SIAM J. APPLIED DYNAMICAL SYSTEMS
Vol. 18, No. 2, pp. 939–972

Bifurcation Analysis of a Mosquito Population Model with a Saturated Release Rate of Sterile Mosquitoes*

Jicai Huang†, Shigui Ruan‡, Pei Yu§, and Yuyue Zhang†

Abstract. Releasing sterile mosquitoes is a method of mosquito control that uses area-wide inundative releases of sterile male mosquitoes to reduce reproduction in a field population of wild mosquitoes. In this paper, we consider a mosquito population model with a nonlinear saturated release rate of sterile mosquitoes and study the complex dynamics and bifurcations of the model. It is shown that there are a weak focus of multiplicity 3 and a nilpotent cusp of codimension 4 for various parameter values and the model exhibits Hopf bifurcation of codimension 3 and Bogdanov–Takens bifurcation of codimension 2 as the parameter values vary. Our analysis also shows that there exists a critical release rate coefficient of sterile mosquitoes, above which the mosquito population can be eliminated and below which the interacting sterile and wild mosquitoes coexist in the form of multiple periodic oscillations and steady states for some initial populations. Numerical simulations are presented to demonstrate the coexistence of a homoclinic loop and a limit cycle, the existence of two limit cycles, and the existence of three limit cycles, respectively.

Key words. wild mosquitoes, sterile mosquitoes, release rate, Hopf bifurcation of codimension 3, nilpotent cusp of codimension 4, Bogdanov–Takens bifurcation

AMS subject classifications. 34C23, 34C25, 92D30

DOI. 10.1137/18M1208435

1. Introduction.

Mosquito-borne diseases such as chikungunya, dengue, malaria, yellow fever, Zika, etc., are transmitted to humans by blood-feeding mosquitoes. In tropical and subtropical areas, mosquito-borne diseases are still rampant and are becoming serious public health problems worldwide. Mosquito control, such as reducing the adult mosquito populations and controlling mosquito larvae, is a major measure to prevent mosquito-borne diseases. However, use of large amounts of insecticides will inevitably give rise to resistance of mosquitoes and some chemicals in pesticides affect human health (Blayneha and Mohammed-Awel [10], Lees et al. [34]). In recent years, the sterile insect technique has been proven to be useful and effective to reduce or eradicate mosquitoes (Alphey [2], Boete, Agusto, and Reeves [11], Dufourd and Dumont [17], Fister and McCarthy [23], Lees et al. [34], Li [38]).

*Received by the editors August 20, 2018; accepted for publication (in revised form) by L. Billings March 13, 2019; published electronically May 21, 2019.
http://www.siam.org/journals/siads/18-2/M120843.html
Funding: The research of the first author was partially supported by NSFC grants 11471133 and 11871235. The research of the second author was partially supported by NSF grant DMS-1853622 and CDC Southeastern Regional Center of Excellency in Vector-Borne Diseases–The Gateway Program (1U01-CK000510). The research of the third author was partially supported by NSERC grant R2686A02.
†School of Mathematics and Statistics, Central China Normal University, Wuhan, Hubei 430079, People's Republic of China (hjc@mail.ccnu.edu.cn, 2548244602@qq.com).
‡Department of Mathematics, University of Miami, Coral Gables, FL 33146 (ruan@math.miami.edu).
§Department of Applied Mathematics, Western University, London, ON N6A 5B7, Canada (pyu@uwo.ca).

The sterile insect technique (SIT) (Knipling [32], Dyck, Hendrichs, and Robinson [21]) is a method of biological insect control, whereby overwhelming numbers of sterile insects are released into the wild. The sterile males compete with wild males to mate with the females, and females that mate with a sterile male produce no offspring, thus reducing their reproductive output. Sterile insects are not self-replicating and cannot become established in the environment. This approach has achieved some success in controlling several insect pest species, including screwworm, Mediterranean fruit flies, and tsetse flies (Krafsur [33], Benedict and Robinson [8]). Applying SIT (by irradiation) to mosquitoes is not as simple as it looks. First, it is necessary to have a very good knowledge of the biology of the mosquito species to be controlled. Second, the irradiation dose is not the same for an *Anopheles* mosquito as for an *Aedes* mosquito. If the irradiation dose is too strong it can greatly affect the lifespan and the competitivity of the male (Benedict and Robinson [8], Dumont and Tchuenche [18]). It is generally agreed that classical SIT strategies have had sporadic success, leading to the recent development of transgenic technologies (Alphey [2], Carvalho, Costa-da-Silva, and Lees [14]).

One such transgenic strategy is the release of insects carrying a dominant lethal (RIDL) (Thomas et al. [45], Fu et al. [24]), in which the released male mosquitoes are homozygous for a repressible dominant lethal gene or genetic system. The repressors are something that could be provided during mass-rearing but are not found in the wild. These RIDL male mosquitoes mate with wild females and produce heterozygous progeny that die under predetermined conditions (Thomas et al. [45], Alphey [2]). Sterility can also be created by artificial infection with various strains of *Wolbachia*, a diverse group of intracellular bacteria (Werren, Baldo, and Clark [42]). *Wolbachia* are not infectious between insects on normal timescales; rather they are maternally transmitted, being passed from mother to her offspring. Infected males are useless to the maternally inherited *Wolbachia* for propagation; instead they produce modified sperm that produce viable zygotes only with eggs from infected females (Alphey [2]).

Mathematical models have been used extensively to gain insights into investigation and assessment of the impact of releasing sterile mosquitoes (Anguelov, Dumont, and Lubuma [3], Atkinson et al. [4], Barclay [5, 6], Berryman [9], Barclay and Mackauer [7], Dufourd and Dumont [17], Dumont and Tchuenche [18], Esteva and Yang [22], Fister and McCarthy [23], Knipling [32], Lewis and van den Driessche [35], Li [38], White, Rohani, and Sait [43]), and the development of appropriate mathematical models can potentially answer important ecological, epidemiological, and pest control problems more generally. Depending on the features under consideration, various types of models such as difference equation models (Knipling [32]), ordinary differential equation models (Anguelov, Dumont, and Lubuma [3], Barclay [5, 6], Berryman [9], Barclay and Mackauer [7], Dufourd and Dumont [17], Dumont and Tchuenche [18], Esteva and Yang [22], Li [38]), delay differential equation models (Atkinson et al. [4], White, Rohani, and Sait [43]), and partial differential equation models (Lewis and van den Driessche [35]) have been used to investigate the effects of releasing sterile mosquitoes on the population dynamics of mosquitoes and the transmission dynamics of some mosquito-borne diseases.

In the process of releasing sterile mosquitoes, whether SIT or transgenic, the most important and difficult problem is to determine suitable releasing strategies, since different releasing methods will produce different dynamic results of interactive wild and sterile mosquitoes as well as different transmission outcomes. So far various release rates of sterile/transgenic male mosquitoes have been used in the literature, including (i) constant rate (Anguelov, Dumont, and Lubuma [3], Berryman [9], Barclay and Mackauer [7], Dufourd and Dumont [17], Dumont and Tchuenche [18], Esteva and Yang [22], Li [38]); (ii) periodic rate (Barclay [6]); (iii) pulsed rate (White, Rohani, and Sait [43]); (iv) proportional rate (Atkinson et al. [4]); (v) trajectory rate (Atkinson et al. [4]); and (vi) nonlinear saturated rate (Cai, Ai, and Li [13], Li, Cai, and Li [39]).

Let $w(t)$ and $g(t)$ be the number of wild mosquitoes and sterile mosquitoes at time t, respectively, and let $N(t)(= w(t) + g(t))$ be the total number of wild mosquitoes and sterile mosquitoes. Assume that the dynamics of both wild and sterile mosquitoes, in the absence of interactions, follow logistic growth, and the birth rate of sterile mosquitoes is their release rate. After the sterile mosquitoes are released into the wild mosquito population, the interactive model takes the following form (Cai, Ai, and Li [13] and Li, Cai, and Li [39]):

$$\begin{aligned} \frac{dw}{dt} &= \left[C(N)\frac{aw}{w+g} - \big(\mu_1 + \xi_1(w+g)\big)\right] w, \\ \frac{dg}{dt} &= B(\cdot) - \big[\mu_2 + \xi_2(w+g)\big] g, \end{aligned} \tag{1.1}$$

where $C(N)$ is the number of matings per individual per unit of time, $a > 0$ is the number of wild offspring produced per mate, $\mu_i > 0$ and $\xi_i > 0 (i = 1, 2)$ are the density independent and dependent death rates of the wild and sterile mosquitoes, respectively, and $B(\cdot)$ is the release rate of the sterile mosquitoes. Considering the possible difficulty in finding mates when the mosquito population size is small, Cai, Ai, and Li [13] assumed an Allee effect such that the mating rate takes the form of $C(N) = c_0N/(1 + N)$, where c_0 is the maximum mating rate. In the meantime, Cai, Ai, and Li [13] supposed that the release rate $B(\cdot)$ is a nonlinear saturated function such that it is proportional to the wild mosquito population size when the wild mosquito population size is small but is saturated and approaches a constant b when the wild mosquito population size is big enough, i.e., $B(\cdot) = \frac{bw}{1+w}$, where $b > 0$ is the release rate coefficient. Under these assumptions and writing c_0a as a, system (1.1) becomes a mosquito population model with nonlinear saturated releasing rate of sterile mosquitoes

$$\begin{aligned} \frac{dw}{dt} &= \left[\frac{aw}{1+w+g} - \big(\mu_1 + \xi_1(w+g)\big)\right] w, \\ \frac{dg}{dt} &= \frac{bw}{1+w} - \big[\mu_2 + \xi_2(w+g)\big] g, \end{aligned} \tag{1.2}$$

where all parameters are positive. Cai, Ai, and Li [13] and Li, Cai, and Li [39] provided some basic analysis on the dynamics and gave some numerical examples to reveal the rich dynamical features of model (1.2). While their findings seem exciting and promising, the model could exhibit much more complex dynamics than has been found and the model deserves further theoretical analysis on its complex dynamics and bifurcation phenomena. Moreover, under the hypothesis that the sterile and wild mosquitoes have the same fitness, or the same death rates, i.e.,

$$\mu_1 = \mu_2 \quad \text{and} \quad \xi_1 = \xi_2,$$

Cai, Ai, and Li [13] studied the existence and local stability of positive equilibria of system (1.2). They fulfilled a relatively complete analysis for this special case, obtained a threshold release value implicitly, and showed that if there exist two positive equilibria, one must be a saddle point and the other is always an asymptotically stable node or spiral. Thus there are no complex bifurcations and dynamical phenomena for system (1.2) under the above hypothesis. However, in general, the sterile and wild mosquitoes should have different fitness, or different death rates, i.e.,

$$\mu_1 \neq \mu_2 \quad \text{and} \quad \xi_1 \neq \xi_2.$$

Hence, in this paper, in order to better understand the effect of a nonlinear saturated releasing rate of sterile mosquitoes to wild mosquitoes, we further consider system (1.2) with different fitnesses (i.e., $\mu_1 \neq \mu_2$ and $\xi_1 \neq \xi_2$) for the sterile and wild mosquitoes and find some complex bifurcation phenomena, such as Hopf bifurcation of codimension 3, nilpotent cusp of codimension 4, and Bogdanov–Takens bifurcation of codimension 2. Numerical simulations are presented to demonstrate the coexistence of a homoclinic loop and a limit cycle, the existence of two limit cycles, and the existence of three limit cycles, respectively.

The paper is organized as follows. In section 2, we first show that there are at most two positive equilibria in system (1.2) with different fitnesses for the sterile and wild mosquitoes, one is a saddle, and the other may be a stable or an unstable focus when there are two positive equilibria; the unique positive equilibrium may be a cusp. Hopf bifurcation of codimension 3, nilpotent cusp of codimension 4, and Bogdanov–Takens bifurcation of codimension 2 are discussed in section 3. The paper ends with a brief discussion in section 4.

2. Equilibria and their stability.

Define

$$\Omega := \left\{(w,g) : 0 \leq w \leq \frac{a}{\xi_1}, 0 \leq g \leq \frac{ba}{\xi_1\mu_2}\right\},$$

which is a positively invariant and attracting set for the flows of (1.2) in the first quadrant.

In fact, first, it is easy to see that $\{(w,g) : w \geq 0, g \geq 0\}$ is positively invariant since $w = 0$ is an invariant line and $\frac{dg}{dt}\big|_{g=0} = \frac{bw}{1+w} > 0$ if $w > 0$.

Second, from the first equation of system (1.2), we can get that

$$\frac{dw}{dt} < (a - \xi_1 w)w,$$

and a standard comparison argument shows that

$$\limsup_{t\to\infty} w(t) \leq \frac{a}{\xi_1}.$$

Then we can observe that there exists a $T > 0$ such that for $t > T$, $w(t) \leq \frac{a}{\xi_1}$. From the second equation of system (1.2), we can see that for $t > T$, we have

$$\frac{dg}{dt} \leq \frac{ba}{\xi_1} - \mu_2 g,$$

which shows that

$$\limsup_{t\to\infty} g(t) \leq \frac{ba}{\xi_1\mu_2}.$$

The above arguments imply that Ω is a positively invariant and attracting set for the flows of (1.2) in the positive quadrant.

We next introduce a new time variable τ by $dt = \frac{d\tau}{a}$ and still denote τ by t for convenience; then system (1.2) can be rewritten as

$$\begin{aligned}\frac{dw}{dt} &= \left[\frac{w}{1+w+g} - \left(\frac{\mu_1}{a} + \frac{\xi_1}{a}(w+g)\right)\right] w,\\ \frac{dg}{dt} &= \frac{\frac{b}{a}w}{1+w} - \left[\frac{\mu_2}{a} + \frac{\xi_2}{a}(w+g)\right] g.\end{aligned}$$

Still denoting $\frac{\mu_1}{a}, \frac{\xi_1}{a}, \frac{b}{a}, \frac{\mu_2}{a}, \frac{\xi_2}{a}$ by $\mu_1, \xi_1, b, \mu_2, \xi_2$, respectively, we have the following system:

$$\begin{aligned}\frac{dw}{dt} &= \left[\frac{w}{1+w+g} - \big(\mu_1 + \xi_1(w+g)\big)\right] w,\\ \frac{dg}{dt} &= \frac{bw}{1+w} - [\mu_2 + \xi_2(w+g)]g,\end{aligned} \tag{2.1}$$

where all parameters, b, μ_1, μ_2, ξ_1, and ξ_2 are positive.

2.1. Existence of equilibria.

Note that the origin $(0, 0)$ is always an equilibrium for system (2.1) and there is no other boundary equilibrium.

Let $N = w + g$ at a positive equilibrium. We have

$$w = (1+N)(\mu_1 + \xi_1 N), \quad g = \frac{b(1+N)(\mu_1+\xi_1 N)}{[1+(1+N)(\mu_1+\xi_1 N)](\mu_2+\xi_2 N)}. \tag{2.2}$$

Then N satisfies

$$\big[N - (1+N)(\mu_1+\xi_1 N)\big]\big[1+(1+N)(\mu_1+\xi_1 N)\big](\mu_2+\xi_2 N) = b(1+N)(\mu_1+\xi_1 N), \tag{2.3}$$

that is,

$$\big[1+(1+N)(\mu_1+\xi_1 N)\big](\mu_2+\xi_2 N) = \frac{b(1+N)(\mu_1+\xi_1 N)}{N-(1+N)(\mu_1+\xi_1 N)}, \tag{2.4}$$

which is a polynomial equation of degree five and may have up to five positive roots. In fact, we will show that (2.3) has at most two positive roots.

