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Bifurcations in a discrete predator–prey model with nonmonotonic functional response

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A B S T R A C T

The predator–prey/consumer–resource interaction is the most fundamental and important process in population dynamics. Many species, such as monocarpic plants and semelparous animals, have discrete nonoverlapping generations and their births occur in regular breeding seasons. Their interactions are described by difference equations or formulated as discrete-time mappings. In this paper we study bifurcations in a discrete predator–prey model with nonmonotone functional response described by a simplified Holling IV function. It is shown that the model exhibits various bifurcations of codimension 1, including fold bifurcation, transcritical bifurcation, flip bifurcations and Neimark–Sacker bifurcation, as the values of parameters vary. Moreover, we establish the existence of Bogdanov–Takens bifurcation of codimension 2 and calculate the approximate expressions of bifurcation curves. Numerical simulations are also presented to illustrate the theoretical analysis. These results demonstrate that the Bogdanov–Takens bifurcation of codimension 2 at the degenerate singularity persists in all three versions of the predator–prey model with nonmonotone functional response: continuous-time, time-delayed, and discrete-time.

1. Introduction

The predator–prey/consumer–resource interaction is the most fundamental and important process in population dynamics. For populations with overlapping generations, the birth processes occur continuously, so the predator–prey interaction is usually modeled by ordinary differential equations. Very rich and com-

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plex dynamics and bifurcations have been observed in continuous-time predator–prey systems (see e.g., Collings [6], Huang and Xiao [11], Ruan and Xiao [21] and references there). Many other species, such as monocarpic plants and semelparous animals, have discrete nonoverlapping generations and their births occur in regular breeding seasons. Their interactions are described by difference equations or formulated as discrete-time mappings. Discrete-time predator–prey models can exhibit even more complicated dynamics than the corresponding continuous-time models (see e.g., Huang [10], Li and Zhang [15], Liu and Xiao [16], May [17], Maynard Smith [18] and references there).

The pioneer model describing the discrete time predator–prey/host–parasitoid interactions is the now well-known Nicholson and Bailey model (Nicholson and Bailey [20]):

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \lambda x e^{-ay} \\ cx(1-e^{-ay}) \end{pmatrix}, \tag{1.1}$$

where x and y are numbers of prey (hosts) and predators (parasitoids) in generation t, λ is the net reproductive rate of prey (hosts), c is the clutch size of the predators (parasitoids), a is the area of discovery with units of area. The Nicholson and Bailey model is the canonical model for predator–prey/host–parasitoid interactions (Kot [12]). It generates large oscillations which can drive both species to extinction. To stabilize the model, Beddinton et al. [1] introduced self-limitation (density dependence) to the prey (host) population and proposed the following model:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \lambda x \exp[r(1-\frac{x}{K}) - ay] \\ cx(1-e^{-ay}) \end{pmatrix}, \tag{1.2}$$

where r is the intrinsic growth rate and K is the carrying capacity of the prey population in the absence of predators. Model (1.2) can exhibit transcritical, flip, or Neimark–Sacker bifurcations (Beddinton et al. [1,2], May [17]).

The Lotka–Volterra type discrete-time predator–prey model was first proposed by Maynard Smith [18] and has been analyzed by Levine [14] and Liu and Xiao [16]. It has been shown that such systems undergo fold bifurcation, flip bifurcation and Neimark–Sacker bifurcation. The discrete-time predator–prey model with Holling type II functional response was first derived from the original continuous-time model by Hadeler and Gerstmann [8], see also Neubert and Kot [19]. For such a model, Li and Zhang [15] clarified the parameter conditions for non-hyperbolicity and then completely discussed bifurcations of codimension 1.

Most discrete-time predator–prey models possess three fixed points or equilibria that correspond to (i) extinction of both species, (ii) extinction of the predator with survival of the prey at its carrying capacity, and (iii) coexistence of both species. To study the nonlinear dynamics of discrete-time predator–prey models with more than three fixed points or equilibria, in this article we consider a discrete-time predator–prey (resource–consumer) model with nonmonotone or Holling type IV functional response.

Nonmonotonic functional response appears naturally in the cases of "inhibition" in microbial dynamics and "group defence" in population dynamics. In microbial dynamics, there are experiments that indicate that nonmonotonic responses occur at the microbial level: when the nutrient concentration reaches a high level an inhibitory effect on the specific growth rate may occur. In population dynamics, group defense is a term used to describe the phenomenon whereby predation is decreased, or even prevented altogether, due to the increased ability of the prey to better defend or disguise themselves when their numbers are large enough. In experiments on the uptake of phenol by pure culture of *Pseudomonas putida* growing on phenol in continuous culture, Sokol and Howell [22] proposed a nonmonotone function of the form

$$\frac{mx}{a+x^2}. \tag{1.3}$$

Collings [6] also used this function in a mite predator–prey interaction model and called it a *Holling type-IV* function.

Ruan and Xiao [21] studied the following continuous-time predator–prey system with nonmonotonic functional response or Holling type-IV response function

$$\begin{aligned} \tfrac{dX}{dT} &= RX\left(1-\tfrac{X}{K}\right)-\tfrac{MXY}{A+X^2} \\ \tfrac{dY}{dT} &= Y\left[-D+\tfrac{CX}{A+X^2}\right], \end{aligned} \tag{1.4}$$

where $X(T)$ and $Y(T)$ represent the densities of the prey and predator populations at (continuous) time T, respectively. $R>0$ and $K>0$ are the intrinsic growth rate and carrying capacity of the prey in absence of predators. $M>0$ is the maximal growth rate of predators, $A>0$ is the half-saturation constant, $C>0$ is the conversion rate, and $D>0$ is the death rate of predators. The bifurcation analysis of the model depending on all parameters was performed which indicates that it exhibits numerous kinds of bifurcation phenomena, including saddle-node bifurcation, supercritical and subcritical Hopf bifurcations, and homoclinic bifurcation. It was shown that there are different parameter values for which the model has a limit cycle or a homoclinic loop, or exhibits the so-called paradox of enrichment phenomenon. Moreover, it was shown that a limit cycle cannot coexist with a homoclinic loop for all parameters. In the generic case, they proved that the model has the bifurcation of cusp type of codimension 2 (i.e., Bogdanov–Takens bifurcation) but for some specific parameter values it has a multiple focus of multiplicity at least 2.

To derive a discrete-time model from (1.4), let

$$\frac{dX}{dT}=\frac{x_{t+h}-x_t}{h},\quad \frac{dY}{dT}=\frac{y_{t+h}-y_t}{h},$$

where x_t and y_t are the densities of the prey and predator populations in discrete time (generation) t. Moreover, let $h\to 1$ and $D=1$. We have the equations for the $(t+1)$th generation of the prey and predator populations

$$\begin{aligned} x_{t+1} &= (R+1)x_t\left[1-\tfrac{R}{K(1+R)}x_t\right]-\tfrac{Mx_ty_t}{A+x_t^2} \\ y_{t+1} &= \tfrac{Cx_ty_t}{A+x_t^2}. \end{aligned} \tag{1.5}$$

Letting

$$x_t \longrightarrow \frac{R}{K(1+R)}x_t$$

and

$$a=R+1,\quad b=\frac{M}{A},\quad d=\frac{CK(1+R)}{AR},\quad \epsilon=\frac{K^2(1+R)^2}{AR^2},$$

rewriting (1.5) as a mapping, we obtain the following discrete-time predator–prey (resource–consumer) system with nonmonotonic functional response

$$F:\begin{pmatrix} x \\ y\end{pmatrix}\to\begin{pmatrix} ax(1-x)-\frac{bxy}{1+\epsilon x^2} \\ \frac{dxy}{1+\epsilon x^2}\end{pmatrix}, \tag{1.6}$$

where a, b, d and ϵ are all positive constants. By the biological meaning of the model variables, we only consider system (1.6) in the region $\Omega=\{(x,y): x\geq 0, y\geq 0\}$ in the (x,y)-plane.

Recall that the continuous-time predator–prey system (1.4) undergoes Bogdanov–Takens bifurcation (Ruan and Xiao [21]), a codimension-two bifurcation. An interesting question is whether system (1.6), the discrete-time version of system (1.4), still exhibits Bogdanov–Takens bifurcation. The theory on Bogdanov–Takens bifurcation for generic diffeomorphisms can be found in Broer et al. [4,5] and Kuznetsov [13]. In particular, if the Jacobian matrix of a planar diffeomorphism at a fixed point has a double unit eigenvalue but is not the identity (1:1 resonance), then the diffeomorphism can be approximated by the time-one flow of a vector field which has a singularity with nilpotent linear part. Yagasaki [27] studied Bogdanov–Takens bifurcation for subharmonics in periodic perturbations of planar Hamiltonian systems and gave estimations of the bifurcations sets near the Bogdanov–Takens bifurcation points of diffeomorphisms. We will use the results in Broer et al. [4,5] and Kuznetsov [13] and techniques in Yagasaki [27] to prove the existence of Bogdanov–Takens bifurcation and calculate the approximate expressions of bifurcation curves in system (1.6). To the best of our knowledge, we believe that this is the first study showing the existence of Bogdanov–Takens bifurcation in discrete-time predator–prey systems.

The paper is organized as follows. In section 2, we study the number and stability of fixed points for model (1.6). In section 3, we discuss bifurcations of codimension 1, including fold bifurcation, transcritical bifurcation, flip bifurcation and Neimark–Sacker bifurcation, for model (1.6). The Bogdanov–Takens bifurcation of codimension 2 is discussed in section 4. Numerical simulations are given in section 5. The paper ends with a brief discussion in section 6.

2. Fixed points

In this section, we present results on the existence and stability of fixed points of the discrete model (1.6), detailed derivations are given in the Appendix. By simple calculations, we can see that the map (1.6) has at most four fixed points under various conditions: the trivial fixed point $O(0,0)$, a semitrivial fixed point $A(\frac{a-1}{a},0)$ if $a>1$, and two positive fixed points

$$E_1(x_1,y_1)=E_1\Big(\frac{d-\sqrt{d^2-4\epsilon}}{2\epsilon},\frac{dx_1}{b}\Big(a(1-x_1)-1\Big)\Big),$$
$$E_2(x_2,y_2)=E_2\Big(\frac{d+\sqrt{d^2-4\epsilon}}{2\epsilon},\frac{dx_2}{b}\Big(a(1-x_2)-1\Big)\Big).$$

The two positive fixed points may coalesce into a unique positive fixed point

$$E_0(x_0,y_0)=E_0\Big(\frac{2}{d},\frac{2a(d-2)-2d}{bd}\Big).$$

The existence conditions of these fixed points are summarized in Table 1.

We have the following results on the linear stability of these fixed points.

2.1. The trivial fixed point $O(0,0)$ and semitrivial fixed point $A(\frac{a-1}{a},0)$

The fixed point O is a stable node if $0<a<1$, or a saddle if $a>1$, or non-hyperbolic if $a=1$. The fixed point A arises when $a>1$. The properties of A are given in Table 2.

2.2. The first positive fixed point $E_1(x_1,y_1)$

Combining with the existence conditions of the positive fixed point E_1, we know that when $\epsilon<\frac{d^2}{4}, x_1<1$ and $y_1>0$, the positive fixed point $E_1(x_1,y_1)$ exists. The properties of E_1 are given in Table 3, where $f(x)=5dx^2-(3d+2)x+2$, $x_{11}=\frac{3d+2-\sqrt{\Delta}}{10d}$, $x_{12}=\frac{3d+2+\sqrt{\Delta}}{10d}$, $\Delta=(3d+2)^2-40d$.

Table 1
Existence of fixed points.

<table>
<tr><td colspan="4">Conditions</td><td>Fixed points</td></tr>
<tr><td colspan="4">$a,\ b,\ \epsilon,\ d > 0$</td><td>O</td></tr>
<tr><td>$\epsilon > \dfrac{d^2}{4}$</td><td colspan="3">$a > 1$</td><td>$O,\ A$</td></tr>
<tr><td rowspan="3">$\epsilon = \dfrac{d^2}{4}$</td><td>$0 < d \leq 2$</td><td colspan="2">$a > 1$</td><td rowspan="2">$O,\ A$</td></tr>
<tr><td>$d > 2$</td><td colspan="2">$1 < a \leq \frac{d}{d-2}$</td></tr>
<tr><td>$d > 2$</td><td colspan="2">$a > \frac{d}{d-2}$</td><td>$O,\ A,\ E_0$</td></tr>
<tr><td rowspan="7">$\epsilon < \dfrac{d^2}{4}$</td><td>$0 < d < 2$</td><td>$\epsilon \geq d - 1$</td><td>$a > 1$</td><td rowspan="3">$O,\ A$</td></tr>
<tr><td>$1 < d \leq 2$</td><td>$\epsilon < d - 1$</td><td>$1 < a \leq \frac{2\epsilon}{2\epsilon - d + \sqrt{d^2 - 4\epsilon}}$</td></tr>
<tr><td>$d > 2$</td><td colspan="2">$1 < a \leq \frac{2\epsilon}{2\epsilon - d + \sqrt{d^2 - 4\epsilon}}$</td></tr>
<tr><td>$1 < d \leq 2$</td><td>$\epsilon < d - 1$</td><td>$a > \frac{2\epsilon}{2\epsilon - d + \sqrt{d^2 - 4\epsilon}}$</td><td rowspan="3">$O,\ A,\ E_1$</td></tr>
<tr><td>$d > 2$</td><td>$\epsilon \leq d - 1$</td><td>$a > \frac{2\epsilon}{2\epsilon - d + \sqrt{d^2 - 4\epsilon}}$</td></tr>
<tr><td>$d > 2$</td><td>$\epsilon > d - 1$</td><td>$\frac{2\epsilon}{2\epsilon - d + \sqrt{d^2 - 4\epsilon}} < a \leq \frac{2\epsilon}{2\epsilon - d - \sqrt{d^2 - 4\epsilon}}$</td></tr>
<tr><td>$d > 2$</td><td>$\epsilon > d - 1$</td><td>$a > \frac{2\epsilon}{2\epsilon - d - \sqrt{d^2 - 4\epsilon}}$</td><td>$O,\ A,\ E_1,\ E_2$</td></tr>
</table>

Table 2
Properties of the semitrivial fixed point A.

<table>
<tr><td colspan="2" rowspan="2">Conditions</td><td colspan="2">Eigenvalues</td><td rowspan="2">Properties</td></tr>
<tr><td>$\lambda_1 = 2 - a$</td><td>$\lambda_2 = \frac{ad(a-1)}{a^2 + \epsilon(a-1)^2}$</td></tr>
<tr><td rowspan="3">$1 < a < 3$</td><td>$d > \frac{a}{a-1},\ 0 < \epsilon < \frac{a(ad-d-a)}{(a-1)^2}$</td><td rowspan="3">$|\lambda_1| < 1$</td><td>$\lambda_2 > 1$</td><td>unstable</td></tr>
<tr><td>$d > \frac{a}{a-1},\ \epsilon = \frac{a(ad-d-a)}{(a-1)^2}$</td><td>$\lambda_2 = 1$</td><td>non-hyperbolic</td></tr>
<tr><td>$\epsilon > \frac{a(ad-d-a)}{(a-1)^2}$</td><td>$0 < \lambda_2 < 1$</td><td>stable</td></tr>
<tr><td rowspan="3">$a = 3$</td><td>$d > \frac{3}{2},\quad 0 < \epsilon < \frac{3(2d-3)}{4}$</td><td rowspan="3">$\lambda_1 = -1$</td><td>$\lambda_2 > 1$</td><td rowspan="3">non-hyperbolic</td></tr>
<tr><td>$d > \frac{3}{2},\quad \epsilon = \frac{3(2d-3)}{4}$</td><td>$\lambda_2 = 1$</td></tr>
<tr><td>$\epsilon > \frac{3(2d-3)}{4}$</td><td>$0 < \lambda_2 < 1$</td></tr>
<tr><td rowspan="3">$a > 3$</td><td>$d > \frac{a}{a-1},\ 0 < \epsilon < \frac{a(ad-d-a)}{(a-1)^2}$</td><td rowspan="3">$\lambda_1 < -1$</td><td>$\lambda_2 > 1$</td><td>unstable</td></tr>
<tr><td>$d > \frac{a}{a-1},\ \epsilon = \frac{a(ad-d-a)}{(a-1)^2}$</td><td>$\lambda_2 = 1$</td><td>non-hyperbolic</td></tr>
<tr><td>$\epsilon > \frac{a(ad-d-a)}{(a-1)^2}$</td><td>$0 < \lambda_2 < 1$</td><td>unstable</td></tr>
</table>

Table 3
Properties of the positive fixed point E_1.

<table>
<tr><td colspan="3">Conditions</td><td>Eigenvalues</td><td>Properties</td></tr>
<tr><td>$\frac{3}{2} < d < \frac{9}{4}$</td><td>$\frac{1}{d} < x_1 < \frac{2}{3}$</td><td>$\frac{1}{1-x_1} < a < \frac{2+dx_1}{f(x_1)}$</td><td rowspan="6">$|\lambda_{1,2}| < 1$</td><td rowspan="6">stable</td></tr>
<tr><td rowspan="2">$\frac{9}{4} \leq d \leq 3$</td><td>$\frac{1}{d} < x_1 \leq \frac{4d-2}{7d}$</td><td>$\frac{1}{1-x_1} < a < \frac{1}{1-2x_1}$</td></tr>
<tr><td>$\frac{4d-2}{7d} < x_1 < \frac{2}{3}$</td><td>$\frac{1}{1-x_1} < a < \frac{2+dx_1}{f(x_1)}$</td></tr>
<tr><td rowspan="2">$3 < d < 4$</td><td>$\frac{1}{d} < x_1 \leq \frac{4d-2}{7d}$</td><td>$\frac{1}{1-x_1} < a < \frac{1}{1-2x_1}$</td></tr>
<tr><td>$\frac{4d-2}{7d} < x_1 < \frac{2}{d}$</td><td>$\frac{1}{1-x_1} < a < \frac{2+dx_1}{f(x_1)}$</td></tr>
<tr><td>$d \geq 4$</td><td>$\frac{1}{d} < x_1 < \frac{2}{d}$</td><td>$\frac{1}{1-x_1} < a < \frac{1}{1-2x_1}$</td></tr>
<tr><td>$\frac{3}{2} < d < \frac{14+4\sqrt{10}}{9}$</td><td>$\frac{1}{d} < x_1 < \frac{2}{3}$</td><td rowspan="4">$a = \frac{2+dx_1}{f(x_1)}$</td><td rowspan="4">$\lambda_1 = -1$
$\lambda_2 = -q(x_1)$</td><td rowspan="7">non-hyperbolic</td></tr>
<tr><td rowspan="2">$\frac{14+4\sqrt{10}}{9} \leq d \leq 3$</td><td>$\frac{1}{d} < x_1 < x_{11}$</td></tr>
<tr><td>$x_{12} \leq x_1 < \frac{2}{3}$</td></tr>
<tr><td>$3 < d < 4$</td><td>$x_{12} < x_1 < \frac{2}{d}$</td></tr>
<tr><td>$\frac{9}{4} < d < 4$</td><td>$\epsilon = \frac{7(4d^3-9d^2)}{4(2d-1)^2}$</td><td>$a = \frac{7d}{4-d}$</td><td>$\lambda_{1,2} = -1$</td></tr>
<tr><td>$\frac{9}{4} < d < 4$</td><td>$\frac{1}{d} < x_1 < \frac{4d-2}{7d}$</td><td rowspan="2">$a = \frac{1}{1-2x_1}$</td><td rowspan="2">$|\lambda_i| = 1$
$\lambda_1 = \overline{\lambda_2}$</td></tr>
<tr><td>$d \geq 4$</td><td>$\frac{1}{d} < x_1 < \frac{2}{d}$</td></tr>
<tr><td colspan="4">others</td><td>unstable</td></tr>
</table>

Table 4
Properties of the positive fixed point E_2.

