}\|u(t)\|^r_{L^r(\Omega)} + \frac{4(r-1)}{r^2}\int_\Omega \left|\nabla\left(|u|^{\frac{r}{2}}\right)\right|^2 dx + \int_\Omega f(x,u)|u|^{r-2}u\,dx \\ &= \int_\Omega g(x,t)|u|^{r-2}u\,dx \\ &\leqslant \|g(x,t)\|_{L^r(\Omega)}\|u\|^{r-1}_{L^r(\Omega)}. \end{aligned} \tag{4.20}$$

By the dissipative condition (4.18), there exist positive constants C_{14} and C_{15} such that

$$f(x,u)|u|^{r-2}u \geqslant C_{14}|u|^r - C_{15}. \tag{4.21}$$

Using (4.19), we have

$$\begin{aligned} &\|g(x,t)\|_{L^r(\Omega)}\|u\|^{r-1}_{L^r(\Omega)} \\ &\leqslant \frac{C_{14}}{2}\|u\|^r_{L^r(\Omega)} + \frac{1}{r}\left(\frac{C_{14}r}{2(r-1)}\right)^{-(r-1)}\|g(x,t)\|^r_{L^r(\Omega)} \\ &\leqslant \frac{C_{14}}{2}\|u\|^r_{L^r(\Omega)} + \frac{p-r}{p}\left[\frac{1}{r}\left(\frac{C_{14}r}{2(r-1)}\right)^{-(r-1)}\right]^{\frac{p}{p-r}} + \frac{r}{p}\|g(x,t)\|^p_{L^r(\Omega)}. \end{aligned} \tag{4.22}$$

Let $M_5 = \frac{p-r}{p}[\frac{1}{r}(\frac{C_{14}r}{2(r-1)})^{-(r-1)}]^{\frac{p}{p-r}}$. It follows from (4.20)–(4.22) that

$$\frac{d}{dt}\|u(t)\|^r_{L^r(\Omega)} + \frac{C_{14}r}{2}\|u\|^r_{L^r(\Omega)} \leqslant C_{15}r + M_5 r + \frac{r^2}{p}\|g(x,t)\|^p_{L^r(\Omega)}.$$

Using Lemma 2.2 and (4.2), we obtain that

$$\|u(t)\|^r_{L^r(\Omega)} \leqslant \|u(\tau)\|^r_{L^r(\Omega)} e^{-\frac{C_{14}}{2}r(t-\tau)} + \left(C_{15}r + M_5 r + \frac{r^2}{p}\|g_0(x,t)\|^p_{L^p_b(\mathbb{R};L^r(\Omega))}\right)\left(1 + \frac{2}{C_{14}r}\right).$$

Therefore, problem (1.1) generates a family of processes $\{U_g(t,\tau)\}$, $g \in \mathcal{H}_{L^r(\Omega)}(g_0)$, acting in $L^r(\Omega)$. Moreover, let

$$R_0^r = 2\left(C_{15}r + M_5 r + \frac{r^2}{p}\|g_0(x,t)\|^p_{L^p_b(\mathbb{R};L^r(\Omega))}\right)\left(1 + \frac{2}{C_{14}r}\right),$$

the set

$$B_0 = \left\{u \in L^r(\Omega) \;\middle|\; \|u\|_{L^r(\Omega)} \leqslant R_0\right\} \tag{4.23}$$

is uniformly (w.r.t. $g \in \mathcal{H}_{L^r(\Omega)}(g_0)$) absorbing, i.e., for any bounded set $B \subset L^r(\Omega)$ and for all $g \in \mathcal{H}_{L^r(\Omega)}(g_0)$, there exists $T_1(\tau, B)$ such that for all $t > T_1(\tau, B)$, $U_g(t,\tau)B \subseteq B_0$.

Theorem 4.2. *Assume that $f(x,u)$ satisfies* (1.2)–(1.3) *with $a(x) \in L^\beta$, $\beta > 1$ and exponent $\rho > 1$ such that*

$$\frac{1}{\beta} + \frac{\rho - 1}{r} < \frac{2}{N},$$

and (4.18)–(4.19) *hold. Let $g_0(t) \in L^p_b(\mathbb{R}; L^r(\Omega))$. Then the family of processes $\{U_g(t,\tau)\}$, $g \in \mathcal{H}_{L^r(\Omega)}(g_0)$, corresponding to problem* (1.1) *possesses a compact uniform* (*w.r.t. $g \in \mathcal{H}_{L^r(\Omega)}(g_0)$*) *attractor $\mathcal{A}_{\mathcal{H}_{L^r(\Omega)}(g_0)}$ in $L^r(\Omega)$ satisfying*:

$$\mathcal{A}_{\mathcal{H}_{L^r(\Omega)}(g_0)} = \omega_{0,\mathcal{H}_{L^r(\Omega)}(g_0)}(B_0) = \bigcup_{g \in \mathcal{H}_{L^r(\Omega)}(g_0)} \mathcal{K}_g(0), \tag{4.24}$$

where B_0 is the uniformly (*w.r.t. $g \in \mathcal{H}_{L^r(\Omega)}(g_0)$*) *absorbing set in $L^r(\Omega)$.*

Proof. By the fact that $p > 2$ and the regularity of solutions of (1.1), choosing $\theta = \frac{1}{2}$, we know that

$$B_1 = \bigcup_{g \in \mathcal{H}_{L^r(\Omega)}(g_0)} \bigcup_{\tau \in \mathbb{R}} U_g(\tau + 1, \tau)B_0 \tag{4.25}$$

is also a uniformly absorbing set and bounded in $W_0^{1,r}(\Omega)$. Thus, there exists a compact uniformly (w.r.t. $g \in \mathcal{H}_{L^r(\Omega)}(g_0)$) absorbing set in $L^r(\Omega)$ for the family of processes $\{U_g(t,\tau)\}$, $g \in \mathcal{H}_{L^r(\Omega)}(g_0)$, acting in $L^r(\Omega)$. This implies that the family of processes $\{U_g(t,\tau)\}$ possesses a uniform attractor $\mathcal{A}_{\mathcal{H}_{L^r(\Omega)}(g_0)}$ in $L^r(\Omega)$. To show (4.24), by Theorem 2.3, we only need to check the weak continuity of the family of processes $\{U_g(t,\tau)\}$, $g \in \mathcal{H}_{L^r(\Omega)}(g_0)$.

