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Bifurcations in a predator–prey system of Leslie type with generalized Holling type III functional response ☆

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Abstract

We consider a predator–prey system of Leslie type with generalized Holling type III functional response $p(x) = \frac{mx^2}{ax^2+bx+1}$. By allowing b to be negative ($b > -2\sqrt{a}$), $p(x)$ is monotonic for $b > 0$ and nonmonotonic for $b < 0$ when $x \geq 0$. The model has two non-hyperbolic positive equilibria (one is a multiple focus of multiplicity one and the other is a cusp of codimension 2) for some values of parameters and a degenerate Bogdanov–Takens singularity (focus or center case) of codimension 3 for other values of parameters. When there exist a multiple focus of multiplicity one and a cusp of codimension 2, we show that the model exhibits subcritical Hopf bifurcation and Bogdanov–Takens bifurcation simultaneously in the corresponding small neighborhoods of the two degenerate equilibria, respectively. Different phase portraits of the model are obtained by computer numerical simulations which demonstrate that the model can have: (i) a stable limit cycle enclosing two non-hyperbolic positive equilibria; (ii) a stable limit cycle enclosing an unstable homoclinic loop; (iii) two limit cycles enclosing a hyperbolic positive equilibrium; (iv) one stable limit cycle enclosing three hyperbolic positive equilibria; or (v) the coexistence of three stable states (two stable equilibria and a stable limit cycle). When the model has a Bogdanov–Takens singularity of codimension 3, we prove that the model exhibits degenerate focus type Bogdanov–Takens bifurcation of codimension 3. These results not only demonstrate that the dynamics of this model when $b > -2\sqrt{a}$ are much more complex and far richer than the case when $b > 0$ but also provide new bifurcation phenomena for predator–prey systems.

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1. Introduction

Let $x(t)$ and $y(t)$ denote densities of the prey and predators at time t, respectively. The classical Gause type predator–prey system takes the following form (Freedman [14]):

$$\begin{aligned} \dot{x} &= xg(x,K) - yp(x), \\ \dot{y} &= y\big(-d + cq(x)\big), \end{aligned} \tag{1.1}$$

where $g(x,K)$ is a continuous and differentiable function describing the specific growth rate of the prey in the absence of predators and satisfying $g(0,K) = r > 0$, $g(K,K) = 0$, $g_x(K,K) < 0$, $g_x(x,K) \le 0$, and $g_K(x,K) > 0$ for any $x > 0$. The logistic growth $g(x,K) = r(1 - x/K)$ is considered as a prototype and satisfies all assumptions.

The functional response $p(x)$ of predators to the prey describes the change in the density of the prey attacked per unit time per predator as the prey density changes. It is continuous and differentiable and satisfies $p(0) = 0$. In general the functional response depends on many factors, for example, the various prey densities, the efficiency with which predators can search out and kill the prey, the handling time, etc. The following functional response functions have been extensively used in modeling population dynamics.

(i) Lotka–Volterra type:

$$p(x) = mx,$$

where $m > 0$ is a constant, which is an unbounded function.

(ii) Holling type II:

$$p(x) = \frac{mx}{a + x},$$

where $m > 0$ and $a > 0$ are constants and a is called the half-saturation constant, which is bounded, $p'(x) > 0$ for $x \ge 0$, and $\lim_{x\to\infty} p(x) = m$.

(iii) Generalized Holling type III or sigmoidal:

$$p(x) = \frac{mx^2}{ax^2 + bx + 1},$$

where m and a are positive constants and b is a constant. When $b = 0$, it is called the Holling type III response function. When $b > -2\sqrt{a}$ (so that $ax^2 + bx + 1 > 0$ for all $x \ge 0$ and hence $p(x) > 0$ for all $x > 0$), it is called the generalized Holling type III or sigmoidal

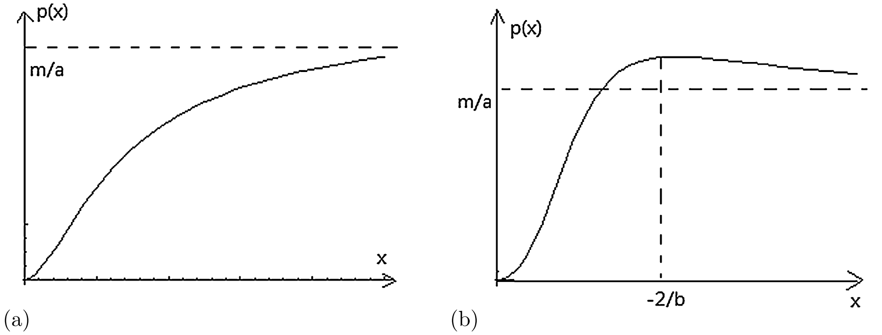

Fig. 1.1. Generalized Holling type-III functional response. (a) $b \geq 0$; (b) $b < 0$.

functional response (Bazykin [2]). The difference between $b \geq 0$ and $b < 0$ can be seen from Fig. 1.1, when the prey density is below a certain level of threshold density, both figures indicate that the predators show some form of learning behavior and will not utilize the prey for food at any great intensity (Hsu and Huang [18]). However, above that density level, the predators increase their feeding rates until some saturation level is reached when $b \geq 0$ (Hsu and Huang [18]), while when $b < 0$, the predation increases to a maximum and then decreases, approaching $\frac{m}{a}$ as x approaches infinity, thus $p(x)$ describes the situation where the prey can better defend or disguise themselves when their population density becomes large enough (see the next case (iv)).

(iv) Generalized Holling type IV or Monod–Haldane:

$$p(x) = \frac{mx}{ax^2 + bx + 1},$$

where m and a are positive constants and b is a constant. This function is called the Monod–Haldane (Andrews [1], Ruan and Xiao [32]) or the generalized Holling type IV (Taylor [37], Collings [10]) functional response and can be used to describe the phenomenon of “inhibition” in microbial dynamics and “group defence” in population dynamics. In microbial dynamics, there are experiments indicating that when the nutrient concentration reaches a high level an inhibitory effect on the specific growth rate may occur. In population dynamics, field observations demonstrate that predation is decreased, or even prevented altogether, due to the increased ability of the prey to better defend or disguise themselves when their numbers are large enough. When $b = 0$, the function is called the Holling type IV functional response in the literature (Ruan and Xiao [32], Li and Xiao [27]).

The generalized Holling type III functional response with $b < 0$ and the generalized Holling type IV functional response are nonmonotone functions (see Fig. 1.1 (b), Fig. 1.1 in Ruan and Xiao [32], and Fig. 1.1 in Zhu, Campbell and Wolkowicz [45]). Predator–prey systems with nonmonotone functional response have been extensively studied by many authors (see Wolkowicz [38], Ruan and Xiao [32], Zhu, Campbell and Wolkowicz [45], Xiao and Zhu [43], Broer, Naudot and Roussarie [6], Lamontagne, Coutu and Rousseau [23], Etoua and Rousseau [13], and references cited therein). In these systems, predator functional responses have played an important role in inducing more complex bifurcation phenomena and dynamical behaviors, such as homoclinic bifurcation and Hopf bifurcation of codimension 1 in Wolkowicz [38], Bogdanov–Takens bifurcation of codimension 2 in Ruan and Xiao [32], Bogdanov–Takens bifurcation of codimension 3 (cusp) in Zhu, Campbell and Wolkowicz [45], Broer, Naudot and Roussarie [6] and Lamontagne, Coutu and Rousseau [23], Hopf bifurcation of codimension 2 in Lamontagne, Coutu and Rousseau [23], existence of two limit cycles in Xiao and Zhu [43], and canard phenomenon in Li and Zhu [26].