Conditions			Eigenvalues	Properties
$3 < d < 4$	$\frac{2}{d} < x_2 < \frac{2}{3}$	$a = \frac{2+dx_2}{f(x_2)}$	$\lambda_1 = -1$	non-hyperbolic
$d \geq 4$	$x_{12} < x_2 < \frac{2}{3}$		$\lambda_2 = -q(x_2) \neq 1, -1$	
others				unstable

Table 5
Properties of the fixed point E_0.

Conditions	Eigenvalues	Properties
	$\lambda_1 = 1, \quad \lambda_2 = \dfrac{a(d-4)}{d}$	
$a \neq \pm\dfrac{d}{d-4}$, $d > 2$	$\lambda_1 = 1, \quad \lvert\lambda_2\rvert \neq 1$	non-hyperbolic
$a = \dfrac{d}{4-d}$, $3 < d < 4$	$\lambda_1 = 1, \quad \lambda_2 = -1$	non-hyperbolic
$a = \dfrac{d}{d-4}$, $d > 4$	$\lambda_1 = 1, \quad \lambda_2 = 1$	non-hyperbolic

2.3. The second positive fixed point $E_2(x_2, y_2)$

When $d > 2, d - 1 < \epsilon < \frac{d^2}{4}$ and $a > \frac{1}{1-x_2}$, the positive fixed point $E_2(x_2, y_2)$ exists and the properties of E_2 are given in Table 4, where $f(x) = 5dx^2 - (3d+2)x + 2$, $x_{12} = \frac{3d+2+\sqrt{\Delta}}{10d}$, $\Delta = (3d+2)^2 - 40d$, $q(x_2) = a(1 - 2x_2)$.

2.4. The unique positive fixed point $E_0(x_0, y_0)$

When $\epsilon = \frac{d^2}{4}$, $d > 2$ and $a > \frac{d}{d-2}$, the map (1.6) has a unique positive fixed point E_0, and the properties of E_0 are given in Table 5.

3. Bifurcations of codimension 1

3.1. Bifurcations around the trivial fixed point $O(0, 0)$

Theorem 3.1. *As a passes through 1, the map (1.6) undergoes a transcritical bifurcation at the fixed point O.*

Proof. By the results in Section 2.1 (see also Theorem A.1), O is non-hyperbolic if $a = 1$. We choose a as the bifurcation parameter, and let $a = 1 + \mu$, where $\mu = a - 1$ is sufficiently small. We can easily deduce that the center manifold of F is $y = 0$ when $a = 1$ and F restricted to this center manifold is the map $x \mapsto x + \mu x - x^2 - \mu x^2$. Thus, as a passes through 1, the map (1.6) undergoes a transcritical bifurcation at the fixed point O, and the fixed point A appears when $a > 1$. □

3.2. Bifurcations around the semitrivial fixed point $A(\frac{a-1}{a}, 0)$

Theorem 3.2. *When $a > 1$, the fixed point A arises. Moreover, when $a = 3$ or $\epsilon = \frac{a(ad-a-d)}{(a-1)^2}(d > \frac{a}{a-1})$, A is non-hyperbolic.*

(i) *If $\epsilon \neq \frac{a(ad-d-a)}{(a-1)^2}$, then as a passes through 3, a flip bifurcation occurs at A;*
(ii) *If $a \neq 3$ and $d \neq \frac{2a}{a-1}$, then as ϵ passes through $\frac{a(ad-a-d)}{(a-1)^2}(d > \frac{a}{a-1})$, a transcritical bifurcation occurs at A.*

Proof. (i) By the results in Section 2.1 (also Theorem A.1), if $a = 3$, $\epsilon \neq \frac{a(ad-d-a)}{(a-1)^2}$, $A(\frac{2}{3}, 0)$ is non-hyperbolic and the eigenvalues of the Jacobian matrix $J(A)$ are $\lambda_1 = -1$ and $|\lambda_2| \neq 1$.

In order to analyze fold bifurcation around the fixed point A, we choose a as the bifurcation parameter. Let $u = x - \frac{2}{3}$, $v = y - 0$ and $r = a - 3$, we transform the fixed point $A(\frac{2}{3}, 0)$ to the origin and expand the right-hand side of map (1.6) around the origin. Then map (1.6) becomes

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} -u - \frac{6b}{9+4\epsilon}v + \frac{2r}{9} - \frac{ru}{3} - (3+r)u^2 - \frac{9b(9-4\epsilon)}{(9+4\epsilon)^2}uv + \frac{54b\epsilon(-27+4\epsilon)}{(9+4\epsilon)^3}u^2v + \mathcal{O}(|u,v|^4) \\ \frac{6d}{9+4\epsilon}v + \frac{9d(9-4\epsilon)}{(9+4\epsilon)^2}uv + \frac{54d\epsilon(-27+4\epsilon)}{(9+4\epsilon)^3}u^2v + \mathcal{O}(|u,v|^4) \end{pmatrix}, \tag{3.1}$$

where r is the new variable and is sufficient small.

Consider the following system

$$\begin{pmatrix} u \\ v \\ r \end{pmatrix} \to \begin{pmatrix} -u - \frac{6b}{9+4\epsilon}v + \frac{2r}{9} - \frac{ru}{3} - (3+r)u^2 - \frac{9b(9-4\epsilon)}{(9+4\epsilon)^2}uv + \frac{54b\epsilon(-27+4\epsilon)}{(9+4\epsilon)^3}u^2v + \mathcal{O}(|u,v|^4) \\ \frac{6d}{9+4\epsilon}v + \frac{9d(9-4\epsilon)}{(9+4\epsilon)^2}uv + \frac{54d\epsilon(-27+4\epsilon)}{(9+4\epsilon)^3}u^2v + \mathcal{O}(|u,v|^4) \\ r \end{pmatrix}. \tag{3.2}$$

Linearizing map (3.2) at $(0, 0, 0)$, we obtain the associated Jacobian matrix

$$J = \begin{pmatrix} -1 & -\frac{6b}{9+4\epsilon} & \frac{2}{9} \\ 0 & \frac{6d}{9+4\epsilon} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Letting

$$T = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -\frac{9+4\epsilon+6d}{6b} & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

and using the transformation

$$\begin{pmatrix} u \\ v \\ r \end{pmatrix} = T \begin{pmatrix} X \\ Y \\ Z \end{pmatrix},$$

then the map (3.2) becomes

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{6d}{9+4\epsilon} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \begin{pmatrix} f_1(X,Y,Z) \\ g_1(X,Y,Z) \\ 0 \end{pmatrix}, \tag{3.3}$$

where

$$\begin{aligned} f_1(X,Y,Z) = {} & -6Z^2 - 9Z^3 - (9Z + 18Z^2)X - \left(\frac{3(45+28\epsilon)Z}{18+8\epsilon} + \frac{9(162+171\epsilon+28\epsilon^2)Z^2}{(9+4\epsilon)^2}\right)Y - (3+9Z)X^2 \\ & - \left(\frac{81+60\epsilon}{18+8\epsilon} + \frac{54(27+33\epsilon+4\epsilon^2)Z}{(9+4\epsilon)^2}\right)XY - \left(\frac{9(3+4\epsilon)}{18+8\epsilon} + \frac{9(81+126\epsilon+8\epsilon^2)Z}{(9+4\epsilon)^2}\right)Y^2 + \frac{9\epsilon(-27+4\epsilon)}{(9+4\epsilon)^2}Y^3 \\ & + \frac{18\epsilon(-27+4\epsilon)}{(9+4\epsilon)^2}XY^2 + \frac{9\epsilon(-27+4\epsilon)}{(9+4\epsilon)^2}X^2Y + \mathcal{O}(|X,Y,Z|^4), \\ g_1(X,Y,Z) = {} & \left(\frac{9d(9-4\epsilon)Z}{(9+4\epsilon)^2} + \frac{54d\epsilon(-27+4\epsilon)Z^2}{(9+4\epsilon)^3}\right)Y + \left(\frac{9d(9-4\epsilon)}{(9+4\epsilon)^2} + \frac{108d\epsilon(-27+4\epsilon)Z}{(9+4\epsilon)^3}\right)Y^2 + \frac{54d\epsilon(-27+4\epsilon)}{(9+4\epsilon)^3}Y^3 \\ & + \left(\frac{9d(9-4\epsilon)}{(9+4\epsilon)^2} + \frac{108d\epsilon(-27+4\epsilon)Z}{(9+4\epsilon)^3}\right)XY + \frac{108d\epsilon(-27+4\epsilon)}{(9+4\epsilon)^3}XY^2 + \frac{54d\epsilon(-27+4\epsilon)}{(9+4\epsilon)^3}X^2Y \\ & + \mathcal{O}(|X,Y,Z|^4). \end{aligned}$$

By the center manifold theory, the stability of $(X, Y) = (0, 0)$ near $Z = 0$ can be determined by studying a one-parameter family of reduced equations on a center manifold, which can be represented as follows

$$W^c(0) = \{(X, Y, Z) \in R^3 | Y = h(X, Z), h(0, 0) = 0, Dh(0, 0) = 0\}$$

for X and Z sufficiently small. We assume that $h(X, Z)$ takes the form

$$h(X, Z) = h_1 Z^2 + h_2 XZ + h_3 X^2 + \mathcal{O}(|X, Z|^3). \tag{3.4}$$

Then $h(X, Z)$ must satisfy

$$\mathcal{N}(h(X, Z)) = h(-X + f_1(X, h(X, Z), Z), Z) - \frac{6d}{9 + 4\epsilon} h(X, Z) - g_1(X, h(X, Z), Z) = 0. \tag{3.5}$$

Substituting (3.4) into (3.5) and equating coefficients of powers to zero in (3.5), we obtain

$$h_1 = h_2 = h_3 = 0.$$

Thus the map restricted to the center manifold is given by

$$X \to \tilde{f}(X, Z) = \quad -X - (9Z + 18Z^2)X - (3 + 9Z)X^2 - 6Z^2 - 9Z^3 + \mathcal{O}(|X, Z|^4).$$

It is easy to see that

$$\frac{\partial \tilde{f}}{\partial X}(0, 0) = -1, \quad -3\Big(\frac{\partial^2 \tilde{f}}{\partial X^2}(0, 0)\Big)^2 - 2\frac{\partial^3 \tilde{f}}{\partial X^3}(0, 0) = -108, \quad \frac{\partial^2 \tilde{f}}{\partial Z \partial X}(0, 0) = -9.$$

Therefore, by [13], map (1.6) undergoes a flip bifurcation at the fixed point A when $a = 3$.

(ii) Similarly, if $\epsilon = \frac{a(ad-a-d)}{(a-1)^2}$ and $a \neq 3$, $A(\frac{a-1}{a}, 0)$ is non-hyperbolic because the eigenvalues of the Jacobian matrix $J(A)$ are $\lambda_1 = 2 - a$ and $\lambda_2 = 1$.

In order to analyze transcritical bifurcation of the fixed point A, we choose ϵ as the bifurcation parameter. Let $u = x - \frac{a-1}{a}$, $v = y - 0$ and $\mu = \epsilon - \frac{a(ad-a-d)}{(a-1)^2}$, we transform the fixed point $(\frac{a-1}{a}, 0)$ to the origin and expand the right-hand side of map (1.6) around the origin. Then map (1.6) becomes

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} (2-a)u - \frac{b}{d}v + \frac{b(a-1)\mu v}{ad^2} - \frac{ab(a(d-2)-d)uv}{(a-1)^2 d^2} - au^2 + \mathcal{O}(|u, v|^3) \\ v + \frac{(1-a)\mu v}{ad} + \frac{a(-a(d-2)+d)uv}{(a-1)^2 d} + \mathcal{O}(|u, v|^3) \end{pmatrix}, \tag{3.6}$$

where μ is the new variable and is sufficient small.

Linearizing map (3.6) at $(0, 0)$, we obtain the associated Jacobian matrix

$$J = \begin{pmatrix} 2-a & -\frac{b}{d} \\ 0 & 1 \end{pmatrix}.$$

Let

$$T = \begin{pmatrix} 1 & -\frac{b}{d(a-1)} \\ 0 & 1 \end{pmatrix}$$

and use the transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} X \\ Y \end{pmatrix},$$

then the map (3.6) becomes

$$\begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} 2-a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} f_1(X,Y,\mu) \\ g_1(X,Y,\mu) \end{pmatrix}, \tag{3.7}$$

where

$$\begin{aligned} f_1(X,Y,\mu) = {} & \tfrac{(a-2)b\mu}{ad^2}Y - aX^2 + \tfrac{ab(a(4-7d)+4d+a^2(3d-2))}{(a-1)^3d^2}XY + \tfrac{ab^2(-2a^2(d-1)-3d+a(5d-4))}{(a-1)^4d^3}Y^2 \\ & +\mathcal{O}(|X,Y|^3), \\ g_1(X,Y,\mu) = {} & \tfrac{(1-a)\mu}{ad}Y + \tfrac{a(-a(d-2)+d)}{(a-1)^2d}XY + \tfrac{ab(a(d-2)-d)}{(a-1)^3d^2}Y^2 + \mathcal{O}(|X,Y|^3). \end{aligned}$$

Once again, the stability of $(X,Y) = (0,0)$ near $\mu = 0$ can be determined by studying a one-parameter family of equations restricted on a center manifold, which can be represented as follows

$$W^c(0) = \{(X,Y,\mu) \in R^3 | X = h(Y,\mu), h(0,0) = 0, Dh(0,0) = 0\}$$

for Y and μ sufficiently small. Assume that

$$h(Y,\mu) = h_1\mu^2 + h_2Y\mu + h_3Y^2 + \mathcal{O}(|Y,\mu|^3). \tag{3.8}$$

Then we have

$$\mathcal{N}(h(Y,\mu)) = h(Y + g_1(h(Y,\mu),Y,\mu),\mu) - (2-a)h(Y,\mu) - f_1(h(Y,\mu),Y,\mu) = 0. \tag{3.9}$$

Substituting (3.8) into (3.9) and comparing the coefficients of (3.9), we obtain

$$h_1 = 0, h_2 = \frac{b(a-2)}{(a^2-a)d^2}, h_3 = -\frac{ab^2(4a-2a^2+3d-5ad+2a^2d)}{(a-1)^5d^3}.$$

Thus the map restricted to the center manifold is given by

$$\begin{aligned} Y \to \tilde{g}(Y,\mu) = {} & Y\Bigg(1 + \tfrac{\mu(1-a)}{ad} + \tfrac{ab(a(d-2)-d)}{(a-1)^3d^2}Y - \tfrac{b(a(d-2)-d)Y}{(a-1)^7d^4}\Big(2(a-2)a^3bY \\ & +(a-1)d((a-2)(a-1)^3\mu + (3-2a)a^2bY)\Big) + \mathcal{O}(|Y,\mu|^3)\Bigg). \end{aligned} \tag{3.10}$$

We can see that $\tilde{g}(0,0) = 0$, $\frac{\partial\tilde{g}}{\partial Y}(0,0) = 1$, $\frac{\partial\tilde{g}}{\partial\mu}(0,0) = 0$, $\frac{\partial^2\tilde{g}}{\partial\mu\partial Y}(0,0) = \frac{1-a}{ad} \neq 0$, and $\frac{\partial^2\tilde{g}}{\partial Y^2}(0,0) = \frac{2ab(ad-d-2a)}{(a-1)^3d^2} \neq 0$ if $a > 1$ and $d \neq \frac{2a}{a-1}$. By [25], when $a \neq 3$ and $d \neq \frac{2a}{a-1}$, the fixed point A undergoes a transcritical bifurcation at $\epsilon = \frac{a(ad-a-d)}{(a-1)^2}(d > \frac{a}{a-1})$. $\square$

3.3. Bifurcation analysis of the first positive fixed point $E_1(x_1, y_1)$

Firstly, we discuss the flip bifurcation at the fixed point $E_1(x_1, y_1)$. Let

$$
\begin{aligned}
&p_1 \equiv p(x_1) = a_0(2-3x_1) - \tfrac{2}{dx_1}\Big(a_0(1-x_1)-1\Big),\\
&q_1 \equiv q(x_1) = a_0(1-2x_1),\\
&a_0 = \tfrac{2+dx_1}{5dx_1^2-(3d+2)x_1+2},\\
&\tilde{x} = \tfrac{2d^7+10d^5\epsilon+d^6\epsilon-166d^3\epsilon^2-19d^4\epsilon^2+216d\epsilon^3+88d^2\epsilon^3-48\epsilon^4}{2d^8+8d^6\epsilon+d^7\epsilon-178d^4\epsilon^2-20d^5\epsilon^2+368d^2\epsilon^3+106d^3\epsilon^3-80\epsilon^4-120d\epsilon^4},\\
&\Gamma_1 \equiv \{(d, x_1, a) : \tfrac{3}{2} < d < \tfrac{14+4\sqrt{10}}{9}, \tfrac{1}{d} < x_1 < \tfrac{2}{3}, a = a_0\},\\
&\Gamma_2 \equiv \{(d, x_1, a) : \tfrac{14+4\sqrt{10}}{9} \le d \le 3, x_1 \in (\tfrac{1}{d}, x_{11}) \cup (x_{12}, \tfrac{2}{3}), a = a_0\},\\
&\Gamma_3 \equiv \{(d, x_1, a) : 3 < d < 4, x_{12} < x_1 < \tfrac{2}{d}, a = a_0\},
\end{aligned}
$$

where x_{11} and x_{12} are given in Table 3 or in Theorem A.2.