Let $u_n(t)$ and $u(t)$ be the solutions of (4.10) and (1.1), respectively. Choosing $0 < \epsilon \leqslant \frac{1}{2}$ and $\gamma(\epsilon) \geqslant \frac{1}{2}$ as in Claim 1 of the proof of Theorem 3.1, we have

$$\begin{aligned}\big\| f\big(x, u_n(t)\big)\big\|_{W^{-1,r}(\Omega)} &\leqslant C_{17}\big\| f\big(x, u_n(t)\big)\big\|_{E_r^{\gamma(\epsilon)-1}} \\ &\leqslant C_{18}\big\|a(x)\big\|_{L^\beta(\Omega)}\big(\big\|u_n(t)\big\|_{E_r^\epsilon}^\rho + 1\big) \\ &\leqslant C_{19}\big\|a(x)\big\|_{L^\beta(\Omega)}\big(\big\|u_n(t)\big\|_{W^{1,r}(\Omega)}^\rho + 1\big).\end{aligned} \tag{4.26}$$

For every $\upsilon \in W^{1,r'}(\Omega)$, $\frac{1}{r'} + \frac{1}{r} = 1$,

$$\begin{aligned}\bigg|\int_\Omega u_n'(t)\upsilon\, dx\bigg| &\leqslant \int_\Omega \big|\nabla u_n(t)\big||\nabla\upsilon|\, dx + \int_\Omega \big|f\big(x, u_n(t)\big)\upsilon\big|\, dx + \int_\Omega \big|g(x,t)\big||\upsilon|\, dx \\ &\leqslant \big\|u_n(t)\big\|_{W_0^{1,r}(\Omega)}\|\upsilon\|_{W^{1,r'}(\Omega)} + \big\|f(x,u)\big\|_{W^{-1,r}(\Omega)}\|\upsilon\|_{W^{1,r'}(\Omega)} \\ &\quad + C_{19}\big\|g(x,t)\big\|_{L^r(\Omega)}\|\upsilon\|_{W^{1,r'}(\Omega)}.\end{aligned} \tag{4.27}$$

By (4.25)–(4.27), we obtain that

$$\int_\tau^t \big\|u_n'(t)\big\|_{W^{-1,r}}^r\, ds \leqslant C_{20}\big(\tau, \big\|g_0(x,t)\big\|_{L_b^p(\mathbb{R};L^r(\Omega))}, \big\|a(x)\big\|_{L^\beta(\Omega)}, R_0\big).$$

Thus, by the fact

$$\int_\tau^t \big\|\nabla u_n(s)\big\|_{L^r(\Omega)}^r \leqslant C_{21},$$

we obtain that

$$\big\{u_n(t)\big\} \quad \text{is precompact in } L^r_{loc}\big(\mathbb{R}_\tau; L^r(\Omega)\big).$$

The rest of the proof is similar to that of Theorem 4.1 after (4.15). This completes the proof. □

Remark 4.2. If $g(x,t) \in L_b^p(\mathbb{R}; W^{1,r}(\Omega))$, by Remark 4.1 we notice that the assumption (4.19) in Theorem 4.2 can be relaxed, i.e., $p \geqslant r$, and the result of Theorem 4.2 still holds with $\mathcal{H}_{L^r(\Omega)}(g_0)$ being replaced by $\mathcal{H}_{W^{1,r}(\Omega)}(g_0)$. For $1 < p < r$, substituting assumption (4.18) and $g_0(x,t) \in L_b^p(\mathbb{R}; L^r(\Omega))$ in Theorem 4.2 with (5.21) and $g_0(x,t) \in L_b^p(\mathbb{R}; W^{1,r}(\Omega))$, respectively, we have the same conclusion as in Theorem 4.2 with $\mathcal{H}_{L^r(\Omega)}(g_0)$ being replaced by $\mathcal{H}_{W^{1,r}(\Omega)}(g_0)$. □

4.3. $\frac{1}{\beta} + \frac{\rho-1}{r} = \frac{2}{N}$, $r > 1$

In this case, from Theorem 3.1 we know that the time of existence for the solutions of (1.1) is only uniform on compact $\mathcal{S} \subset L^r(\Omega)$. Thus, we cannot obtain existence of uniform attractor in $L^r(\Omega)$. However, we have:

Theorem 4.3. *Assume that $f(x,u)$ satisfies (1.2)–(1.3) with $a(x) \in L^{\beta}$, $\beta > 1$ and exponent $\rho > 1$ such that*

$$\frac{1}{\beta} + \frac{\rho - 1}{r} = \frac{2}{N},$$

and (4.17)–(4.18) hold. Let $g_0(t) \in L_b^p(\mathbb{R}; L^r(\Omega))$, $p > 2$. Then there exists a compact set $\mathcal{A}$ such that $\mathcal{A}$ uniformly (w.r.t. $g \in \mathcal{H}_{L^r(\Omega)}(g_0)$) attracts every compact set B in $L^r(\Omega)$.

5. Existence of attractors in $W^{1,r}(\Omega)$

5.1. $r = 2$, $\frac{N-2}{N+2}\rho + \frac{2N}{\beta(N+2)} < 1$, $N > 2$

Suppose that the nonlinear function f satisfies the following conditions:

$$\frac{\partial f}{\partial s}(x,s) \geqslant -c_1, \qquad \left|\frac{\partial f}{\partial x}(x,s)\right| \leqslant b(x), \tag{5.1}$$

where c_1 is a positive constant such that $c_1 < \frac{1}{2}(1 - 2\alpha_1)\lambda_1$, $\alpha_1 < \frac{1}{2}$, and $b(x) \in L^2(\Omega)$.

From Theorem 3.2 and Remark 3.1 we know that if $g(x,t)$ belongs to $L_b^p(\mathbb{R}; L^r(\Omega))$ and $L_b^p(\mathbb{R}; W^{1,r}(\Omega))$, respectively, the solutions of (1.1) cannot enter in $H^2(\Omega) := W^{2,2}(\Omega)$ in both cases. Therefore, this brings some difficulties in priori estimates on solutions. To overcome this, using standard Faedo–Galerkin method, we first show that there exists a new type of solutions of (1.1).