Let $G_1(N) = N - (1+N)(\mu_1 + \xi_1 N)$. Then $G_1(0) = -\mu_1 < 0$, and it is easy to see that $G_1(N) = 0$ has two positive roots

$$N_{1,2} = \frac{1}{2\xi_1}\left[(1-\mu_1-\xi_1) \pm \sqrt{(1-\mu_1-\xi_1)^2 - 4\mu_1\xi_1}\right]$$

if

$$1-\mu_1-\xi_1 > 0 \quad \text{and} \quad (1-\mu_1-\xi_1)^2 - 4\mu_1\xi_1 > 0,$$

or, equivalently, if

$$\sqrt{\mu_1} + \sqrt{\xi_1} < 1. \tag{2.5}$$

Since $G_1(N) > 0$ if $N \in (N_1, N_2)$, we consider (2.4) when $N \in (N_1, N_2)$. Let

$$F_1(N) = \big[1 + (1+N)(\mu_1 + \xi_1 N)\big](\mu_2 + \xi_2 N), \quad F_2(N) = \frac{(1+N)(\mu_1 + \xi_1 N)}{N - (1+N)(\mu_1 + \xi_1 N)}.$$

Then (2.4) can be rewritten as

$$F_1(N) = bF_2(N).$$

By some trivial computation, we can get the asymptotic features of $F_1(N)$ and $F_2(N)$. $F_1(N)$ is a monotone increasing positive function when $N > 0$ and $\lim_{N\to+\infty} F_1(N) = +\infty$. $F_2(N)$ is positive and decreases first and then increases when $N \in (N_1, N_2)$, $\lim_{N\to N_1^+} F_2(N) = +\infty$, $\lim_{N\to N_2^-} F_2(N) = +\infty$. Furthermore, it is easy to see that $F_1(N)$ and $F_2(N)$ have no inflection points when $N > 0$. Then $F_1(N) = bF_2(N)$ (i.e., (2.4)) has at most two positive roots.

We also rewrite (2.4) as

$$\frac{N - (1+N)(\mu_1 + \xi_1 N)}{(1+N)(\mu_1 + \xi_1 N)}\big[1 + (1+N)(\mu_1 + \xi_1 N)\big](\mu_2 + \xi_2 N) = b,$$

that is, $\frac{1}{F_2(N)} F_1(N) = b$. Define $F_3(N) = \frac{1}{F_2(N)}$; then (2.4) becomes

$$F_3(N)F_1(N) = b. \tag{2.6}$$

It can be seen that the positive equilibria of system (2.1) are determined by the positive roots of (2.6) when $N \in (N_1, N_2)$.

By some simple calculation, we can see that $F_3(N_1) = F_3(N_2) = 0$ and $F_3(N)$ increases first and then decreases when $N \in (N_1, N_2)$. Since (2.4) has at most two positive roots when $N \in (N_1, N_2)$, it follows that the curve for $F_3(N)F_1(N)$ must have a shape similar to that for $F_3(N)$. Thus, the function $F_3(N)F_1(N)$ has a unique maximum value on the interval (N_1, N_2), which determines the threshold release value

$$b_0 := \max_{N\in(N_1,N_2)} F_3(N)F_1(N). \tag{2.7}$$

From the above analysis, we have the following result, which is similar to Theorem 4.1 in [13], in which only a special case that $\mu_1 = \mu_2$ and $\xi_1 = \xi_2$ was considered.

Theorem 2.1. *The origin $(0, 0)$ is a unique boundary equilibrium for system (2.1). Moreover,*

(i) *if $b > b_0$ or $\sqrt{\mu_1} + \sqrt{\xi_1} \geq 1$, then system (2.1) has no positive equilibrium;*
(ii) *if $b = b_0$ and $\sqrt{\mu_1} + \sqrt{\xi_1} < 1$, then system (2.1) has a unique positive equilibrium $E^*(w^*, g^*)$ with*

$$w^* = (1+N^*)(\mu_1 + \xi_1 N^*), \qquad g^* = \frac{b(1+N^*)(\mu_1 + \xi_1 N^*)}{[1 + (1+N^*)(\mu_1 + \xi_1 N^*)](\mu_2 + \xi_2 N^*)}, \tag{2.8}$$

where N^* is the unique positive root of (2.6);

(iii) *if* $b < b_0$ *and* $\sqrt{\mu_1} + \sqrt{\xi_1} < 1$, *then system* (2.1) *has two positive equilibria* $E_1^*(w_1^*, g_1^*)$ *and* $E_2^*(w_2^*, g_2^*)$ *with*

$$w_{1,2}^* = (1 + N_{1,2}^*)(\mu_1 + \xi_1 N_{1,2}^*), \quad g_{1,2}^* = \frac{b(1 + N_{1,2}^*)(\mu_1 + \xi_1 N_{1,2}^*)}{[1 + (1 + N_{1,2}^*)(\mu_1 + \xi_1 N_{1,2}^*)](\mu_2 + \xi_2 N_{1,2}^*)}, \tag{2.9}$$

where $N_1^* < N_2^*$ *are the two positive roots of* (2.6).

Remark 2.2. From Theorem 2.1, we can see that there exists a critical release rate coefficient of sterile mosquitoes, above which the mosquito population can be eliminated.

2.2. Stability of the equilibria. The Jacobian matrix of system (2.1) at the equilibrium $(0, 0)$ has the form

$$\begin{bmatrix} -\mu_1 & 0 \\ b & -\mu_2 \end{bmatrix}.$$

It is easy to see that the eigenvalues of the above matrix are $-\mu_1$ and $-\mu_2$, which are all negative, so the origin $(0, 0)$ is locally asymptotically stable.

The Jacobian matrix of system (2.1) at a positive equilibrium has the form

$$J = \begin{bmatrix} \frac{w(1+g)}{(1+N)^2} - \xi_1 w & -\left(\frac{w}{(1+N)^2} + \xi_1\right) w \\ \frac{b}{(1+w)^2} - \xi_2 g & -\mu_2 - \xi_2 w - 2\xi_2 g \end{bmatrix},$$

from which we have

$$\mathrm{tr} J = -\frac{w(\mu_1 + \xi_1 N)}{1 + N} + \mu_1 - \mu_2 + (\xi_1 - \xi_2) g - \xi_2 N.$$

Define

$$G_1(N) = N - (1 + N)(\mu_1 + \xi_1 N), \quad G_2(N) = \frac{F_1(N)}{(1 + N)(\mu_1 + \xi_1 N)} = Q_2 \frac{\mu_2 + \xi_2 N}{\mu_1 + \xi_1 N},$$

where $Q_2 = \frac{1+(1+N)(\mu_1+\xi_1 N)}{1+N}$; then (2.3) can be rewritten as

$$G_1(N) G_2(N) = b. \tag{2.10}$$

Through some calculations, we can derive that

$$\det J = -\frac{w(\mu_1 + \xi_1 N)}{1 + w} \frac{\mathrm{d}(G_1 G_2)}{\mathrm{d}N}; \tag{2.11}$$

the detailed computations are given in Appendix A.

If there is a unique positive equilibrium E^* for system (2.1) associated with the unique positive root N^* of (2.10), then we have

$$\left.\frac{\mathrm{d}(G_1 G_2)}{\mathrm{d}N}\right|_{N=N^*} = 0,$$

which implies that

$$\det J\big|_{N=N^*} = 0.$$

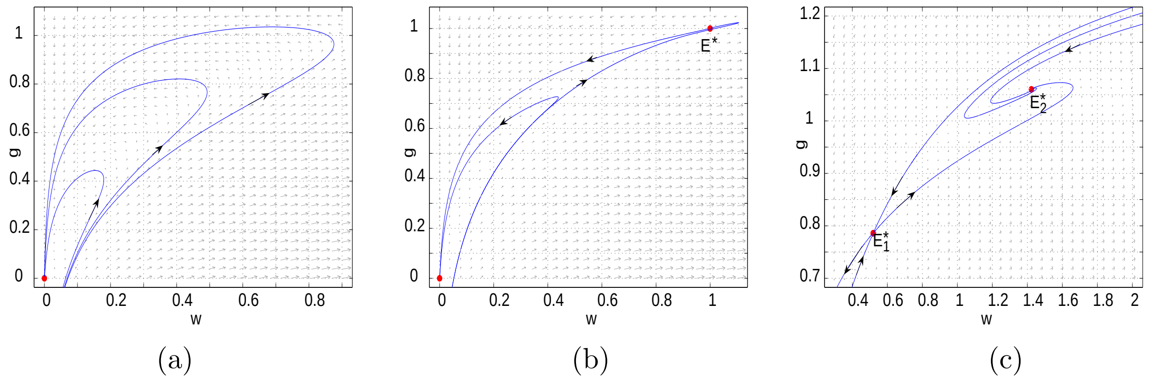

Figure 2.1. *The phase portraits of system* (2.1) *with* $\mu_1 = \frac{1}{45}, \xi_1 = \frac{7}{45}, \mu_2 = \frac{1}{30}, \xi_2 = \frac{1}{90}$. (a) *No positive equilibrium when* $b = \frac{2}{15}$. (b) *A unique positive equilibrium which is a cusp when* $b = \frac{1}{9}$. (c) *Two positive equilibria when* $b = \frac{11}{100}$, E_1^* *is a saddle,* E_2^* *is a stable node.*

If there are two positive equilibria E_1^* and E_2^* for system (2.1) associated with the two positive roots $N_1^* < N_2^*$ of (2.10), then we have

$$\left.\frac{\mathrm{d}(G_1G_2)}{\mathrm{d}N}\right|_{N=N_2^*} < 0 < \left.\frac{\mathrm{d}(G_1G_2)}{\mathrm{d}N}\right|_{N=N_1^*}.$$

Therefore, we have

$$\det J\big|_{N=N_1^*} < 0 < \det J\big|_{N=N_2^*}.$$

By the above analysis and Theorems 7.1–7.3 in Zhang et al. [50], we have the following result.

Theorem 2.3.

(i) *The origin* $(0,0)$ *is always locally asymptotically stable.*

(ii) *If there exists a unique positive equilibrium* $E^*(w^*, g^*)$ *for system* (2.1), *then it is a saddle-node if* $\operatorname{tr}J\,|_{E^*} \neq 0$ *and a cusp if* $\operatorname{tr}J\,|_{E^*} = 0$.

(iii) *If there exist two positive equilibria* $E_1^*(w_1^*, g_1^*)$ *and* $E_2^*(w_2^*, g_2^*)$ *for system* (2.1), *then* E_1^* *is a saddle;* E_2^* *is a locally asymptotically stable node or focus when* $\operatorname{tr}J\,|_{E_2^*} < 0$, *an unstable node or focus when* $\operatorname{tr}J\,|_{E_2^*} > 0$, *and a linear center when* $\operatorname{tr}J\,|_{E_2^*} = 0$.

The phase portraits are shown in Figure 2.1.

3. Bifurcation analysis.

In this section, we are interested in various possible bifurcations in system (2.1). From Theorem 2.3, we know that system (2.1) may exhibit Hopf bifurcation around the equilibrium E_2^* and Bogdanov–Takens bifurcation around the equilibrium E^*. From (2.4), we can see that the positive equilibrium depends on a polynomial equation of degree five, which makes the full bifurcation analysis very difficult and challenging.

3.1. Hopf bifurcation of codimension 3.

We first consider Hopf bifurcation. For convenience, introducing the time scaling $t = (1+w)(1+w+g)\tau$ to system (2.1), we obtain

$$\begin{aligned} \frac{dw}{d\tau} &= w(1+w)\big[w - (1+w+g)\big(\mu_1 + \xi_1(w+g)\big)\big], \\ \frac{dg}{d\tau} &= (1+w+g)\big[bw - (1+w)\big(\mu_2 + \xi_2(w+g)\big)g\big]. \end{aligned} \tag{3.1}$$

Solving μ_1 and μ_2 from the equations $\frac{dw}{d\tau} = \frac{dg}{d\tau} = 0$, we obtain

$$\mu_1 = \frac{w_2^*}{1 + w_2^* + g_2^*} - \xi_1(w_2^* + g_2^*), \quad \mu_2 = \frac{bw_2^*}{(1 + w_2^*)g_2^*} - \xi_2(w_2^* + g_2^*), \tag{3.2}$$

which requires (due to $\mu_1 > 0$ and $\mu_2 > 0$)

$$0 < \xi_1 < \frac{w_2^*}{(w_2^* + g_2^*)(1 + w_2^* + g_2^*)}, \quad 0 < \xi_2 < \frac{bw_2^*}{g_2^*(1 + w_2^*)(w_2^* + g_2^*)}. \tag{3.3}$$

To have a Hopf bifurcation at the equilibrium $E_2^*(w_2^*, g_2^*)$, we let $\mathrm{tr}J\,|_{E_2^*} = 0$ and get

$$\xi_1 = \xi_{1\mathrm{H}} = \frac{1 + g_2^*}{(1 + w_2^* + g_2^*)^2} - \frac{bw_2^* + (1 + w_2^*)(g_2^*)^2\xi_2}{w_2^* g_2^*(1 + w_2^*)}, \tag{3.4}$$

which requires

$$0 < \xi_2 < \frac{w_2^*}{g_2^*}\Big[\frac{1 + g_2^*}{(1 + w_2^* + g_2^*)^2} - \frac{b}{g_2^*(1 + w_2^*)}\Big] \tag{3.5}$$

to guarantee $\xi_{1\mathrm{H}} > 0$, which in turn requires

$$0 < b < \frac{g_2^*(1 + w_2^*)(1 + g_2^*)}{(1 + w_2^* + g_2^*)^2}. \tag{3.6}$$

Then at the critical point $\xi_1 = \xi_{1\mathrm{H}}$, the eigenvalues of the linearized system of (3.1) around (w_2^*, g_2^*) are $\lambda_{1,2} = \pm i\omega_c$, where

$$\begin{aligned} \omega_c = \ & \Big\{ \tfrac{w_2^*}{g_2^*} \big[(1 + w_2^*)\big((1 + w_2^* + g_2^*)^2\xi_1 - 1 - g_2^*\big)\big(bw_2^* + (g_2^*)^2(1 + w_2^*)\xi_2\big) \\ & + g_2^*\big(w_2^* + (1 + w_2^* + g_2^*)^2\xi_1\big)\big(b - g_2^*(1 + w_2^*)^2\xi_2\big)\big] \Big\}^{1/2}. \end{aligned} \tag{3.7}$$

In addition to the conditions given in (3.3), (3.5), and (3.6), it is further required that w_2^*, g_2^*, and b are chosen such that ω_c is real and positive.