Theorem 3.3. *If $(d, x_1, a) \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and $x_1 \ne \frac{4d-2}{7d}$, then the eigenvalues of the Jacobian matrix $J(E_1)$ are $\lambda_1 = -1$ and $\lambda_2 = -q_1 \ne -1, 1$. Moreover, if $x_1 \ne \tilde{x}$, then the map (1.6) undergoes a flip bifurcation at the fixed point E_1.*

Proof. By the results in Section 2.2 (see also Theorem A.2), if $(d, x_1, a) \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and $x_1 \ne \frac{4d-2}{7d}$, then the positive fixed point $E_1(x_1, y_1)$ of map (1.6) is non-hyperbolic and the eigenvalues of the Jacobian matrix $J(E_1)$ are $\lambda_1 = -1$ and $\lambda_2 = -q_1 \ne -1, 1$. Thus, $1 + p_1 + q_1 = 0$ and $p_1 \ne -2, 0$.

In order to analyze flip bifurcation at $E_1(x_1, y_1)$, we choose a as the bifurcation parameter. Let $u = x-x_1$, $v = y - y_1$ and $r = a - a_0$, we transform the fixed point (x_1, y_1) to the origin and expand the right-hand side of map (1.6) around the origin. Then map (1.6) becomes

$$
\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} (p_1 - 1 + (1-2x_1)r)u - \frac{b}{d}v + (1-x_1)x_1 r - \frac{b}{d}Q_1 uv + (-\frac{b}{d}Q_2 - a_0 - r)u^2 \\ \quad - \frac{b}{d}Q_3 u^2 v - \frac{b}{d}Q_4 u^3 + \mathcal{O}(|u,v|^4) \\ \frac{-2p_1 d}{b}u + v + Q_1 uv + Q_2 u^2 + Q_3 u^2 v + Q_4 u^3 + \mathcal{O}(|u,v|^4) \end{pmatrix}, \tag{3.11}
$$

where $Q_1 = \frac{d-dx_1^2\epsilon}{(1+x_1^2\epsilon)^2}, Q_2 = \frac{dx_1y_1\epsilon(-3+x_1^2\epsilon)}{(1+x_1^2\epsilon)^3}, Q_3 = \frac{dx_1\epsilon(-3+x_1^2\epsilon)}{(1+x_1^2\epsilon)^3}, Q_4 = -\frac{dy_1\epsilon(1-6x_1^2\epsilon+x_1^4\epsilon^2)}{(1+x_1^2\epsilon)^4}$, and r is the new variable and is sufficient small.

Consider the following system

$$
\begin{pmatrix} u \\ v \\ r \end{pmatrix} \to \begin{pmatrix} (p_1 - 1 + (1-2x_1)r)u - \frac{b}{d}v + (1-x_1)x_1 r - \frac{b}{d}Q_1 uv + (-\frac{b}{d}Q_2 - a_0 - r)u^2 \\ \quad - \frac{b}{d}Q_3 u^2 v - \frac{b}{d}Q_4 u^3 + \mathcal{O}(|u,v|^4) \\ \frac{-2p_1 d}{b}u + v + Q_1 uv + Q_2 u^2 + Q_3 u^2 v + Q_4 u^3 + \mathcal{O}(|u,v|^4) \\ r \end{pmatrix}. \tag{3.12}
$$

Linearizing map (3.12) at $(0, 0, 0)$, we obtain the associated Jacobian matrix

$$
J = \begin{pmatrix} p_1 - 1 & -\frac{b}{d} & (1-x_1)x_1 \\ \frac{-2p_1 d}{b} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
$$

Letting

$$
T = \begin{pmatrix} \frac{b}{dp_1} & -\frac{b}{2d} & 0 \\ 1 & 1 & \frac{d(1-x_1)x_1}{b} \\ 0 & 0 & 1 \end{pmatrix}
$$

and using the transformation

$$\begin{pmatrix} u \\ v \\ r \end{pmatrix} = T \begin{pmatrix} X \\ Y \\ r \end{pmatrix},$$

then map (3.12) becomes (omitting the third equation)

$$\begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} -1 & 0 \\ 0 & -q_1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} f_1(X,Y,r) \\ g_1(X,Y,r) \end{pmatrix}, \tag{3.13}$$

where

$$\begin{aligned} f_1(X,Y,r) &= k_1 rX - \tfrac{p_1}{2} k_1 rY + k_2 X^2 + k_3 XY + k_4 Y^2 + \mathcal{O}(|X,Y|^3), \\ g_1(X,Y,r) &= -k_5 rX + \tfrac{p_1}{2} k_5 rY + k_6 X^2 + k_7 XY + k_8 Y^2 + \mathcal{O}(|X,Y|^3), \end{aligned}$$

and

$$\begin{aligned} k_1 &= \tfrac{2-(4+Q_1)x_1+Q_1x_1^2}{2+p_1}, k_2 = -\tfrac{b(2a_0d+dp_1Q_1+bQ_2)}{d^2p_1(2+p_1)}, k_3 = \tfrac{b(4a_0d+d(p_1-2)Q_1+2bQ_2)}{2d^2(2+p_1)}, \\ k_4 &= -\tfrac{bp_1(2a_0d-2dQ_1+bQ_2)}{4d^2(2+p_1)}, k_5 = \tfrac{2(Q_1(x_1-1)x_1+p_1(1-(2+Q_1)x_1+Q_1x_1^2))}{p_1(2+p_1)}, k_6 = \tfrac{2b(a_0dp_1+(1+p_1)(dp_1Q_1+bQ_2))}{d^2p_1^2(2+p_1)}, \\ k_7 &= -\tfrac{b(2a_0dp_1+(1+p_1)(d(p_1-2)Q_1+2bQ_2))}{d^2p_1(2+p_1)}, k_8 = \tfrac{b(a_0dp_1-(1+p_1)(2dQ_1-bQ_2))}{2d^2(2+p_1)}. \end{aligned}$$

To discuss the stability of $(X,Y)=(0,0)$ near $r=0$, consider the center manifold

$$W^c(0) = \{(X,Y,r) \in R^3 | Y = h(X,r), h(0,0)=0, Dh(0,0)=0\}$$

for X and r sufficiently small. Assume that

$$h(X,r) = h_1 r^2 + h_2 Xr + h_3 X^2 + \mathcal{O}(|X,r|^3). \tag{3.14}$$

Then

$$\mathcal{N}(h(X,r)) = h(-X + f_1(X,h(X,r),r),r) + q_1 h(X,r) - g_1(X,h(X,r),r) = 0. \tag{3.15}$$

Substituting (3.14) into (3.15) and comparing coefficients of (3.15), we obtain

$$h_1 = 0,\ \ h_2 = \frac{k_5}{2+p_1},\ \ h_3 = -\frac{k_6}{p_1}.$$

Thus the map (3.13) restricted to the center manifold is given by

$$X \to \tilde{f}(X,r) = -X + k_1 rX - \tfrac{k_1k_5p_1}{2(2+p_1)} r^2 X + \left(\tfrac{k_1k_6}{2} + \tfrac{k_3k_5}{2+p_1}\right) rX^2 + k_2 X^2 - \tfrac{k_3k_6}{p_1} X^3 + \mathcal{O}(|r,X|^4). \tag{3.16}$$

By lengthy calculations, and using the second equation of (A.1), we can see that

$$\begin{aligned} &\tfrac{\partial \tilde{f}}{\partial X}(0,0) = -1, \quad \tfrac{\partial^2 \tilde{f}}{\partial r \partial X}(0,0) = k_1 = -\tfrac{(2-(2+3d)x_1+5dx_1^2)^2}{dx_1^2(2-4d+7dx_1)}, \\ &-3\big(\tfrac{\partial^2 \tilde{f}}{\partial X^2}(0,0)\big)^2 - 2\tfrac{\partial^3 \tilde{f}}{\partial X^3}(0,0) = -12k_2^2 + \tfrac{12k_3k_6}{p_1} \equiv \ell_1 \ell_2, \end{aligned}$$

where

$$\begin{aligned}
\ell_1 &= \tfrac{12b^2(2-(2+3d)x_1+5dx_1^2)^2}{d^4x_1^5(2-3x_1)^2(dx_1-2)^3(2-4d+7dx_1)},\\
\ell_2 &= 48-40(2+3d)x_1+8d(27+11d)x_1^2-2d^2(76+9d)x_1^3+d^3(14+d)x_1^4+2d^4x_1^5\\
&= 48-\tfrac{2d^7}{\epsilon^4}-\tfrac{10d^5}{\epsilon^3}-\tfrac{d^6}{\epsilon^3}+\tfrac{166d^3}{\epsilon^2}+\tfrac{19d^4}{\epsilon^2}-\tfrac{216d}{\epsilon}-\tfrac{88d^2}{\epsilon}\\
&\quad+(-80-120d+\tfrac{2d^8}{\epsilon^4}+\tfrac{8d^6}{\epsilon^3}+\tfrac{d^7}{\epsilon^3}-\tfrac{178d^4}{\epsilon^2}-\tfrac{20d^5}{\epsilon^2}+\tfrac{368d^2}{\epsilon}+\tfrac{106d^3}{\epsilon})x_1.
\end{aligned}$$

When $(d,x_1,a)\in\Gamma_1\cup\Gamma_2\cup\Gamma_3$ and $x_1\neq\frac{4d-2}{7d}$, we have $\frac{1}{d}<x_1<\min\{\frac{2}{d},\frac{2}{3}\}$ and $f(x_1)=2-(2+3d)x_1+5dx_1^2\neq 0$, it is easy to see that $k_1\neq 0$ and $\ell_1\neq 0$, and $\ell_2\neq 0$ if $x_1\neq\tilde{x}\equiv\frac{2d^7+10d^5\epsilon+d^6\epsilon-166d^3\epsilon^2-19d^4\epsilon^2+216d\epsilon^3+88d^2\epsilon^3-48\epsilon^4}{2d^8+8d^6\epsilon+d^7\epsilon-178d^4\epsilon^2-20d^5\epsilon^2+368d^2\epsilon^3+106d^3\epsilon^3-80\epsilon^4-120d\epsilon^4}$. By [25], the fixed point E_1 undergoes a flip bifurcation if $(d,x_1,a)\in\Gamma_1\cup\Gamma_2\cup\Gamma_3$, $x_1\neq\frac{4d-2}{7d}$ and $x_1\neq\tilde{x}$. $\square$

Secondly, we discuss the Neimark–Sacker bifurcation at the fixed point $E_1(x_1,y_1)$. Let

$$\begin{aligned}
&a_1\equiv\tfrac{1}{1-2x_1},\\
&\bar{x}\equiv\tfrac{4+3d\pm\sqrt{5d^2-8d+16}}{d(8+d)},\\
&\Lambda_1\equiv\{(d,x_1):\tfrac{9}{4}<d\le\tfrac{5}{2},\tfrac{1}{d}<x_1<\tfrac{4d-2}{7d}\},\\
&\Lambda_2\equiv\{(d,x_1):\tfrac{5}{2}<d\le 3,\tfrac{1}{d}<x_1<\tfrac{4d-2}{7d}\},\\
&\Lambda_3\equiv\{(d,x_1):3<d<4,\tfrac{1}{d}<x_1<\tfrac{4d-2}{7d}\},\\
&\Lambda_4\equiv\{(d,x_1):d\ge 4,\tfrac{1}{d}<x_1<\tfrac{2}{d}\}.
\end{aligned}$$

Theorem 3.4. *The map (1.6) undergoes a Neimark–Sacker bifurcation at the fixed point E_1 if one of the following conditions holds:*

(i) $(d,x_1)\in\Lambda_1\cup\Lambda_4$, $a=a_1, x_1\neq\bar{x}$;
(ii) $(d,x_1)\in\Lambda_2$, $a=a_1, x_1\neq\frac{2d-2}{3d}$, $x_1\neq\bar{x}$;
(iii) $(d,x_1)\in\Lambda_3$, $a=a_1, x_1\neq\frac{d-2}{d}, x_1\neq\frac{2d-2}{3d}$, $x_1\neq\bar{x}$.

Proof. By the results in Section 2.2 (also Theorem A.2), if $(d,x_1)\in\Lambda_1\cup\Lambda_2\cup\Lambda_3\cup\Lambda_4$ and $a=a_1$, the positive fixed point $E_1(x_1,y_1)$ of map (1.6) is non-hyperbolic and the eigenvalues of the Jacobian matrix $J(E_1)$ are complex conjugate with module 1.

When a is sufficiently closing to a_1, we denote the eigenvalues by $\lambda(a)=\bar{\lambda}(a)=\frac{p(a)+\sqrt{4q(a)-p^2(a)}i}{2}$, where $p(a)=a(2-3x_1)-\frac{2}{dx_1}(a(1-x_1)-1), q(a)=a(1-2x_1)$; When $a=a_1$, we let $\lambda\equiv\lambda(a_1),\bar{\lambda}\equiv\bar{\lambda}(a_1), p\equiv p(a_1)=\frac{2d-2-3dx_1}{d(2x_1-1)}$, $q\equiv q(a_1)=1$. Then we have $\lambda=\frac{p+\sqrt{4-p^2}i}{2}, |\lambda(a)|=\sqrt{q(a)}, |\lambda|=1$, and the transversality condition $\frac{d|\lambda(a)|}{da}\Big|_{a=a_1}=\frac{1-2x_1}{2}>0$ due to $x_1<\frac{1}{2}$ when $(d,x_1)\in\Lambda_1\cup\Lambda_2\cup\Lambda_3\cup\Lambda_4$.

Obviously, $\lambda^j\neq 1$ $(j=1,2,3,4)$ if and only if $p^2-4<0$ and $p\neq 0,-1$. Clearly, we have $p^2-4=\frac{(2-dx_1)(2-4d+7dx_1)}{d^2(2x_1-1)^2}<0$ due to $x_1<\frac{4d-2}{7d}$ under the existence of E_1 with a pair of conjugate imaginary roots.

By computation, $p=0\Leftrightarrow x_1=\frac{2d-2}{3d}$, and $p=-1\Leftrightarrow x_1=\frac{d-2}{d}$. (i) When $\frac{9}{4}<d\le\frac{5}{2}$, we have $\frac{2d-2}{3d}$(or $\frac{d-2}{d}$) $\le\frac{1}{d}<\frac{4d-2}{7d}<\frac{1}{2}<\frac{2}{d}$, then $p\neq 0,-1$ if $x_1\in(\frac{1}{d},\frac{4d-2}{7d})$; (ii) When $\frac{5}{2}<d\le 3$, we have $\frac{d-2}{d}\le\frac{1}{d}<\frac{2d-2}{3d}<\frac{4d-2}{7d}<\frac{1}{2}<\frac{2}{d}$, then $p\neq 0,-1$ if $x_1\in(\frac{1}{d},\frac{4d-2}{7d})$ and $x_1\neq\frac{2d-2}{3d}$; (iii) When $3<d<4$, we have $\frac{1}{d}<\frac{2d-2}{3d}$(or $\frac{d-2}{d}$) $<\frac{4d-2}{7d}<\frac{1}{2}<\frac{2}{d}$, then $p\neq 0,-1$ if $x_1\in(\frac{1}{d},\frac{4d-2}{7d})$ and $x_1\neq\frac{d-2}{d}$ and $x_1\neq\frac{2d-2}{3d}$; (iv) When $d\ge 4$, we have $\frac{1}{d}<\frac{2}{d}\le\frac{1}{2}\le\frac{4d-2}{7d}\le\frac{2d-2}{3d}$(or $\frac{d-2}{d}$), then $p\neq 0,-1$ if $x_1\in(\frac{1}{d},\frac{2}{d})$.

So we have the nondegeneracy condition $\lambda^j\neq 1, j=1,2,3,4$, if one of the following conditions holds: (i) $(d,x_1)\in\Lambda_1\cup\Lambda_4$; (ii) $(d,x_1)\in\Lambda_2$, $x_1\neq\frac{2d-2}{3d}$; (iii) $(d,x_1)\in\Lambda_3$, $x_1\neq\frac{d-2}{d}$, $x_1\neq\frac{2d-2}{3d}$.

Next we calculate the other nondegeneracy condition for Neimark–Sacker bifurcation.

We transform the fixed point (x_1,y_1) to the origin and expand the right-hand side of map (1.6) around the origin by the translations $u=x-x_1$, $v=y-y_1$. Then map (1.6) becomes

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} (p-1)u - \frac{b}{d}v - \frac{b}{d}Q_1uv + (-\frac{b}{d}Q_2 - a_0)u^2 - \frac{b}{d}Q_3u^2v - \frac{b}{d}Q_4u^3 + \mathcal{O}(|u,v|^4) \\ \frac{d}{b}(2-p)u + v + Q_1uv + Q_2u^2 + Q_3u^2v + Q_4u^3 + \mathcal{O}(|u,v|^4) \end{pmatrix}, \tag{3.17}$$

where p, q, Q_1, Q_2, Q_3, Q_4 are same as those in map (3.11) by replacing a_0 by a_1 and r by 0.