Claim 3. *$u \in C([\tau,T]; V) \cap C((\tau,T]; E_2^{\frac{1}{2}+\epsilon})$ is a solution of (1.1) in the sense of (1.4), and*

$$u \in C\big([\tau,T]; V\big) \cap L^2_{loc}\big([\tau,T]; D(-A)\big) \cap L^{\infty}(\mathbb{R}_{\tau}; V) \tag{5.2}$$

is the weak solution of (1.1).

Proof of Claim 3. Since the injection of V in H is compact, $(-A)^{-1}$ can be considered as a self-adjoint compact operator in H. By the elementary spectral theory of self-adjoint compact operators in a Hilbert space, there exists a sequence $\{\lambda_j\}_{j=1}^{\infty}$ and a family of elements $\{\omega_j\}_{j=1}^{\infty}$ of $D(-A)$, which are orthonormal in H, such that

$$\begin{cases} -A\omega_j = \lambda_j \omega_j, & j = 1, 2, \ldots, \\ 0 < \lambda_1 \leqslant \lambda_2, \ldots, & \lambda_j \to \infty \text{ as } j \to \infty. \end{cases} \tag{5.3}$$

Let $H_m = span\{\omega_1, \omega_2, \ldots, \omega_m\}$ in H, $P_m : H \to H_m$ be the orthogonal projector. Let $u_m(t) = \sum_{j=1}^{m} b_{j,m}(t)\omega_j(x)$ be a solution of the following ordinary differential equation

$$\begin{cases} \dfrac{du_m}{dt} - P_m \Delta u_m + P_m f(x, u_m) = P_m g(x,t), \\ u_m(\tau) = P_m u_{\tau}, \end{cases} \tag{5.4}$$

where $b_{j,m}(t)$ are absolutely continuous scalar functions on $[\tau, T]$.

Multiplying (5.4) by $-\Delta u_m$ and integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla u_m(t)\|^2 + \|\Delta u_m(t)\|^2 + \big\langle P_m f(x,u_m), -\Delta u_m\big\rangle = \big\langle P_m g(x,t), -\Delta u_m(t)\big\rangle. \tag{5.5}$$

Note that the third term on the left-hand side of (5.5) can be rewritten as

$$\left\langle P_m f(x,u_m), -\Delta u_m\right\rangle = \int_\Omega \frac{\partial f}{\partial x}(x,u_m)\nabla u_m + \int_\Omega \frac{\partial f}{\partial u_m}(x,u_m)|\nabla u_m|^2, \tag{5.6}$$

and by (5.1),

$$\begin{aligned}\left|\int_\Omega \frac{\partial f}{\partial x}(x,u_m)\nabla u_m\right| &\leqslant \int_\Omega \left|b(x)\right||\nabla u_m| \\ &\leqslant \left\|b(x)\right\|\|\nabla u_m\| \\ &\leqslant \frac{1}{4\alpha_1\lambda_1}\left\|b(x)\right\|^2 + \alpha_1\lambda_1\|\nabla u_m\|^2.\end{aligned} \tag{5.7}$$

The term on the right-hand side of (5.5) satisfies

$$\left\langle P_m g(x,t), -\Delta u_m(t)\right\rangle \leqslant \frac{1}{2}\left\|g(x,t)\right\|^2 + \frac{1}{2}\|\Delta u_m\|^2. \tag{5.8}$$

By (5.1), from (5.5)–(5.8) we have that

$$\begin{aligned}&\frac{d}{dt}\left\|\nabla u_m(t)\right\|^2 + \left\|\Delta u_m(t)\right\|^2 \\ &\quad\leqslant 2(c_1+\alpha_1\lambda_1)\left\|\nabla u_m(t)\right\|^2 + \frac{1}{2\alpha_1\lambda_1}\left\|b(x)\right\|^2 + \left\|g(x,t)\right\|^2,\end{aligned} \tag{5.9}$$

which yields

$$\begin{aligned}&\frac{d}{dt}\left\|\nabla u_m(t)\right\|^2 + \big((1-2\alpha_1)\lambda_1 - 2c_1\big)\left\|\nabla u_m(t)\right\|^2 \\ &\quad\leqslant \frac{1}{2\alpha_1\lambda_1}\left\|b(x)\right\|^2 + \left\|g(x,t)\right\|^2 \\ &\quad\leqslant \frac{1}{2\alpha_1\lambda_1}\left\|b(x)\right\|^2 + \left\|g(x,t)\right\|_H^p + C_{22}.\end{aligned} \tag{5.10}$$

Thus, applying Gronwall's inequality to (5.10) and integrating (5.9) from τ to T, respectively, we obtain that

$$\begin{aligned}u_m(t)\quad &\text{is bounded in } L^\infty\big((\tau,\infty);V\big), \\ &\text{bounded in } L^2\big([\tau,T];H^2(\Omega)\big).\end{aligned}$$

Since $u_m(t)\in C([\tau,T];V)$ and $P_m u_\tau \to u_\tau$ $(m\to\infty)$ strongly in V, we have

$$\begin{aligned}u_m(t) &\rightharpoonup \widetilde{u}(t)\quad *\text{-weakly in } L^\infty\big((\tau,T];V\big), \\ &\rightharpoonup \widetilde{u}(t)\quad \text{weakly in } L^2\big([\tau,T];H^2(\Omega)\big), \\ &\to \widetilde{u}(t)\quad \text{strongly in } C\big([\tau,T];V\big).\end{aligned}$$

Similar to the proof of (4.8), we obtain the result. □

In the rest of this subsection, the solutions of (1.1) are meant the solutions in sense of Claim 3. In obtaining the compact uniformly attracting set in V for the family of processes $\{U_g(t,\tau)\}$, $g \in \mathcal{H}_H(g_0)$, we use the idea in [31]. We first introduce a new class of external forces which are similar to Definition 3.1 in [31].