The function $q(x)$ in system (1.1) describes how predators convert the consumed prey into the growth of predators and the parameter c indicates the efficiency of predators in converting consumed prey into their growth, while d is the predator mortality rate. In most classical predator–prey models, $q(x) = p(x)$. The dynamics of the predator–prey model (1.1) when $q(x) = p(x)$ taking one of the above four types have been studied extensively, see for example, Seo and DeAngelis [34] for Holling type I, Bazykin [2], Freedman [14], Kuang and Freedman [22], and May [30] for Holling type II, Lamontagne, Coutu and Rousseau [23] for generalized Holling type III, and Huang and Xiao [21] for Holling type IV.

An interesting case is when the predator growth function is different from the predator predation function. Moreover, the predator growth term is described by a function of not the prey density only, instead it is assumed to be dependent on the ratio of predators and their prey, y/x, see for example, Leslie [24], Leslie and Gower [25]. The new predator–prey system takes the following form (Freedman and Mathsen [15], Hsu and Huang [18]):

$$\begin{aligned} \dot{x} &= xg(x, K) - yp(x), \\ \dot{y} &= yq\left(\frac{y}{x}\right), \end{aligned} \tag{1.2}$$

where $q(z)$ is continuous and differentiable and satisfies $q(0) > 0$ and $q'(0) < 0$. System (1.2) includes some very interesting special cases.

(a) When $g(x, K) = r(1 - x/K)$, $p(x) = mx$, and $q(\frac{y}{x}) = s(1 - \frac{y}{hx})$, where s and h are positive constants, it becomes the so-called Leslie–Gower model (Leslie and Gower [25]):

$$\begin{aligned} \dot{x} &= rx\left(1 - \frac{x}{K}\right) - mxy, \\ \dot{y} &= sy\left(1 - \frac{y}{hx}\right), \end{aligned} \tag{1.3}$$

which has been studied extensively, for example, Hsu and Huang [18] showed that the unique positive equilibrium of system (1.3) is globally asymptotically stable under all biologically admissible parameters.

(b) When $g(x, K) = r(1 - x/K)$, $p(x) = \frac{mx}{a+x}$, and $q(\frac{y}{x}) = s(1 - \frac{y}{hx})$, it becomes the so-called Holling–Tanner model (Leslie [24], Holling [17], Tanner [36]):

$$\begin{aligned} \dot{x} &= rx\left(1 - \frac{x}{K}\right) - \frac{mxy}{a + x}, \\ \dot{y} &= sy\left(1 - \frac{y}{hx}\right). \end{aligned} \tag{1.4}$$

Model (1.4) was first proposed in May [30]. Caughley [8] used this system to model the biological control of the prickly-pear cactus by the moth *Cactoblastis cactorum*. Wollkind, Collings and Logan [39] employed this model to study the temperature-mediated stability of the predator–prey mite interaction between *Metaseiulus occidentalis* and the phytophagous spider mite *Tetranychus mcdanieli* on apple trees. The nonlinear dynamics of system (1.4) have been investigated by many researchers (see, for example, Hsu and Huang [18–20], Collings [10], Gasull, Kooij and Torregrosa [16], Braza [5], Sáez and González-Olivares [33]) and many interesting results on the existence and uniqueness of limit cycles have been obtained.

(c) Collings [10] further suggested that the Holling type II functional response in system (1.4) can be replaced by the Holling types III and IV function responses. The predator–prey model of Leslie type with Holling type IV function response was considered by Li and Xiao [27]. For other related cases and their analysis, we refer to Lindström [28], Freedman and Mathsen [15], Li and Xiao [27] and so on.

In this paper, we consider system (1.2) with the following functions, $g(x,K)=r(1-x/K)$, $p(x)=\frac{mx^2}{ax^2+bx+1}$, and $q(\frac{y}{x})=s(1-\frac{y}{hx})$; namely, we study the following predator–prey system of Leslie type with generalized Holling type III or sigmoidal functional response

$$\begin{aligned}\dot{x}&=rx\left(1-\frac{x}{K}\right)-\frac{mx^2y}{ax^2+bx+1},\\ \dot{y}&=sy\left(1-\frac{y}{hx}\right),\end{aligned}\tag{1.5}$$

where the parameters r, K, m, a, s and h are all positive constants, and b is an arbitrary constant. When $b>0$, Hsu and Huang [18] studied the global stability of system (1.5) and showed that the unique positive equilibrium of system (1.5) is globally asymptotically stable for some parameter values if it is local asymptotically stable. However, the nonlinear dynamics of system (1.5) are not well-understood.

Before going into details, we make the following scaling:

$$\bar{t}=rt,\qquad \bar{x}=\frac{x}{K},\qquad \bar{y}=\frac{mKy}{r},\qquad \bar{a}=aK^2,$$
$$\bar{b}=bK,\qquad \delta=\frac{s}{r},\qquad \beta=\frac{s}{hmK^2}.$$

Dropping the bars, model (1.5) becomes

$$\begin{aligned}\dot{x}&=x(1-x)-\frac{x^2y}{ax^2+bx+1},\\ \dot{y}&=y\left(\delta-\frac{\beta y}{x}\right),\end{aligned}\tag{1.6}$$

where $b>-2\sqrt{a}$ (so that $ax^2+bx+1>0$ for all $x\ge 0$). From the point of view of biology, we only restrict our attention to system (1.6) in the closed first quadrant in the (x,y) plane. Since system (1.6) is not well-defined at $x=0$, we consider (1.6) in $\mathbb{R}_2^+=\{(x,y): x>0,\ y\ge 0\}$. It is standard to show that solutions of (1.6) are positive and bounded and for each solution $(x(t),y(t))$ of (1.6) there exists a $T\ge 0$ such that $0<x(t)<1$ and $0\le y(t)<\frac{\delta}{\beta}$ for all $t\ge T$. Thus, system (1.6) is well-defined in a subset of $\mathbb{R}_2^+$.

To study dynamical behaviors of system (1.6), we show that it has at most three positive equilibria for all parameters values and perform qualitative and bifurcation analyses of the system. Firstly, there exist some values of parameters such that the model has two non-hyperbolic positive equilibria, one is a multiple focus of multiplicity one and the other is a cusp of codimension 2. Choosing two parameters of the model as bifurcation parameters, we demonstrate that the model exhibits a subcritical Hopf bifurcation and a Bogdanov–Takens bifurcation simultaneously in the corresponding small neighborhoods of the two degenerate equilibria, respectively. The bifurcation curves divide the plane of bifurcation parameters into six regions. For various parameter values in these regions, different phase portraits of the model are obtained by computer numerical simulations which demonstrate that the model can have: (i) a stable limit cycle enclosing two non-hyperbolic positive equilibria; (ii) a stable limit cycle enclosing an unstable homoclinic loop; (iii) two limit cycles enclosing a hyperbolic positive equilibrium; (iv) one stable limit cycle enclosing three hyperbolic positive equilibria; or (v) the coexistence of three stable states (two stable equilibria and a stable limit cycle). Secondly, when the model has a unique degenerate positive equilibrium we show that it is a Bogdanov–Takens singularity of codimension 3 (focus or center case) for some values of parameters. Choosing three parameters of the model as bifurcation parameters, we prove that the model exhibits degenerate focus type Bogdanov–Takens bifurcation of codimension 3. These results indicate that the dynamics of system (1.6) with $b > -2\sqrt{a}$ are much more complex and richer than the case when $b > 0$.