Linearizing map (3.17) at $(0,0)$, we obtain the associated Jacobian matrix

$$J = \begin{pmatrix} p-1 & -\frac{b}{d} \\ \frac{d(2-p)}{b} & 1 \end{pmatrix}.$$

Let

$$T = \begin{pmatrix} \frac{b}{d} & 0 \\ \frac{p-2}{2} & \frac{\sqrt{4-p^2}}{2} \end{pmatrix}$$

and use the transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} X \\ Y \end{pmatrix}$$

then the map (3.17) becomes

$$\begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} \frac{p}{2} & -\frac{\sqrt{4-p^2}}{2} \\ \frac{\sqrt{4-p^2}}{2} & \frac{p}{2} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} \hat{f}(X,Y) \\ \hat{g}(X,Y) \end{pmatrix}, \tag{3.18}$$

where

$$\hat{f}(X,Y) = m_1X^2 + m_2XY + m_3X^2Y + m_4X^3 + \mathcal{O}(|X,Y|^4),$$
$$\hat{g}(X,Y) = m_5X^2 + m_6XY + m_7X^2Y + m_8X^3 + \mathcal{O}(|X,Y|^4),$$

and

$$m_1 = -\frac{b(2a_1d + d(p-2)Q_1 + 2bQ_2)}{2d^2}, m_2 = -\frac{b\sqrt{4-p^2}Q_1}{2d}, m_3 = -\frac{b^2\sqrt{4-p^2}Q_3}{2d^2}, m_4 = -\frac{b^2(d(p-2)Q_3 + 2bQ_4)}{2d^3},$$
$$m_5 = \frac{b(2a_1d(p-2) + p(d(p-2)Q_1 + 2bQ_2))}{2d^2\sqrt{4-p^2}}, m_6 = \frac{bpQ_1}{2d}, m_7 = \frac{b^2pQ_3}{2d^2}, m_8 = \frac{b^2p(d(p-2)Q_3 + 2bQ_4)}{2d^3\sqrt{4-p^2}}.$$

Noticing that (3.18) is exactly in the form on the center manifold, the additional nondegeneracy condition for Neimark–Sacker bifurcation is given by (see Theorem 3.5.2 in [7])

$$\hat{a} = -\mathrm{Re}\Big(\frac{(1-2\lambda)\bar{\lambda}^2}{1-\lambda}\varrho_{11}\varrho_{20}\Big) - \frac{1}{2}|\varrho_{11}|^2 - |\varrho_{02}|^2 + \mathrm{Re}(\bar{\lambda}\varrho_{21}), \tag{3.19}$$

where

$$\varrho_{20} = \tfrac{1}{8}\big[\hat{f}_{XX} - \hat{f}_{YY} + 2\hat{g}_{XY} + i(\hat{g}_{XX} - \hat{g}_{YY} - 2\hat{f}_{XY})\big]\Big|_{(0,0)},$$
$$\varrho_{11} = \tfrac{1}{4}\big[\hat{f}_{XX} + \hat{f}_{YY} + i(\hat{g}_{XX} + \hat{g}_{YY})\big]\Big|_{(0,0)},$$
$$\varrho_{02} = \tfrac{1}{8}\big[\hat{f}_{XX} - \hat{f}_{YY} - 2\hat{g}_{XY} + i(\hat{g}_{XX} - \hat{g}_{YY} + 2\hat{f}_{XY})\big]\Big|_{(0,0)},$$
$$\varrho_{21} = \tfrac{1}{16}\big[\hat{f}_{XXX} - \hat{f}_{XYY} + \hat{g}_{XXY} + \hat{g}_{YYY} + i(\hat{g}_{XXX} - \hat{g}_{XYY} - \hat{f}_{XXY} - \hat{f}_{YYY})\big]\Big|_{(0,0)}.$$

By lengthy computations and using the second equation of (A.1), we get

$$\begin{aligned}\hat{a} = &\tfrac{1}{16(p-2)}\Big((-p^3+3p^2+p-6)(m_1^2+m_1m_6-m_5^2+m_2m_5)\\ &-(p^2-3p+1)(2m_1m_5-m_1m_2+m_5m_6)\sqrt{4-p^2}\Big)-\tfrac{1}{8}(m_1^2+m_5^2)\\ &-\tfrac{1}{16}\Big((m_1-m_6)^2+(m_2+m_5)^2\Big)+\tfrac{1}{16}\Big(p(3m_4+m_7)+(3m_8-m_3)\sqrt{4-p^2}\Big)\\ = &-b^2\tfrac{4-2(4+3d)x_1+d(8+d)x_1^2}{8d^3x_1^2(1-2x_1)^2(dx_1-2)}.\end{aligned}$$

When $(d,x_1)\in\Lambda_1\cup\Lambda_2\cup\Lambda_3\cup\Lambda_4$ and $a=a_1$, we have $\frac{1}{d}<x_1<\min\left\{\frac{2}{d},\frac{1}{2}\right\}$, and it is easy to see that $\hat{a}\neq 0$ if $x_1\neq\bar{x}\equiv\frac{4+3d\pm\sqrt{5d^2-8d+16}}{d(8+d)}$.

Summarizing the above results, by [7], the map (1.6) undergoes a Neimark–Sacker bifurcation at the fixed point (x_1,y_1) if one of the following conditions holds: (i) $(d,x_1)\in\Lambda_1\cup\Lambda_4$, $a=a_1, x_1\neq\bar{x}$; (ii) $(d,x_1)\in\Lambda_2$, $a=a_1, x_1\neq\frac{2d-2}{3d}$, $x_1\neq\bar{x}$; (iii) $(d,x_1)\in\Lambda_3$, $a=a_1, x_1\neq\frac{d-2}{d}, x_1\neq\frac{2d-2}{3d}$, $x_1\neq\bar{x}$. □

3.4. Bifurcation analysis at the second positive fixed point $E_2(x_2,y_2)$

Theorem 3.5. *When $(d,x_2,a)\in\{(d,x_2,a):3<d<4,\frac{2}{d}<x_2<\frac{2}{3},a=a_2\}\cup\{(d,x_2,a):d\geq 4,x_{12}<x_2<\frac{2}{3},a=a_2\}$, and $x_2\neq\tilde{x}$, then the map (1.6) undergoes a flip bifurcation at the fixed point E_2, where $a_2=\frac{2+dx_2}{5dx_2^2-(3d+2)x_2+2}$, x_{12} is given in Table 4 and $\tilde{x}$ is given in Theorem 3.3.*

Proof. The proof is completely similar to that of Theorem 3.3 by replacing all x_1 and a_0 by x_2 and a_2, respectively, so we omit it here. □

3.5. Bifurcations around the unique positive fixed point $E_0(\frac{2}{d},\frac{2a(d-2)-2d}{bd})$

By the results in Section 2.4 (see also Theorem A.4), when $\epsilon=\frac{d^2}{4}, d>2$ and $a>\frac{d}{d-2}$, the map (1.6) has a unique positive fixed point E_0, and the eigenvalues of $J(E_0)$ are $\lambda_1=1$ and $|\lambda_2|\neq 1$ if $a\neq\pm\frac{d}{d-4}$. Thus fold bifurcation may occur at E_0. We have the following results.

Theorem 3.6. *If $\epsilon=\frac{d^2}{4}$, $a>\frac{d}{d-2}$, $d>2$ and $a\neq\pm\frac{d}{d-4}$, a fold bifurcation occurs at E_0.*

Proof. When $\epsilon=\frac{d^2}{4}$, $d>2$ and $a>\frac{d}{d-2}$, $E_0(x_0,y_0)$ arises, where $x_0=\frac{2}{d}$, $y_0=\frac{2a(d-2)-2d}{bd}$. The eigenvalues of $J(E_0)$ are $\lambda_1=1$ and $|\lambda_2|\neq 1$ if $a\neq\pm\frac{d}{d-4}$.

In order to analyze the fold bifurcation of the fixed point E_0, we choose ϵ as the bifurcation parameter. Let $u=x-x_0$, $v=y-y_0$ and $r=\epsilon-\frac{d^2}{4}$. We transform the fixed point (x_0,y_0) to the origin and expand the right-hand side of map (1.6) around the origin, where r is sufficient small. Then map (1.6) becomes

$$\begin{pmatrix}u\\v\\r\end{pmatrix}\to\begin{pmatrix}\frac{a(d-4)}{d}u-\frac{b}{d}v+\frac{4a(d-2)-4d}{d^4}r+\frac{8(-a(d-2)+d)}{d^6}r^2+\frac{2a(d-2)-2d}{d^3}ru\\ \quad+\frac{2b}{d^3}rv+\frac{a(d-6)-d}{4}u^2+\mathcal{O}(|u,v|^3)\\ v+\frac{8a+4d-4ad}{bd^3}r+\frac{8(a(d-2)-d)}{bd^5}r^2-\frac{2}{d^2}rv+\frac{4a+2d-2ad}{bd^2}ru\\ \quad+\frac{d(-a(d-2)+d)}{4b}u^2+\mathcal{O}(|u,v|^3)\\ r\end{pmatrix},\tag{3.20}$$

where $r=\epsilon-\frac{d^2}{4}$ is the new dependent variable.

Linearizing map (3.20) at $(0,0)$, we obtain the associated Jacobian matrix

$$J=\begin{pmatrix}\frac{a(d-4)}{d} & -\frac{b}{d} & \frac{4a(d-2)-4d}{d^4}\\ 0 & 1 & \frac{8a+4d-4ad}{bd^3}\\ 0 & 0 & 1\end{pmatrix}.$$

Let

$$T=\begin{pmatrix} 1 & \frac{b}{ad-4a-d} & \frac{bd}{(ad-4a-d)^2} \\ 0 & 1 & -1 \\ 0 & 0 & \frac{bd^3}{4(2a+d-ad)} \end{pmatrix}$$

and use the transformation

$$\begin{pmatrix} u \\ v \\ r \end{pmatrix} = T \begin{pmatrix} X \\ Y \\ \mu \end{pmatrix},$$

then we obtain from (3.20) that

$$\begin{pmatrix} X \\ Y \\ \mu \end{pmatrix} \to \begin{pmatrix} \frac{a(d-4)}{d} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ \mu \end{pmatrix} + \begin{pmatrix} f_2(X,Y,\mu) \\ g_2(X,Y,\mu) \\ 0 \end{pmatrix}, \tag{3.21}$$

where

$$\begin{aligned} f_2(X,Y,\mu) &= \bar{a}_{00}\mu^2 + \bar{a}_{10}X\mu + \bar{a}_{01}Y\mu + \bar{a}_{20}X^2 + \bar{a}_{11}XY + \bar{a}_{02}Y^2 + \mathcal{O}(|X,Y|^3), \\ g_2(X,Y,\mu) &= \bar{b}_{00}\mu^2 + \bar{b}_{10}X\mu + \bar{b}_{01}Y\mu + \bar{b}_{20}X^2 + \bar{b}_{11}XY + \bar{b}_{02}Y^2 + \mathcal{O}(|X,Y|^3), \end{aligned}$$

and

$$\begin{aligned}
&\bar{a}_{00} = \tfrac{ab^2d((16-3d)d^2-2a^2(d-4)^3+ad(88-42d+5d^2))}{4(ad-4a-d)^5},\ \bar{a}_{10} = -\tfrac{ab(a^2(d-4)^3+2d^2(d-6)+ad(-56+26d-3d^2))}{2(ad-4a-d)^3}, \\
&\bar{a}_{01} = -\tfrac{ab^2(2a^3(d-4)^3(d-3)+(16-3d)d^3+2ad^2(64-33d+4d^2)+a^2d(368-300d+80d^2-7d^3))}{2(ad-2a-d)(4a-ad+d)^4}, \\
&\bar{a}_{20} = \tfrac{a(a(24-10d+d^2)-d(d-8))}{4(ad-4a-d)},\ \bar{a}_{11} = \tfrac{ab(a(24-10d+d^2)-d(d-8))}{2(ad-4a-d)^2},\ \bar{a}_{02} = \tfrac{ab^2(a(24-10d+d^2)-d(d-8))}{4(ad-4a-d)^3}, \\
&\bar{b}_{00} = \tfrac{bd^2(2a^2(d-4)^2+ad(18-5d)+3d^2)}{4(ad-4a-d)^4},\ \bar{b}_{10} = \tfrac{d(a^2(d-4)^2+ad(10-3d)+2d^2)}{2(ad-4a-d)^2}, \\
&\bar{b}_{01} = \tfrac{bd(2a^3(d-4)^2(d-3)-3d^3+2ad^2(-13+4d)+a^2d(-84+48d-7d^2))}{2(ad-4a-d)^3(ad-2a-d)},\ \bar{b}_{20} = \tfrac{d(2a+d-ad)}{4b}, \\
&\bar{b}_{11} = \tfrac{d(a^2(d-4)^2+ad(10-3d)+2d^2)}{2(4a+d-ad)^2},\ \bar{b}_{02} = \tfrac{bd(2a+d-ad)}{4(4a+d-ad)^2}.
\end{aligned}$$

The stability of $(X,Y)=(0,0)$ near $\mu=0$ is determined by a one-parameter family of equations on a center manifold represented by

$$W^c(0) = \{(X,Y,\mu) \in R^3 | X = h(Y,\mu), h(0,0)=0, Dh(0,0)=0\}$$

for Y and μ sufficiently small. Assume that

$$h(Y,\mu) = h_1\mu^2 + h_2 Y\mu + h_3 Y^2 + \mathcal{O}(|Y,\mu|^3). \tag{3.22}$$

We have

$$\mathcal{N}(h(Y,\mu)) = h(Y+\mu+g_2(h(Y,\mu),Y,\mu),\mu) - \frac{a(d-4)}{d}h(Y,\mu) - f_2(h(Y,\mu),Y,\mu) = 0. \tag{3.23}$$

Substituting (3.22) into (3.23) and comparing coefficients of (3.23), we obtain

$$
\begin{aligned}
h_1 &= \tfrac{ab^2d^2(d^3(56-10d)+a^3(d-4)^2(52-32d+5d^2)+ad^2(384-204d+25d^2)-4a^2d(-232+198d-55d^2+5d^3)}{4(ad-2a-d)(ad-4a-d)^6},\\
h_2 &= \tfrac{ab^2d(a^3(d-4)^3(d-3)-2d^3(d-6)+a^2d(84-43d+5d^2)+2a^2d(104-86d+23d^2-2d^3))}{(ad-4a-d)^5(ad-2a-d)},\\
h_3 &= \tfrac{-ab^2d(24a+8d-10ad-d^2+ad^2)}{4(ad-4a-d)^4}.
\end{aligned}
$$

Thus the map restricted to the center manifold is given by

$$
\begin{aligned}
Y \to \tilde{f}(Y,\mu) = \quad & Y+\mu+\tfrac{bd(2a+d-ad)}{4(4a+d-ad)^2}Y^2+\tfrac{bd^2(2a^2(d-4)^2)+ad(18-5d)+3d^2}{4(4a+d-ad)^4}\mu^2 \\
& +\tfrac{bd(2a^3(d-4)^2(d-3)-3d^3+2ad^2(-13+4d)+a^2d(-84+48d-7d^2))}{2(ad-4a-d)^3(ad-2a-d)}Y\mu+\mathcal{O}(|Y,\mu|^3).
\end{aligned}
\tag{3.24}
$$

We have $\tilde{f}(0,0)=0$, $\frac{\partial \tilde{f}}{\partial Y}(0,0)=1$, $\frac{\partial \tilde{f}}{\partial \mu}(0,0)=1$, $\frac{\partial^2 \tilde{f}}{\partial Y^2}(0,0)=\frac{bd(2a+d-ad)}{2(4a+d-ad)^2}\neq 0$ because $a>\frac{d}{d-2}$ and $a\neq \pm\frac{d}{d-4}$. By [25], if $\epsilon=\frac{d^2}{4}$, $a>\frac{d}{d-2}$, $d>2$ and $a\neq\pm\frac{d}{d-4}$, a fold bifurcation occurs at E_0. □

Remark 3.7. By the results in Section 2.4 (also Theorem A.4), when $\epsilon=\frac{d^2}{4}, a>\frac{d}{d-2}, 3<d<4$ and $a=\frac{d}{4-d}$, the eigenvalues of $J(E_0)$ are 1 and -1. By computation we obtain a normal form of map (1.6) at E_0:

$$
\begin{bmatrix} x_1\\ x_2\end{bmatrix}\longmapsto\begin{bmatrix} -x_1+g_{20}x_1^2+g_{11}x_1x_2+g_{02}x_2^2+g_{30}x_1^3+g_{21}x_1^2x_2+g_{12}x_1x_2^2+g_{03}x_2^3\\ x_2+e_{20}x_1^2+e_{11}x_1x_2+e_{02}x_2^2+e_{30}x_1^3+e_{21}x_1^2x_2+e_{12}x_1x_2^2+e_{03}x_2^3\end{bmatrix}+\mathcal{O}(|x|^4),
\tag{3.25}
$$

where

$$
\begin{aligned}
&x=(x_1,x_2),\ g_{20}=-\tfrac{d(d-7)}{4(d-4)},\ g_{11}=-\tfrac{d(d-7)}{2(d-4)},\ g_{02}=-\tfrac{d(d-7)}{4(d-4)},\ g_{30}=\tfrac{d^2(d-3)}{8(d-4)},\\
&g_{21}=\tfrac{d^2(2d-5)}{8(d-4)},\ g_{12}=-\tfrac{d^2(d-1)}{8(d-4)},\ g_{03}=\tfrac{d^2}{8(d-4)},\ e_{20}=-\tfrac{d(d-3)}{4(d-4)},\ e_{11}=-\tfrac{d(d-3)}{2(d-4)},\\
&e_{02}=-\tfrac{d(d-3)}{4(d-4)},\ e_{30}=\tfrac{d^2(d-3)}{8(d-4)},\ e_{21}=\tfrac{d^2(2d-5)}{8(d-4)},\ e_{12}=\tfrac{d^2(d-1)}{8(d-4)},\ e_{03}=\tfrac{d^2}{8(d-4)}.
\end{aligned}
$$

Since all of the above coefficients are not zero for $3<d<4$, a fold-flip bifurcation may occur at E_0, which will be considered in the future.

4. Bogdanov–Takens bifurcation around the unique positive fixed point $E_0(\frac{2}{d}, \frac{2a(d-2)-2d}{bd})$

By the results in Section 2.4 (also Theorem A.4), when $\epsilon=\frac{d^2}{4}, d>2$ and $a>\frac{d}{d-2}$, the map (1.6) has a unique positive fixed point E_0 and $J(E_0)$ has an eigenvalue 1 with multiplicity 2 if $a=\frac{d}{d-4}(d>4)$. Thus Bogdanov–Takens bifurcation may occur at E_0.