Now introducing the transformation

$$\begin{aligned} w &= w_2^* + \frac{w_2^*\big[b(1 + w_2^*) + (b - 1 - w_2^*)g_2^*\big] + (g_2^*)^2(1 + w_2^*)(1 + w_2^* + g_2^*)\xi_2}{g_2^*}\, u, \\ g &= g_2^* - \frac{(1 + w_2^* + g_2^*)\big[bw_2^* + (g_2^*)^2(1 + w_2^*)\xi_2\big]}{g_2^*}\, u + \omega_c\, v \end{aligned}$$

and the time rescaling $\tau_1 = \omega_c\tau$ into (3.1), we obtain

$$\frac{du}{d\tau_1} = v + \sum_{i+j=2}^{4} a_{ij}u^i v^j, \quad \frac{dv}{d\tau_1} = -u + \sum_{i+j=2}^{4} b_{ij}u^i v^j, \tag{3.8}$$

where the coefficients a_{ij} and b_{ij} are given in terms of b, ξ_2, w_2^*, and g_2^*. Next, using the formal series method (see p. 93 in Chapter 2 in [50]) and applying the Maple program for computing the normal forms of Hopf and generalized Hopf bifurcations in Yu [47] we obtain the following normal form:

$$\frac{dr}{d\tau_1} = r(v_0\mu + v_1 r^2 + v_2 r^4 + v_3 r^6 + \cdots),$$
$$\frac{d\theta}{d\tau_1} = 1 + \nu_0\mu + \nu_1 r^2 + \nu_2 r^4 + \nu_3 r^6 + \cdots,$$

where r and θ represent the amplitude and phase of the periodic motions, respectively, and $\mu = \xi_1 - \xi_{1H}$ is the unfolding. v_0 and ν_0 are obtained from linear analysis, while v_k and $\nu_k (k \geq 1)$ must be derived using nonlinear analysis with the aid of a computer algebra system such as Maple or Mathematica. The v_k's are usually called *focus values*, which can be used to determine the number of limit cycles bifurcating from the Hopf critical point as well as the center conditions on the singular point, while ν_k's can be applied to determine the critical periods of the bifurcating limit cycles. It should be noted that the formal series method [50] can be used to obtain only the focus values. However, the coefficients ν_k are not needed in this paper. The Maple output of the focus values is given as follows:

$$v_1 = -\frac{w_2^*[b(1+w_2^*+g_2^*)-g_2^*(1+w_2^*)]+(g_2^*)^2(1+w_2^*)(1+w_2^*+g_2^*)\xi_2}{8w_2^*(g_2^*)^4(1+w_2^*)^2(1+w_2^*+g_2^*)^3\omega_c^2}\,F_1,$$

$$v_2 = \frac{\{w_2^*[b(1+w_2^*+g_2^*)-g_2^*(1+w_2^*)]+(g_2^*)^2(1+w_2^*)(1+w_2^*+g_2^*)\xi_2\}^2}{288(w_2^*)^3(g_2^*)^{10}(1+w_2^*)^6(1+w_2^*+g_2^*)^6\omega_c^6}\,F_2,$$

$$v_3 = -\frac{1}{663552(w_2^*)^{10}(g_2^*)^{18}(1+w_2^*)^{14}(1+w_2^*+g_2^*)^8\omega_c^{12}\{w_2^*[b(1+w_2^*+g_2^*)-g_2^*(1+w_2^*)]+(g_2^*)^2(1+w_2^*)(1+w_2^*+g_2^*)\xi_2\}^6}\,F_3,$$

$$v_4 = \cdots,$$

where F_i, $i = 1, 2, 3$, and v_4 are lengthy polynomials in b, ξ_2, w_2^*, and g_2^*.

It is extremely difficult or impossible to solve $v_1 = v_2 = v_3 = v_4 = 0$ for the four parameters. We consider a slightly simpler case, $w_2^* = g_2^*$, which implies that the equilibrium (w_2^*, g_2^*) is restricted on the 45^{o} straight line in the first quadrant of the w-g plane. Biologically this means that the wild and sterile mosquitoes are balanced at the equilibrium. We will see that the system still exhibits very complex dynamics under this restriction. In fact, we have the following theorem.

Theorem 3.1. *If $b < b_0$, $\sqrt{\mu_1} + \sqrt{\xi_1} < 1$ and the equilibrium $E_2^*(w_2^*, g_2^*)$ satisfies $w_2^* = g_2^*$, then system* (3.1) *(i.e., system* (2.1)*) undergoes a Hopf bifurcation of codimension* 3 *around the equilibrium $E_2^*(w_2^*, w_2^*)$, three limit cycles bifurcate from $E_2^*(w_2^*, w_2^*)$, and the outer bifurcating limit cycle is unstable.*

Proof. Let $g_2^* = w_2^*$. Then we use the three equations $F_1 = F_2 = F_3 = 0$ to eliminate ξ_2 and obtain a solution $\xi_2 = \xi_2(b, w_2^*)$ and two resultants:

$$\mathrm{R}_1 = \mathrm{R}_1(b, w_2^*) \quad \text{and} \quad \mathrm{R}_2 = \mathrm{R}_2(b, w_2^*).$$

Solving $\mathrm{R}_1 = \mathrm{R}_2 = 0$, we obtain 10 real solutions $w_2^* > 0$, $b > 0$. However, none of them yields all parameters being positive and $\omega_c^2 > 0$. Thus, it is not possible to have four small limit cycles arising from the Hopf bifurcation at the equilibrium (w_2^*, w_2^*) with the restrictions on the parameters.

The next best possibility is to have three limit cycles arising from the Hopf bifurcation at the equilibrium (w_2^*, w_2^*), which requires that $v_1 = v_2 = 0$ but $v_3 \neq 0$. Since there are three free parameters to be used, it may have an infinite number of solutions. We take the special case $w_2^* = g_2^* = 1$ to show that it indeed has three limit cycles around the equilibrium $E_2^*(w_2^*, g_2^*) = (1,1)$.

When $w_2^* = g_2^* = 1$, the focus values v_1, v_2, and v_3 are reduced to

$$\begin{aligned}
v_1 &= -\frac{3b + 6\xi_2 - 2}{432(9b^2 + 18b\xi_2 - 2b + 8\xi_2)}\, G_1,\\
v_2 &= -\frac{(3b + 6\xi_2 - 2)^2}{15116544(9b^2 + 18b\xi_2 - 2b + 8\xi_2)^3}\, G_2,\\
v_3 &= -\frac{(3b + 6\xi_2 - 2)^3}{33853318889472(9b^2 + 18b\xi_2 - 2b + 8\xi_2)^5}\, G_3,
\end{aligned}$$

where

$$\begin{aligned}
G_1 &= 58320b\xi_2^3 + 36(1053b^2 + 54b + 112)\xi_2^2 + 4(729b^3 + 162b^2 + 234b - 128)\xi_2\\
&\quad - 3b(243b^3 - 594b^2 + 276b - 32),\\
G_2 &= 326517350400b(9b + 4)\xi_2^7 + (5652627555456b^3 + 1230154117632b^2\\
&\quad - 233641304064b + 90296156160)\xi_2^6 + (5356772224992b^4 - 992646757440b^3\\
&\quad - 494477270784b^2 + 122343229440b - 17951735808)\xi_2^5 + (2828380020336b^5\\
&\quad - 1861727105088b^4 + 213989796864b^3 - 410884462208b^2 - 80167698432b\\
&\quad - 800243712)\xi_2^4 + (676120497840b^6 - 729715259808b^5 + 304777870560b^4\\
&\quad - 187985142144b^3 + 3568810752b^2 + 10306787328b + 43646976)\xi_2^3\\
&\quad + (14081060736b^7 + 30491958816b^6 - 144618086688b^5 - 62097293088b^4\\
&\quad + 31379425920b^3 - 2248839936b^2 - 438829056b + 20971520)\xi_2^2\\
&\quad - 2b(7179236469b^7 - 27051409782b^6 + 30140735364b^5 - 10457184240b^4\\
&\quad - 290713536b^3 + 754523424b^2 - 118374912b + 4587520)\xi_2\\
&\quad - 3b^2(307704339b^7 - 1511418204b^6 + 2493993564b^5 - 1772397288b^4\\
&\quad + 627476544b^3 - 114646752b^2 + 10123008b - 327680),\\
G_3 &= \cdots .
\end{aligned}$$

Eliminating ξ_2 from the two equations $G_1 = G_2 = 0$ we obtain a solution $\xi_2 = -3b\frac{\xi_{2n}}{\xi_{2d}}$, where

$$\begin{aligned}
\xi_{2n} &= 1215766545905692880100b^{19} + 11428355626148810072061b^{18}\\
&\quad + 9762866639469341936784 0b^{17} - 7159975365687912355747 68b^{16}\\
&\quad + 1211735866715961350229198b^{15} - 5435595028828370787773 04b^{14}\\
&\quad - 2559897255123412714076 04b^{13} + 2206724808807947979424 80b^{12}\\
&\quad + 121512908642961148453 44b^{11} - 325741909191943259468 16b^{10}\\
&\quad + 141784614534492620006 4b^{9} + 173370169007719072358 4b^{8}\\
&\quad + 324186225479331803136b^{7} - 187110581980086190080b^{6}\\
&\quad + 13019602777240043 52b^{5} + 875303182018805760b^{4}\\
&\quad + 147833877654090547 2b^{3} - 282357186133229568b^{2}\\
&\quad + 13910323048218624b - 85418309582848,
\end{aligned}$$

$$
\begin{aligned}
\xi_{2d} = {} & 729459927543415728060 0b^{19} + 653280896344776794187 66b^{18} \\
& + 21170010447033134768623 44b^{17} - 9009507952299734170657740b^{16} \\
& + 7164657795528717547554108b^{15} + 3337639940500031251056024b^{14} \\
& - 4346676042279950693824344b^{13} - 2381000782849485434694 72b^{12} \\
& + 7423263173323867172238 72b^{11} + 906319196962203079802 88b^{10} \\
& - 875602370781416236262 40b^{9} - 846542183963446134374 4b^{8} \\
& + 290043673288823638425 6b^{7} + 165028470179956103577 6b^{6} \\
& - 28910866770142278451 2b^{5} - 26871389454145683456b^{4} \\
& - 7078702070790881280b^{3} + 3411874036366442496b^{2} \\
& - 215293935168258048b + 1366692953325568,
\end{aligned}
$$

and a resultant:

$$
\begin{aligned}
\mathrm{R}_1 = {} & b(405b^2 - 72b + 8)(2187b^3 - 81b^2 + 36b - 64)(83385908498332845 00b^{15} \\
& + 106384977408965611545b^{14} - 851807047085209819260b^{13} \\
& + 2083240023585151729374b^{12} - 2328544331618373048240b^{11} \\
& + 1290985483343789350512b^{10} - 314786318806183991544b^{9} \\
& - 61236581296755843 36b^{8} + 33488339014669159296b^{7} \\
& - 25233272486194035456b^{6} + 13708773822817963008b^{5} \\
& - 4299563347822319616b^{4} + 725893086283628544b^{3} \\
& - 61341366413721600b^{2} + 2107325685694464b - 12178782420992).
\end{aligned}
$$

Solving $\mathrm{R}_1 = 0$ yields eight real positive solutions:

$$
\begin{aligned}
b = {} & 0.00714474\cdots, \quad 0.07301690\cdots, \quad 0.15234730\cdots, \quad 0.19515215\cdots, \\
& 0.30247351\cdots, \quad 0.73356398\cdots, \quad 1.60445318\cdots, \quad 2.43448704\cdots.
\end{aligned}
$$

Among these eight solutions, only the third one is a feasible solution under which all parameters and ω_c are positive, as given below:

$$
\begin{aligned}
& w_2^* = g_2^* = 1, && b = 0.15234730\cdots, && \mu_1 = 0.04939357\cdots, && \mu_2 = 0.06801627\cdots, \\
& \xi_1 = 0.14196988\cdots, && \xi_2 = 0.00407868\cdots, && \omega_c = 0.27926600\cdots,
\end{aligned}
$$

under which

$$
v_1 = v_2 = 0, \quad v_3 = 0.02803338\cdots.
$$

Moreover, a direct computation shows that at the above critical point,

$$
\det\left[\frac{\partial(v_1, v_2)}{\partial(b, \xi_2)}\right] = 18.68187323\cdots \neq 0,
$$

which implies that three small-amplitude limit cycles bifurcate from the Hopf critical point and the outer one is unstable since $v_3 > 0$. The proof is complete. ■

Based on Theorem 3.1, we give some numerical simulations to illustrate the existence of multiple limit cycles.

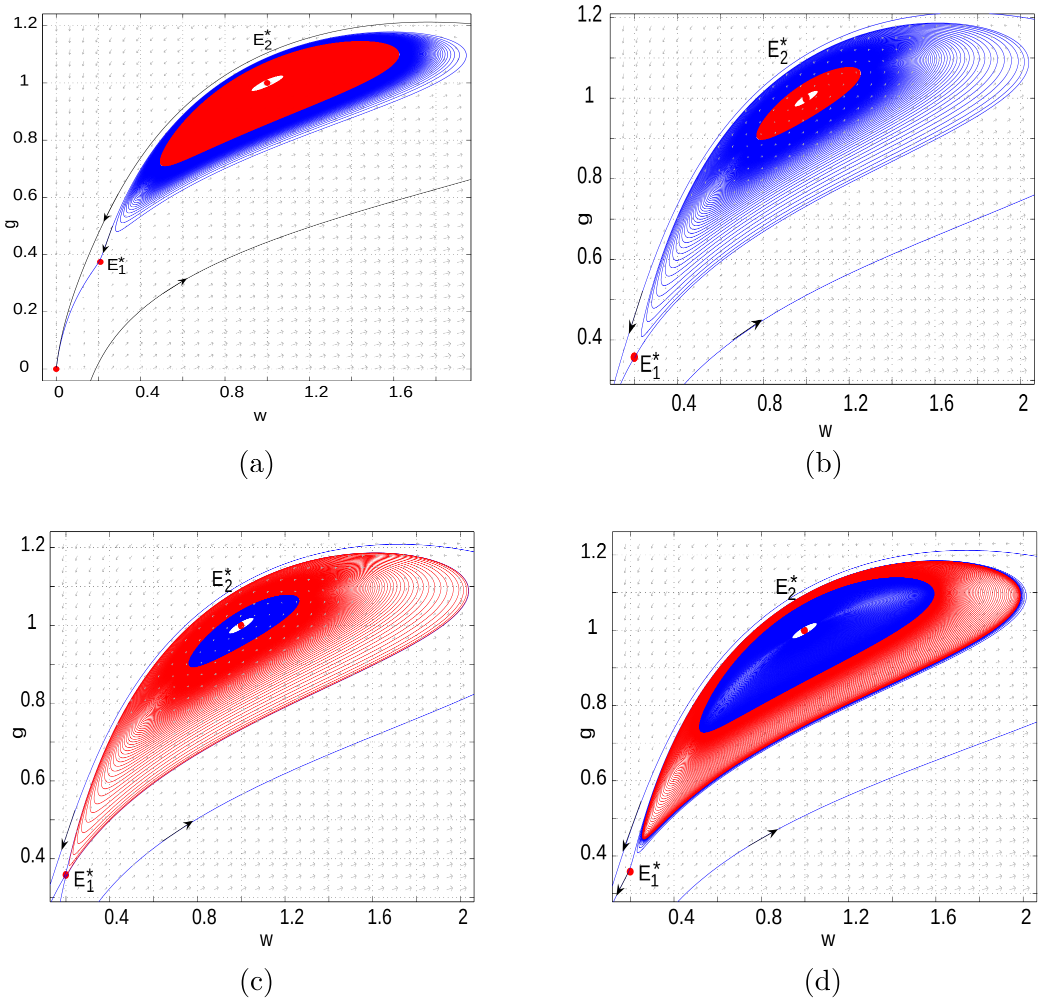

Figure 3.1. (a) *An unstable limit cycle in system* (2.1) *with* $\xi_2 = 0.0041$, $b = 0.152$, $\xi_1 = \frac{2}{9} - 0.080101$, $\mu_1 = -\frac{1}{9} + 0.1602, \mu_2 = 0.0678$. (b) *A stable limit cycle in system* (2.1) *with* $\xi_2 = 0.0038$, $b = 0.152$, $\xi_1 = \frac{2}{9} - 0.07979, \mu_1 = -\frac{1}{9} + 0.1596, \mu_2 = 0.0684$. (c) *The coexistence of one homoclinic loop and one stable limit cycle in system* (2.1) *with* $\xi_2 = 0.003821$, $b = 0.152$, $\xi_1 = \frac{2}{9} - 0.079811, \mu_1 = -\frac{1}{9} + 0.159642, \mu_2 = 0.068358$. (d) *Two limit cycles in system* (2.1) *with* $\xi_2 = 0.003812$, $b = 0.152$, $\xi_1 = 0.142412, \mu_1 = 0.0485289, \mu_2 = 0.06836$.