Definition 5.1. A function $g \in L^p_{loc}(\mathbb{R}; X)$ $(p > 1)$ is said to be p-normal if for any $\epsilon > 0$, there exists $\eta > 0$ such that

$$\sup_{t\in\mathbb{R}} \int_t^{t+\eta} \|g(x,s)\|_X^p \leqslant \epsilon,$$

where X is a Banach space.

Denote by $L^p_n(\mathbb{R}; X)$ the set of all p-normal functions in $L^p_{loc}(\mathbb{R}; X)$. It is easy to see that $L^p_n(\mathbb{R}; X) \subset L^p_b(\mathbb{R}; X)$.

Lemma 5.1. *If $g_0 \in L^p_n(\mathbb{R}; X)$ then for any $\tau \in \mathbb{R}$,*

$$\lim_{\lambda\to+\infty} \sup_{t\geqslant\tau} \int_\tau^t e^{-\lambda(t-s)} \|g(x,s)\|_X^p \, ds = 0$$

uniformly w.r.t. $g \in \mathcal{H}_X(g_0)$.

The proof is similar to Lemma 3.1 in [31].

Theorem 5.2. *Assume that $f(x,u)$ satisfies (1.2)–(1.3) with $a(x) \in L^\beta$, $\beta > 1$ and exponent $\rho > 1$ such that*

$$\frac{N-2}{N+2}\rho + \frac{2N}{\beta(N+2)} < 1, \quad N > 2,$$

and (5.1) holds. Suppose that $g_0(x,t) \in L^p_{loc}(\mathbb{R}; H)$ is p-normal, $p > 2$. Then the family of processes $\{U_g(t,\tau)\}$, $g \in \mathcal{H}_H(g_0)$, corresponding to problem (1.1) possesses a compact uniform (w.r.t. $g \in \mathcal{H}_H(g_0)$) attractor $\mathcal{A}_{\mathcal{H}_H(g_0)}$ in V satisfying:

$$\mathcal{A}_{\mathcal{H}_H(g_0)} = \omega_{0,\mathcal{H}_H(g_0)}(B_0) = \bigcup_{g\in\mathcal{H}_H(g_0)} \mathcal{K}_g(0), \tag{5.11}$$

where B_0 is the uniformly (w.r.t. $g \in \mathcal{H}_H(g_0)$) absorbing set in V.

Proof. Taking the scalar product in H of (1.1) with $-\Delta u$, we have that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2 + \langle -\Delta u, -\Delta u\rangle + \langle f(x,u), -\Delta u\rangle \\
&\quad = \langle g(x,t), -\Delta u\rangle \\
&\quad \leqslant \frac{1}{2}\|g(x,t)\|^2 + \frac{1}{2}\|-\Delta u\|^2.
\end{aligned} \tag{5.12}$$

Proceeding as in the derivation of (5.10), it follows from (5.12) that

$$\frac{d}{dt}\|\nabla u(t)\|^2 + \big((1-2\alpha_1)\lambda_1 - 2c_1\big)\|\nabla u(t)\|^2 \leqslant \frac{1}{2\alpha_1\lambda_1}\|b(x)\|^2 + \|g(x,t)\|_H^p + C_{23}.$$

Applying Lemma 2.2 and (4.2), from above we have that

$$\|\nabla u(t)\|^2 \leqslant \|\nabla u(\tau)\|^2 e^{-((1-2\alpha_1)\lambda_1-2c_1)(t-\tau)} + \left(1+\frac{1}{(1-2\alpha_1)\lambda_1-2c_1}\right)\left(\|g_0(x,t)\|^p_{L_b^p(\mathbb{R};H)} + C_{23}\right).$$

Thus, the family of processes $\{U_g(t,\tau)\}$, $g \in \mathcal{H}_H(g_0)$, acting in V is well defined. This estimate also implies that the set

$$B_0 = \left\{u(t) \in V \;\middle|\; \|u(t)\|_V \leqslant R_0\right\}$$

is the uniformly (w.r.t. $\mathcal{H}_H(g_0)$) absorbing set, where

$$R_0^2 = 2\left(1+\frac{1}{(1-2\alpha_1)\lambda_1-2c_1}\right)\left(\|g_0(x,t)\|^p_{L_b^p(\mathbb{R};H)} + C_{23}\right).$$

For any bounded set $B \subset V$, let $T_0(\tau, B) > \tau$ such that $\bigcup_{g\in\mathcal{H}_H(g_0)} U(t,\tau)B \subset B_0$, $t > T_0$.

Let $V_m = span\{\omega_1, \omega_2, \ldots, \omega_m\}$ in V and $P_m : V \to V_m$ be an orthogonal projector. For any $u \in D(-A)$, write

$$u(t) = P_m u(t) + (I - P_m)u(t) \triangleq u_1(t) + u_2(t).$$

Taking the scalar product in H of (1.1) with $-\Delta u_2$, proceeding as in the derivation of (5.12), we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla u_2(t)\|^2 + \langle -\Delta u_2, -\Delta u_2\rangle + \big\langle f(x,u), -\Delta u_2\big\rangle \leqslant \frac{1}{2}\|g(x,t)\|^2 + \frac{1}{2}\|-\Delta u_2\|^2. \tag{5.13}$$

Similar to (5.7),

$$\left|\frac{\partial f}{\partial x}(x,u)\nabla u_2\right| \leqslant \frac{1}{2}\|b(x)\|^2 + \frac{1}{2}\|\nabla u_2\|^2. \tag{5.14}$$

By (5.1) and (5.14), proceeding as in the derivation of (5.9), from (5.13) we get that

$$\frac{d}{dt}\|\nabla u_2(t)\|^2 + \|\Delta u_2(t)\|^2 \leqslant (2c_1+1)\|\nabla u_2(t)\|^2 + \|b(x)\|^2 + \|g(x,t)\|^2, \tag{5.15}$$

which implies that

$$\frac{d}{dt}\|\nabla u_2(t)\|^2 + (\lambda_m - 2c_1 - 1)\|\nabla u_2(t)\|^2 \leqslant \|b(x)\|^2 + \|g(x,t)\|^2 \leqslant \|b(x)\|^2 + \|g(x,t)\|^p + C_{24}. \tag{5.16}$$