2. Equilibria and their types

Notice that system (1.6) always has a boundary equilibrium $E_0 = (1, 0)$ for all parameters which is a hyperbolic saddle. The biological interpretation of this boundary equilibrium is that the prey population reaches its carrying capacity in the absence of predators. E_0 divides the positive x-axis into two parts which are two stable manifolds of E_0 and there exists a unique unstable manifold of E_0 in the interior of $\mathbb{R}^2_+$.

If $\bar{E}(\bar{x}, \bar{y})$ is a positive equilibrium of system (1.6), then $\bar{x}$ is a root of the equation

$$a\bar{x}^3 + \left(\frac{\delta}{\beta} + b - a\right)\bar{x}^2 + (1-b)\bar{x} - 1 = 0 \tag{2.1}$$

in the interval $(0, 1)$. Note that the third-order algebraic equation (2.1) can have one, two, or three positive roots in the interval $(0, 1)$ which can be evaluated by using the root formula of the third-order algebraic equation. Correspondingly, system (1.6) can have one, two, or three positive equilibria. The Jacobian matrix of system (1.6) at $\bar{E}(\bar{x}, \bar{y})$ takes the form

$$J(\bar{E}) = \begin{pmatrix} 1 - 2\bar{x} - \frac{(b\bar{x}^2+2\bar{x})\bar{y}}{(a\bar{x}^2+b\bar{x}+1)^2} & \frac{-\bar{x}^2}{a\bar{x}^2+b\bar{x}+1} \\ \frac{\beta\bar{y}^2}{\bar{x}^2} & -\delta \end{pmatrix},$$

and

$$\operatorname{Det}\big(J(\bar{E})\big) = -\delta\left(1 - 2\bar{x} - \frac{\delta(b\bar{x}^3 + 2\bar{x}^2)}{\beta(a\bar{x}^2 + b\bar{x} + 1)^2}\right) + \frac{\delta^2\bar{x}^2}{\beta(a\bar{x}^2 + b\bar{x} + 1)},$$

$$\operatorname{Tr}\big(J(\bar{E})\big) = 1 - 2\bar{x} - \frac{\delta(b\bar{x}^3 + 2\bar{x}^2)}{\beta(a\bar{x}^2 + b\bar{x} + 1)^2} - \delta.$$

It implies that $\bar{E}(\bar{x}, \bar{y})$ is an elementary equilibrium if $\mathrm{Det}(J(\bar{E})) \neq 0$, a hyperbolic saddle if $\mathrm{Det}(J(\bar{E})) < 0$, or a degenerate equilibrium if $\mathrm{Det}(J(\bar{E})) = 0$, respectively.

Regarding the number of positive equilibria of system (1.6), similar to Lemma 2.1 in Li and Xiao [27], we have the following results.

Lemma 2.1. *Let* $A = (\frac{\delta}{\beta} + b - a)^2 + 3a(b-1)$ *and* $\Delta = -4A^3 + (27a^2 + 9a(1-b)(\frac{\delta}{\beta} + b - a) - 2(\frac{\delta}{\beta} + b - a)^3)^2$.

(a) *If* $\Delta > 0$, *then system* (1.6) *has a unique positive equilibrium* $E^* = (x^*, y^*)$, *which is an elementary and anti-saddle equilibrium;*

(b) *If* $\Delta = 0$ *and*

  (b1) $A > 0$, *then system* (1.6) *has two different positive equilibria: an elementary anti-saddle equilibrium* $E_2^*(x_2^*, y_2^*) = (\frac{\beta\delta}{\beta\delta+(1-\delta)^2}, \frac{\delta^2}{\beta\delta+(1-\delta)^2})$ *and a degenerate equilibrium* $E^*(x^*, y^*) = (1-\delta, \frac{\delta}{\beta}(1-\delta))$;

  (b2) $A = 0$, *then system* (1.6) *has a unique positive equilibrium* $E^*(x^*, y^*) = (\frac{3}{1-b}, \frac{\delta}{\beta}\frac{3}{1-b})$, *which is a degenerate equilibrium;*

(c) *If* $\Delta < 0$, $\frac{\delta}{\beta} \leq a - b - \sqrt{3a(1-b)}$ *and* $-2\sqrt{a} < b < 1$, *then system* (1.6) *has three different positive equilibria* $E_1^*(x_1^*, y_1^*)$, $E_2^*(x_2^*, y_2^*)$, *and* $E_3^*(x_3^*, y_3^*)$, *which are all elementary equilibria and* E_3^* *is a saddle.*

In case (a) of Lemma 2.1, the stability of the unique positive equilibrium E^* can be determined easily. In the following we consider the other three cases (b1), (b2), and (c) of Lemma 2.1.

2.1. Two non-hyperbolic positive equilibria

We first discuss case (b1) of Lemma 2.1 and look for some parameter values such that system (1.6) has a nonhyperbolic equilibrium $E_2^*(x_2^*, y_2^*)$ with $\mathrm{Det}(J(E_2^*)) > 0$ and $\mathrm{Tr}(J(E_2^*)) = 0$ and a degenerate equilibrium $E^*(x^*, y^*)$ with $\mathrm{Det}(J(E^*)) = 0$ and $\mathrm{Tr}(J(E^*)) = 0$.

We can verify that if $\mathrm{Det}(J(E^*)) = 0$ and $\mathrm{Tr}(J(E^*)) = 0$, then

$$x^* = 1 - \delta, \qquad y^* = \frac{\delta}{\beta}(1-\delta), \qquad a = \frac{(\delta-1)^2 + \beta\delta}{\beta\delta(1-\delta)^2}, \qquad b = \frac{(\delta-1)^3 - 2\beta\delta}{\beta\delta(1-\delta)}, \tag{2.2}$$

where $0 < \delta < 1$. If $a = \frac{(\delta-1)^2+\beta\delta}{\beta\delta(1-\delta)^2}$, $b = \frac{(\delta-1)^3-2\beta\delta}{\beta\delta(1-\delta)}$ and $0 < \delta < 1$, then system (1.6) reduces to

$$\begin{aligned} \dot{x} &= x(1-x) - \frac{x^2 y}{\frac{(\delta-1)^2+\beta\delta}{\beta\delta(1-\delta)^2}x^2 + \frac{(\delta-1)^3-2\beta\delta}{\beta\delta(1-\delta)}x + 1}, \\ \dot{y} &= y\left(\delta - \frac{\beta y}{x}\right). \end{aligned} \tag{2.3}$$

Lemma 2.1 (b1) then implies that system (2.3) has two positive equilibria

$$E_2^*(x_2^*, y_2^*) = \left(\frac{\beta\delta}{\beta\delta + (1-\delta)^2}, \frac{\delta^2}{\beta\delta + (1-\delta)^2}\right), \qquad E^*(x^*, y^*) = \left(1-\delta, \frac{\delta}{\beta}(1-\delta)\right).$$