(i) Two limit cycles. An unstable limit cycle and a stable limit cycle around E_2^* created from the subcritical and supercritical Hopf bifurcations are given in Figures 3.1(a) and 3.1(b), respectively. Figure 3.1(c) shows the coexistence of one homoclinic loop and one stable limit cycle in system (2.1) with $\xi_2 = 0.003821$, $b = 0.152$, $\xi_1 = \frac{2}{9} - 0.079811, \mu_1 = -\frac{1}{9} + 0.159642, \mu_2 = 0.068358$. Figure 3.1(d) presents the existence of two limit cycles in system (2.1) with $\xi_2 = 0.003812$, $b = 0.152$, $\xi_1 = 0.142412, \mu_1 = 0.0485289, \mu_2 = 0.06836$, where the outer limit cycle is unstable.

(ii) Three limit cycles. To numerically simulate the existence of three limit cycles in system (3.1), we use the normal form to determine the parameter values. First, we perturb the parameters b and ξ_2 such that the perturbed focus values satisfy

$$v_1 \ll -v_2 \ll v_3 = 0.02803338\cdots,$$

and then we perturb ξ_1 so that $0 < -v_0\mu \ll v_1$. The perturbed parameter values we obtained are

$$b = 0.151428459565, \quad \mu_1 = 0.048698952305, \quad \mu_2 = 0.0673326199307,$$
$$\xi_1 = 0.142317190514, \quad \xi_2 = 0.00419080492586,$$

under which the perturbed focus values are given by

$$v_0\mu = -0.9\times 10^{-8}, \quad v_1 = 0.5790344618\times 10^{-5}, \quad v_2 = -0.8572860349\times 10^{-3}, \quad v_3 = 0.0244464229.$$

Next, from the truncated normal form equation

$$\dot r = r(v_0\mu + v_1 r^2 + v_2 r^4 + v_3 r^6),$$

we obtain the approximated amplitudes for the three limit cycles as follows:

$$r_1 = 0.0476, \quad r_2 = 0.0780, \quad r_3 = 0.1635.$$

We use the fourth-order Runge–Kutta method to run the simulations on a PC machine. Since the model is a two-dimensional differential equation system, we apply negative time steps in the integration scheme to simulate the unstable limit cycles. Since $v_1 > 0$, $v_2 < 0$, and $v_3 > 0$, the innermost and outermost limit cycles are unstable while the one between these two unstable limit cycles is stable. All three limit cycles enclose the equilibrium $(1, 1)$, which is stable since $v_0 < 0$, and convergence of trajectories to the equilibrium can be verified by simulations. Our simulations show that the convergent speed is extremely slow and the process is very time consuming. For each limit cycle, we choose two initial points, one lying outside the limit cycle and one lying inside the cycle, and have trajectories initiated from both points converging to the limit cycle. (Note that convergence also appears for the unstable limit cycles since negative time steps are used.) The two initial points for simulating the outermost unstable limit cycle are $(1.3, 1.0)$ (outside) and $(0.85, 0.93)$ (inside) with time step -0.05; those for the middle stable limit cycle are $(0.85, 0.93)$ (outside) and $(1.0, 0.975)$ (inside) with time step 0.2; and those for the innermost unstable limit cycle are $(1.0, 0.975)$ (outside) and $(1.002, 1.002)$ (inside) with time step -0.2. Finally, the simulation of the trajectory starting from the point $(1.002, 1.002)$ with time step 0.2 shows that it converges to the equilibrium $(1, 1)$. The simulations of three limit cycles are shown in Figure 3.2, where we only present the very last portion of each trajectory in order to avoid massive data plotting. The solid and dashed curves represent stable and unstable limit cycles, respectively. It is seen from Figure 3.2 that the simulations agree very well with the analytical predictions.

Remark 3.2. Finding a predator-prey or other interacting system in nature with at least two ecologically stable cycles is very challenging and "almost impossible" (Coleman [16]). Ecological stability means that a natural cycle persevering over a long period of time must be somewhat insensitive to the perturbations of the real world, so a system describing this situation must have at least three limit cycles (González-Olivares et al. [28]). There are very few papers presenting two-dimensional real-world biological models with three limit cycles. Aguirre, González-Olivares, and Sáez [1] showed that a Leslie–Gower predator-prey model

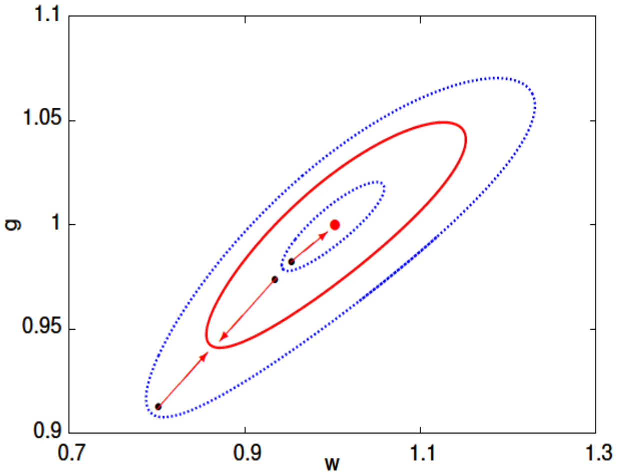

Figure 3.2. *Simulations of three limit cycles where the inner most and outer most limit cycles (blue dotted curves) are unstable and the middle limit cycle (red solid curve) is stable. The initial points for the simulations are also shown with arrows indicating the spiral directions of trajectories toward the stable limit cycle and the stable focus (in red color). The unstable limit cycles are obtained using negative time steps.*

with additive Allee effect and Holling type IV functional response has three limit cycles, two of which are infinitesimal ones generated by Hopf bifurcation, and the third one arises from homoclinic bifurcation. Here we provide a different ecological example which has three limit cycles.

Remark 3.3. From Theorem 3.1, we can see that the release rate coefficient of sterile mosquitoes is the most important and sensitive parameter in affecting the nonlinear dynamics of the model and in determining the success of the sterile mosquito release program. There exists a critical release rate coefficient of sterile mosquitoes, below which the interacting sterile and wild mosquitoes will coexist in the form of multiple periodic oscillations and steady states.

3.2. Nilpotent cusp of codimension 4 and Bogdanov–Takens bifurcation. Now we turn to consider Bogdanov–Takens bifurcation around the equilibrium $E^*(w^*, g^*)$ in system (3.1) (i.e., system (2.1)). Again assume that $w^* = g^*$. Define

$$
\begin{aligned}
C_{20} &= (2+w^*)^2(1+2w^*)^2(3w^{*3}+6w^{*2}+5w^*+1)\,b^2\\
&\quad - w^*(2+w^*)(1+2w^*)(1+w^*)^2(3w^{*2}+2w^*+1)\,b\\
&\quad + w^{*3}(1+w^*)^3(w^{*3}+w^*+1),\\
C_{11} &= 2(2+w^*)^2(1+2w^*)^5\,b^3\\
&\quad + w^*(2+w^*)(1+w^*)(1+2w^*)^2(4w^{*3}-w^{*2}-5w^*-1)\,b^2\\
&\quad + w^{*2}(1+2w^*)(1+w^*)^3(2w^{*3}+w^{*2}-w^*+1)\,b\\
&\quad - 2w^{*4}(1+w^*)^6,
\end{aligned}
\tag{3.9}
$$

which are the key factors of the coefficients in the normal form for a nilpotent cusp with codimension 4 (see (3.15) and (3.16)). Note that C_{11} is a cubic polynomial in b. We can use the discriminant of the polynomial to determine the number of real roots of the polynomial equation $C_{11} = 0$. In fact, the discriminant is obtained as

$$\mathrm{Disc} = \frac{w^{*6}(1+w^*)^8}{1728(2+w^*)^6(1+2w^*)^{12}} \Big[(2224w^{*10}+18008w^{*9}+62343w^{*8}+122528w^{*7}+151642w^{*6} +120900w^{*5}+59089w^{*4}+15044w^{*3}+\frac{1706}{7}w^{*2}+\frac{1}{7}(7-74w^*)^2 \Big] > 0,$$

implying that the cubic polynomial equation $C_{11} = 0$ has one real root. Moreover, since C_{11} has positive coefficient for the term b^3 and a negative constant term, the unique real root is positive. Denote the unique positive root by $b = b^*(w^*)$. Further, define

$$\begin{aligned} & b_{\mathrm{L}} = \frac{w^*(1+w^*)^2}{(2+w^*)(1+2w^*)(1+4w^*)}, && b_{\mathrm{U}} = \frac{w^*(1+w^*)}{(2+w^*)(1+2w^*)}, \\ & w_{\mathrm{L}}^* = 0.15732625\cdots, && w_{\mathrm{U}}^* = \tfrac{1}{2}(\sqrt{3}-1) = 0.36602540\cdots, \\ & w^{**} = 0.19309073\cdots, && b^{**} = 0.05936122\cdots. \end{aligned} \tag{3.10}$$

Then, we have the following result.

Theorem 3.4. *When the unique positive equilibrium $E^*(w^*, g^*)$ of system (2.1) satisfies $w^* = g^*$, then $E^*(w^*, w^*)$ is a nilpotent cusp with codimensions 2, 3, and 4, respectively, corresponding to the following conditions:*

(i) *codimension 2 if*

$$w^* \in (0, w_{\mathrm{L}}^*] \cup [w_{\mathrm{U}}^*, \infty), \quad \text{or} \quad w^* \in (w_{\mathrm{L}}^*, w_{\mathrm{U}}^*) \ \text{ and } \ b \neq b^*(w^*);$$

(ii) *codimension 3 if*

$$w^* \in (w_{\mathrm{L}}^*, w^{**}) \cup (w^{**}, w_{\mathrm{U}}^*) \ \text{ and } \ b = b^*(w^*);$$

(iii) *codimension 4 if*

$$w^* = w^{**} \ \text{ and } \ b = b^{**},$$

where $b^*(w^*)$ is the unique positive real root of $C_{11} = 0$.

Proof. When $g^* = w^*$, from the equilibrium equations for $E^*(w^*, g^*)$, i.e., equations (3.2), and the necessary conditions for (w^*, g^*) being a cusp, i.e., $\mathrm{tr}(J(E^*)) = \det(J(E^*)) = 0$, we obtain

$$\begin{aligned} \xi_1 &= \frac{w^*\,\xi_1^*}{(1+2w^*)^2\big[w^*(1+w^*)^2 + b(2+w^*)(1+2w^*)\big]}, \\ \xi_2 &= \frac{b\,\xi_2^*}{w^*(1+w^*)\big[w^*(1+w^*)^2 + b(2+w^*)(1+2w^*)\big]}, \\ \mu_1 &= \frac{w^*\,\mu_1^*}{(1+2w^*)^2\big[w^*(1+w^*)^2 + b(2+w^*)(1+2w^*)\big]}, \\ \mu_2 &= \frac{b\,\mu_2^*}{(1+w^*)\big[w^*(1+w^*)^2 + b(2+w^*)(1+2w^*)\big]}, \end{aligned} \tag{3.11}$$

where

$$\begin{aligned}\xi_1^* &= (1+w^*)^3 - b(2+w^*)(1+2w^*),\\ \xi_2^* &= w^*(1+w^*) - b(2+w^*)(1+2w^*),\\ \mu_1^* &= b(2+w^*)(1+2w^*)(1+4w^*) - w^*(1+w^*)^2,\\ \mu_2^* &= 3b(2+w^*)(1+2w^*) + w^*(w^{*2}-1).\end{aligned} \tag{3.12}$$

Hence, ξ_1, ξ_2, μ_1, and μ_2 have the same sign as that of ξ_1^*, ξ_2^*, μ_1^*, and μ_2^*, respectively.

To determine the conditions on b and w^* satisfying $\xi_1^* > 0$, $\xi_2^* > 0$, $\mu_1^* > 0$, and $\mu_2^* > 0$, first note that $\xi_1^* > \xi_2^*$ for $w^* > 0$. So we may ignore $\xi_1^* > 0$. Then, $\xi_2^* > 0$ and $\mu_1^* > 0$ yield

$$b_{\mathrm{L}} < b < b_{\mathrm{U}} \quad \text{for} \quad w^* > 0, \tag{3.13}$$

where b_{L} and b_{U} are given in (3.10). For $\mu_2^* > 0$, it is true for any $b > 0$ if $w^* \geq 1$. If $0 < w^* < 1$, then

$$b > \frac{w^*(1-w^{*2})}{3(2+w^*)(1+2w^*)} \quad \text{for} \quad 0 < w^* < 1.$$

Further, it is easy to show that

$$\frac{w^*(1-w^{*2})}{3(2+w^*)(1+2w^*)} < b_L \quad \text{for} \quad w^* > 0.$$

Summarizing the above results shows that the necessary and sufficient conditions for $\xi_1^* > 0$, $\xi_2^* > 0$, $\mu_1^* > 0$, and $\mu_2^* > 0$ are given in (3.13).

Next, introducing the transformation

$$\begin{aligned} w &= w^* - \frac{w^{*2}(1+w^*)^2}{(1+2w^*)\big[b(2+w^*)(1+2w^*) + w^*(1+w^*)^2\big]}\, u,\\ g &= g^* - \frac{bw^*(2+w^*)}{b(2+w^*)(1+2w^*) + w^*(1+w^*)^2}\, u + v \end{aligned}$$

into system (2.1) and then expanding the equations around the origin $(u,v) = (0,0)$ up to the fifth order, we obtain the following system:

$$\begin{aligned} \frac{du}{dt} &= v + \sum_{i+j=2}^{5} a_{ij}\, u^i v^j,\\ \frac{dv}{dt} &= \sum_{i+j=2}^{5} b_{ij}\, u^i v^j, \end{aligned} \tag{3.14}$$

where a_{ij} and b_{ij} are expressed in terms of b and w^*.