By Gronwall's lemma, (5.16) yields

$$\|\nabla u_2(t)\|^2 \leqslant \|\nabla u_2(T_0+1)\|^2 e^{-(\lambda_m-2c_1-1)(t-(T_0+1))} + \int_{T_0+1}^{t} e^{-(\lambda_m-2c_1-1)(t-s)}\|g(x,t)\|^p\,ds + \left(\|b(x)\|^2 + C_{24}\right)\int_{T_0+1}^{t} e^{-(\lambda_m-2c_1-1)(t-s)}\,ds, \quad \forall t > T_0+1. \tag{5.17}$$

For any $\epsilon > 0$, by Lemma 5.1 we can choose λ_m ($\lambda_m > 2c_1 + 1$) large enough such that

$$\int_{T_0+1}^{t} e^{-(\lambda_m-2c_1-1)(t-s)}\|g(x,t)\|^p\,ds < \epsilon, \quad \forall t > T_0+1, \tag{5.18}$$

and

$$\left(\|b(x)\|^2 + C_{24}\right)\int_{T_0+1}^{t} e^{-(\lambda_m-2c_1-1)(t-s)}\,ds \leqslant \left(\|b(x)\|^2 + C_{24}\right)\frac{1}{\lambda_m - 2c_1 - 1} < \epsilon, \quad \forall t > T_0+1. \tag{5.19}$$

Let $T_1 = T_0 + 1 + \frac{1}{\lambda_m - 2c_1 - 1}\ln\frac{R_0^2}{\epsilon}$. For any $t > T_1$, we have

$$\|\nabla u_2(T_0+1)\|^2 e^{-(\lambda_m-2c_1-1)(t-(T_0+1))} < \epsilon. \tag{5.20}$$

Therefore, (5.17)–(5.20) imply that

$$\|\nabla u_2(t)\|^2 \leqslant 3\epsilon, \quad \forall t \geqslant T_1,\ g \in \mathcal{H}_H(g_0).$$

Using the properties of the Kuratowski measure of non-compactness and Theorem 2.4, we know that the family of processes $\{U_g(t,\tau)\}$, $g \in \mathcal{H}_H(g_0)$, corresponding to problem (1.1) has the uniform (w.r.t. $g \in \mathcal{H}_H(g_0)$) attractor $\mathcal{A}_{\mathcal{H}_H(g_0)}$ in V.

To show (5.11), according to Theorem 2.3, we only need to verify the weak continuity of the family of processes $\{U_g(t,\tau)\}$, $g \in \mathcal{H}_H(g_0)$, acting in V. Let $u_{\tau_n} \rightharpoonup u_\tau$ weakly in V, $g_n \rightharpoonup g_0$ weakly in $L^p_{loc}(\mathbb{R}; H)$. Let $u_n(t) = U_{g_n}(t,\tau)u_{\tau_n}$ be the solution of Eq. (4.10), $u(t) = U_{g_0}(t,\tau)u_\tau$. Similar to the proof of Theorem 4.1, we can obtain

$$\{u_n(t)\} \quad \text{is bounded in } L^\infty(\mathbb{R}_\tau; V) \cap L^2_{loc}\big(\mathbb{R}_\tau; D(-A)\big),$$
$$\{u_n'(t)\} \quad \text{is bounded in } L^2_{loc}\big(\mathbb{R}_\tau; H^{-1}(\Omega)\big).$$

Similar to the proof of the weak continuity of the family of processes in Theorem 4.1, we can obtain that $U_{g_n}(t,\tau)u_{\tau_n} \rightharpoonup U_{g_0}(t,\tau)u_\tau$ weakly in V.

This completes the proof. $\square$

5.2. $1 < r < N$, $\frac{N-2}{N+2}\rho + \frac{2N}{\beta(N+2)} < 1$

We first recall a result in [37, Proposition 48.5].

Lemma 5.3. *Let Ω be an arbitrary bounded domain and let $\{e^{At}\}_{t\geqslant 0}$ be the Dirichlet heat semigroup in Ω. For all $1 \leqslant p \leqslant \infty$ and all $\Phi \in L^p(\Omega)$, there holds*

$$\left\|e^{At}\Phi\right\|_{L^p(\Omega)} \leqslant M_6(\Omega)e^{-\lambda_1 t}\|\Phi\|_{L^p(\Omega)}, \quad t \geqslant 0.$$

We need the following dissipative condition:

$$f(x,u) \geqslant 0 \quad \text{for } a.e.\ x \in \Omega. \tag{5.21}$$

Suppose that $g(x,t) \in L_b^p(\mathbb{R}; W^{1,r}(\Omega))$. Notice that $u(t)$ satisfies Eq. (3.1), by Theorem 3.2 and (5.21), we have

$$\begin{aligned} u(t) &= e^{A(t-\tau)}u(\tau) + \int_\tau^t e^{A(t-s)}\big[-f\big(x,u(s)\big) + g(x,s)\big]\,ds \\ &\leqslant e^{A(t-\tau)}u(\tau) + \int_\tau^t e^{A(t-s)}g(x,s)\,ds. \end{aligned} \tag{5.22}$$

Using Lemma 5.3, from (5.22) we have

$$\begin{aligned} \left\|u(t)\right\|_{W^{1,r}(\Omega)} &\leqslant \left\|e^{A(t-\tau)}u(\tau)\right\|_{W^{1,r}(\Omega)} + \int_\tau^t \left\|e^{A(t-s)}g(x,s)\right\|_{W^{1,r}(\Omega)}\,ds \\ &\leqslant M_6(\Omega)e^{-\lambda_1(t-\tau)}\left\|u(\tau)\right\|_{W^{1,r}(\Omega)} + C_{24}M_6(\Omega)\int_\tau^t e^{-\lambda_1(t-s)}\left\|g(x,s)\right\|_{W^{1,r}(\Omega)}\,ds \\ &\leqslant M_6(\Omega)e^{-\lambda_1(t-\tau)}\left\|u(\tau)\right\|_{W^{1,r}(\Omega)} \\ &\quad + C_{24}M_6(\Omega)\left(\int_\tau^t e^{-\frac{\lambda_1}{2}(t-s)q}\,ds\right)^{\frac{1}{q}}\left(\int_\tau^t e^{-\frac{\lambda_1}{2}(t-s)p}\left\|g(x,s)\right\|^p_{W^{1,r}(\Omega)}\,ds\right)^{\frac{1}{p}} \\ &\leqslant M_6(\Omega)e^{-\lambda_1(t-\tau)}\left\|u(\tau)\right\|_{W^{1,r}(\Omega)} \\ &\quad + C_{24}M_6(\Omega)\left(\frac{2}{\lambda_1 q}\right)^{\frac{1}{q}}\left(\frac{e^{\frac{\lambda_1}{2}p}}{e^{\frac{\lambda_1}{2}p}-1}\right)\left\|g_0(x,t)\right\|^p_{L_b^p(\mathbb{R};W^{1,r}(\Omega))}. \end{aligned} \tag{5.23}$$