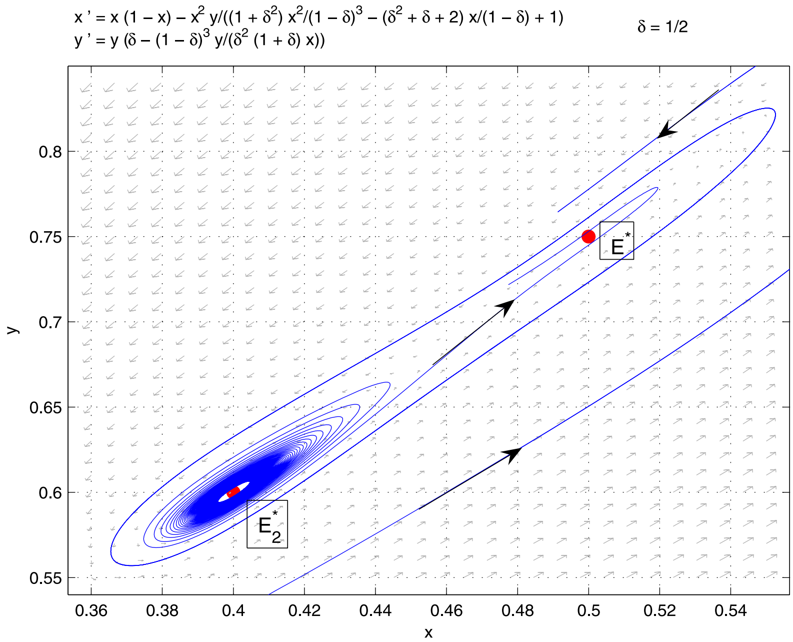

Fig. 2.1. The coexistence of an unstable multiple focus with multiplicity one (E_2^*) and a cusp of codimension 2 (E^*).

Theorem 2.2. *If* $(a, b, \beta) = (\frac{1+\delta^2}{(1-\delta)^3}, -\frac{\delta^2+\delta+2}{1-\delta}, \frac{(1-\delta)^3}{\delta^2(1+\delta)})$ *and* $0 < \delta < 1$, *then system (1.6) has two positive equilibria:*

(i) $E_2^*(\frac{1-\delta}{1+\delta^2}, \frac{\delta^3(1+\delta)}{(1-\delta)^2(1+\delta^2)})$ *is an unstable multiple focus with multiplicity one;*
(ii) $E^*(1-\delta, \frac{\delta^3(1+\delta)}{(1-\delta)^2})$ *is a cusp of codimension* 2.

The phase portrait is given in Fig. 2.1.

Proof. Under the assumptions of Theorem 2.2 system (1.6) becomes

$$\dot{x} = x(1-x) - \frac{x^2 y}{\frac{1+\delta^2}{(1-\delta)^3}x^2 - \frac{\delta^2+\delta+2}{1-\delta}x + 1},$$
$$\dot{y} = y\left(\delta - \frac{(1-\delta)^3}{\delta^2(1+\delta)}\frac{y}{x}\right), \tag{2.4}$$

which has two positive equilibria: $E_2^*(\frac{1-\delta}{1+\delta^2}, \frac{\delta^3(1+\delta)}{(1-\delta)^2(1+\delta^2)})$ and $E^*(1-\delta, \frac{\delta^3(1+\delta)}{(1-\delta)^2})$.

(i) We first verify that $E_2^*(\frac{1-\delta}{1+\delta^2}, \frac{\delta^3(1+\delta)}{(1-\delta)^2(1+\delta^2)})$ is an unstable multiple focus with multiplicity one. Translate E_2^* to the origin by letting $u = x - \frac{1-\delta}{1+\delta^2}$ and $v = y - \frac{\delta^3(1+\delta)}{(1-\delta)^2(1+\delta^2)}$, the Taylor expansion of system (2.4) around the origin takes the form

$$\begin{aligned}
\dot{u} &= \delta u - \frac{(1-\delta)^3}{\delta^2(1+\delta^2)}v + \big(1+\delta+\delta^3\big)u^2 + \frac{(1-\delta)^4}{\delta^3}uv \\
&\quad + \frac{(1+\delta)(1+\delta^2)(1+\delta-\delta^2+3\delta^3-\delta^4+\delta^5)}{(-1+\delta)\delta^2}u^3 \\
&\quad + \frac{(1-\delta)^2(2-\delta+3\delta^2-\delta^3+\delta^4)}{\delta^3}u^2v + O\big(|u,v|^4\big), \\
\dot{v} &= \frac{\delta^4(1+\delta)}{(1-\delta)^3}u - \delta v - \frac{\delta^4(1+\delta)(1+\delta^2)}{(1-\delta)^4}u^2 + \frac{2\delta(1+\delta^2)}{1-\delta}uv - \frac{(1-\delta)^2(1+\delta^2)}{\delta^2(1+\delta)}v^2 \\
&\quad + \frac{\delta^4(1+\delta)(1+\delta^2)^2}{(1-\delta)^5}u^3 - \frac{2\delta(1+\delta^2)^2}{(1-\delta)^2}u^2v + \frac{(1-\delta)(1+\delta^2)^2}{\delta^2(1+\delta)}uv^2 + O\big(|u,v|^4\big). \qquad (2.5)
\end{aligned}$$

Make a change of variables as follows:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{(1-\delta)^7}{\delta^5(1+\delta)^2(1+\delta^2)}} & \frac{(1-\delta)^3}{\delta^3(1+\delta)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

then system (2.5) can be written as

$$\begin{aligned}
\dot{x} &= -\sqrt{\frac{\delta^3(1-\delta)}{1+\delta^2}}y + f(x,y), \\
\dot{y} &= \sqrt{\frac{\delta^3(1-\delta)}{1+\delta^2}}x + g(x,y), \qquad (2.6)
\end{aligned}$$

where

$$\begin{aligned}
f(x,y) &= \frac{\sqrt{(1-\delta)^7}\sqrt{\delta^5(1+\delta)^2(1+\delta^2)}(-1-\delta+\delta^2-2\delta^3+\delta^4)}{\delta^5(-1-\delta+\delta^4+\delta^5)}x^2 \\
&\quad + \frac{(1-\delta)^3(3+2\delta-\delta^2+2\delta^3)}{\delta^3(1+\delta)}xy \\
&\quad + \frac{\sqrt{\delta^5(1+\delta)^2(1+\delta^2)}(1-\delta)^6(2+\delta-\delta^2+\delta^3)}{\sqrt{(1-\delta)^7}\delta^6(1+\delta)^2}y^2 \\
&\quad + \frac{(1-\delta)^5(-1-\delta+2\delta^2-3\delta^3+\delta^5-\delta^6+\delta^7)}{\delta^7(1+\delta)^2}x^3 \\
&\quad + \frac{\sqrt{\delta^5(1+\delta)^2(1+\delta^2)}(1-\delta)(-3-3\delta+8\delta^2-8\delta^3-\delta^4+6\delta^5-4\delta^6+3\delta 7)}{\delta^{10}(1+\delta)^3}x^2y \\
&\quad - \frac{(1-\delta)^5(3+3\delta-4\delta^2+14\delta^3-12\delta^4+14\delta^5-5\delta^6+3\delta^7)}{\delta^8(1+\delta)}xy^2 \\
&\quad - \frac{\sqrt{(1-\delta)^7}(1-\delta)(1+\delta^2)^2(1+\delta-3\delta^2+4\delta^3-2\delta^4+\delta^5)}{\delta^6\sqrt{\delta^5(1+\delta)^2(1+\delta^2)}}y^3 + O\big(|x,y|^4\big),
\end{aligned}$$

$$g(x,y) = -\frac{(1-\delta)^3}{\delta(1+\delta)}x^2 + \frac{(-1+\delta)^2\sqrt{(1-\delta)^7}\sqrt{\delta^5(1+\delta)^2(1+\delta^2)}}{\delta^6(1+\delta)^3}x^3 + \frac{(1-\delta)^5(1+\delta^2)}{\delta^4(1+\delta)^2}x^2y + O\big(|x,y|^4\big).$$