Now we apply the normal form theory (e.g., see [48, 49, 25, 26, 27]) to system (3.14) with the transformations

$$u = x_1 + \sum_{i+j=2}^{5} h_{1ij}\, x_1^i x_2^j, \quad v = x_2 + \sum_{i+j=2}^{5} h_{2ij}\, x_1^i x_2^j, \quad t = \big(1 + t_{10}\, x_1\big)\,\tau,$$

and obtain the following normal form:

$$\begin{aligned}\frac{dx_1}{d\tau} &= x_2,\\ \frac{dx_2}{d\tau} &= c_{20}\,x_1^2 + c_{11}\,x_1x_2 + c_{31}\,x_1^3x_2 + c_{41}\,x_1^4x_2.\end{aligned} \tag{3.15}$$

Here, all the coefficients h_{1ij}, h_{2ij}, t_{10}, and c_{ij} are expressed in terms of b and w^*. In particular,

$$t_{10} = \frac{t_{10n}}{t_{10d}},$$

where

$$\begin{aligned}t_{10n} = w^*\big[&(2+w^*)^3(1+2w^*)^3(w^{*3}+3w^{*2}+4w^*+1)\,b^3\\ &-(1+w^*)(2+w^*)^2(1+2w^*)^2(3w^{*5}+15w^{*4}+19w^{*3}+15w^{*2}+7w^*+1)\,b^2\\ &+w^*(1+w^*)^3(2+w^*)(1+2w^*)(3w^{*5}+10w^{*4}+20w^{*3}+13w^{*2}+4w^*+1)\,b\\ &-w^{*3}(1+w^*)^4(w^{*5}+8w^{*4}+11w^{*3}+10w^{*2}+7w^*+2)\big],\\ t_{10d} = 2(1&+2w^*)^2\big[w^*(1+w^*)^2+b(2+w^*)(1+2w^*)\big]\\ &\times\big[(2+w^*)^2(1+2w^*)^2(3w^{*3}+6w^{*2}+5w^*+1)\,b^2\\ &-w^*(1+w^*)^2(2+w^*)(1+2w^*)(3w^{*2}+2w^*+1)\,b\\ &+w^{*3}(1+w^*)^3(w^{*3}+w^*+1)\big],\end{aligned}$$

and

$$\begin{aligned}c_{20} &= \frac{bw^*}{(1+w^*)^2(1+2w^*)^4\big[w^*(1+w^*)^2+b(2+w^*)(1+2w^*)\big]^2}\,C_{20},\\ c_{11} &= \frac{-1}{(1+w^*)(1+2w^*)^4\big[w^*(1+w^*)^2+b(2+w^*)(1+2w^*)\big]^2}\,C_{11},\\ c_{31} &= -\,\frac{bw^{*2}}{40(1+w^*)^5(1+2w^*)^{18}\,c_{20}^2\big[w^*(1+w^*)^2+b(2+w^*)(1+2w^*)\big]^9}\,C_{31},\end{aligned} \tag{3.16}$$

where C_{20} and C_{11} are given in (3.9) and C_{31} is given in Appendix B, but C_{41} is very lengthy, so we omit it for the sake of brevity. Thus, $c_{20}=0$ and $c_{11}=0$ are equivalent to $C_{20}=0$ and $C_{11}=0$, respectively.

First, we consider the possibility of $C_{20}=0$. Since C_{20} is a quadratic polynomial in b, we solve $C_{20}=0$ for b to obtain

$$\begin{aligned}b_\pm = &\frac{w^*(1+w^*)}{2(2+w^*)(1+2w^*)(3w^{*3}+6w^{*2}+5w^*+1)}\\ &\times\Big[3w^{*3}+5w^{*2}+3w^*+1\pm(1+2w^*)\sqrt{(1+w^*)(1-3w^*-2w^{*2}-2w^{*3}-3w^{*4}-3w^{*5})}\Big].\end{aligned}$$

When

$$P(w^*) \equiv 1-3w^*-2w^{*2}-2w^{*3}-3w^{*4}-3w^{*5} \ge 0, \quad \text{i.e.,} \quad 0 < w^* \le 0.26679924\cdots,$$

there are two positive solutions $b_+ > b_- > 0$ ($b_+ = b_-$ if $w^* = 0.26679924\cdots$). Instead of directly verifying if $b_\pm \in (b_L, b_U)$, we compute μ_1 at $b = b_\pm$ and obtain

$$\mu_{1\pm} = -\frac{w^{*2}(1+w^*)}{2(1+2w^*)(3w^{*3}+6w^{*2}+5w^*+1)\big[w^*(1+w^*)^2+b_\pm(2+w^*)(1+2w^*)\big]} \times\Big[(1-w^{*2})(1+3w^*) \mp (1+4w^*)\sqrt{(1+w^*)P(w^*)}\Big].$$

It is obvious that $\mu_{1-} < 0$ for $0 < w^* \le 0.26679924\cdots$. For $b = b_+$, it can be shown that

$$\begin{aligned} &(1-w^{*2})(1+3w^*) - (1+4w^*)\sqrt{(1+w^*)P(w^*)} > 0 \\ \Longleftrightarrow\quad & w^{*2}\big[2(1+w^*)+(1-2w^*)^2\big](3w^{*3}+6w^{*2}+5w^*+1) > 0. \end{aligned}$$

Therefore, $\mu_{1+} < 0$, i.e., $b_\pm \notin (b_\mathrm{L}, b_\mathrm{U})$, then $c_{20} \neq 0$ under the feasible condition $b \in (b_\mathrm{L}, b_\mathrm{U})$.

Next, we discuss the possibility of $C_{11} = 0$ (i.e., $c_{11} = 0$). As shown above, $C_{11} = 0$ has a unique real positive solution $b = b^*(w^*)$. In order to have this solution in the interval $(b_\mathrm{L}, b_\mathrm{U})$, it needs

$$C_{11}(b_\mathrm{L}, w^*)C_{11}(b_\mathrm{U}, w^*) < 0,$$

which is equivalent to

$$(2w^{*2}+2w^*-1)(16w^{*4}+40w^{*3}+21w^{*2}+2w^*-1) < 0.$$

Since

$$2w^{*2}+2w^*-1 \begin{cases} > 0 & \text{if} \quad w^* > \frac{1}{2}(\sqrt{3}-1) = 0.36602540\cdots, \\ < 0 & \text{if} \quad w^* < \frac{1}{2}(\sqrt{3}-1) = 0.36602540\cdots, \end{cases}$$

$$16w^{*4}+40w^{*3}+21w^{*2}+2w^*-1 \begin{cases} > 0 & \text{if} \quad w^* > 0.15732625\cdots, \\ < 0 & \text{if} \quad w^* < 0.15732625\cdots, \end{cases}$$

we know that for

$$w^*_\mathrm{L} \triangleq 0.15732625\cdots < w^* < \tfrac{1}{2}(\sqrt{3}-1) = 0.36602540\cdots \triangleq w^*_\mathrm{U}, \tag{3.17}$$

$C_{11} = 0$ has a unique positive solution $b^*(w^*) \in (b_\mathrm{L}, b_\mathrm{U})$. Hence, the nilpotent cusp $E^*(w^*, w^*)$ is codimension 2 if

$$w^* \in (0, w^*_\mathrm{L}] \cup [w^*_\mathrm{U}, \infty), \quad \text{or} \quad w^* \in (w^*_\mathrm{L}, w^*_\mathrm{U}) \;\; \text{and} \;\; b \neq b^*(w^*).$$

So when $w^* = 1$ (and so $g^* = w^* = 1$), it is easy to see that the nilpotent cusp $E^*(1,1)$ is codimension 2, which will be used in the following unfolding process.

To study Bogdanov–Takens singularity of codimension 3, we first verify if there exist solutions such that c_{11} and c_{31} equal zero simultaneously. To achieve this, eliminating b from the two equations $C_{11} = C_{31} = 0$ yields a solution $b = \tilde{b}(w^*)$ and a resultant

$$
\begin{aligned}
\mathrm{R}_2 = {} & w^*(1+w^*)(1+2w^*)(19w^{*12}+160w^{*11}+695w^{*10}+1850w^{*9}+3516w^{*8}+5138w^{*7} \\
& +5752w^{*6}+4704w^{*5}+2654w^{*4}+944w^{*3}+155w^{*2}-12w^*-6) \\
& \times(8772w^{*29}+146446w^{*28}+1084168w^{*27}+4727481w^{*26}+14441605w^{*25} \\
& +38087268w^{*24}+105837095w^{*23}+304927780w^{*22}+785978012w^{*21}+1665060927w^{*20} \\
& +2834444190w^{*19}+3867211998w^{*18}+4215316877w^{*17}+3619345869w^{*16} \\
& +2352432058w^{*15}+1027429156w^{*14}+144447875w^{*13}-191144790w^{*12} \\
& -183218482w^{*11}-80812223w^{*10}-12409164w^{*9}+8160325w^{*8}+6938467w^{*7} \\
& +2713619w^{*6}+617679w^{*5}+58193w^{*4}-11357w^{*3}-4976w^{*2}-756w^*-46),
\end{aligned}
$$

which has only one real solution $w^* = w^{**} \in (w^*_{\mathrm{L}}, w^*_{\mathrm{U}})$, for which $b = b^{**} \triangleq \tilde{b}(w^{**}) = b^*(w^{**})$. When $w^* = w^{**}$ and $b = b^{**}$, we can get that $c_{41} = 0.00006181\cdots > 0$.

Thus, the system (2.1) has a nilpotent cusp $E^*(w^*, w^*)$ of codimension 3 if the following conditions hold:

$$w^* \in (w^*_{\mathrm{L}}, w^{**}) \cup (w^{**}, w^*_{\mathrm{U}}) \quad \text{and} \quad b = b^*(w^*);$$

the system has a nilpotent cusp $E^*(w^*, w^*)$ of codimension 4 if

$$w^* = w^{**} \quad \text{and} \quad b = b^{**}.$$

This completes the proof. ∎

Next we further investigate if system (2.1) can exhibit Bogdanov–Takens bifurcation around the unique positive equilibrium $E^*(w^*, g^*)$ as the bifurcation parameters are chosen suitably. In order to simplify the calculation in the bifurcations analysis, we let $E^*(w^*, g^*) = (1, 1)$ in the following analysis.

We prove the existence of Bogdanov–Takens bifurcation by following the process in Xiao and Ruan [46] (see also Huang et al. [31, 29]). The necessary conditions for Bogdanov–Takens bifurcation to occur are $\det(J(E^*)) = 0$ and $\operatorname{tr}(J(E^*)) = 0$.

Combining $E^* = (1, 1)$ with $\operatorname{tr}(J(E^*)) = 0$, we have

$$\xi_1 = \frac{2}{9} - \mu_2 - 3\xi_2, \quad \mu_1 = 2\mu_2 + 6\xi_2 - \frac{1}{9}, \quad \mu_2 = \frac{b}{2} - 2\xi_2, \tag{3.18}$$

and from $\det(J(E^*)) = 0$, we get

$$\xi_2 = \frac{2b - 9b^2}{2(9b+4)}, \tag{3.19}$$

then combining (3.18) and (3.19), we can represent parameters ξ_1, ξ_2, μ_1, and μ_2 by b as follows:

$$\xi_1 = \frac{8 - 9b}{9(9b+4)}, \quad \xi_2 = \frac{2b - 9b^2}{2(9b+4)}, \quad \mu_1 = \frac{45b - 4}{9(9b+4)}, \quad \mu_2 = \frac{27b^2}{2(9b+4)}, \tag{3.20}$$

where $\frac{4}{45} < b < \frac{2}{9}$.

First, it is easy to see that the nilpotent cusp $E^*(1, 1)$ is codimension 2 from the proof of Theorem 3.4.

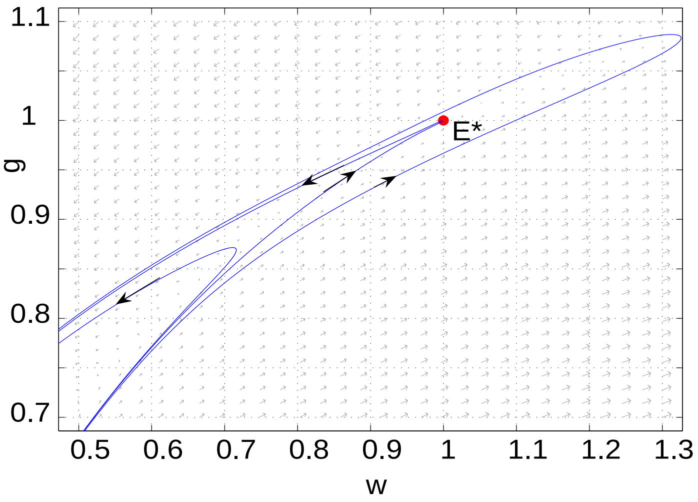

Figure 3.3. *A codimension 2 cusp E^* of system (2.1) under the conditions (3.20) and $b = \frac{7}{45}$.*

Theorem 3.5. *Under the conditions (3.20), the unique positive equilibrium $E^*(1,1)$ of system (2.1) is a cusp of codimension 2 (i.e., the Bogdanov–Takens singularity). The phase portrait for a codimension 2 cusp is shown in Figure 3.3.*

Thus system (2.1) may exhibit Bogdanov–Takens bifurcation under a small parameter perturbation if the bifurcation parameters are chosen suitably. In order to make sure such a bifurcation can be fully unfolded, for system (2.1) we choose ξ_1 and μ_1 as bifurcation parameters. Actually we have the following theorem.

Theorem 3.6. *Under conditions (3.20), system (2.1) undergoes repelling Bogdanov–Takens bifurcation of codimension 2 in a small neighborhood of the unique positive equilibrium $E^*(1,1)$ as (ξ_1, μ_1) varies near $(\frac{8-9b}{9(9b+4)}, \frac{45b-4}{9(9b+4)})$, where $\frac{4}{45} < b < \frac{2}{9}$. Hence, there exist some parameter values such that system (2.1) has an unstable limit cycle, and there exist some other parameter values such that system (2.1) has an unstable homoclinic loop. The bifurcation portrait and corresponding phase portraits are shown in Figure 3.4.*

Proof. Under condition (3.20), we consider the following unfolding system:

$$\begin{aligned}\frac{dw}{dt} &= w\Big[\frac{w}{1+w+g} - \Big(\frac{45b-4}{9(9b+4)} + \lambda_1\Big) - \Big(\frac{8-9b}{9(9b+4)} + \lambda_2\Big)(w+g)\Big],\\ \frac{dg}{dt} &= \frac{bw}{1+w} - \Big[\frac{27b^2}{2(9b+4)} + \frac{2b-9b^2}{2(9b+4)}(w+g)\Big]g,\end{aligned} \tag{3.21}$$

where (λ_1, λ_2) is a parameter vector in a small neighborhood of $(0,0)$.