Therefore, the family of processes $\{U_g(t,\tau)\}$, $g \in \mathcal{H}_{W^{1,r}(\Omega)}(g_0)$, acting in the space $W^{1,r}(\Omega)$ is well defined. Let

$$R_0 = 2C_{24}M_6(\Omega)\left(\frac{2}{\lambda_1 q}\right)^{\frac{1}{q}}\left(\frac{e^{\frac{\lambda_1}{2}p}}{e^{\frac{\lambda_1}{2}p}-1}\right)\|g_0(x,t)\|^p_{L^p_b(\mathbb{R};W^{1,r}(\Omega))},$$

the set

$$B_0 = \left\{u(t) \in W^{1,r}(\Omega) \;\middle|\; \|u(t)\|_{W^{1,r}(\Omega)} \leqslant R_0\right\} \tag{5.24}$$

is the uniformly (w.r.t. $g \in \mathcal{H}_{W^{1,r}(\Omega)}(g_0)$) absorbing set for the family of processes $\{U_g(t,\tau)\}$. We know from Remark 3.1 that the solutions of (1.1) cannot enter in $W^{2,r}(\Omega)$. However, we can obtain the existence of uniform attractor in the weakly topological space $W^{-1,r}(\Omega)$.

Theorem 5.4. *Assume that $f(x,u)$ satisfies* (1.2)–(1.3) *with $a(x) \in L^\beta$, $\beta > 1$ and exponent $\rho > 1$ such that*

$$\frac{N-r}{N+r}\rho + \frac{Nr}{\beta(N+r)} < 1, \quad 1 < r < N,$$

and (5.21) *holds. Let $g_0(x,t) \in L^p_b(\mathbb{R}; W^{1,r}(\Omega))$, $p > 2$. Then the family of processes $\{U_g(t,\tau)\}$, $g \in \mathcal{H}_{W^{1,r}(\Omega)}(g_0)$, is well defined in $W^{1,r}(\Omega)$ and possesses a uniformly (w.r.t. $\mathcal{H}_{W^{1,r}(\Omega)}(g_0)$) absorbing set in $W^{1,r}(\Omega)$. Moreover, there exists a compact set $\mathcal{A}$ in $W^{-1,r}(\Omega)$ such that $\mathcal{A}$ uniformly (w.r.t. $\mathcal{H}_{W^{1,r}(\Omega)}(g_0)$) attracts bounded set B of $W^{1,r}(\Omega)$ in the topology of $W^{-1,r}(\Omega)$.*

Proof. The existence of a uniformly absorbing set in $W^{1,r}(\Omega)$ is obtained in (5.24). The proof of continuity of the process in the topology $W^{-1,r}(\Omega)$ is similar to the proof of Theorem 4.2. □

6. Properties of attractors

In this section, we investigate the relationship between pullback, forward attractors corresponding to problem (1.1) and uniform attractors obtained in Theorems 4.1, 4.2 and 5.2. Lots of work have been done on studying the existence of pullback attractors for non-autonomous dynamical systems, e.g. [13–15,38,45,49]. Non-autonomous dynamical systems can often be formulated in terms of a *cocycle mapping* ϕ on a state space E for the dynamics in E that is driven by an autonomous dynamical system $\{\theta_t\}_{t\in\mathbb{R}}$ in what is called a parameter space Σ. Let Σ be a metric space and $\{\theta_t\}_{t\in\mathbb{R}}$ be a group acting on Σ satisfying:

(1) $\theta_0(\sigma) = \sigma$ for all $\sigma \in \Sigma$;
(2) $\theta_{t+s}(\sigma) = \theta_t(\theta_s(\sigma))$ for all $t, s \in \mathbb{R}$;
(3) the mapping $(t,\sigma) \to \theta_t(\sigma)$ is continuous.

A mapping $\phi: \mathbb{R}_+ \times \Sigma \times E \to E$ is called a *cocycle* on E if it satisfies:

(1) $\phi(0,\sigma,x) = x$ for all $(\sigma,x) \in \Sigma \times E$;
(2) $\phi(t+s,\sigma,x) = \phi(t,\theta_s(\sigma),\phi(s,\sigma,x))$ for all $t, s \in \mathbb{R}_+$ and all $(\sigma,x) \in \Sigma \times E$;
(3) the mapping $(t,\sigma,x) \to \phi(t,\sigma,x)$ is continuous.

A family of nonempty compact subsets $\{\mathcal{A}_\sigma\}_{\sigma\in\Sigma}$ of E is called a *pullback* (or *cocycle*) *attractor* of ϕ with respect to θ, if for all $\sigma \in \Sigma$, it satisfies:

(1) $\phi(t,\sigma,\mathcal{A}_\sigma)=\mathcal{A}_{\theta_t(\sigma)}$ for all $t\in\mathbb{R}_+$ (ϕ-invariance);
(2) $\lim_{t\to+\infty}\operatorname{dist}_E(\phi(t,\theta_{-t}(\sigma),B),\mathcal{A}_\sigma)=0$ for all bounded $B\subset E$.