The Liapunov number (Perko [31, p. 219]) can be expressed as

$$\begin{aligned}\operatorname{Re} c_1 &= \frac{1}{16}\bigg\{(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}) \\ &\quad + \frac{1}{\omega}\big[f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}\big]\bigg\}\bigg|_{x=y=0} \\ &= \frac{(-1+\delta)^5(-3+2\delta-8\delta^2+\delta^3-\delta^5+\delta^6)}{8\delta^8(1+\delta)} > 0\end{aligned}$$

because $0<\delta<1$. Therefore, E_2^* is an unstable multiple focus with multiplicity one.

(ii) Next we show that the degenerate equilibrium E^* is a cusp of codimension 2. Using the change of variables $x_1 = x-(1-\delta)$, $x_2 = y - \frac{\delta^3(1+\delta)}{(1-\delta)^2}$, system (2.4) is transformed into

$$\begin{aligned}\dot{x}_1 &= \delta x_1 + \frac{(-1+\delta)^3}{\delta^2(1+\delta)}x_2 - \frac{2+\delta}{1+\delta}x_1^2 + \frac{(1-\delta)^3}{\delta^3(1+\delta)}x_1x_2 + O\big(|x_1,x_2|^3\big), \\ \dot{x}_2 &= \frac{\delta^4(1+\delta)}{(1-\delta)^3}x_1 - \delta x_2 - \frac{\delta^4(1+\delta)}{(1-\delta)^4}x_1^2 + \frac{2\delta}{1-\delta}x_1x_2 - \frac{(1-\delta)^2}{\delta^2(1+\delta)}x_2^2 + O\big(|x_1,x_2|^3\big).\end{aligned} \tag{2.7}$$

Let $y_1 = x_1$, $y_2 = \delta x_1 + \frac{(-1+\delta)^3}{\delta^2(1+\delta)}x_2$. Then system (2.7) can be rewritten as

$$\begin{aligned}\dot{y}_1 &= y_2 - \frac{1}{1+\delta}y_1^2 - \frac{1}{\delta}y_1y_2 + O\big(|y_1,y_2|^3\big), \\ \dot{y}_2 &= -\frac{\delta}{1+\delta}y_1^2 - y_1y_2 + \frac{1}{1-\delta}y_2^2 + O\big(|y_1,y_2|^3\big).\end{aligned} \tag{2.8}$$

Making a C^∞-change of variables $z_1 = y_1 - \frac{2\delta-1}{2\delta(1-\delta)}y_1^2$, $z_2 = y_2 - \frac{1}{1+\delta}y_1^2 - \frac{1}{1-\delta}y_1y_2$ in a small neighborhood of $(0,0)$, system (2.8) becomes

$$\begin{aligned}\dot{z}_1 &= z_2 + O\big(|z_1,z_2|^3\big), \\ \dot{z}_2 &= \mu_1 z_1^2 + \mu_2 z_1 z_2 + O\big(|z_1,z_2|^3\big),\end{aligned} \tag{2.9}$$

where $\mu_1 = \frac{-\delta}{1+\delta}$, $\mu_2 = -\frac{\delta+3}{1+\delta}$. Notice that $\mu_1\mu_2 = \frac{\delta(\delta+3)}{(1+\delta)^2} \neq 0$, hence the equilibrium E^* is a cusp of codimension 2. $\square$

2.2. The degenerate positive equilibrium

Next we consider case (b2) of Lemma 2.1, that is the unique degenerate positive equilibrium E^*, and have the following result.

Theorem 2.3.

(i) *If* $(a,\delta)=(\frac{(1-b)^3}{27},-\frac{\beta(2+b)^3}{27})$ *and* $\beta\neq\frac{27}{(1-b)(2+b)^2}$, $-2\sqrt{a}<b<-2$, $a>1$, *then system (1.6) has a unique degenerate positive equilibrium* $E^*(\frac{3}{1-b},\frac{\delta}{\beta}\frac{3}{1-b})$, *which is a stable (an unstable) degenerate node if* $\beta>\frac{27}{(1-b)(2+b)^2}$ *(if* $\beta<\frac{27}{(1-b)(2+b)^2}$, *respectively);*

(ii) *If* $(a,\delta,\beta)=(\frac{(1-b)^3}{27},\frac{2+b}{b-1},\frac{27}{(1-b)(2+b)^2})$, *then the unique degenerate positive equilibrium* $E^*(\frac{3}{1-b},-\frac{(2+b)^3}{9(1-b)})$ *of system (1.6) is a codimension* 3 *Bogdanov–Takens singularity (focus or center case).*

The phase portraits are given in Fig. 2.2.

Proof. We have $\Delta=0$ and $A=0$ when $a=\frac{(1-b)^3}{27}$ and $\delta=-\frac{\beta(2+b)^3}{27}$. Moreover,

$$\mathrm{Det}\big(J\big(E^*\big)\big)=0,\qquad \mathrm{Tr}\big(J\big(E^*\big)\big)=\frac{(2+b)(27+(b-1)(2+b)^2\beta)}{27(b-1)}.$$

(i) When $\beta\neq\frac{27}{(1-b)(2+b)^2}$, we have $\mathrm{Tr}(J(E^*))\neq 0$; that is, there is only one zero eigenvalue for the Jacobian matrix $J(E^*)$, Theorem 7.1 in Zhang et al. [44] implies that the unique degenerate positive equilibrium $E^*(\frac{3}{1-b},\frac{\delta}{\beta}\frac{3}{1-b})$ of system (1.6) is a stable (an unstable) degenerate node if $\beta>\frac{27}{(1-b)(2+b)^2}$ (if $\beta<\frac{27}{(1-b)(2+b)^2}$, respectively).

(ii) When $\beta=\frac{27}{(1-b)(2+b)^2}$, we have $\mathrm{Tr}(J(E^*))=0$. It follows that $E^*(\frac{3}{1-b},-\frac{(2+b)^3}{9(1-b)})$ is nilpotent (with double-zero eigenvalue). To determine the exact type of this equilibrium, we provide a series of explicitly smooth transformations to derive a normal form with terms up to the fourth order.