We first let $x = w - 1$, $y = g - 1$; then system (3.21) becomes

$$\begin{aligned}\frac{dx}{dt} &= a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2 + O(|(x,y)|^3),\\ \frac{dy}{dt} &= b_1 x + b_2 y + b_3 x^2 + b_4 xy + b_5 y^2 + \frac{b}{16}x^3 + O(|(x,y)|^4),\end{aligned} \tag{3.22}$$

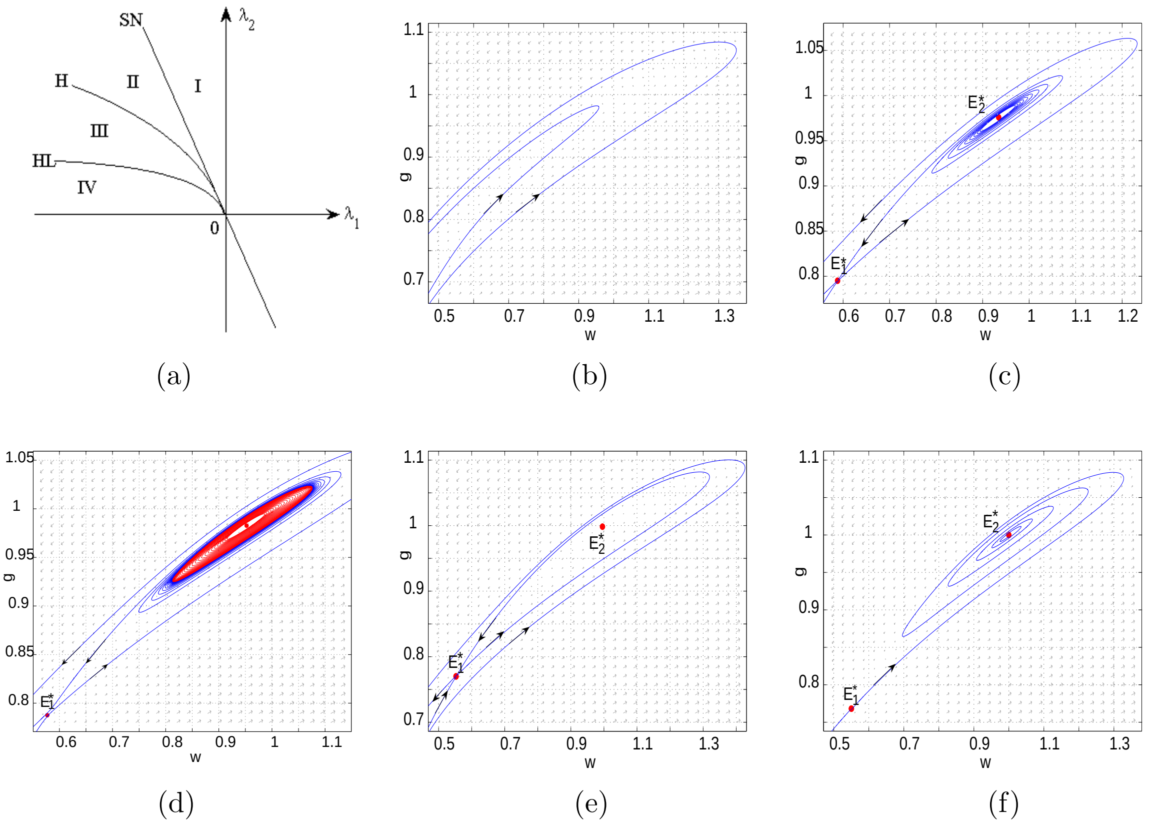

Figure 3.4. *The bifurcation diagram and the corresponding phase portraits of system* (3.21) *when* $b = \frac{7}{45}$. (a) *Bifurcation diagram.* (b) *No equilibria when* $(\lambda_1, \lambda_2) = (-0.01, 0.0058)$ *lies in region* I. (c) *An unstable focus* E_2^* *and a saddle* E_1^* *when* $(\lambda_1, \lambda_2) = (-0.01, 0.0052)$ *lies in region* II. (d) *An unstable limit cycle around a stable focus* E_2^* *when* $(\lambda_1, \lambda_2) = (-0.01, 0.00515)$ *lies in region* III. (e) *An unstable homoclinic loop when* $(\lambda_1, \lambda_2) = (-0.01, 0.005015)$ *lies on the curve HL.* (f) *A stable focus* E_2^* *and a saddle* E_1^* *when* $(\lambda_1, \lambda_2) = (-0.01, 0.005)$ *lies in region* IV.

where

$$
\begin{aligned}
&a_0 = -\lambda_1 - 2\lambda_2,\ a_1 = \frac{3b}{9b+4} - \lambda_1 - 3\lambda_2,\ a_2 = -\frac{4}{3(9b+4)} - \lambda_2,\\
&a_3 = \frac{63b-8}{27(9b+4)} - \lambda_2,\ a_4 = -\frac{9b+40}{27(9b+4)} - \lambda_2,\ a_5 = \frac{1}{27},\ b_1 = \frac{27b^2}{4(9b+4)},\\
&b_2 = -\frac{3b}{9b+4},\ b_3 = -\frac{b}{8},\ b_4 = b_5 = \frac{9b^2-2b}{2(9b+4)}.
\end{aligned}
$$

Next we let

$$
x_1 = y,\ x_2 = b_1 x + b_2 y + b_3 x^2 + b_4 xy + b_5 y^2 + \frac{b}{16}x^3 + O(|(x,y)|^4).
$$

System (3.22) can be written as

$$\begin{aligned}\frac{dx_1}{dt} &= x_2,\\ \frac{dx_2}{dt} &= \gamma_0 + \gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_1^2 + \gamma_4 x_1 x_2 + \gamma_5 x_2^2 + O(|(x_1, x_2)|^3),\end{aligned} \tag{3.23}$$

where

$$\begin{aligned}\gamma_0 &= -\frac{27b^2(\lambda_1 + 2\lambda_2)}{4(9b+4)},\\ \gamma_1 &= \frac{(4-18b)\lambda_1 + (8-63b)\lambda_2}{36},\\ \gamma_2 &= \frac{(4-18b)\lambda_1 + (8-63b)\lambda_2}{27b},\\ \gamma_3 &= \frac{1}{8748b^2(9b+4)}[-27b^2(8-72b+405b^2) + 4(32+180b-81b^2-4374b^3+6561b^4)\lambda_1\\ &\quad + 8(32+288b-810b^2-6561b^3+6561b^4)\lambda_2],\\ \gamma_4 &= \frac{1}{6561b^3(9b+4)}[27b^2(-64+36b-81b^2+2187b^3) - 2(4+9b)^2(-8-9b+81b^2)\lambda_1\\ &\quad - 2(-256-2304b+5832b^2+36450b^3+32805b^4)\lambda_2],\\ \gamma_5 &= \frac{1}{19683b^4}[(4+9b)(54b^2(-7+27b) + (32+36b-81b^2)\lambda_1 - 32(-2-9b+81b^2)\lambda_2)].\end{aligned}$$

We next introduce a new time variable τ by $dt = (1-\gamma_5 x_1)d\tau$ and let $x = x_1$, $y = x_2(1-\gamma_5 x_1)$ (still denoting τ as t); then system (3.23) becomes

$$\begin{aligned}\frac{dx}{dt} &= y,\\ \frac{dy}{dt} &= \psi_1 + \psi_2 x + \psi_3 y + \psi_4 x^2 + \psi_5 xy + O(|(x,y)|^3),\end{aligned} \tag{3.24}$$

where

$$\psi_1 = \gamma_0,\ \psi_2 = \gamma_1 - 2\gamma_5\gamma_0,\ \psi_3 = \gamma_2,\ \psi_4 = \gamma_3 - 2\gamma_1\gamma_5 + \gamma_5^2\gamma_0,\ \psi_5 = \gamma_4 - \gamma_2\gamma_5.$$

By simple calculation, we have

$$\psi_4 = \frac{-8+72b-405b^2}{324(4+9b)} + h_1(\lambda_1, \lambda_2),$$

where $h_1(\lambda_1, \lambda_2)$ is a function with respect to (λ_1, λ_2), whose coefficients depend smoothly on b, and we can see that $\psi_4 < 0$ when $\frac{4}{45} < b < \frac{2}{9}$ and λ_i are small. Let

$$x_1 = x, x_2 = \frac{y}{\sqrt{-\psi_4}}, \tau = \sqrt{-\psi_4}t,$$

and still denote τ as t; then system (3.24) becomes

$$\begin{aligned}\frac{dx_1}{dt} &= x_2,\\ \frac{dx_2}{dt} &= -\frac{\psi_1}{\psi_4} - \frac{\psi_2}{\psi_4}x_1 + \frac{\psi_3}{\sqrt{-\psi_4}}x_2 - x_1^2 + \frac{\psi_5}{\sqrt{-\psi_4}}x_1x_2 + O(|(x_1,x_2)|^3).\end{aligned} \tag{3.25}$$

Under the transformation

$$x = x_1 + \frac{\psi_2}{2\psi_4}, y = x_2,$$

system (3.25) becomes

$$\begin{aligned}\frac{dx}{dt} &= y,\\ \frac{dy}{dt} &= -\frac{\psi_1}{\psi_4} + \frac{\psi_2^2}{4\psi_4^2} + \left(\frac{\psi_3}{\sqrt{-\psi_4}} - \frac{\psi_2\psi_5}{2\psi_4\sqrt{-\psi_4}}\right) y - x^2 + \frac{\psi_5}{\sqrt{-\psi_4}}xy + O(|(x,y)|^3).\end{aligned} \tag{3.26}$$

Notice that

$$\psi_5 = \frac{-64 + 36b - 81b^2 + 2187b^3}{243b(4+9b)} + h_2(\lambda_1,\lambda_2),$$

where $h_2(\lambda_1,\lambda_2)$ is a function with respect to (λ_1,λ_2), and the coefficients of h_2 depend smoothly on b; we can see that $\psi_5 < 0$ when $\frac{4}{45} < b < \frac{2}{9}$ and λ_i are small.

Finally, making the change of variables

$$x_1 = \frac{\psi_5^2}{\psi_4}x, x_2 = \frac{\psi_5^3}{-\psi_4\sqrt{-\psi_4}}y, \tau = -\frac{\sqrt{-\psi_4}}{\psi_5}t,$$

and rewriting τ as t, we obtain the universal unfolding of system (3.21)

$$\begin{aligned}\dot{x_1} &= x_2,\\ \dot{x_2} &= \tau_1 + \tau_2x_2 + x_1^2 + x_1x_2 + O(|(x_1,x_2)|^3),\end{aligned} \tag{3.27}$$

where

$$\tau_1 = \frac{\psi_1\psi_5^4}{\psi_4^3} - \frac{\psi_2^2\psi_5^4}{4\psi_4^4}, \tau_2 = \frac{\psi_3\psi_5}{\psi_4} - \frac{\psi_2\psi_5^2}{2\psi_4^2}.$$

By a complex calculation, we obtain that

$$\left|\frac{\partial(\tau_1,\tau_2)}{\partial(\lambda_1,\lambda_2)}\right|_{\lambda=0} = \frac{32(-16-36b+243b^2+729b^3)(-64+36b-81b^2+2187b^3)^5}{243b^3(4+9b)^2(8-72b+405b^2)^5} \neq 0$$

when $\frac{4}{45} < b < \frac{2}{9}$, then τ_1 and τ_2 are independent parameters, and the above parameter transformation is a homeomorphism in a small neighborhood of the origin.

Since the time transformations we have made are all positve, then by the results in Bogdanov [12] and Takens [44] (see also Chow, Li, and Wang [15] and Perko [41]), we know that system (3.21) is the versal unfolding of the repelling Bogdanov–Takens bifurcation of codimension 2, i.e., system (2.1) undergoes the repelling Bogdanov–Takens bifurcation of codimension

2 in a small neighborhood of $E^*(1,1)$ as (ξ_1, μ_1) varies near $(\frac{8-9b}{9(9b+4)}, \frac{45b-4}{9(9b+4)})$, and the Hopf bifurcation within the Bogdanov–Takens bifurcation is subcritical. Moreover, we can obtain the following local representations of the bifurcation curves up to second-order approximations:

(i) the saddle-node bifurcation curve $SN = \{(\tau_1, \tau_2) | \tau_1 = 0, \tau_2 \neq 0\}$, i.e.,

$$\Big\{(\lambda_1,\lambda_2)|\frac{16(64-36b+81b^2-2187b^3)^4(\lambda_1+2\lambda_2)}{243b^2(4+9b)^2(8-72b+405b^2)^3}+\frac{4(-64+36b-81b^2+2187b^3)^3}{59049b^4(4+9b)^2(8-72b+405b^2)^4}$$
$$[4(-425984-370944b+11352960b^2+445214888b^3-171058392b^4-183524292b^5-301327047b^6$$
$$+2970223749b^7)\lambda_1^2+4(-1703936+4571136b+54276480b^2+136515456b^3-1104190056b^4$$
$$-1656678744b^5+731794257b^6+16917361353b^7)\lambda_1\lambda_2+(-6815744+42504192b+259282944b^2$$
$$+406233792b^3-6071077008b^4-10527491916b^5+9685512225b^6+86653049373b^7)\lambda_2^2]=0\Big\};$$

(ii) the subcritical Hopf bifurcation curve $H = \{(\tau_1, \tau_2) | \tau_2 = \sqrt{-\tau_1}, \tau_1 < 0\}$, i.e.,

$$\Big\{(\lambda_1,\lambda_2)|\frac{16(64-36b+81b^2-2187b^3)^4(\lambda_1+2\lambda_2)}{243b^2(4+9b)^2(8-72b+405b^2)^3}+\frac{16(64-36b+81b^2-2187b^3)^2}{2187b^4(4+9b)^2(8-72b+405b^2)^4}$$
$$[4(307200-272384b-5446656b^2-359095568b^3+105734160b^4+276270588b^5+1090162638b^6$$
$$-4886068554b^7-30419682284b^8-96855122225b^9+627621192186^{10})\lambda_1^2+4(1228800-4327424b$$
$$-24565248b^2-127495296b^3+754380864b^4+1691793216b^5+2452777362b^6-28485237600b^7$$
$$-287408607216b^8-73609892916b^9+345191655699b^{10})\lambda_1\lambda_2+(4915200-30261248b$$
$$-111117312b^2-438690816b^3+4331962944b^4+8802237600b^5+13760778966b^6-147122000676b^7$$
$$-166275134316b^8+113514203277b^9+1757339338104b^{10})\lambda_2^2]=0\Big\};$$

(iii) the homoclinic bifurcation curve $HL = \big\{(\tau_1, \tau_2) | \tau_2 = \frac{5}{7}\sqrt{-\tau_1}, \tau_1 < 0\big\}$, i.e.,

$$\Big\{(\lambda_1,\lambda_2)|\frac{16(64-36b+81b^2-2187b^3)^4(\lambda_1+2\lambda_2)}{243b^2(4+9b)^2(8-72b+405b^2)^3}+\frac{16(64-36b+81b^2-2187b^3)^2}{492075b^4(4+9b)^2(8-72b+405b^2)^4}$$
$$[4(80949248-136931328b-991097856b^2-9151621056b^3+249894318246^4+68215058172b^5$$
$$+232957162350b^6-1034393573754b^7-801443851578b^8-2472130140309b^9$$
$$+14686335897012b^{10})\lambda_1^2+4(323796992-1200605184b-4491044352b^2-34465813632b^3$$
$$+170315161920b^4+398237793408b^5+518627957490b^6-5887911366504b^7-6527906099487b^8$$
$$-3706451818263b^9+78232981605237b^{10})\lambda_1\lambda_2+(1295187968-7413940224b-19978970112b^2$$
$$-123435353088b^3+960139469376b^4+1903091638176b^5-1558057466616b^6$$
$$-29637131715972b^7-31950051167178b^8+217331271714336^9+395966210146362b^{10})\lambda_2^2]=0\Big\}.$$

This completes the proof. ∎

The Bogdanov–Takens bifurcation diagram and corresponding phase portraits of system (3.21) are given in Figure 3.4, where we let $b = \frac{7}{45}$. From Figure 3.4, we can see that

(a) these bifurcation curves SN, H, and HL divide the small neighborhood of the origin in the parameter (λ_1, λ_2)-plane into four regions (see Figure 3.4(a));

(b) when $(\lambda_1, \lambda_2) = (0, 0)$, the unique positive equilibrium is a cusp of codimension 2 (see Figure 3.3);

(c) there are no equilibria when the parameters lie in region I (see Figure 3.4(b));

(d) when the parameters pass region I and lie on the curve SN, there is a positive equilibrium, which is a saddle-node;

(e) when the parameters cross SN into region II, there are two positive equilibria through the saddle-node bifurcation, one an unstable focus and the other a saddle (see Figure 3.4(c));

(f) when the parameters cross H into region III, an unstable limit cycle appears through the subcritical Hopf bifurcation (see Figure 3.4(d)), where the focus is stable, whereas the focus is an unstable one with multiplicity one when the parameters lie on the curve H;

(g) when the parameters pass region III and lie on the curve HL, an unstable homoclinic cycle appears through the homoclinic bifurcation (see Figure 3.4(e));

(h) when the parameters cross III into region IV, the relative locations of one stable and one unstable manifold of the saddle E_1^* are reversed (compare Figures 3.4(c) and 3.4(f)).