Before discussing the Bogdanov–Takens bifurcation around the unique positive fixed point E_0 in map (1.6), we firstly state some known results about Bogdanov–Takens bifurcation for diffeomorphisms (see Broer et al. [4,5], Kuznetsov [13] and Yagasaki [27]).

Consider an analytic family of planar diffeomorphisms $f_\lambda:\mathbb{R}^2\longrightarrow\mathbb{R}^2, \lambda\in\mathbb{R}^2$. We assume that f_λ has a fixed point $x=0$ at $\lambda=0$ such that the Jacobian matrix $D_xf_0(0)$ has a double unit eigenvalue but is not the identity (1:1 resonance); that is, $D_xf_0(0)$ has the nilpotent form

$$
D_xf_0(0)=\begin{pmatrix}1&1\\0&1\end{pmatrix}.
$$

In appropriate coordinates f_λ has the form

$$f_\lambda(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} f_1(x,\lambda) \\ f_2(x,\lambda) \end{pmatrix} + \mathcal{O}((|x|)^3), \tag{4.1}$$

where $x = (x_1, x_2)^T, \lambda = (\lambda_1, \lambda_2)$,

$$\begin{aligned} f_1(x,\lambda) &= a_{00}(\lambda) + a_{10}(\lambda)x_1 + a_{01}(\lambda)x_2 + \tfrac{1}{2}a_{20}(\lambda)x_1^2 + a_{11}(\lambda)x_1x_2 + \tfrac{1}{2}a_{02}(\lambda)x_2^2, \\ f_2(x,\lambda) &= b_{00}(\lambda) + b_{10}(\lambda)x_1 + b_{01}(\lambda)x_2 + \tfrac{1}{2}b_{20}(\lambda)x_1^2 + b_{11}(\lambda)x_1x_2 + \tfrac{1}{2}b_{02}(\lambda)x_2^2 \end{aligned} \tag{4.2}$$

with $a_{00}(0) = a_{10}(0) = a_{01}(0) = b_{00}(0) = b_{10}(0) = b_{01}(0) = 0$.

Diffeomorphism (4.1) can be approximated by the time-one flow of a planar vector field, which has a singularity with nilpotent linear part, the following lemma is Lemma 3.1 in Yagasaki [27] (see also Lemma 9.6 in Kuznetsov [13] or Theorem 1 in Broer et al. [4]).

Lemma 4.1. *For $|\lambda|$ sufficiently small diffeomorphism (4.1) can be represented as*

$$x \mapsto \psi_\lambda^1(x) + \mathcal{O}((|x|)^3), \tag{4.3}$$

where $\psi_\lambda^1(x)$ is the time-one flow of the following planar vector field

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} g_1(x,\lambda) \\ g_2(x,\lambda) \end{pmatrix}, \tag{4.4}$$

where

$$\begin{aligned} g_1(x,\lambda) &= c_{00}(\lambda) + c_{10}(\lambda)x_1 + c_{01}(\lambda)x_2 + \tfrac{1}{2}c_{20}(\lambda)x_1^2 + c_{11}(\lambda)x_1x_2 + \tfrac{1}{2}c_{02}(\lambda)x_2^2, \\ g_2(x,\lambda) &= d_{00}(\lambda) + d_{10}(\lambda)x_1 + d_{01}(\lambda)x_2 + \tfrac{1}{2}d_{20}(\lambda)x_1^2 + d_{11}(\lambda)x_1x_2 + \tfrac{1}{2}d_{02}(\lambda)x_2^2, \end{aligned} \tag{4.5}$$

in which the coefficients can be expressed by those in (4.2) as follows:

$$\begin{aligned} c_{00}(\lambda) &= a_{00}(\lambda) - (\frac{1}{2}a_{10}(\lambda) - \frac{1}{3}b_{10}(\lambda))a_{00}(\lambda) \\ &\quad -(\frac{1}{2} - \frac{1}{3}a_{10}(\lambda) + \frac{1}{2}a_{01}(\lambda) + \frac{1}{4}b_{10}(\lambda) - \frac{1}{3}b_{01}(\lambda))b_{00}(\lambda), \\ c_{10}(\lambda) &= a_{10}(\lambda) - \frac{1}{2}b_{10}(\lambda), \\ c_{01}(\lambda) &= a_{01}(\lambda) - \frac{1}{2}a_{10}(\lambda) + \frac{1}{3}b_{10}(\lambda) - \frac{1}{2}b_{01}(\lambda), \\ c_{20}(\lambda) &= a_{20}(\lambda) - \frac{1}{2}b_{20}(\lambda), \\ c_{11}(\lambda) &= a_{11}(\lambda) - \frac{1}{2}a_{20}(\lambda) + \frac{1}{3}b_{20}(\lambda) - \frac{1}{2}b_{11}(\lambda), \\ c_{02}(\lambda) &= a_{02}(\lambda) + \frac{1}{6}a_{20}(\lambda) - a_{11}(\lambda) - \frac{1}{6}b_{20}(\lambda) + \frac{2}{3}b_{11}(\lambda) - \frac{1}{2}b_{02}(\lambda), \\ d_{00}(\lambda) &= b_{00}(\lambda) - \frac{1}{2}b_{10}(\lambda)a_{00}(\lambda) + (\frac{1}{3}b_{10}(\lambda) - \frac{1}{2}b_{01}(\lambda))b_{00}(\lambda), \\ d_{10}(\lambda) &= b_{10}(\lambda), \\ d_{01}(\lambda) &= b_{01}(\lambda) - \frac{1}{2}b_{10}(\lambda), \\ d_{20}(\lambda) &= b_{20}(\lambda), \end{aligned}$$

$$d_{11}(\lambda) = b_{11}(\lambda) - \frac{1}{2}b_{20}(\lambda),$$
$$d_{02}(\lambda) = b_{02}(\lambda) + \frac{1}{6}b_{20}(\lambda) - b_{11}(\lambda).$$

In particular, $c_{00}(0) = c_{10}(0) = c_{01}(0) = d_{00}(0) = d_{10}(0) = d_{01}(0) = 0$.

Under some nondegeneracy and transversality conditions, system (4.4) can be transformed to the versal unfolding of a Bogdanov–Takens singularity of codimension 2 by a series of near-identity transformations. Consequently, the versal unfolding of a Bogdanov–Takens singularity of codimension 2 for diffeomorphism (4.1) can be obtained as follows (see Lemma 3.2 and Proposition 3.1 in Yagasaki [27]).

Lemma 4.2. *Suppose that the following nondegeneracy conditions*

$$d_{20}(0) \neq 0 \ (i.e., b_{20}(0) \neq 0), \qquad c_{20}(0) + d_{11}(0) \neq 0 \ (i.e., a_{20}(0) + b_{11}(0) - b_{20}(0) \neq 0) \tag{4.6}$$

are satisfied, then under analytic near-identity transformations of coordinates and scaling of time system (4.4) (and in turn system (4.1)) becomes (up to second order of coordinates)

$$\dot{y}_1 = y_2, \qquad \dot{y}_2 = \nu_1(\lambda) + \nu_2(\lambda)y_1 + y_1^2 + sy_1y_2, \tag{4.7}$$

where

$$s = \mathrm{sign}[b_{20}(0)(a_{20}(0) + b_{11}(0) - b_{20}(0))] = \pm 1$$

and $\nu_1(\lambda)$ *and* $\nu_2(\lambda)$ *can be expressed by the coefficients in (4.5) (and in turn by those in (4.2)) as follows:*

$$\nu_1(\lambda) = \frac{8\beta_0^4}{b_{20}^3(0)}\beta_1(\lambda) - \frac{8\beta_0^3}{b_{20}^3(0)}\beta_2(\lambda)\beta_3(\lambda) + \frac{4\beta_0^2}{b_{20}^2(0)}\beta_2^2(\lambda),$$
$$\nu_2(\lambda) = \frac{4\beta_0^2}{b_{20}^2(0)}\beta_4(\lambda) - \frac{4\beta_0}{b_{20}(0)}\beta_2(\lambda)$$

in which

$$\begin{aligned}
\beta_0 &= a_{20}(0) + b_{11}(0) - b_{20}(0),\\
\beta_1(\lambda) &= b_{00}(\lambda) + \frac{1}{2}(\frac{1}{6}b_{20}(0) - b_{11}(0) + b_{02}(0))a_{00}^2(\lambda)\\
&\quad -(\frac{1}{6}a_{20}(0) - a_{11}(0) + a_{02}(0) - \frac{1}{12}b_{20}(0) + \frac{1}{6}b_{11}(0))a_{00}(\lambda)b_{00}(\lambda)\\
&\quad +\frac{1}{2}(\frac{1}{6}a_{20}(0) - a_{11}(0) + a_{02}(0) - \frac{1}{8}b_{20}(0) + \frac{5}{12}b_{11}(0) - \frac{1}{4}b_{02}(0))b_{00}^2(\lambda)\\
&\quad -a_{00}(\lambda)b_{01}(\lambda) - \frac{1}{2}a_{10}(\lambda)b_{00}(\lambda) + a_{01}(\lambda)b_{00}(\lambda) + \frac{5}{12}b_{00}(\lambda)b_{10}(\lambda) - \frac{1}{2}b_{00}(\lambda)b_{01}(\lambda),\\
\beta_2(\lambda) &= a_{10}(\lambda) - b_{10}(\lambda) + b_{01}(\lambda) + (\frac{1}{2}a_{20}(0) - a_{11}(0) - \frac{1}{2}b_{20}(0) + \frac{3}{2}b_{11}(0) - b_{02}(0))a_{00}(\lambda)\\
&\quad -(\frac{1}{12}a_{20}(0) + \frac{1}{2}a_{11}(0) - a_{02}(0) - \frac{1}{12}b_{20}(0) + \frac{1}{12}b_{11}(0))b_{00}(\lambda),\\
\beta_3(\lambda) &= b_{10}(\lambda) + (\frac{1}{2}b_{20}(0) - b_{11}(0))a_{00}(\lambda) - (\frac{1}{2}a_{20}(0) - a_{11}(0) - \frac{1}{12}b_{20}(0))b_{00}(\lambda),
\end{aligned}$$

$$\beta_4(\lambda) = b_{10}(\lambda) + (\frac{1}{2}b_{20}(0) - b_{11}(0))a_{00}(\lambda)$$
$$+(\frac{1}{2}a_{20}(0) - a_{11}(0) - \frac{3}{4}b_{20}(0) + 2b_{11}(0) - b_{02}(0))b_{00}(\lambda).$$

Furthermore, if the following transversality condition

$$\det D_\lambda \nu(0) \neq 0 \tag{4.8}$$

is satisfied, then system (4.7) is the versal unfolding of the Bogdanov–Takens singularity of codimension 2.

Remark 4.3. The bifurcation sets of system (4.7) is now well known (i.e. Bogdanov–Takens bifurcation for vector fields, see Bogdanov [3] and Takens [23]): A saddle-node bifurcation occurs at $\nu_1 = \frac{\nu_2^2}{4}$; a Hopf bifurcation of codimension 1 occurs at $\nu_1 = 0, \nu_2 < 0$; and a homoclinic bifurcation of codimension 1 occurs near $\nu_1 = -\frac{6\nu_2^2}{25}, \nu_2 < 0$.

Remark 4.4. The dynamical behaviors of the approximate map ψ_λ^1 in Lemma 4.1 are described by those of system (4.7), the bifurcation diagram of system (4.7) therefore describes the bifurcation sets of the approximate map ψ_λ^1, where equilibria correspond to fixed points, a limit cycle corresponds to a normally hyperbolic invariant cycle, etc.

Remark 4.5. By Lemma 4.1, the generic diffeomorphism $f_\lambda(x)$ in (4.1) can be seen as the perturbation of the approximate map ψ_λ^1, and some certain features of the bifurcation diagram for ψ_λ^1 do persist, such as the saddle-node and Hopf bifurcations, which correspond to the fold and Neimark–Sacker bifurcations for $f_\lambda(x)$, respectively. However, the orbit structure on the closed invariant cycle for f_λ is generically different from that for ψ_λ^1, phase-locking or quasi-periodic phenomena occur for f_λ. Even less persistence is that the homoclinic bifurcation curve in ψ_λ^1 generically extends to an exponentially narrow horn, which is bounded by two smooth bifurcation curves, corresponding to homoclinic tangencies, and transversal homoclinic intersection (homoclinic tangle) occurs between these two curves (Broer et al. [5] and Kuznetsov [13]).

From the above known results, we can prove the existence of Bogdanov–Takens bifurcation and calculate the bifurcation curves of diffeomorphism (1.6) as follows.

Theorem 4.6. *Suppose that $d > 4$, diffeomorphism (1.6) undergoes Bogdanov–Takens bifurcation in a small neighborhood of the unique interior fixed point $E_0\left(\frac{2}{d}, \frac{4}{b(d-4)}\right)$ as (ϵ, a) varies near $(\frac{d^2}{4}, \frac{d}{d-4})$. Furthermore, the bifurcation sets of diffeomorphism (1.6) are as follows for sufficiently small $\mid \lambda \mid$:*

(i) *A fold bifurcation occurs on the curve*

$$f^\pm : \nu_1(\lambda) = \frac{1}{4}\nu_2^2(\lambda) + \mathcal{O}((|\lambda|)^3);$$

(ii) *A Neimark–Sacker bifurcation around one of the fixed points born at the fold bifurcation of (i) occurs on the curve*

$$NS : \nu_1(\lambda) = \mathcal{O}((|\lambda|)^3), \qquad \nu_2(\lambda) + \mathcal{O}((|\lambda|)^2) < 0.$$

The invariant circle created at the Neimark–Sacker bifurcation is stable;

(iii) *A homoclinic bifurcation at which the stable and unstable manifolds of the saddle point born at the fold bifurcation of (i) have homoclinic tangencies occurs on two curves (denoted by h_1 and h_2) with the asymptotic forms*

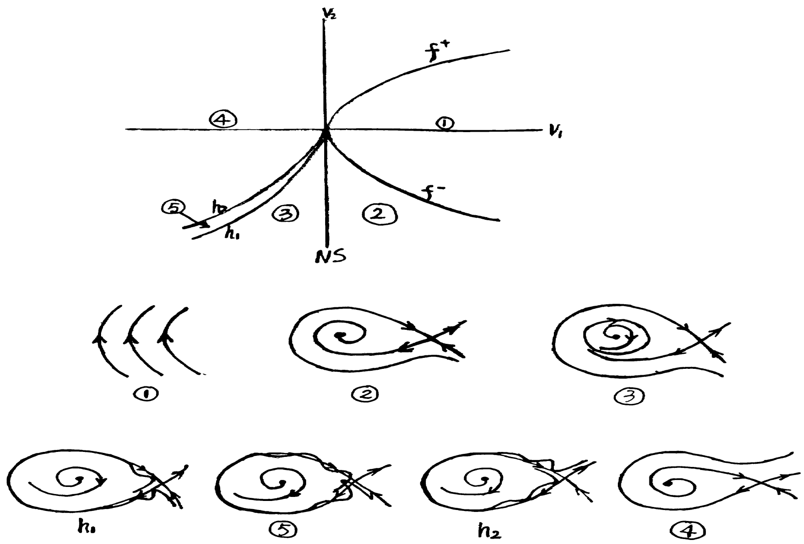

Fig. 4.1. Bogdanov–Takens bifurcation diagram and phase portraits of $f_\lambda(U)$ near $(U, \nu(\lambda)) = (0, 0)$.

$$\nu_1(\lambda) = -\frac{6}{25}\nu_2^2(\lambda) + \mathcal{O}((|\lambda|)^3), \qquad \nu_2(\lambda) + \mathcal{O}((|\lambda|)^2) < 0.$$

The distance between the two homoclinic tangencies bifurcation curves is exponentially small with respect to $\sqrt{|\lambda|}$ *and the invariant manifolds intersect transversally (homoclinic tangle) inside the parameter region between the curves and do not intersect outside, where*

$$\begin{aligned}\nu_1(\lambda_1,\lambda_2) = {} & \tfrac{1024}{(d-4)^2d^2}\lambda_2 + \tfrac{16(752-752d+284d^2-48d^3+3d^4)}{3(d-4)^2d^2}\lambda_1^2 - \tfrac{64(2600-2308d+704d^2-84d^3+3d^4)}{3(d-4)^3d^3}\lambda_1\lambda_2 \\ & + \tfrac{64(1340-1376d+496d^2-72d^3+3d^4)}{3(d-4)^4d^4}\lambda_2^2 + \mathcal{O}((|\lambda|)^3), \\ \nu_2(\lambda_1,\lambda_2) = {} & \tfrac{8(d-4)}{d}\lambda_1 - \tfrac{16(124-36d+3d^2)}{3(d-4)^4d^2}\lambda_2 - \tfrac{112(d-2)}{3d^2}\lambda_1^2 + \tfrac{16(-24+88d-36d^2+3d^3)}{3(d-4)^2d^3}\lambda_1\lambda_2 \\ & + \tfrac{32(-856+392d-60d^2+3d^3)}{3(d-4)^3d^4}\lambda_2^2 + \mathcal{O}((|\lambda|)^3).\end{aligned}$$

The bifurcation diagram and phase portraits of $f_\lambda(U)$ *near* $(U, \nu(\lambda)) = (0, 0)$ *are given in Fig. 4.1.*

Proof. By the results in Section 2.4 (see also Theorem A.4), we know that the unique positive fixed point $E_0(\frac{2}{d}, \frac{4}{b(d-4)})$ of map (1.6) is a nilpotent fixed point when $(\epsilon, a) = (\frac{d^2}{4}, \frac{d}{d-4})$, where $d > 4$.

In order to establish the existence of Bogdanov–Takens bifurcation and calculate the bifurcation curves of diffeomorphism (1.6), we choose the parameters a and ϵ as bifurcation parameters and consider the following unfolding map

$$f_\lambda : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} (a_0+\lambda_1)x(1-x) - \frac{bxy}{1+(\epsilon_0+\lambda_2)x^2} \\ \frac{dxy}{1+(\epsilon_0+\lambda_2)x^2} \end{pmatrix}, \tag{4.9}$$

where λ_1 and λ_2 are parameters in a small neighborhood of $(0, 0)$. We are interested only in the dynamics of map (4.9) when x and y are in a small neighborhood of the nilpotent fixed point when $E_0(\frac{2}{d}, \frac{4}{b(d-4)})$.