Firstly, we translate the unique positive equilibrium $E^*(\frac{3}{1-b},-\frac{(2+b)^3}{9(1-b)})$ to the origin and expand system (1.6) in power series up to the fourth order around the origin. Let

$$(I):\quad X=x-\frac{3}{1-b},\qquad Y=y-\left(-\frac{(2+b)^3}{9(1-b)}\right).$$

Then system (1.6) can be rewritten as

$$\begin{aligned}\dot{X}&=F\left(X+\frac{3}{1-b},Y-\frac{(2+b)^3}{9(1-b)}\right),\\ \dot{Y}&=G\left(X+\frac{3}{1-b},Y-\frac{(2+b)^3}{9(1-b)}\right).\end{aligned}\tag{2.10}$$

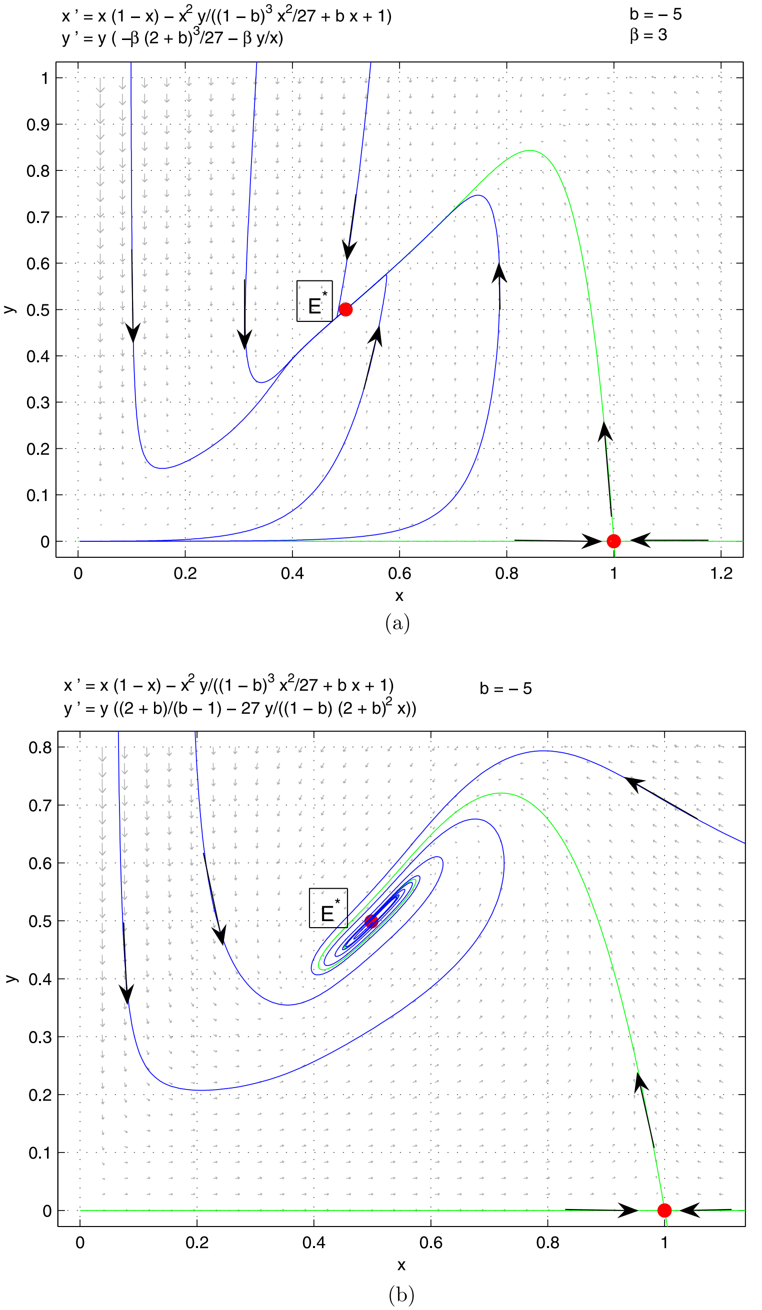

Fig. 2.2. The unique degenerate positive equilibrium E^*. (a) Stable degenerate node; (b) codimension 3 Bogdanov–Takens singularity (focus case).

Secondly, we transform the linear part of system (2.10) to the Jordan canonical form. To do so, let

$$(II):\quad X=-\frac{27}{(2+b)^3}x-\frac{27(b-1)}{(2+b)^4}y,\qquad Y=x.$$

Then system (2.10) becomes

$$\begin{aligned}\dot{x}&=y+a_{02}y^2+a_{12}xy^2+a_{03}y^3+a_{13}xy^3+a_{22}x^2y^2+a_{04}y^4+R_1(x,y),\\ \dot{y}&=b_{11}xy+b_{02}y^2+b_{30}x^3+b_{12}xy^2+b_{21}x^2y+b_{03}y^3+b_{40}x^4\\ &\quad+b_{13}xy^3+b_{22}x^2y^2+b_{31}x^3y+b_{04}y^4+R_2(x,y),\end{aligned}\tag{2.11}$$

where $b_{11}a_{02}b_{02}\neq 0$, a_{ij}, b_{ij}, $i,j=0,\dots,4$, $2\le i+j\le 4$, are functions of parameter b, and $R_1(x,y)$ and $R_2(x,y)$ are smooth functions of order at least five in (x,y). Thirdly, to eliminate the y^2 terms in system (2.11), we make the following near identity transformation

$$(III):\quad x=X+\frac{b_{02}}{2}X^2,\qquad y=Y+b_{02}XY-a_{02}Y^2,$$

which transforms system (2.11) into

$$\begin{aligned}\dot{X}&=Y+c_{30}X^3+c_{12}XY^2+c_{21}X^2Y+c_{03}Y^3+c_{40}X^4+c_{13}XY^3\\ &\quad+c_{31}X^3Y+c_{22}X^2Y^2+c_{04}Y^4+Q_1(X,Y),\\ \dot{Y}&=d_{11}XY+d_{30}X^3+d_{12}XY^2+d_{21}X^2Y+d_{03}Y^3+d_{40}X^4+d_{13}XY^3\\ &\quad+d_{31}X^3Y+d_{22}X^2Y^2+d_{04}Y^4+Q_2(X,Y),\end{aligned}\tag{2.12}$$

where $Q_1(X,Y)$ and $Q_2(X,Y)$ are smooth functions of order at least five in (X,Y), and

$$c_{30}=0,\qquad c_{12}=-\frac{81(8+b)(b-1)^3}{(2+b)^8},\qquad c_{21}=0,\qquad c_{03}=-\frac{81(b-1)^4}{(2+b)^8},\qquad c_{40}=0,$$

$$c_{13}=\frac{1458(b-1)^5(5+b)}{(2+b)^{12}},\qquad c_{31}=0,\qquad c_{22}=\frac{2187(b-1)^4(24+11b+b^2)}{2(2+b)^{12}},$$

$$c_{04}=\frac{729(b-1)^6}{(2+b)^{12}},\qquad d_{11}=\frac{27}{(2+b)^3},\qquad d_{30}=\frac{243(b-1)^2}{(2+b)^7},$$

$$d_{12}=\frac{81(b-1)^2(89+48b+24b^2+b^3)}{(2+b)^9},\qquad d_{21}=\frac{243(-16+21b-12b^2+7b^3)}{2(2+b)^8},$$

$$d_{03}=\frac{81(b-1)^3(35+30b+15b^2+b^3)}{(2+b)^{10}},\qquad d_{40}=-\frac{2187(b-7)(b-1)^3}{2(2+b)^{11}},$$

$$d_{13}=-\frac{2187(b-1)^4(72+137b+42b^2+18b^3+b^4)}{(2+b)^{14}},\qquad d_{31}=-\frac{2187(4b-13)(b-1)^4}{(2+b)^{12}},$$

$$d_{22}=\frac{2187(1-b)^3(143+287b+24b^2+31b^3+b^4)}{2(2+b)^{13}},$$
$$d_{04}=\frac{729(1-b)^5(77+145b+75b^2+25b^3+2b^4)}{(2+b)^{15}}.$$