Remark 3.7. From Theorem 3.4, we can see that system (2.1) may exhibit nilpotent cusp bifurcation of codimension 3 (usually called Bogdanov–Takens bifurcation of codimension 3) and nilpotent cusp bifurcation of codimension 4; for the bifurcation diagrams and corresponding phase portraits refer to Dumortier et al. [19, 20] and Li and Rousseau [37] (see also the applications of Bogdanov–Takens bifurcation of codimension 3 in predator-prey systems and epidemic models in Huang et al. [30], Li, Li, and Ma [36], and Zhu, Campbell, and Wolkowicz [51]). These results indicate that the dynamical behavior of the model is very sensitive to the initial densities of the wild and sterile mosquitoes. When the initial values lie inside the homoclinic loop, the densities of wild and sterile mosquitoes fluctuate periodically about the endemic levels. When the initial values lie outside the homoclinic loop, either wild mosquitoes or sterile mosquitoes will die out depending on their initial population densities.

4. Discussion.

It is possible to reduce or eradicate some mosquito-borne diseases by making full use of the SIT (Alphey [2], Anguelov, Dumont, and Lubuma [3], Cai, Ai, and Li [13], Dumont and Tchuenche [18], Lees, Gilles, and Hendrichs [34]). In this paper, we revisited a mosquito population model with a nonlinear saturated release rate of sterile mosquitoes, which was proposed by Cai, Ai, and Li [13]. After removing the restriction that the sterile and wild mosquitoes have the same fitness, i.e., $\mu_1 = \mu_2$ and $\xi_1 = \xi_2$, assumed in [13], our qualitative and bifurcation analysis reveals that model (2.1) exhibits complex dynamics and bifurcations, such as Hopf bifurcation of codimension 3, nilpotent cusp of codimension 4, and Bogdanov–Takens bifurcation of codimension 2. Thus, there exist some parameter values such that model (2.1) exhibits multiple stable or unstable limit cycles, and there exist some other parameter values such that model (2.1) exhibits an unstable homoclinic cycle. We presented numerical examples which have two and three limit cycles, respectively. The limit cycles represent sustained oscillations for the interacting wild and sterile mosquitoes.

We also found that there exists a critical release rate of sterile mosquitoes, above which the mosquito population can be eliminated, and the constant b in the release rate plays an important role in determining the dynamics and bifurcations of system (2.1); it affects not only the number and the type of equilibria (Theorems 2.1, 2.3, and 3.4) but also the type of bifurcations (Hopf bifurcation of codimensions 1, 2, and 3 in Theorem 3.1 and Bogdanov–Takens bifurcation in Theorem 3.6). In detail, we obtained the threshold release value b_0 in (2.7) implicitly. In Theorem 2.1, we showed that there is a unique equilibrium $(0, 0)$ when the release parameter b is above the threshold b_0, and it is locally asymptotically stable such that the two interactional mosquito populations go extinct, which means that mosquito-borne diseases can be controlled or eradicated with enough release of sterile mosquitoes. However, releasing sterile mosquitoes in large quantities represents one of those great unknowns in the SIT and this will consume a lot of money and result in tremendous social impact, and so on (Okorie et al. [40]).

There are two positive equilibria E_1^* and E_2^* when the release parameter b is below the threshold b_0, and multiple unstable or stable limit cycles appear from the Hopf bifurcations of codimensions 1, 2, and 3. Especially, the stable limit cycle represents the sustained oscillations, which provides useful guidelines in mosquito control and disease prevention. There is only one positive equilibrium E^* when the release parameter b is equal to the threshold b_0, which is a cusp of codimension 4; we proved that an unstable limit cycle or unstable homoclinic loop arises from the Bogdanov–Takens bifurcation of codimension 2.

It is worth noting that we have assumed that the equilibrium (w^*, g^*) of system (2.1) satisfies $w^* = g^*$ in the bifurcation analysis. The complete bifurcation analysis will be very interesting and challenging if we take this hypothesis away, and bifurcations with codimension more than 4 may occur. On the other hand, the model may undergo Bogdanov–Takens bifurcation of codimensions 3 and 4 for suitable bifurcation parameters; then for some various parameter values, the model may have the coexistence of limit cycles and homoclinic loops, the existence of two or three limit cycles, etc., which means that the nonlinear dynamics of the interactive wild and sterile mosquito population not only depend on more bifurcation parameters but also are very sensitive to parameter perturbations, which are important for the control of the wild population. In detail, in Figure 3.2, for example, when the initial mosquito population is outside the outer unstable limit cycle, both the interactive wild and the sterile mosquitoes will tend to extinction; when the initial mosquito population is inside the outer unstable limit cycle and outside the inner unstable limit cycle, the interactive wild and sterile mosquitoes will tend to periodic fluctuations (i.e., the inner stable limit cycle); when the initial mosquito population is inside the inner unstable limit cycle, the interactive wild and sterile mosquitoes will tend to a positive steady state. These results will be useful in designing reasonable sterile mosquito releasing policies.

Appendix A. Computation of the determinant in (2.11). By direct computation, we can get

$$\begin{aligned}\frac{1}{w}\det J &= \left[\xi_1 - \frac{1+g}{(1+N)^2}\right](\mu_2 + \xi_2 N + \xi_2 g) + \left[\frac{w}{(1+N)^2} + \xi_1\right]\left[\frac{b}{(1+w)^2} - \xi_2 g\right] \\ &= \left[\xi_1 - \frac{1+g}{(1+N)^2}\right](\mu_2 + \xi_2 N) - \frac{\xi_2 g(1+g)}{(1+N)^2} + \left[\frac{w}{(1+N)^2} + \xi_1\right]\frac{b}{(1+w)^2} - \frac{\xi_2 g w}{(1+N)^2}\end{aligned}$$

$$
\begin{aligned}
&= \Big[\xi_1 - \frac{1+g}{(1+N)^2}\Big](\mu_2+\xi_2 N) + \Big[\frac{w}{(1+N)^2}+\xi_1\Big]\frac{b}{(1+w)^2} - \frac{\xi_2 g}{1+N}\\
&= \Big[\xi_1 - \frac{1+g}{(1+N)^2}\Big](\mu_2+\xi_2 N) + \frac{b(\xi_1\mu_2-\xi_2\mu_1) - b\xi_2\mu_1 w - b\xi_1\xi_2 N w}{(1+w)^2(\mu_2+\xi_2N)}\\
&\quad + \frac{w}{(1+N)^2}\frac{b}{(1+w)^2}\\
&= \Big[\xi_1 - \frac{1+g}{(1+N)^2}\Big](\mu_2+\xi_2 N) + \Big[\frac{1}{(1+N)^2} - \frac{\xi_2(\mu_1+\xi_1N)}{\mu_2+\xi_2N}\Big]\frac{g(\mu_2+\xi_2N)}{1+w}\\
&\quad + \frac{b(\xi_1\mu_2-\xi_2\mu_1)}{(1+w)^2(\mu_2+\xi_2N)},
\end{aligned}
$$

then

$$
\frac{\det J}{(\mu_2+\xi_2N)w} = \xi_1 - \frac{1+g}{(1+N)^2} + \Big[\frac{1}{(1+N)^2} - \frac{\xi_2(\mu_1+\xi_1N)}{\mu_2+\xi_2N}\Big]\frac{g}{1+w} + \frac{b(\xi_1\mu_2-\xi_2\mu_1)}{(1+w)^2(\mu_2+\xi_2N)^2}.
$$

Next, it is easy to get that

$$
1+w = (1+N)Q_2, \quad g = N-w = N-((1+N)Q_2-1) = (1+N)(1-Q_2),
$$

$$
G_1 = N-w = (1+N)(1-Q_2) = (1+N)\left(1-\frac{\mu_1+\xi_1N}{\mu_2+\xi_2N}G_2\right),
$$

then

$$
\xi_1 - \frac{1}{(1+N)^2} = \frac{\mathrm{d}Q_2}{\mathrm{d}N} = \frac{\mathrm{d}}{\mathrm{d}N}(\frac{\mu_1+\xi_1N}{\mu_2+\xi_2N}G_2), \quad \frac{-g}{(1+N)^2} = \frac{(1+N)(Q_2-1)}{(1+N)^2} = \frac{Q_2-1}{1+N},
$$

$$
\frac{g}{1+w} = \frac{(1+N)(1-Q_2)}{(1+N)Q_2} = \frac{1-Q_2}{Q_2}.
$$

Notice that

$$
\frac{\mathrm{d}G_2}{\mathrm{d}N} = \xi_2 + \frac{(1+N)(\xi_2\mu_1-\xi_1\mu_2)}{[(1+N)(\mu_1+\xi_1N)]^2} - \frac{(\mu_2+\xi_2N)(\mu_1+\xi_1N)}{[(1+N)(\mu_1+\xi_1N)]^2},
$$

we have

$$
\frac{1}{(1+N)^2} - \frac{\xi_2(\mu_1+\xi_1N)}{\mu_2+\xi_2N} = \frac{\xi_2\mu_1-\xi_1\mu_2}{(1+N)(\mu_1+\xi_1N)(\mu_2+\xi_2N)} - \frac{\mu_1+\xi_1N}{\mu_2+\xi_2N}\frac{\mathrm{d}G_2}{\mathrm{d}N},
$$

then

$$
\begin{aligned}
\frac{(1+w)\det J}{(\mu_2+\xi_2N)w} &= (1+w)\Big[\frac{\mathrm{d}Q_2}{\mathrm{d}N} + \frac{Q_2-1}{1+N} + \frac{b\xi_1\mu_2-b\xi_2\mu_1}{(1+w)^2(\mu_2+\xi_2N)^2}\\
&\quad + \frac{1-Q_2}{Q_2}\Big(\frac{\xi_2\mu_1-\xi_1\mu_2}{(1+N)(\mu_1+\xi_1N)(\mu_2+\xi_2N)} - \frac{\mu_1+\xi_1N}{\mu_2+\xi_2N}\frac{\mathrm{d}G_2}{\mathrm{d}N}\Big)\Big]\\
&= (1+w)\Big[\frac{\mathrm{d}Q_2}{\mathrm{d}N} + \frac{Q_2-1}{1+N} - \frac{1-Q_2}{Q_2}\frac{\mu_1+\xi_1N}{\mu_2+\xi_2N}\frac{\mathrm{d}G_2}{\mathrm{d}N} + \frac{b\xi_1\mu_2-b\xi_2\mu_1}{(1+w)^2(\mu_2+\xi_2N)^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{g}{1+w}\frac{\xi_2\mu_1-\xi_1\mu_2}{(1+N)(\mu_1+\xi_1 N)(\mu_2+\xi_2 N)}\Big]\\
&=(1+w)\Big[\frac{\mathrm{d}Q_2}{\mathrm{d}N}+\frac{Q_2-1}{1+N}-\frac{1-Q_2}{Q_2}\frac{\mu_1+\xi_1 N}{\mu_2+\xi_2 N}\frac{\mathrm{d}G_2}{\mathrm{d}N}\Big]\\
&=(1+w)\Big[\frac{\mathrm{d}}{\mathrm{d}N}\Big(\frac{\mu_1+\xi_1 N}{\mu_2+\xi_2 N}G_2\Big)+\frac{\frac{\mu_1+\xi_1 N}{\mu_2+\xi_2 N}G_2-1}{1+N}-\frac{1-\frac{\mu_1+\xi_1 N}{\mu_2+\xi_2 N}G_2}{G_2}\frac{\mathrm{d}G_2}{\mathrm{d}N}\Big]\\
&=(1+w)\Big[\frac{\xi_1\mu_2-\xi_2\mu_1}{(\mu_2+\xi_2 N)^2}G_2+\frac{\mu_1+\xi_1 N}{\mu_2+\xi_2 N}\frac{\mathrm{d}G_2}{\mathrm{d}N}+\frac{\frac{\mu_1+\xi_1 N}{\mu_2+\xi_2 N}G_2-1}{1+N}\\
&\quad-\frac{1-\frac{\mu_1+\xi_1 N}{\mu_2+\xi_2 N}G_2}{G_2}\frac{\mathrm{d}G_2}{\mathrm{d}N}\Big]\\
&=(1+w)\frac{\xi_1\mu_2-\xi_2\mu_1}{(\mu_2+\xi_2 N)^2}G_2+\frac{\mu_1+\xi_1 N}{\mu_2+\xi_2 N}\frac{\mathrm{d}G_2}{\mathrm{d}N}(1+N)\Big[2\frac{\mu_1+\xi_1 N}{\mu_2+\xi_2 N}G_2-1\Big]\\
&\quad+\frac{\mu_1+\xi_1 N}{\mu_2+\xi_2 N}G_2\Big[\frac{\mu_1+\xi_1 N}{\mu_2+\xi_2 N}G_2-1\Big].
\end{aligned}
$$

Finally we can compute that

$$
\begin{aligned}
\frac{\mathrm{d}(G_1G_2)}{\mathrm{d}N}&=\frac{\mathrm{d}}{\mathrm{d}N}\Big[(1+N)\Big(1-\frac{\mu_1+\xi_1 N}{\mu_2+\xi_2 N}G_2\Big)G_2\Big]\\
&=\Big[1-\frac{\mu_1+\xi_1 N}{\mu_2+\xi_2 N}G_2\Big]G_2+(1+N)\Big[1-2\frac{\mu_1+\xi_1 N}{\mu_2+\xi_2 N}G_2\Big]\frac{\mathrm{d}G_2}{\mathrm{d}N}\\
&\quad-(1+N)\frac{\xi_1\mu_2-\xi_2\mu_1}{(\mu_2+\xi_2 N)^2}G_2^2
\end{aligned}
$$

and

$$\frac{\mu_2+\xi_2 N}{\mu_1+\xi_1 N}(1+w)\frac{\xi_1\mu_2-\xi_2\mu_1}{(\mu_2+\xi_2 N)^2}G_2=(1+N)\frac{\xi_1\mu_2-\xi_2\mu_1}{(\mu_2+\xi_2 N)^2}G_2^2.$$