We firstly expand map (4.9) into a power series around the fixed point $E_0(\frac{2}{d}, \frac{4}{b(d-4)})$ and translate $E_0(\frac{2}{d}, \frac{4}{b(d-4)})$ to the origin. Let

$$X = x - \frac{2}{d}, \quad Y = y - \frac{4}{b(d-4)},$$

then we have

$$\begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} X + \frac{2(-2+d)\lambda_1}{d^2} + \frac{8\lambda_2}{(-4+d)d^3} + c_1(\lambda_1,\lambda_2) + \Big(\frac{(d-4)\lambda_1}{d} + \frac{4\lambda_2}{(d-4)d^2} + c_2(\lambda_1,\lambda_2)\Big)X \\ +\Big(-\frac{b}{d} + \frac{2b\lambda_2}{d^3} + c_3(\lambda_1,\lambda_2)\Big)Y + \Big(-\lambda_1 + \frac{d}{8-2d} - \frac{2\lambda_2}{(-4+d)d} + c_4(\lambda_1,\lambda_2)\Big)X^2 \\ +\Big(\frac{b\lambda_2}{d^2} + c_5(\lambda_1,\lambda_2)\Big)XY + P_3(X,Y,\lambda_1,\lambda_2) \\ Y - \frac{8\lambda_2}{b(-4+d)d^2} + d_1(\lambda_1,\lambda_2) + \Big(\frac{4\lambda_2}{4bd-bd^2} + d_2(\lambda_1,\lambda_2)\Big)X + \Big(-\frac{2\lambda_2}{d^2} + d_3(\lambda_1,\lambda_2)\Big)Y \\ +\Big(\frac{d^2}{8b-2bd} + \frac{2\lambda_2}{b(-4+d)} + d_4(\lambda_1,\lambda_2)\Big)X^2 + \Big(-\frac{\lambda_2}{d} + d_5(\lambda_1,\lambda_2)\Big)XY \\ +Q_3(X,Y,\lambda_1,\lambda_2) \end{pmatrix}, \tag{4.10}$$

where $c_i(\lambda_1,\lambda_2),\ d_i(\lambda_1,\lambda_2)(i=1,2,3,4,5)$ are functions of at least the second order in $\lambda_1,\ \lambda_2$. P_3 and Q_3 are functions of at least the third order with respect to (X,Y).

Secondly, under the parameter-dependent affine translation

$$u = X, \quad v = \Big(\frac{(d-4)\lambda_1}{d} + \frac{4\lambda_2}{(d-4)d^2} + c_2(\lambda_1,\lambda_2)\Big)X + \Big(-\frac{b}{d} + \frac{2b\lambda_2}{d^3} + c_3(\lambda_1,\lambda_2)\Big)Y,$$

map (4.10) becomes

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} u + v + \frac{2(-2+d)\lambda_1}{d^2} + \frac{8\lambda_2}{(-4+d)d^3} + e_1(\lambda_1,\lambda_2) \\ +\Big(\frac{d}{8-2d} - \lambda_1 - \frac{2\lambda_2}{d(d-4)} + e_4(\lambda_1,\lambda_2)\Big)u^2 + \Big(-\frac{\lambda_2}{d} + e_5(\lambda_1,\lambda_2)\Big)uv \\ +P_4(u,v,\lambda_1,\lambda_2) \\ v + \frac{8\lambda_2}{(-4+d)d^3} + f_1(\lambda_1,\lambda_2) + \Big(\frac{4\lambda_2}{(d-4)d^2} + f_2(\lambda_1,\lambda_2)\Big)u + \Big(\frac{(d-4)\lambda_1}{d} - \frac{2(d-6)\lambda_2}{d^2(d-4)} \\ +f_3(\lambda_1,\lambda_2)\Big)v + \Big(\frac{d}{2d-8} - \frac{\lambda_1}{2} + \frac{(10-3d)\lambda_2}{(-4+d)^2 d} + f_4(\lambda_1,\lambda_2)\Big)u^2 \\ +\Big(-\frac{\lambda_2}{d} + f_5(\lambda_1,\lambda_2)\Big)uv + Q_4(u,v,\lambda_1,\lambda_2) \end{pmatrix}, \tag{4.11}$$

where $e_i(\lambda_1,\lambda_2), f_i(\lambda_1,\lambda_2)(i=1,2,3,4,5)$ are functions of at least the second order with respect to (λ_1,λ_2), P_4 and Q_4 are functions of at least the third order with respect to (u,v).

We next rewrite the above map (4.11) in the following form

$$f_\lambda(U) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} U + \begin{pmatrix} a(U,\lambda) \\ b(U,\lambda) \end{pmatrix} + \mathcal{O}((|U|)^3), \tag{4.12}$$

where

$$\begin{aligned}
&U = (u,v)^T, \lambda = (\lambda_1,\lambda_2),\\
&a(U,\lambda) = a_{00}(\lambda) + a_{10}(\lambda)u + a_{01}(\lambda)v + \tfrac{1}{2}a_{20}(\lambda)u^2 + a_{11}(\lambda)uv + \tfrac{1}{2}a_{02}(\lambda)v^2,\\
&b(U,\lambda) = b_{00}(\lambda) + b_{10}(\lambda)u + b_{01}(\lambda)v + \tfrac{1}{2}b_{20}(\lambda)u^2 + b_{11}(\lambda)uv + \tfrac{1}{2}b_{02}(\lambda)v^2,\\
&a_{00}(\lambda) = \tfrac{2(-2+d)\lambda_1}{d^2} + \tfrac{8\lambda_2}{(-4+d)d^3} + e_1(\lambda), \quad a_{10}(\lambda) = 0, \quad a_{01}(\lambda) = 0,\\
&a_{20}(\lambda) = 2\Big(\tfrac{d}{8-2d} - \lambda_1 - \tfrac{2\lambda_2}{d(d-4)} + e_4(\lambda)\Big), \quad a_{11}(\lambda) = -\tfrac{\lambda_2}{d} + e_5(\lambda),\\
&b_{00}(\lambda) = \tfrac{8\lambda_2}{(-4+d)d^3} + f_1(\lambda), \quad b_{10}(\lambda) = \tfrac{4\lambda_2}{(d-4)d^2} + f_2(\lambda), b_{11}(\lambda) = -\tfrac{\lambda_2}{d} + f_5(\lambda),\\
&b_{01}(\lambda) = \tfrac{(d-4)\lambda_1}{d} - \tfrac{2(-6+d)\lambda_2}{d^2(d-4)} + f_3(\lambda), \quad b_{20}(\lambda) = 2\Big(\tfrac{d}{2d-8} - \tfrac{\lambda_1}{2} - \tfrac{(10-3d)\lambda_2}{(-4+d)^2 d} + f_4(\lambda)\Big),\\
&a_{02}(\lambda) = 0, \quad b_{02}(\lambda) = 0, \text{ and } a_{00}(0) = a_{10}(0) = a_{01}(0) = b_{00}(0) = b_{10}(0) = b_{01}(0) = 0.
\end{aligned}$$

By Lemma 4.2, after tedious calculations, we obtain the nondegeneracy conditions, if $d > 4$,

$$d_{20}(0) = b_{20}(0) = \frac{d}{d-4} > 0,$$

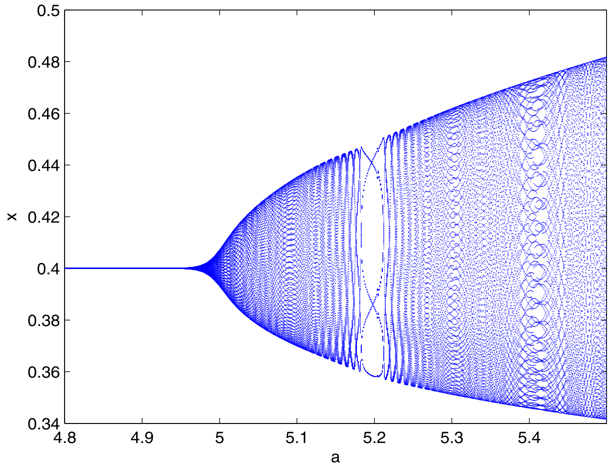

Fig. 5.1. Neimark–Sacker bifurcation diagram in the (a, x)-plane when $d = 3, b = 1, \epsilon = \frac{5}{4}$ in diffeomorphism (1.6).

$$c_{20}(0) + d_{11}(0) = a_{20}(0) + b_{11}(0) - b_{20}(0) = \frac{2d}{4-d} < 0,$$

and the transversality condition

$$\det D_\lambda \nu(0) = -\frac{8192}{(d-4)d^3} < 0,$$

where

$$\begin{aligned}
\nu_1(\lambda_1, \lambda_2) &= \tfrac{1024}{(d-4)^2d^2}\lambda_2 + \tfrac{16(752-752d+284d^2-48d^3+3d^4)}{3(d-4)^2d^2}\lambda_1^2 - \tfrac{64(2600-2308d+704d^2-84d^3+3d^4)}{3(d-4)^3d^3}\lambda_1\lambda_2 \\
&\quad + \tfrac{64(1340-1376d+496d^2-72d^3+3d^4)}{3(d-4)^4d^4}\lambda_2^2 + \mathcal{O}((|\lambda|)^3), \\
\nu_2(\lambda_1, \lambda_2) &= \tfrac{8(d-4)}{d}\lambda_1 - \tfrac{16(124-36d+3d^2)}{3(d-4)^4d^2}\lambda_2 - \tfrac{112(d-2)}{3d^2}\lambda_1^2 + \tfrac{16(-24+88d-36d^2+3d^3)}{3(d-4)^2d^3}\lambda_1\lambda_2 \\
&\quad + \tfrac{32(-856+392d-60d^2+3d^3)}{3(d-4)^3d^4}\lambda_2^2 + \mathcal{O}((|\lambda|)^3).
\end{aligned}$$

By Proposition 3.1 in [27] and corresponding results in Broer et al. [4] and Kuznetsov [13], diffeomorphism (1.6) undergoes Bogdanov–Takens bifurcation in a small neighborhood of the unique interior fixed point $E_0\left(\frac{2}{d}, \frac{4}{b(d-4)}\right)$ as (ϵ, a) varies near $(\frac{d^2}{4}, \frac{d}{d-4})$ if $d > 4$. We also can calculate the approximate expressions of fold and Neimark–Sacker bifurcation curves and the asymptotic forms of homoclinic tangency curves according to Proposition 3.1 in Yagasaki [27]. Furthermore, the invariant circle created at the Neimark–Sacker bifurcation is stable because $b_{20}(0)(a_{20}(0) + b_{11}(0) - b_{20}(0)) < 0$. $\square$

5. Numerical simulations

In order to use Theorem 3.4, we choose $(d, x_1) = (3, \frac{2}{5}) \in \Lambda_2$ and let $b = 1$, then $\epsilon = \frac{dx_1-1}{x_1^2} = \frac{5}{4}, a_1 = \frac{1}{1-2x_1} = 5$, it is easy to see that $x_1 = \frac{2}{5} \neq \frac{2d-2}{3d}(= \frac{4}{9})$ and $x_1 = \frac{2}{5} \neq \bar{x}(= \frac{13\pm\sqrt{37}}{33})$. We can calculate $\hat{a} = -\frac{875}{864} < 0$. By Theorem 3.4, the fixed point E_1 is stable when $a < a_1 = 5$, E_1 loses its stability and becomes unstable, and an attractive invariant cycle occurs near E_1 when $a > a_1 = 5$ slightly. The bifurcation diagram in the (a, x)-plane for the above parameters is given in Fig. 5.1, and the corresponding phase portraits are given in Fig. 5.2, which depict how a smooth invariant cycle bifurcates from the fixed point E_1. Fig. 5.2(d) shows the existence of a period-5 orbit.

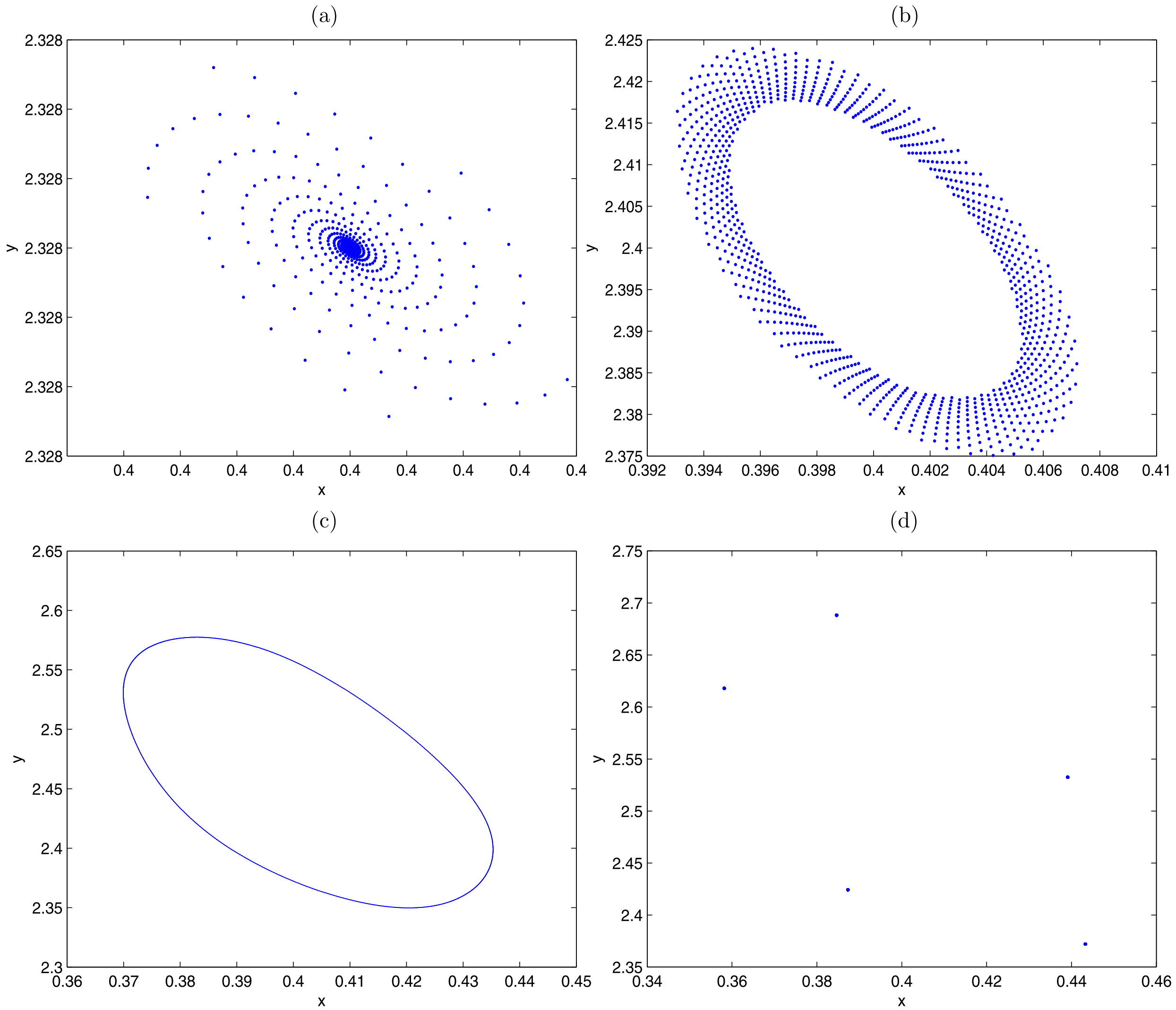

Fig. 5.2. Phase portraits corresponding to Fig. 5.1, (a) $a = 4.9$; (b) $a = 5$; (c) $a = 5.1$; (d) $a = 5.2$.

6. Conclusions

The Bogdanov–Takens bifurcation theory for generic diffeomorphisms developed by Broer et al. [4,5] and Kuznetsov [13] indicates that if the Jacobian matrix of a planar diffeomorphism at a fixed point has a double unit eigenvalue but is not the identity (1:1 resonance), then the diffeomorphism can be approximated by the time-one flow of a vector field which has a singularity with nilpotent linear part. Using this fact, Yagasaki [27] studied Bogdanov–Takens bifurcation for subharmonics in periodic perturbations of planar Hamiltonian systems and gave estimations of the bifurcation sets near the Bogdanov–Takens bifurcation points of diffeomorphisms.

In this paper we considered a discrete predator–prey model (1.6) with nonmonotone functional response. It was shown that the model exhibits various bifurcations of codimension 1, including fold bifurcation, transcritical bifurcation, flip bifurcations and Neimark–Sacker bifurcation, as the values of parameters vary. Moreover, we employed the results of Broer et al. [4,5] and Kuznetsov [13] and techniques of Yagasaki [27] to prove the existence of Bogdanov–Takens bifurcation and calculate the approximate expressions of bifurcation curves in system (1.6). To the best of our knowledge, this is the first study showing the existence of Bogdanov–Takens bifurcation in discrete-time predator–prey systems.

We would like to mention that other researchers have studied the discrete version of the predator–prey model with nonmonotone functional response of Ruan and Xiao [21], for example, Hu et al. [9] and Wang and Li [24]. However, there are two key differences between these studies and ours. Firstly, we gave detailed derivation of our model (1.6) which is different from theirs. Secondly, we proved that model (1.6) exhibits bifurcations of codimension 1, including fold bifurcation, transcritical bifurcation, flip bifurcations Neimark–Sacker bifurcation, as well as Bogdanov–Takens bifurcation of codimension 2 whereas Hu et al. [9] and Wang and Li [24] only considered bifurcations of codimension 1 of their models.

Notice that the original continuous-time predator–prey system (1.4) with nonmonotone functional response undergoes Bogdanov–Takens bifurcation (Ruan and Xiao [21]). Moreover, the delayed version of this predator–prey model with nonmonotone functional response still exhibits Bogdanov–Takens bifurcation (Xiao and Ruan [26]). Thus, it is interesting to see that Bogdanov–Takens bifurcation persists in the three versions of the predator–prey model with nonmonotone functional response: continuous-time, discrete time, and time-delayed. This indicates that the nonlinearity that induces the codimension 2 bifurcations near the singularity might be more dominated than the structure of the systems.

Theorems A.1, A.4 and 3.6 show that fold-flip bifurcation may occur at A and E_0, and Theorem A.2 indicates that 1:2 resonance bifurcation may occur at E_1. It will be very interesting to study these bifurcations and we leave these cases for future consideration.