Notice that

$$d_{11}d_{30}=\frac{6561(b-1)^2}{(2+b)^{10}}\neq 0,$$

by Lemma 3.1 in Cai, Chen and Xiao [7], there exists a small neighborhood U of $(0,0)$ such that in this neighborhood U system (2.12) is locally topologically equivalent to

$$\begin{aligned}\dot{x}&=y,\\ \dot{y}&=d_{11}xy+d_{30}x^3+(d_{21}+3c_{30})x^2y+(d_{40}-d_{11}c_{30})x^4\\ &\quad+\left(4c_{40}+d_{31}+\frac{1}{3}d_{11}c_{21}+\frac{1}{6}d_{11}d_{12}\right)x^3y+Q(x,y),\end{aligned}\tag{2.13}$$

where $Q(x,y)$ is a smooth function of order at least five in (x,y). Moreover, we have

$$5d_{30}(d_{21}+3c_{30})-3d_{11}(d_{40}-d_{11}c_{30})=\frac{59\,049(b-1)^3(19-20b+19b^2)}{(2+b)^{15}}\neq 0$$

and

$$d_{30}=\frac{243(b-1)^2}{(2+b)^7}<0,\qquad d_{11}^2+8d_{30}=\frac{243(14-13b+8b^2)}{(2+b)^7}<0$$

since $-2\sqrt{a}<b<-2$. Again by Lemma 3.1 in Cai, Chen and Xiao [7], we know that the equilibrium $(0,0)$ of system (2.12) is a degenerate focus or center of codimension 3, that is, the unique degenerate positive equilibrium $E^*(\frac{3}{1-b},-\frac{(2+b)^3}{9(1-b)})$ of system (1.6) is a codimension 3 Bogdanov–Takens singularity (focus or center case). □

3. Bifurcations

In this section, we first discuss the existence of a subcritical Hopf bifurcation and a Bogdanov–Takens bifurcation in case (b1) of Lemma 2.1 and then consider the degenerate focus type Bogdanov–Takens bifurcation of codimension 3 in case (b2) of Lemma 2.1.

3.1. Hopf bifurcation and Bogdanov–Takens bifurcation

Consider case (b1) of Lemma 2.1 when system (1.6) has a nonhyperbolic equilibrium $E_2^*(x_2^*,y_2^*)$ and a degenerate equilibrium $E^*(x^*,y^*)$ with $\mathrm{Det}(J(E^*))=0$ and $\mathrm{Tr}(J(E^*))=0$. We study the existence of a subcritical Hopf bifurcation in a small neighborhood of E_2^* and a Bogdanov–Takens bifurcation in a small neighborhood of E^* in system (1.6). Choosing δ and β as bifurcation parameters, we carry out a bifurcation analysis of system (1.6) as (δ,β) varies near

(δ_0, β_0), where δ_0 and β_0 satisfy $(a, b, \beta) = (\frac{1+\delta^2}{(1-\delta)^3}, -\frac{\delta^2+\delta+2}{1-\delta}, \frac{(1-\delta)^3}{\delta^2(1+\delta)})$. That is, we consider the following unfolding system of system (1.6):

$$\begin{aligned}\dot{x} &= x(1-x) - \frac{x^2 y}{\frac{1+\delta^2}{(1-\delta)^3}x^2 - \frac{\delta^2+\delta+2}{1-\delta}x + 1},\\ \dot{y} &= y\left(\delta + \lambda_1 - \left(\frac{(1-\delta)^3}{\delta^2(1+\delta)} + \lambda_2\right)\frac{y}{x}\right),\end{aligned} \tag{3.1}$$

where $\lambda = (\lambda_1, \lambda_2)$ are small parameters and $0 < \delta < 1$. Actually, we have the following results.

Theorem 3.1. *When parameters (λ_1, λ_2) vary in a small neighborhood of the origin, system (3.1) undergoes a subcritical Hopf bifurcation in a small neighborhood of E_2^* and a Bogdanov–Takens bifurcation in a small neighborhood of E^*. Hence, there exist some parameter values such that system (1.6) has an unstable limit cycle around E_2^*, and there exist some other parameter values such that system (1.6) has an unstable limit cycle or an unstable homoclinic loop around E^*.*

Proof. By Theorem 2.2, if $\lambda_1 = \lambda_2 = 0$ then system (3.1) has two equilibria: $E_2^*(x_2^*, y_2^*) = (\frac{1-\delta}{1+\delta^2}, \frac{\delta^3(1+\delta)}{(1-\delta)^2(1+\delta^2)})$ which is an unstable multiple focus with multiplicity one and $E^*(x^*, y^*) = (1-\delta, \frac{\delta^3(1+\delta)}{(1-\delta)^2})$ which is a cusp of codimension 2.

(i) First we study the existence of a subcritical Hopf bifurcation of system (3.1) in a small neighborhood of the equilibrium E_2^* when parameters (λ_1, λ_2) vary in a small neighborhood of the origin. When $(\lambda_1, \lambda_2) \neq (0, 0)$, system (3.1) has an equilibrium $E_2(x_2, y_2)$ with $x_2 = \frac{1-\delta}{1+\delta^2} + w$, $|w| \ll 1$, $y_2 = \frac{\delta+\lambda_1}{\frac{(1-\delta)^3}{\delta^2(1+\delta)}+\lambda_2}x_2$. We have

$$\begin{aligned}\operatorname{Tr}\big(J(E_2)\big) &= 1 - 2x_2 - \delta + \frac{\delta^3(-1+\delta)^2(1+\delta)[2(-1+\delta) + (2+\delta+\delta^2)x_2]x_2^2}{[(-1+\delta)^3 - (1+\delta^2)x_2^2 + (1-\delta)^2(2+\delta+\delta^2)x_2]^2}\\ &\quad - \left(1 - \frac{\delta^2(1-\delta)^2(1+\delta)[2(-1+\delta) + (2+\delta+\delta^2)x_2]x_2^2}{[(-1+\delta)^3 - (1+\delta^2)x_2^2 + (1-\delta)^2(2+\delta+\delta^2)x_2]^2}\right)\lambda_1\\ &\quad + \frac{\delta^5(1+\delta)^2[2(-1+\delta) + (2+\delta+\delta^2)x_2]x_2^2}{(-1+\delta)[(-1+\delta)^3 - (1+\delta^2)x_2^2 + (1-\delta)^2(2+\delta+\delta^2)x_2]^2}\lambda_2\\ &\quad + O\big(|\lambda_1, \lambda_2|^2\big),\\ \operatorname{Det}\big(J(E_2)\big) &= \frac{\delta^2(1+\delta)(1-\delta)^3(\delta+\lambda_1)^2 x_2^2}{[(1-\delta)^3 - (1-\delta)^2(2+\delta+\delta^2)x_2 + (1+\delta^2)x_2^2][(1-\delta)^3 + \delta^2(1+\delta)\lambda_2]}\\ &\quad - (\delta+\lambda_1)\Bigg(1 - 2x_2\\ &\quad - \frac{\delta^2(1+\delta)(-1+\delta)^5[2(-1+\delta) + (2+\delta+\delta^2)x_2](\delta+\lambda_1)x_2^2}{[(1-\delta)^3 - (1-\delta)^2(2+\delta+\delta^2)x_2 + (1+\delta^2)x_2^2]^2[(1-\delta)^3 + \delta^2(1+\delta)\lambda_2]}\Bigg).\end{aligned}$$