Hence, we have

$$\det J=-\frac{w(\mu_1+\xi_1 N)}{1+w}\frac{\mathrm{d}(G_1G_2)}{dN}.$$

Appendix B. The normal form coefficient C_{31} in (3.15). We have

$$
\begin{aligned}
C_{31}=&\frac{10935}{268435456}(2+w^*)^2(1+2w^*)^2(400w^{*2}+532w^*+157)\,b^7\\
&+\frac{405}{2147483648}(2+w^*)(1+2w^*)(2040164w^{*4}+7345120w^{*3}+8021037w^{*2}\\
&+3467155w^*+519434)\,b^6+\frac{15}{17179869184}(220872892w^{*6}+2012558376w^{*5}\\
&+5427607587w^{*4}+5882428747w^{*3}+2964209802w^{*2}+681948588w^*+55434712)\,b^5\\
&+\frac{5}{549755813888}(904911086255471128l6w^{*6}+40720998862133l167912w^{*5}\\
&+6786833139092574734 88w^{*4}+5655694277950639766l0w^{*3}\\
&+2545062422814201940 99w^{*2}+593847898131047037 96w^*
\end{aligned}
$$

$$
\begin{aligned}
&+ 5655694263387712684)\, b^4 + \frac{5}{8796093022208}(8077328936067004964944 w^{*6} \\
&+ 3744895536616453049663 2w^{*5} + 6355863267343800940367 2w^{*4} \\
&+ 5371082204936678910264 6w^{*3} + 2445001531129388653499 1w^{*2} \\
&+ 5757915734560851036324 w^* + 551963844104437159740)\, b^3 \\
&- \frac{5}{17592186044416} w^*(1+w^*)(8655355533852672 w^{*38} + 199768068627365888 w^{*37} \\
&+ 2159549963481120768 w^{*36} + 14582987096181440512 w^{*35} \\
&+ 69206571313501569024 w^{*34} + 246153237170967543808 w^{*33} \\
&+ 683156440426334912512 w^{*32} + 1520307706192552001536 w^{*31} \\
&+ 2766134478953842212864 w^{*30} + 4175908204348337815552 w^{*29} \\
&+ 5299626061744327098368 w^{*28} + 5744032827241217916928 w^{*27} \\
&+ 5454910034146028945408 w^{*26} + 4738805020638991876096 w^{*25} \\
&+ 3977925921437590224896 w^{*24} + 3323023712636786180096 w^{*23} \\
&+ 2673422691564344836096 w^{*22} + 1918044394160951918592 w^{*21} \\
&+ 1126269787115186421760 w^{*20} + 486845242821262114816 w^{*19} \\
&+ 117907898900226703360 w^{*18} - 16138810339937550336 w^{*17} \\
&- 30125056780022939648 w^{*16} - 14517915728681869312 w^{*15} \\
&- 3953315709850659840 w^{*14} - 432976326466850304 w^{*13} \\
&- 111122577743033600 w^{*12} + 325306578545902208 w^{*11} \\
&- 592687303277461760 w^{*10} + 1193208278628159424 w^{*9} \\
&- 2385998862171245120 w^{*8} + 4771988508665800896 w^{*7} \\
&- 9543980176694203312 w^{*6} - 45625231535755942288 36w^{*5} \\
&- 3561104918995743382620 w^{*4} + 5298123749676963193727 w^{*3} \\
&+ 6773237545745039974837 w^{*2} + 2453673595675322222989 w^* \\
&+ 275981922052218579870)\, b^2 - \frac{5}{17592186044416} w^{*2}(w^*+1)^3(4860940906397696 w^{*36} \\
&+ 106008588958302208 w^{*35} + 1116160776441167872 w^{*34} \\
&+ 7499130598099779584 w^{*33} + 35958874053736923136 w^{*32} \\
&+ 130752328119263166464 w^{*31} \\
&+ 374521547118367211520 w^{*30} + 867353185622826156032 w^{*29} \\
&+ 1654892060187629191168 w^{*28} + 2638276342638239023104 w^{*27} \\
&+ 3552881543132586442752 w^{*26} + 4076474357220806492160 w^{*25} \\
&+ 4011774879235838050304 w^{*24} + 3401707588884656816128 w^{*23}
\end{aligned}
$$

$$
\begin{aligned}
&+ 2487877654355254444032w^{*22} + 1561255170333249896448w^{*21}\\
&+ 826718954032604643328w^{*20} + 355261871049785606144w^{*19}\\
&+ 112787860602429833216w^{*18} + 186127599915335024464w^{*17}\\
&- 446754365466430668 8w^{*16} - 4455443629760708608w^{*15}\\
&- 2000166245611663360w^{*14} - 246544647902320 64w^{*13}\\
&- 942135910358966784w^{*12} + 1777709168327884544w^{*11}\\
&- 3672638787394217344w^{*10} + 7541881550721502208w^{*9}\\
&- 15481602030482666560w^{*8} + 31758563129409142336w^{*7}\\
&- 65107809065257493888w^{*6} + 133396955003178748192w^{*5}\\
&- 137419905964873192092w^{*4} - 529807387244153981484w^{*3}\\
&+ 220562102355633587327w^{*2} - 65225017642507572415w^{*}\\
&- 13799096102610928 9935)\, b\\
&+ \frac{5}{8796093022208} w^{*4}(w^{*}+1)^{6}(540684842958848w^{*32} + 11467974997180416w^{*31}\\
&+ 112289738110533632w^{*30} + 677359202057846784w^{*29} + 2834784882230558720w^{*28}\\
&+ 8781226322820071424w^{*27} + 20945216454665437184w^{*26}\\
&+ 39429696594607341568w^{*25}\\
&+ 59402844017289854976w^{*24} + 71903787981100548096w^{*23}\\
&+ 69338317915118632960w^{*22}\\
&+ 51657822937064407040w^{*21} + 27128234990123352064w^{*20}\\
&+ 6430713621765947392w^{*19}\\
&- 4473822265756614656w^{*18} - 6551056013710196736w^{*17} - 4429594232603213824w^{*16}\\
&- 1986042127752691712w^{*15} - 641065886359126016w^{*14} - 137920606478221312w^{*13}\\
&- 53658929621835776w^{*12} + 47946203369582592w^{*11} - 109831747771763712w^{*10}\\
&+ 225203595050309888w^{*9} - 465365226137065600w^{*8} + 960222030190801280w^{*7}\\
&- 1979368825414780096w^{*6} + 4076565316012812672w^{*5} - 8388782409167098432w^{*4}\\
&+ 17248870746396534080w^{*3} - 35440356161179121600w^{*2}\\
&+ 72765943383601717520w^{*} - 137990961026109289935),
\end{aligned}
$$

where $C_{11} = 0$ has been used to simplify C_{31}.

Acknowledgment. We would like to thank the two anonymous reviewers for their helpful comments and suggestions, which really helped us to improve the manuscript.

REFERENCES

[1] P. Aguirre, E. González-Olivares, and E. Sáez, *Three limit cycles in a Leslie-Gower predator-prey model with additive Allee effect*, SIAM J. Appl. Math., 69 (2009), pp. 1244–1262.

[2] L. Alphey, *Genetic control of mosquitoes*, Annu. Rev. Entomol., 59 (2014), pp. 205–224.
[3] R. Anguelov, Y. Dumont, and J. Lubuma, *Mathematical modeling of sterile insect technology for control of anopheles mosquito*, Comput. Math. Appl., 64 (2012), pp. 374–389.
[4] M. P. Atkinson, Z. Su, N. Alphey, L. S. Alphey, P. G. Coleman, and L. M. Wein, *Analyzing the control of mosquito-borne diseases by a dominant lethal genetic system*, Proc. Natl. Acad. Sci. USA, 104 (2007), pp. 9540–9545.
[5] H. J. Barclay, *Models for the sterile insect release method with the concurrent release of pesticides*, Ecol. Model., 11 (1980), pp. 167–177.
[6] H. J. Barclay, *Models for pest control: Complementary effects of periodic releases of sterile pests and parasitoids*, Theoret. Popul. Biol., 32 (1987), pp. 76–89.
[7] H. J. Barclay and M. Mackauer, *The sterile insect release method for pest control: a density dependent model*, Environment. Entomol., 9 (1980), pp. 810–817.
[8] M. Q. Benedict and A. S. Robinson, *The first releases of transgenic mosquitoes: An argument for the sterile insect technique*, Trends Parasitol., 19 (2003), pp. 349–356.
[9] A. A. Berryman, *Mathematical description of the sterile male principle*, Can. Entomol., 99 (1967), pp. 858–865.
[10] K. W. Blayneha and J. Mohammed-Awel, *Insecticide-resistant mosquitoes and malaria control*, Math. Biosci., 252 (2014), pp. 14–26.
[11] C. Boete, F. B. Agusto, and R. G. Reeves, *Impact of mating behaviour on the success of malaria control through a single inundative release of transgenic mosquitoes*, J. Theoret. Biol., 347 (2014), pp. 33–43.
[12] R. I. Bogdanov, *The versal deformations of a singular point on the plane in the case of zero eigenvalues*, Trudy Sem. Petrovsk. Vyp., 2 (1976), pp. 37–65.
[13] L. Cai, S. Ai, and J. Li, *Dynamics of mosquitoes populations with different strategies of releasing sterile mosquitoes*, SIAM, J. Appl. Math., 75 (2014), pp. 1223–1237.
[14] D. O. Carvalho, A. L. Costa-da-Silva, and R. S. Lees, *Two step male release strategy using transgenic mosquito lines to control transmission of vector-borne diseases*, Acta Tropica, 132 (2014), pp. S170–S177.
[15] S.-N. Chow, C. Li, and D. Wang, *Normal Forms and Bifurcation of Planar Vector Fields*, Cambridge University Press, Cambridge, UK, 1994.
[16] C. S. Coleman, *Hilbert's* 16*th problem: How many cycles?* in Differential Equations Model, M. Braun, C. S. Coleman, D. Drew, eds., Springer, New York, 1983, pp. 279–297.
[17] C. Dufourd and Y. Dumont, *Impact of environmental factors on mosquito dispersal in the prospect of sterile insect technique control*, Comput. Math. Appl., 66 (2013), pp. 1695–1715.
[18] Y. Dumont and J. M. Tchuenche, *Mathematical studies on the sterile insect technique for the chikungunya disease and* Aedes albopictus, J. Math. Biol., 65 (2012), pp. 809–854.
[19] F. Dumortier, R. Roussarie, and J. Sotomayor, *Generic* 3*-parameter families of vector fields on the plane, unfolding a singularity with nilpotent linear part. The cusp case of codimension* 3, Ergodic Theory Dynam. Systems, 7 (1987) (3), pp. 375–413.
[20] F. Dumortier, R. Roussarie, J. Sotomayor, and K. Zoladek, *Bifurcation of Planar Vector Fields, Nilpotent Singularities and Abelian Integrals*, Lecture Notes in Math. 1480, Springer, New York, 1991.
[21] V. A. Dyck, J. Hendrichs, and A. S. Robinson, eds., *Sterile Insect Technique—Principles and Practice in Area-Wide Integrated Pest Management*, Springer, New York, 2005.
[22] L. Esteva and M. H. Yang, *Mathematical model to assess the control of* Aedes aegypti *mosquitoes by the sterile insect technique*, Math. Biosci., 198 (2005), pp. 132–147.
[23] K. R. Fister and M. McCarthy, *Optimal control of insects through sterile insect release and habitat modification*, Math. Biosci., 244 (2013), pp. 201–212.
[24] G. Fu, R. S. Lees, D. Nimmo, D. Aw, L. Jin, P. Gray, T. U. Berendonk, H. White-Cooper, S. Scaife, H. Kim Phuc, O. Marinotti, N. Jasinskiene, A. A. James, and L. Alphey, *Female-specific flightless phenotype for mosquito control*, Proc. Natl. Acad. Sci. USA, 107 (2010), pp. 4550–4554.
[25] M. Gazor and P. Yu, *Formal decomposition method and parametric normal form*, Internat. J. Bifur. Chaos, 20 (2010), pp. 3487–3415.

[26] M. Gazor and P. Yu, *Spectral sequences and parametric normal forms*, J. Differential Equations, 252 (2013), pp. 1003–1031.

[27] M. Gazor and M. Moazeni, *Parametric normal forms for Bogdanov-Takens singularity; the generalized saddle-node case*, Discrete Contin. Dynam. Syst., 35 (2015), pp. 205–224.

[28] E. González-Olivares, B. González-Yañez, J. Mena Lorca, A. Rojas-Palma, and J. D. Flores, *Consequences of double Allee effect on the number of limit cycles in a predator-prey model*, Comput. Math. Appl., 62 (2011), pp. 3449–3463

[29] J. Huang, Y. Gong, and S. Ruan, *Bifurcation analysis in a predator-prey model with constant-yield predator harvesting*, Discrete Contin. Dyn. Syst. Ser. B, 18 (2013), pp. 2101–2121.

[30] J. Huang, S. Liu, S. Ruan, and X. Zhang, *Bogdanov-Takens bifurcation of codimension* 3 *in a predator-prey model with constant-yield predator harvesting*, Commun. Pure Appl. Anal., 15 (2016), pp. 1041–1055.

[31] J. Huang, S. Ruan, and J. Song, *Bifurcations in a predator-prey system of Leslie type with generalized Holling type* III *functional response*, J. Differential Equations, 257 (2014), pp. 1721–1752.

[32] E. F. Knipling, *Possibilities of insect control or eradication through the use of sexually sterile males*, J. Econ. Entomol., 48 (1955), pp. 459–462.

[33] E. S. Krafsur, *Sterile insect technique for suppressing and eradicating insect populations:* 55 *years and counting*, J. Agr. Entomol., 15 (1998), pp. 303–317.

[34] R. S. Lees, J. R. L. Gilles, J. Hendrichs, M. J. B. Vreysen, and K. Bourtzis, *Back to the future: The sterile insect technique against mosquito disease vectors*, Curr. Opin. Insect Sci., 10 (2015), pp. 156–162.

[35] M. A. Lewis and P. van den Driessche, *Waves of extinction from sterile insect release*, Math. Biosci., 116 (1993), pp. 221–247.

[36] C. Li, J. Li, and Z. Ma, *Codimension* 3 *B-T bifurcation in an epidemic model with a nonlinear incidence*, Discrete Contin. Dynam. Syst. Ser. B., 20 (2015), pp. 1107–1116.

[37] C. Li and C. Rousseau, *A system with three limit cycles appearing in a Hopf bifurcation and dying in a homoclinic bifurcation: The cusp of order* 4, J. Differential Equations, 79 (1989), pp. 132–167.

[38] J. Li, *Simple mathematical models for interacting wild and transgenic mosquito populations*, Math. Biosci., 189 (2004), pp. 39–59.

[39] J. Li, L. Cai, and Y. Li, *Stage-structured wild and sterile mosquito population models and their dynamics*, J. Biol. Dynam., 11 (2017), pp. 79–101.

[40] P. N Okorie, J. M Marshall, O. M Akpa, and O. G Ademowo, *Perceptions and recommendations by scientists for a potential release of genetically modified mosquitoes in Nigeria*, Malaria J., 13 (2014), 154.

[41] L. Perko, *Differential Equations and Dynamical Systems*, 3rd ed., Springer, New York, 2001.

[42] J. H. Werren, L. Baldo, and M. E. Clark, *Wolbachia: Master manipulators of invertebrate biology*, Nat. Rev. Microbiol., 6 (2008), pp. 741–751.

[43] S. M. White, P. Rohani, and S. M. Sait, *Modelling pulsed releases for sterile insect techniques: Fitness costs of sterile and transgenic males and the effects on mosquito dynamics*, J. Appl. Ecol., 47 (2010), pp. 1329–1339.

[44] F. Takens, *Forced oscillations and bifurcations*, in Applications of Global Analysis I, Comm. Math. Inst. Rijksuniv. Utrecht 3, Mathematisch Instituut der Rijksuniversiteit Utrecht, the Netherlands, 1974, pp. 1–59.

[45] D. D. Thomas, C. A. Donnelly, R. J. Wood, and L. S. Alphey, *Insect population control using a dominant, repressible lethal, genetic system*, Science, 287 (2000), pp. 2474–2476.

[46] D. Xiao and S. Ruan, *Bogdanov-Takens bifurcations in predator-prey systems with constant rate harvesting*, Fields Inst. Commun., 21 (1999), pp. 493–506.

[47] P. Yu, *Computation of normal forms via a perturbation technique*, J. Sound Vib., 211 (1998), pp. 19–38.

[48] P. Yu, *Simplest normal forms of Hopf and generalized Hopf bifurcations*, Internat. J. Bifur. Chaos, 9 (1999), pp. 1917–1939.

[49] P. Yu and A. Y. L. Leung, *The simplest normal form of Hopf bifurcation*, Nonlinearity, 16 (2003), pp. 277–300.

[50] Z. Zhang, T. Ding, W. Huang, and Z. Dong, *Qualitative Theory of Differential Equation*, Transl. Math. Monogr. 101, AMS, Providence, RI, 1992.
[51] H. Zhu, S. A. Campbell, and G. S. K. Wolkowicz, *Bifurcation analysis of a predator-prey system with nonmonotonic functional response*, SIAM J. Appl. Math., 63 (2002), pp. 636–682.