A family of nonempty compact subsets $\{\mathcal{A}_\sigma\}_{\sigma\in\Sigma}$ of E is said to be a *forward attractor* of ϕ, if it satisfies the ϕ-invariance property and if, in addition, $\{\mathcal{A}_\sigma\}_{\sigma\in\Sigma}$ forward attract each bounded set B of E, i.e.,

$$\lim_{t\to+\infty}\operatorname{dist}_E\big(\phi(t,\sigma,B),\mathcal{A}_{\theta_t(\sigma)}\big)=0.$$

Suppose that assumptions of Theorem 4.1 hold. By Theorem 3.1 and (4.6), we can define a continuous cocycle ϕ on H:

$$\phi(t,g,u_\tau)=u(t),\quad \forall(t,g,u_\tau)\in\mathbb{R}_\tau\times\mathcal{H}_H(g_0)\times H, \tag{6.1}$$

where $u(t)$ is the solution of problem (1.1) with initial data $u_\tau\in H$ and external force function $g\in\mathcal{H}_H(g_0)$. By Theorem 3.4 of [45], from the proof of Theorem 4.1 we deduce that the cocycle ϕ defined by (6.1) possesses a pullback attractor $\mathcal{A}^H$ in H:

$$\mathcal{A}^H=\big\{\mathcal{A}^H_g\big\}_{g\in\mathcal{H}_H(g_0)}=\big\{\omega_g(B_0)\big\}_{g\in\mathcal{H}_H(g_0)},$$

where B_0 is the bounded uniformly absorbing set that is the same one as in Theorem 4.1, and

$$\omega_g(B_0)=\bigcap_{s\geqslant 0}\overline{\bigcup_{t\geqslant s}\phi\big(t,\theta_{-t}(g),B_0\big)}^H.$$

Therefore, we have:

Proposition 6.1. *Under the assumptions of Theorem* 4.1, *the cocycle ϕ corresponding to problem* (1.1) *possesses a pullback attractor $\mathcal{A}^H_1=\{\mathcal{A}^H_{1g}\}_{g\in\mathcal{H}_H(g_0)}$ and a forward attractor $\mathcal{A}^H_2=\{\mathcal{A}^H_{2g}\}_{g\in\mathcal{H}_H(g_0)}$ in H. Moreover,*

$$\mathcal{A}_{\mathcal{H}_H(g_0)}=\bigcup_{g\in\mathcal{H}_H(g_0)}\mathcal{A}^H_{1g}=\bigcup_{g\in\mathcal{H}_H(g_0)}\mathcal{A}^H_{2g}.$$

Analogously, we have:

Proposition 6.2. *Under the assumptions of Theorem* 4.2, *the cocycle ϕ corresponding to problem* (1.1) *possesses a pullback attractor $\mathcal{A}^{L^r(\Omega)}_1=\{\mathcal{A}^{L^r(\Omega)}_{1g}\}_{g\in\mathcal{H}_{L^r(\Omega)}(g_0)}$ and a forward attractor $\mathcal{A}^{L^r(\Omega)}_2=\{\mathcal{A}^{L^r(\Omega)}_{2g}\}_{g\in\mathcal{H}_{L^r(\Omega)}(g_0)}$ in $L^r(\Omega)$. Moreover,*

$$\mathcal{A}_{\mathcal{H}_{L^r(\Omega)}(g_0)}=\bigcup_{g\in\mathcal{H}_{L^r(\Omega)}(g_0)}\mathcal{A}^{L^r(\Omega)}_{1g}=\bigcup_{g\in\mathcal{H}_{L^r(\Omega)}(g_0)}\mathcal{A}^{L^r(\Omega)}_{2g}.$$

Proposition 6.3. *Under the assumptions of Theorem* 5.2, *the cocycle ϕ corresponding to problem* (1.1) *possesses a pullback attractor $\mathcal{A}^V_1=\{\mathcal{A}^V_{1g}\}_{g\in\mathcal{H}_H(g_0)}$ and a forward attractor $\mathcal{A}^V_2=\{\mathcal{A}^V_{2g}\}_{g\in\mathcal{H}_H(g_0)}$ in V. Moreover,*

$$\mathcal{A}_{\mathcal{H}_H(g_0)}=\bigcup_{g\in\mathcal{H}_H(g_0)}\mathcal{A}^V_{1g}=\bigcup_{g\in\mathcal{H}_H(g_0)}\mathcal{A}^V_{2g}.$$

Acknowledgments

Firstly, the authors extend the great thanks to anonymous referee, and with his/her comments, the version of the paper has been improved. This work was utterly done while the first author was visiting the Department of Mathematics, University of Miami. The first author would like to acknowledge the great hospitality of the faculty and staff from this Institution.