Appendix

In this Appendix, we provide details in deriving the existence and stability of fixed points of the discrete model (1.6) which were given in section 2.

The fixed points of map (1.6) satisfy the following equations

$$\begin{aligned} x &= ax(1-x) - \tfrac{bxy}{1+\epsilon x^2}, \\ y &= \tfrac{dxy}{1+\epsilon x^2}. \end{aligned} \tag{A.1}$$

By simple calculations, we can see that the map (1.6) has at most four fixed points: $O(0,0)$, $A(\frac{a-1}{a},0)$, two positive fixed points $E_1(x_1,y_1)$ and $E_2(x_2,y_2)$, and the two positive fixed points may coalesce into a unique positive fixed point $E_0(x_0,y_0)$, where

$$\begin{aligned} x_1 &= \tfrac{d-\sqrt{d^2-4\epsilon}}{2\epsilon}, & y_1 &= \tfrac{dx_1}{b}\Big(a(1-x_1)-1\Big), \\ x_2 &= \tfrac{d+\sqrt{d^2-4\epsilon}}{2\epsilon}, & y_2 &= \tfrac{dx_2}{b}\Big(a(1-x_2)-1\Big), \\ x_0 &= \tfrac{2}{d}, & y_0 &= \tfrac{2a(d-2)-2d}{bd}. \end{aligned} \tag{A.2}$$

Clearly, map (1.6) always has the fixed point O, and has the fixed point A if $a > 1$. We now discuss the existence of possible positive fixed points.

(I) When $0 < \epsilon < \frac{d^2}{4}$, we can see that $x_i (i = 1, 2)$ exists and $0 < \frac{1}{d} < x_1 < \frac{2}{d} < x_2$ from the second equation of (A.1), and $a(1-x_1)-1 > a(1-x_2)-1$ from equations (A.2).

(i) If $x_1 \geq 1$, or $x_1 < 1$ and $y_1 \leq 0$, we can deduce that $y_1 \leq 0$ and $y_2 \leq 0$, so both positive fixed points E_1 and E_2 do not exist. By simple calculations, we can see that $x_1 \geq 1$, or $x_1 < 1$ and $y_1 \leq 0$ $\Longleftrightarrow 0 < d < 2,\ d-1 \leq \epsilon < \frac{d^2}{4}$ or $1 < d \leq 2,\ \epsilon < d-1,\ a \leq \frac{1}{1-x_1}$ or $d > 2,\ \epsilon < \frac{d^2}{4},\ a \leq \frac{1}{1-x_1}$.

(ii) If $x_1 < 1 \leq x_2$ and $y_1 > 0$, or $x_2 < 1, y_1 > 0$ and $y_2 \leq 0$, we can see that the positive fixed point E_1 exists and the positive fixed point E_2 does not exist. We can also get that $x_1 < 1 \leq x_2$ and $y_1 > 0$, or $x_2 < 1, y_1 > 0$ and $y_2 \leq 0 \Longleftrightarrow 1 < d \leq 2,\ \epsilon < d-1,\ a > \frac{1}{1-x_1}$ or $d > 2,\ \epsilon \leq d-1,\ a > \frac{1}{1-x_1}$ or $d > 2,\ d-1 < \epsilon < \frac{d^2}{4},\ \frac{1}{1-x_1} < a \leq \frac{1}{1-x_2}$.

(iii) If $x_2 < 1$ and $y_2 > 0$, i.e., $d > 2$, $d - 1 < \epsilon < \frac{d^2}{4}$, $a > \frac{1}{1-x_2}$, both positive fixed points E_1 and E_2 exist.

(II) When $\epsilon = \frac{d^2}{4}$, then $x_2 = x_1 = \frac{2}{d} > 0$.

(i) If $x_0 < 1$ and $y_0 > 0$, i.e., $d > 2$, $a > \frac{2}{d-2}$, then the positive fixed point E_1 coincides with E_2 and a unique positive fixed point $E_0(x_0, y_0)$ arises.

(ii) If $x_0 \geq 1$ or $y_0 < 0$, i.e., $0 < d \leq 2$, or $d > 2$ and $a \leq \frac{d}{d-2}$, then the unique positive fixed point $E_0(x_0, y_0)$ does not exist.

(III) When $\epsilon > \frac{d^2}{4}$, then x_1 and x_2 do not exist, so the map (1.6) only has boundary fixed points.

The existence conditions of these fixed points were given in Table 1.

We next consider the linear stability of the fixed points of map (1.6), the Jacobian matrix of map (1.6) at any fixed point (x, y) is given by

$$J(x,y) = \begin{pmatrix} a(1-2x) - \frac{by(1-\epsilon x^2)}{(1+\epsilon x^2)^2} & -\frac{bx}{1+\epsilon x^2} \\ \frac{dy(1-\epsilon x^2)}{(1+\epsilon x^2)^2} & \frac{dx}{1+\epsilon x^2} \end{pmatrix}. \tag{A.3}$$

Theorem A.1. *The fixed point O is a stable node if $0 < a < 1$, or a saddle if $a > 1$, or non-hyperbolic if $a = 1$. The fixed point A arises when $a > 1$. The properties of A are given in Table 2.*

Proof. The eigenvalues of $J(O(0,0))$ are $\lambda_1 = a, \lambda_2 = 0$, so the type of $O(0,0)$ is simple.

If $a > 1$, the fixed point $A(\frac{a-1}{a}, 0)$ arises. The eigenvalues of the Jacobian matrix of map (1.6) at the fixed point A are

$$\lambda_1 = 2 - a, \ \lambda_2 = \frac{ad(a-1)}{a^2 + \epsilon(a-1)^2}.$$

When $a > 1$, then $\lambda_2 > 0$. Hence, $\lambda_2 > 1$ if $0 < \epsilon < \frac{a(ad-d-a)}{(a-1)^2}$ and $d > \frac{a}{a-1}$; $\lambda_2 = 1$ if $\epsilon = \frac{a(ad-d-a)}{(a-1)^2}$ and $d > \frac{a}{a-1}$; $0 < \lambda_2 < 1$ if $\epsilon > \frac{a(ad-d-a)}{(a-1)^2}$.

On the other hand, we can see that $0 \leq \lambda_1 < 1$ if $1 < a \leq 2$, $-1 < \lambda_1 < 0$ if $2 < a < 3$, $\lambda_1 = -1$ if $a = 3$, and $\lambda_1 < -1$ if $a > 3$.

By the above analysis, we can easily verify all cases in Table 2. $\square$

The characteristic equation of the Jacobian matrix J around any positive fixed point (x, y) can be written as

$$\lambda^2 - p(x)\ \lambda + q(x) = 0, \tag{A.4}$$

where

$$p(x) = a(2-3x) - \tfrac{2}{dx}\Big(a(1-x) - 1\Big), q(x) = a(1-2x), \tag{A.5}$$

and the eigenvalues of the Jacobian matrix J around any positive fixed point (x, y) are

$$\lambda_{1,2} = \frac{p(x) \pm \sqrt{p^2(x) - 4q(x)}}{2}. \tag{A.6}$$

Combining with the existence conditions of the positive fixed point E_1, i.e., $\epsilon < \frac{d^2}{4}, x_1 < 1$ and $y_1 > 0$, we have the stability of E_1 as follows.

Theorem A.2. *When $\epsilon < \frac{d^2}{4}$, $x_1 < 1$ and $y_1 > 0$, the positive fixed point $E_1(x_1, y_1)$ exists. The properties of E_1 are given in Table 3, where $f(x) = 5dx^2 - (3d+2)x + 2$, $x_{11} = \frac{3d+2-\sqrt{\Delta}}{10d}$, $x_{12} = \frac{3d+2+\sqrt{\Delta}}{10d}$, $\Delta = (3d+2)^2 - 40d$.*

Proof. We choose x_1 as a parameter to analyze the stability of $E_1(x_1, y_1)$ and denote $f(x) = 5dx^2 - (3d + 2)x + 2$, $x_{11} = \frac{3d+2-\sqrt{\Delta}}{10d}$, $x_{12} = \frac{3d+2+\sqrt{\Delta}}{10d}$, $\Delta = (3d + 2)^2 - 40d$, which will be used in the following proof.

From the second equation of (A.1) and $\epsilon > 0$, we have $\frac{1}{d} < x_1 < \frac{2}{d}$. Combining with $x_1 < 1$, we get

$$\frac{1}{d} < x_1 < \min\left\{1, \frac{2}{d}\right\}, d > 1, \tag{A.7}$$

and we can deduce that $\epsilon < \frac{d^2}{4}$ provided that x_1 satisfies the second equation of (A.1). When $x_1 < 1$ and $y_1 > 0$, we can also get that

$$a > \frac{1}{1 - x_1}. \tag{A.8}$$

Firstly, we analyze the stability of the positive fixed point E_1. E_1 is stable if and only if

$$\begin{cases} \mathrm{tr}(J(E_1)) - \det(J(E_1)) < 1, \\ \mathrm{tr}(J(E_1)) + \det(J(E_1)) > -1, \\ \det(J(E_1)) < 1. \end{cases} \Longleftrightarrow \begin{cases} a(1 - x_1)(1 - \frac{2}{dx_1}) < 1 - \frac{2}{dx_1}, \\ af(x_1) < dx_1 + 2, \\ a(1 - 2x_1) < 1. \end{cases} \tag{A.9}$$

By (A.7), we have

$$a(1 - x_1)(1 - \frac{2}{dx_1}) + \frac{2}{dx_1} < 1 \Longleftrightarrow a > \frac{1}{1 - x_1},$$

which is the same as (A.8), and

$$a(1 - 2x_1) < 1 \Longleftrightarrow a < \frac{1}{1 - 2x_1} \text{ if } 0 < x_1 < \frac{1}{2}, \text{ or } a > 0 \text{ if } \frac{1}{2} \le x_1 < 1.$$

In order to investigate the second inequality of (A.9), we firstly discuss the sign of $f(x_1)$. Obviously, when $1 < d < \frac{14+4\sqrt{10}}{9}$, then $\Delta < 0$ and $f(x_1) > 0$ for any x_1; when $d \ge \frac{14+4\sqrt{10}}{9}$, then $\Delta \ge 0$, moreover, $f(x_1) \le 0$ if $x_{11} \le x_1 \le x_{12}$, and $f(x_1) > 0$ if $0 < x_1 < x_{11}$ or $x_1 > x_{12}$.

(i) When $1 < d \le \frac{3}{2}$, then $0 < \frac{4d-2}{7d} < \frac{1}{2} < \frac{2}{3} \le \frac{1}{d} < 1 < \frac{2}{d}$. Moreover, $\frac{1}{1-x_1} \ge \frac{2+dx_1}{f(x_1)}$ if $\frac{2}{3} \le x_1 < 1$. Then, it is easy to see that the solution set of inequality system (A.9) is empty by (A.7), so E_1 is unstable or non-hyperbolic if $1 < d < \frac{3}{2}$.

(ii) When $\frac{3}{2} < d \le 2$, then $0 < \frac{4d-2}{7d} < \frac{1}{2} \le \frac{1}{d} < \frac{2}{3} < 1 \le \frac{2}{d}$. Then, the solution set of inequality system (A.9) is $\{(x_1, a) : \frac{1}{d} < x_1 < \frac{2}{3},\ \frac{1}{1-x_1} < a < \frac{2+dx_1}{f(x_1)}\}$. When $2 < d < \frac{9}{4}$, then $0 < \frac{4d-2}{7d} < \frac{1}{d} < \frac{1}{2} < \frac{2}{3} < \frac{2}{d} < 1$, moreover, $\frac{1}{1-2x_1} \ge \frac{2+dx_1}{f(x_1)}$ if $\frac{4d-2}{7d} \le x_1 < \frac{1}{2}$, similarly, the solution set of inequality system (A.9) is also $\{(x_1, a) : \frac{1}{d} < x_1 < \frac{2}{3},\ \frac{1}{1-x_1} < a < \frac{2+dx_1}{f(x_1)}\}$. Hence, when $\frac{3}{2} < d < \frac{9}{4}$, E_1 is stable if $\frac{1}{d} < x_1 < \frac{2}{3}$ and $\frac{1}{1-x_1} < a < \frac{2+dx_1}{f(x_1)}$.

(iii) When $\frac{9}{4} \le d < \frac{14+4\sqrt{10}}{9}$, then $\frac{1}{d} < \frac{4d-2}{7d} < \frac{1}{2} < \frac{2}{3} < \frac{2}{d} < 1$. If $\frac{1}{d} < x_1 \le \frac{4d-2}{7d}$, then $\frac{1}{1-x_1} < \frac{1}{1-2x_1} \le \frac{2+dx_1}{f(x_1)}$, and the solution set of inequality system (A.9) is $\{(x_1, a) : \frac{1}{d} < x_1 \le \frac{4d-2}{7d},\ \frac{1}{1-x_1} < a < \frac{1}{1-2x_1}\}$; If $\frac{4d-2}{7d} < x_1 < \frac{2}{3}$, then $\frac{1}{1-x_1} < \frac{2+dx_1}{f(x_1)} < \frac{1}{1-2x_1}$, and the solution set of inequality system (A.9) is $\{(x_1, a) : \frac{4d-2}{7d} < x_1 < \frac{2}{3},\ \frac{1}{1-x_1} < a < \frac{2+dx_1}{f(x_1)}\}$; If $\frac{2}{3} < x_1 < \frac{2}{d}$, then the solution set of inequality system (A.9) is empty. So, when $\frac{9}{4} \le d < \frac{14+4\sqrt{10}}{9}$, E_1 is stable if $\frac{1}{d} < x_1 \le \frac{4d-2}{7d}$ and $\frac{1}{1-x_1} < a < \frac{1}{1-2x_1}$, or $\frac{4d-2}{7d} < x_1 < \frac{2}{3}$ and $\frac{1}{1-x_1} < a < \frac{2+dx_1}{f(x_1)}$.

When $\frac{14+4\sqrt{10}}{9} \le d \le 3$, then $\frac{1}{d} < x_{11} \le x_{12} < \frac{4d-2}{7d} < \frac{1}{2} < \frac{2}{3} \le \frac{2}{d} < 1$, we have $\frac{1}{1-x_1} < \frac{1}{1-2x_1} \le \frac{2+dx_1}{f(x_1)}$ if $\frac{1}{d} < x_1 \le \frac{4d-2}{7d}$, and $\frac{1}{1-x_1} < \frac{2+dx_1}{f(x_1)} < \frac{1}{1-2x_1}$ if $\frac{4d-2}{7d} < x_1 < \frac{2}{3}$. Similarly, when $\frac{14+4\sqrt{10}}{9} \le d \le 3$, E_1 is stable if $\frac{1}{d} < x_1 \le \frac{4d-2}{7d}$, $\frac{1}{1-x_1} < a < \frac{1}{1-2x_1}$, or $\frac{4d-2}{7d} < x_1 < \frac{2}{3}$, $\frac{1}{1-x_1} < a < \frac{dx_1+2}{f(x_1)}$.

In conclusion, when $\frac{9}{4} \le d \le 3$, E_1 is stable if $\frac{1}{d} < x_1 \le \frac{4d-2}{7d}$ and $\frac{1}{1-x_1} < a < \frac{1}{1-2x_1}$, or $\frac{4d-2}{7d} < x_1 < \frac{2}{3}$ and $\frac{1}{1-x_1} < a < \frac{2+dx_1}{f(x_1)}$.

(iv) When $3 < d < 4$, then $x_{11} < \frac{1}{d} < x_{12} < \frac{4d-2}{7d} < \frac{1}{2} < \frac{2}{d} < \frac{2}{3}$, we have $\frac{1}{1-x_1} < \frac{1}{1-2x_1} \le \frac{2+dx_1}{f(x_1)}$ if $\frac{1}{d} < x_1 \le \frac{4d-2}{7d}$, and $\frac{1}{1-x_1} < \frac{2+dx_1}{f(x_1)} < \frac{1}{1-2x_1}$ if $\frac{4d-2}{7d} < x_1 < \frac{2}{d}$. Similarly, when $3 < d < 4$, E_1 is stable if $\frac{1}{d} < x_1 \le \frac{4d-2}{7d}$ and $\frac{1}{1-x_1} < a < \frac{1}{1-2x_1}$, or $\frac{4d-2}{7d} < x_1 < \frac{2}{d}$ and $\frac{1}{1-x_1} < a < \frac{dx_1+2}{f(x_1)}$.

(v) When $d \ge 4$, then $x_{11} < \frac{1}{d} < \frac{2}{d} \le \frac{1}{2} < \frac{4d-2}{7d} < x_{12} < \frac{2}{3}$, we have $\frac{1}{1-x_1} < \frac{1}{1-2x_1} < \frac{2+dx_1}{f(x_1)}$ if $\frac{1}{d} < x_1 < \frac{2}{d}$. Similarly, when $d \ge 4$, E_1 is stable if $\frac{1}{d} < x_1 < \frac{2}{d}$ and $\frac{1}{1-x_1} < a < \frac{1}{1-2x_1}$.

Secondly, we discuss the non-hyperbolicity of E_1.

(vi) One eigenvalue of the Jacobian matrix at E_1 is $\lambda = 1$ if and only if $\mathrm{tr}(J(E_1)) - \det(J(E_1)) = 1$. But $\mathrm{tr}(J(E_1)) - \det(J(E_1)) = 1 \Leftrightarrow a = \frac{1}{1-x_1}$, which conflicts with the condition (A.7). So $\lambda = 1$ is not an eigenvalue of E_1. Moreover, the Jacobian matrix of map (1.6) at E_1 does not have a unit eigenvalue 1 with multiplicity 2.

(vii) One of eigenvalues of $J(E_1)$ is $\lambda = -1$ if and only if

$$\mathrm{tr}(J(E_1)) + \det(J(E_1)) = -1 \Longleftrightarrow a = \frac{dx_1+2}{f(x_1)} \ (f(x_1) > 0). \tag{A.10}$$

By (A.7) and (A.8), and $\frac{2+dx_1}{f(x_1)} > \frac{1}{1-x_1}$ if $x_1 < \frac{2}{3}$, we have $\frac{1}{d} < x_1 < \min\left\{\frac{2}{d}, \frac{2}{3}\right\}$ $(d > \frac{3}{2})$. By the above analysis about the stability of E_1, we have $\frac{1}{d} < \frac{2}{3} < \frac{2}{d}$ if $\frac{3}{2} < d < \frac{14+4\sqrt{10}}{9}$, and $\frac{1}{d} < x_{11} \le x_{12} < \frac{2}{3} \le \frac{2}{d}$ if $\frac{14+4\sqrt{10}}{9} \le d \le 3$, and $x_{11} < \frac{1}{d} < x_{12} < \frac{2}{d} < \frac{2}{3}$ if $3 < d < 4$, and $x_{11} < \frac{1}{d} < \frac{2}{d} < x_{12} < \frac{2}{3}$ if $d \ge 4$. So, combining with $f(x_1) > 0$, one of eigenvalues of $J(E_1)$ is $\lambda = -1$ if $\frac{3}{2} < d < \frac{14+4\sqrt{10}}{9}$, $\frac{1}{d} < x_1 < \frac{2}{3}$ and $a = \frac{2+dx_1}{f(x_1)}$, or $\frac{14+4\sqrt{10}}{9} \le d \le 3$, $\frac{1}{d} < x_1 < x_{11}$ and $a = \frac{2+dx_1}{f(x_1)}$, or $\frac{14+4\sqrt{10}}{9} \le d \le 3$, $x_{12} < x_1 < \frac{2}{3}$ and $a = \frac{2+dx_1}{f(x_1)}$, or $3 < d < 4$, $x_{12} < x_1 < \frac{2}{d}$ and $a = \frac{2+dx_1}{f(x_1)}$.

(viii) $J(E_1)$ has an eigenvalue $\lambda = -1$ with multiplicity 2 if and only if

$$\begin{cases} \mathrm{tr}(J(E_1)) = -2, \\ \det(J(E_1)) = 1. \end{cases} \Longleftrightarrow \begin{cases} a = \frac{2dx_1+2}{3dx_1^2-(2d+2)x_1+2} & \text{if } d \ge 2+\sqrt{3} \text{ and } 0 < x_1 < x_{13}, \\ & \text{or } d \ge 2+\sqrt{3} \text{ and } x_1 > x_{14}, \\ & \text{or } 2-\sqrt{3} < d < 2+\sqrt{3}, \\ a = \frac{1}{1-2x_1} & \text{if } 0 < x_1 < \frac{1}{2}, \end{cases} \tag{A.11}$$

where $x_{13} = \frac{2d+2-\sqrt{\Delta_1}}{6d}, x_{14} = \frac{2d+2+\sqrt{\Delta_1}}{6d}, \Delta_1 = 4d^2 - 16d + 4$.

When $\frac{2dx_1+2}{3dx_1^2-(2d+2)x_1+2} = \frac{1}{1-2x_1}$, then $x_1 = \frac{4d-2}{7d}$. Noted that $\frac{1}{d} < x_1 = \frac{4d-2}{7d} < \frac{2}{d}$, i.e., $\frac{9}{4} < d < 4$. Moreover, we have $\frac{1}{d} < x_{13} < x_{14} < \frac{4d-2}{7d} < \frac{1}{2} < \frac{2}{d}$ if $\frac{9}{4} < d < 4$. It is easy to see that $x_1 = \frac{4d-2}{7d}$ $(\frac{9}{4} < d < 4)$ satisfies (A.11). Hence, $J(E_1)$ has an eigenvalue -1 with multiplicity 2 if $\frac{9}{4} < d < 4, x_1 = \frac{4d-2}{7d}$ and $a = \frac{1}{1-2x_1}$. From $x_1 = \frac{d-\sqrt{d^2-4\epsilon}}{2\epsilon} = \frac{4d-2}{7d}$, we can get that $\epsilon = \frac{7(4d^3-9d^2)}{4(2d-1)^2}$ and $a = \frac{1}{1-2x_1} = \frac{7d}{4-d}$. Hence, we can obtain that $J(E_1)$ has eigenvalues -1 with multiplicity 2 if $\frac{9}{4} < d < 4, \epsilon = \frac{7(4d^3-9d^2)}{4(2d-1)^2}$ and $a = \frac{7d}{4-d}$.

(ix) $J(E_1)$ has conjugate complex eigenvalues with module 1 if and only if

$$\begin{cases} \det J(E_1) = 1, \\ -2 < \mathrm{tr}J(E_1) < 2. \end{cases} \Leftrightarrow \begin{cases} a = \frac{1}{1-2x_1}, \quad \text{if } 0 < x_1 < \frac{1}{2}, \\ 2 - 2dx_1 < a(3dx_1^2 - (2d+2)x_1 + 2), \\ a(3dx_1^2 - (2d+2)x_1 + 2) < 2 + 2dx_1. \end{cases} \Leftrightarrow \begin{cases} a = \frac{1}{1-2x_1}, \\ 0 < x_1 < \frac{1}{2}, \\ \frac{1}{d} < x_1 < \frac{2}{d}, \\ x_1 < \frac{4d-2}{7d}. \end{cases} \tag{A.12}$$

Obviously, when $\frac{4d-2}{7d} > \frac{1}{d}$, i.e., $d > \frac{9}{4}$, the solution set of (A.12) may be non-empty. We easily obtain that $\frac{1}{d} < \frac{4d-2}{7d} < \frac{1}{2} < \frac{2}{d}$ if $\frac{9}{4} < d < 4$, and $\frac{1}{d} < \frac{2}{d} \leq \frac{1}{2} \leq \frac{4d-2}{7d}$ if $d \geq 4$. Hence, $J(E_1)$ has conjugate complex eigenvalues with module 1 if $\frac{9}{4} < d < 4, \frac{1}{d} < x_1 < \frac{4d-2}{7d}$ and $a = \frac{1}{1-2x_1}$, or $d \geq 4, \frac{1}{d} < x_1 < \frac{2}{d}$ and $a = \frac{1}{1-2x_1}$.

(x) Obviously, E_1 is unstable for other cases. □

Theorem A.3. *When $d > 2, d-1 < \epsilon < \frac{d^2}{4}$ and $a > \frac{1}{1-x_2}$, the positive fixed point $E_2(x_2, y_2)$ exists and the types of E_2 are given in Table 4, where $f(x) = 5dx^2 - (3d+2)x + 2$, $x_{12} = \frac{3d+2+\sqrt{\Delta}}{10d}$, $\Delta = (3d+2)^2 - 40d$, $q(x_2) = a(1-2x_2)$.*

Proof. If $d > 2, d-1 < \epsilon < \frac{d^2}{4}, a > \frac{1}{1-x_2}$, E_2 exists by Table 1 and $\frac{2}{d} < x_2 < 1$.

(i) Firstly, we analyze stability of the positive fixed point E_2. E_2 is stable if and only if

$$\begin{cases} \mathrm{tr}(J(E_2)) - \det(J(E_2)) < 1, \\ \mathrm{tr}(J(E_2)) + \det(J(E_2)) > -1, \\ \det(J(E_2)) < 1. \end{cases} \Longleftrightarrow \begin{cases} a(1-x_2)(1-\frac{2}{dx_2}) < 1 - \frac{2}{dx_2}, \\ af(x_2) < dx_2 + 2, \\ a(1-2x_2) < 1. \end{cases} \tag{A.13}$$

Since $x_2 > \frac{2}{d}$, then $a(1-x_2)(1-\frac{2}{dx_2}) + \frac{2}{dx_2} < 1 \Leftrightarrow a < \frac{1}{1-x_2}$, which contradicts the existence condition $a > \frac{1}{1-x_2}$, so the positive fixed point E_2 is always unstable.

(ii) Secondly, we analyze the non-hyperbolicity of E_2.

• One of the eigenvalues of $J(E_2)$ is $\lambda = 1$ if and only if $\mathrm{tr}(J(E_2)) - \det(J(E_2)) = 1$, but $\mathrm{tr}(J(E_2)) - \det(J(E_2)) = 1 \Leftrightarrow a = \frac{1}{1-x_2}$, which conflicts with the existence condition $a > \frac{1}{1-x_2}$. Hence $\lambda = 1$ is not an eigenvalue of $J(E_2)$, and $J(E_2)$ does not have an eigenvalue 1 with multiplicity 2.

• One of the eigenvalues of $J(E_2)$ is $\lambda = -1$ if and only if

$$\mathrm{tr}(J(E_2)) + \det(J(E_2)) = -1 \Longleftrightarrow a = \frac{dx_2+2}{f(x_2)} \ (f(x_2) > 0). \tag{A.14}$$

Because $\frac{2}{d} < x_2 < 1$ and $a > \frac{1}{1-x_2}$, we have $\frac{2+dx_2}{f(x_1)} > \frac{1}{1-x_2}$ if $\frac{2}{d} < x_2 < \frac{2}{3}(d > 3)$. Similarly, we have $x_{11} < x_{12} < \frac{2}{d} < \frac{2}{3} < 1$ if $3 < d < 4$, and $x_{11} < \frac{2}{d} < x_{12} < \frac{2}{3} < 1$ if $d \geq 4$. Hence, when $3 < d < 4, \frac{2}{d} < x_2 < \frac{2}{3}$ and $a = \frac{2+dx_2}{f(x_2)}$, or when $d \geq 4, x_{12} < x_2 < \frac{2}{3}$ and $a = \frac{2+dx_2}{f(x_2)}$, one of the eigenvalues of $J(E_2)$ is $\lambda = -1$.

• $J(E_2)$ has an eigenvalue $\lambda = -1$ with multiplicity 2 if and only if

$$\begin{cases} \mathrm{tr}J(E_2) = -2, \\ \det J(E_2) = 1. \end{cases} \Longleftrightarrow \begin{cases} a = \frac{2dx_2+2}{3dx_2^2-(2d+2)x_2+2} & \text{if } d \geq 2+\sqrt{3} \text{ and } 0 < x_2 < x_{13}, \\ & \text{or } d \geq 2+\sqrt{3} \text{ and } x_2 > x_{14}, \\ & \text{or } 2-\sqrt{3} < d < 2+\sqrt{3}, \\ a = \frac{1}{1-2x_2} & \text{if } 0 < x_2 < \frac{1}{2}, \end{cases} \tag{A.15}$$

where $x_{13} = \frac{2d+2-\sqrt{\Delta_1}}{6d}$, $x_{14} = \frac{2d+2+\sqrt{\Delta_1}}{6d}$, $\Delta_1 = 4d^2 - 16d + 4$.

When $\frac{2dx_2+2}{3dx_2^2-(2d+2)x_2+2} = \frac{1}{1-2x_2}$, then $x_2 = \frac{4d-2}{7d}$. Noted that $x_2 = \frac{4d-2}{7d} > \frac{2}{d}$, i.e., $d > 4$. However, we have $\frac{2}{d} < \frac{1}{2} < \frac{4d-2}{7d}$ if $d > 4$, that is $x_2 = \frac{4d-2}{7d} \notin (\frac{2}{d}, \frac{1}{2})$, which shows that the solution set of system (A.15) is empty. Hence, $J(E_2)$ does not have an eigenvalue -1 with multiplicity 2.

• $J(E_2)$ has conjugate complex eigenvalues with module 1 if and only if

$$\begin{cases} \det(J(E_2)) = 1, \\ -2 < \mathrm{tr}J(E_2) < 2. \end{cases} \Leftrightarrow \begin{cases} a = \frac{1}{1-2x_2}, \quad \text{if } 0 < x_2 < \frac{1}{2}, \\ 2 - 2dx_2 < a(3dx_2^2 - (2d+2)x_2 + 2), \\ a(3dx_2^2 - (2d+2)x_2 + 2) < 2 + 2dx_2. \end{cases} \Leftrightarrow \begin{cases} 0 < x_2 < \frac{1}{2}, \\ a = \frac{1}{1-2x_2}, \\ x_2 < \frac{2}{d}, \\ x_2 < \frac{4d-2}{7d}. \end{cases} \tag{A.16}$$

Noted that $x_2 > \frac{2}{d}$, which conflicts with (A.16), so the solution set of system (A.16) is empty, hence $J(E_2)$ does not have conjugate complex eigenvalues with module 1.

(iii) Obviously, E_2 is unstable for other cases. □

For the unique positive fixed point E_0, we have the following results.

Theorem A.4. *When $\epsilon = \frac{d^2}{4}$, $d > 2$ and $a > \frac{d}{d-2}$, the map (1.6) has a unique positive fixed point E_0, and the types of E_0 are given in Table 5.*

Proof. When $\epsilon = \frac{d^2}{4}$, $d > 2$ and $a > \frac{d}{d-2}$, a unique positive fixed point $E_0(x_0, y_0)$ arises by Table 1, where $x_0 = \frac{2}{d}$, $y_0 = \frac{2a(d-2)-2d}{bd}$. The eigenvalues of the Jacobian matrix J around the fixed point E_0 of map (1.6) are

$$\lambda_1 = 1,\ \lambda_2 = \frac{a(d-4)}{d}. \tag{A.17}$$

Then E_0 is always non-hyperbolic. Concretely,

(i) When $a \neq \pm\frac{d}{d-4}$ and $d > 2$, then $|\lambda_2| \neq 1$. Hence, a fold bifurcation may occur at E_0.

(ii) When $a = \frac{d}{4-d}$ and $3 < d < 4$, then $\lambda_2 = \frac{a(d-4)}{d} = -1$, $x_0 = \frac{2}{d}$, $y_0 = \frac{4(d-3)}{b(4-d)} > 0$, the degenerate fixed point E_0 has eigenvalues 1 and -1 and therefore a fold-flip bifurcation may occur at E_0.

(iii) When $a = \frac{d}{d-4}$ and $d > 4$, then $\lambda_2 = \frac{a(d-4)}{d} = 1$, the degenerate fixed point $E_0\left(\frac{2}{d}, \frac{4}{b(d-4)}\right)$ has an eigenvalue 1 with multiplicity 2, and a Bogdanov–Takens bifurcation of codimension 2 may occur around E_0. □

References

[1] J.R. Beddington, C.A. Free, J.H. Lawton, Dynamic complexity in predator–prey models framed in difference equations, Nature 255 (1975) 58–60.

[2] J.R. Beddington, C.A. Free, J.H. Lawton, Characteristics of successful natural enemies in models of biological control of insect pests, Nature 273 (1978) 513–519.

[3] I.R. Bogdanov, Versal deformation of a singularity of a vector field on the plane in the case of zero eigenvalues, Sel. Math. Sov. 1 (1981) 389–421.

[4] H.W. Broer, R. Roussarie, C. Simó, On the Bogdanov–Takens bifurcation for planar diffeomorphisms, in: International Conference on Differential Equations, vol. 1, 2, Barcelona, 1991, World Sci. Publ., River Edge, NJ, 1993, pp. 81–92.

[5] H.W. Broer, R. Roussarie, C. Simó, Invariant circles in the Bogdanov–Takens bifurcation for diffeomorphisms, Ergodic Theory Dynam. Systems 16 (1996) 1147–1172.

[6] J.B. Collings, The effects of the functional response on the bifurcation behavior of a mite predator–prey interaction model, J. Math. Biol. 36 (1997) 149–168.

[7] J. Guckenheimer, P. Holmes, Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields, Springer-Verlag, New York, 1983.

[8] K.P. Hadeler, I. Gerstmann, The discrete Rosenzweig model, Math. Biosci. 98 (1990) 49–72.

[9] Z. Hu, Z. Teng, L. Zhang, Stability and bifurcation analysis of a discrete predator–prey model with nonmonotonic functional response, Nonlinear Anal. Real World Appl. 12 (2011) 2356–2377.

[10] J. Huang, Bifurcations and chaos in a discrete predator–prey system with Holling type-IV functional response, Acta Math. Appl. Sin. Engl. Ser. 21 (2005) 157–176.

[11] J. Huang, D. Xiao, Analyses of bifurcations and stability in a predator–prey system with Holling type-IV functional response, Acta Math. Appl. Sin. Engl. Ser. 20 (1) (2004) 167–178.

[12] M. Kot, Elements of Mathematical Ecology, Cambridge University Press, Cambridge, 2001.

[13] Y. Kuznetsov, Elements of Applied Bifurcation Theory, Springer-Verlag, New York, 1998.

[14] S.H. Levine, Discrete time modeling of ecosystems with applications in environmental enrichment, Math. Biosci. 24 (1975) 307–317.

[15] S. Li, W. Zhang, Bifurcations of a discrete prey–predator model with Holling type II functional response, Discrete Contin. Dyn. Syst. Ser. B 14 (2010) 159–176.

[16] X. Liu, D. Xiao, Bifurcations in a discrete time Lotka–Volterra predator–prey system, Discrete Contin. Dyn. Syst. Ser. B 6 (2006) 559–572.

[17] R.M. May, Biological populations with nonoverlapping generations: stable points, stable cycles, and chaos, Science 186 (1974) 645–647.

[18] J. Maynard Smith, Mathematical Ideas in Biology, Cambridge University Press, Cambridge, 1968.
[19] M.G. Neubert, M. Kot, The subcritical collapse of predator populations in discrete-time predator–prey models, Math. Biosci. 110 (1992) 45–66.
[20] A.J. Nicholson, V.A. Bailey, The balance of animal populations, Proc. Zool. Soc. Lond. (1935) 551–598.
[21] S. Ruan, D. Xiao, Global analysis in a predator–prey system with nonmonotonic functional response, SIAM J. Appl. Math. 61 (2001) 1445–1472.
[22] W. Sokol, J.A. Howell, Kinetics of phenol oxidation by washed cells, Biotechnol. Bioeng. 23 (1980) 2039–2049.
[23] F. Takens, Forced oscillations and bifurcations, in: Applications of Global Analysis I, in: Comm. Math. Inst. Rijksuniv. Utrecht, vol. 3, 1974, pp. 1–59.
[24] C. Wang, X. Li, Further investigations into the stability and bifurcation of a discrete predator–prey model, J. Math. Anal. Appl. 422 (2015) 920–939.
[25] S. Wiggins, Introduction to Applied Nonlinear Dynamical Systems and Chaos, Springer-Verlag, New York, 1990.
[26] D. Xiao, S. Ruan, Multiple bifurcations in a delayed predator–prey system with nonmonotonic functional response, J. Differential Equations 176 (2001) 494–510.
[27] K. Yagasaki, Melnikov's method and codimension-two bifurcations in forced oscillations, J. Differential Equations 185 (2002) 1–24.