References

[1] N. Ackermann, T. Bartsch, Superstable manifolds of nondissipative parabolic problems, J. Dynam. Differential Equations 17 (2005) 115–173.
[2] R. Adams, Sobolev Spaces, Academic Press, Boston, 1978.
[3] H. Amann, Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems, in: Schmeisser/Triebel: Function Spaces, Differential Operators and Nonlinear Analysis, in: Teubner-Texte Math., vol. 133, Teubner, Stuttgart, 1993, pp. 9–126.
[4] H. Amann, J. López-Gómez, A priori bounds and multiple solutions for superlinear indefinite elliptic problems, J. Differential Equations 146 (1998) 336–374.
[5] A. Ambrosetti, H. Brézis, G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, J. Funct. Anal. 122 (1994) 519–543.
[6] J.M. Arrieta, A.N. Carvalho, Abstract parabolic problems with critial nonlinearities and applications to Navier–Stokes and heat equations, Trans. Amer. Math. Soc. 352 (1999) 285–310.
[7] J.M. Arrieta, A.N. Carvalho, A. Rodríguez-Bernal, Attractors of parabolic problems with nonlinear boundary conditions. Uniform bounds, Comm. Partial Differential Equations 25 (1/2) (2000) 1–37.
[8] J.M. Arrieta, A.N. Carvalho, A. Rodríguez-Bernal, Parabolic problems with nonlinear boundary conditions and critical nonlinearities, J. Differential Equations 156 (1999) 376–406.
[9] A.V. Babin, M.I. Vishik, Attractors of Evolution Equations, North-Holland, Amsterdam, 1992.
[10] A.V. Babin, M.I. Vishik, Attractors of partial differential evolution equations in an unbounded domain, Proc. Roy. Soc. Edinburgh Sect. A 116 (1990) 221–243.
[11] J.M. Ball, Global attractors for damped semilinear wave equations, Discrete Contin. Dyn. Syst. 10 (2004) 31–52.
[12] P.W. Bates, K. Lu, B. Wang, Random attractors for stochastic reaction–diffusion equations on unbounded domains, J. Differential Equations 246 (2009) 845–869.
[13] H. Brézis, T. Cazenave, A nonlinear heat equation with singular initial data, J. Anal. Math. 68 (1996) 277–304.
[14] T. Caraballo, J.A. Lange, J.C. Robinson, Attractors for differential equations with variable delays, J. Math. Anal. Appl. 260 (2001) 421–438.
[15] D.N. Cheban, P.E. Kloeden, B. Schmalfuss, The relationship between pullback, forward and global attractors of nonautonomous dynamical systems, Nonlinear Dyn. Syst. Theory 2 (2002) 9–28.
[16] V.V. Chepyzhov, M.I. Vishik, Attractors for Equations of Mathematical Physics, Amer. Math. Soc. Colloq. Publ., vol. 49, American Mathematical Society, Providence, RI, 2002.
[17] V.V. Chepyzhov, M.I. Vishik, Attractors of nonautonomous dynamical systems and their dimension, J. Math. Pures Appl. 73 (1994) 279–333.
[18] V.V. Chepyzhov, M.I. Vishik, Evolution equations and their trajectory attractors, J. Math. Pures Appl. 76 (1997) 913–964.
[19] V.V. Chepyzhov, M.I. Vishik, Non-autonomous 2D Navier–Stokes system with singularly oscillating external force and its global attractor, J. Dynam. Differential Equations 19 (2007) 655–684.
[20] D.G. De Figueiredo, J.P. Gossez, P. Ubilla, Local superlinearity and sublinearity for indefinite semilinear elliptic problems, J. Funct. Anal. 199 (2003) 452–467.
[21] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag, 1985.
[22] M.A. Efendiev, S.V. Zelik, The attractor for a nonlinear reaction–diffusion system in an unbounded domain, Comm. Pure Appl. Math. 54 (2001) 625–688.
[23] M. Fila, P. Souplet, F.B. Weissler, Linear and nonlinear heat equations in L^q_δ spaces and universal bounds for global solutions, Math. Ann. 320 (2001) 87–113.
[24] Y. Giga, Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier–Stokes system, J. Differential Equations 62 (1986) 186–212.
[25] J.K. Hale, Asymptotic Behavior of Dissipative Systems, American Mathematical Society, Providence, RI, 1988.
[26] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Math., vol. 840, Springer-Verlag, Berlin, 1981.
[27] J.L. Lions, Quelques Méthodes de Résolution des Problèmes Aux Limites Non Linéaires, Dunod Gauthier–Villars, Paris, 1969.
[28] M. Loayza, The heat equation with singular nonlinearity and singular initial data, J. Differential Equations 229 (2006) 509–528.
[29] S.S. Lu, Attractors for nonautonomous 2D Navier–Stokes equations with less regular normal forces, J. Differential Equations 230 (2006) 196–212.
[30] S.S. Lu, Attractors for nonautonomous reaction–diffusion systems with symbols without strong translation compactness, Asymptot. Anal. 54 (2007) 197–210.
[31] S.S. Lu, H. Wu, C.K. Zhong, Attractors for nonautonomous 2D Navier–Stokes equations with normal external forces, Discrete Contin. Dyn. Syst. 13 (2005) 701–719.

[32] Q.F. Ma, S.H. Wang, C.K. Zhong, Necessary and sufficient conditions for the existence of global attractors for semigroups and applications, Indiana Univ. Math. J. 51 (2002) 1541–1559.

[33] S. Ma, C.K. Zhong, The attractors for weakly damped non-autonomous hyperbolic equations with a new class of external forces, Discrete Contin. Dyn. Syst. 18 (2007) 53–70.

[34] W.M. Ni, P. Sacks, Singular behavior in nonlinear parabolic equations, Trans. Amer. Math. Soc. 287 (1985) 657–671.

[35] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, 1983.

[36] P. Quittner, F. Simondon, A priori bounds and complete blow-up of positive solutions of indefinite superlinear parabolic problems, J. Math. Anal. Appl. 304 (2005) 614–631.

[37] P. Quittner, P. Souplet, Superlinear Paraboloic Problems, Blow-Up, Global Existence and Steady States, Birkhäuser, Basel, Boston, Berlin, 2007.

[38] J.C. Robinson, A. Rodríguez-Bernal, A. Vidal-López, Pullback attractors and extremal complete trajectories for non-autonomous reaction–diffusion problems, J. Differential Equations 238 (2007) 289–337.

[39] A. Rodríguez-Bernal, Attractors for parabolic equations with nonlinear boundary conditions, critical exponents, and singular initial data, J. Differential Equations 181 (2002) 165–196.

[40] R. Rosa, The global attractor for the 2D Navier–Stokes flow on some unbounded domains, Nonlinear Anal. 32 (1998) 71–213.

[41] G.R. Sell, Y.C. You, Dynamics of Evolutionary Equations, Springer-Verlag, New York, 2002.

[42] S. Snoussi, S. Tayachi, Global existence, asymptotic behavior and self-similar solutions for a class of semilinear parabolic systems, Nonlinear Anal. 48 (2002) 13–35.

[43] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, 2nd ed., Springer-Verlag, New York, 1997.

[44] B. Wang, Attractors for reaction–diffusion equations in unbounded domains, Phys. D 128 (1999) 41–52.

[45] Y.J. Wang, C.K. Zhong, S.F. Zhou, Pullback attractors of nonautonomous dynamical systems, Discrete Contin. Dyn. Syst. 16 (2006) 587–614.

[46] F.B. Weissler, Local existence and nonexistence for semilinear parabolic equation in L^p, Indiana Univ. Math. J. 29 (1980) 79–102.

[47] F.B. Weissler, Semilinear evolution equations in Banach spaces, J. Funct. Anal. 32 (1979) 277–296.

[48] S.V. Zelik, Attractors of reaction–diffusion systems in unbounded domains and their spatial complexity, Comm. Pure Appl. Math. 56 (2003) 584–637.

[49] C. Zhao, S. Zhou, Pullback attractors for a non-autonomous incompressible non-Newtonian fluid, J. Differential Equations 238 (2007) 394–425.