Let

$$\mathrm{Tr}\big(J(E_2)\big)=0, \qquad \mathrm{Det}\big(J(E_2)\big)>0. \tag{3.2}$$

It follows that

$$\begin{aligned}
\lambda_1 &= -\frac{w(2w^2(1+\delta^2)^2-(-1+\delta)^2\delta^2(3+\delta+\delta^2+\delta^3)+w(3-2\delta+\delta^2-3\delta^4+2\delta^5-\delta^6))}{(-1+\delta)^2\delta^2+w^2(1+\delta^2)^2-w(-1+\delta)\delta(1+\delta^2)^2},\\
\lambda_2 &= -\big(w(-1+\delta)^3\big(-w^4\big(1+\delta^2\big)^4\big(-1+\delta^2+2\delta^3\big)+(-1+\delta)^4\delta^4\big(3+4\delta+3\delta^2+2\delta^3+2\delta^4\big)\\
&\quad+w^2\delta^2\big(-1+\delta-\delta^2+\delta^3\big)^2\big(-7-12\delta+5\delta^2+\delta^4+\delta^6\big)+w^3\delta\big(1+\delta^2\big)^3\big(1-10\delta+3\delta^2\\
&\quad+4\delta^3+3\delta^4-2\delta^5+\delta^6\big)-w(-1+\delta)^3\delta^2\big(-3-4\delta+\delta^2+5\delta^3+5\delta^4+15\delta^5+3\delta^6\\
&\quad+7\delta^7+2\delta^8+\delta^9\big)\big)\big)/\big(\delta^5(1+\delta)^2\big(1+w-\delta+w\delta^2\big)^2\big((-1+\delta)^2\delta^2+w^2\big(1+\delta^2\big)^2\\
&\quad-w(-1+\delta)\delta\big(1+\delta^2\big)^2\big)\big),
\end{aligned} \tag{3.3}$$

and $w_1<w<w_2$, where

$$\begin{aligned}
w_1 &= \frac{B_1-\sqrt{\Delta}}{2A_1}, \qquad w_2=\frac{B_1+\sqrt{\Delta}}{2A_1}, \qquad B_1=4\delta^2-2\delta^3+3\delta^4+4\delta^5-6\delta^6-\delta^8-2\delta^9,\\
A_1 &= 3-8\delta-4\delta^2+19\delta^3+2\delta^4+35\delta^5+27\delta^6+18\delta^7+17\delta^8+15\delta^9-\delta^{10}+5\delta^{11},\\
\Delta &= 4\delta^4+28\delta^5+24\delta^6-116\delta^7+29\delta^8+8\delta^9-64\delta^{10}+108\delta^{11}+10\delta^{12}-48\delta^{13}\\
&\quad+56\delta^{14}-8\delta^{15}-43\delta^{16}+28\delta^{17}-16\delta^{18}.
\end{aligned}$$

Thus the Hopf bifurcation curve of system (3.1) at E_2^* is defined by

$$H_2=\big\{(\lambda_1,\lambda_2):(\lambda_1,\lambda_2)\ \text{satisfy (3.3) and } w_1<w<w_2\big\}.$$

Notice that

$$\lim_{w\to 0}\frac{\lambda_2}{\lambda_1}=\frac{(1-\delta)^3(3+4\delta+3\delta^2+2\delta^3+2\delta^4)}{\delta^3(1+\delta)^2(3+\delta+\delta^2+\delta^3)},$$

we can see that the approximate representation of H_2 is a straight line

$$\lambda_2=\frac{(1-\delta)^3(3+4\delta+3\delta^2+2\delta^3+2\delta^4)}{\delta^3(1+\delta)^2(3+\delta+\delta^2+\delta^3)}\lambda_1$$

in a small neighborhood of the origin in the parameter plane (see Fig. 3.1).

(ii) Next we consider the Bogdanov–Takens bifurcation of system (3.1) in a small neighborhood of the equilibrium E^*. We follow the techniques and steps of Xiao and Ruan [41] and Li and Xiao [27] to derive a normal form by using a series of transformations. When (λ_1,λ_2) vary in a small neighborhood of the origin, let $x_1=x-x^*$, $x_2=y-y^*$, where $x^*=1-\delta$, $y^*=\frac{\delta^3(1+\delta)}{(1-\delta)^2}$. Then system (3.1) is transformed into

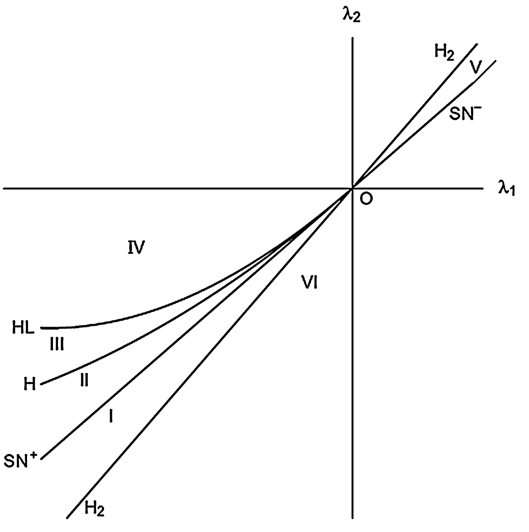

Fig. 3.1. The bifurcation diagram of system (3.1) with $\delta = 0.47$.

$$\begin{aligned}\dot{x}_1 &= \alpha_{10}x_1 + \alpha_{01}x_2 + \alpha_{11}x_1^2 + 2\alpha_{12}x_1x_2 + A_1(x_1,x_2),\\ \dot{x}_2 &= \beta_{00} + \beta_{10}x_1 + \beta_{01}x_2 + \beta_{11}x_1^2 + 2\beta_{12}x_1x_2 + \beta_{22}x_2^2 + A_2(x_1,x_2),\end{aligned} \tag{3.4}$$

where

$$\alpha_{10} = \delta, \qquad \alpha_{01} = \frac{(-1+\delta)^3}{\delta^2(1+\delta)}, \qquad \alpha_{11} = -\frac{2+\delta}{1+\delta}, \qquad \alpha_{12} = \frac{(1-\delta)^3}{2\delta^3(1+\delta)},$$

$$\beta_{00} = \frac{\delta^3(1+\delta)}{(-1+\delta)^5}\left[(-1+\delta)^3\lambda_1 + \delta^3(1+\delta)\lambda_2\right], \qquad \beta_{10} = \frac{\delta^4(1+\delta)}{(1-\delta)^3} + \frac{\delta^6(1+\delta)^2}{(1-\delta)^6}\lambda_2,$$

$$\beta_{01} = -\delta + \lambda_1 - \frac{2\delta^3(1+\delta)}{(1-\delta)^3}\lambda_2, \qquad \beta_{11} = -\frac{\delta^4(1+\delta)}{(1-\delta)^4} - \frac{\delta^6(1+\delta)^2}{(1-\delta)^7}\lambda_2,$$

$$\beta_{12} = \frac{\delta}{1-\delta} + \frac{\delta^3(1+\delta)}{(1-\delta)^4}\lambda_2, \qquad \beta_{22} = \frac{(1-\delta)^3}{\delta^2(-1+\delta^2)} - \frac{1}{1-\delta}\lambda_2,$$

and A_i $(i = 1, 2)$ are C^∞ in (x_1, x_2) and $A_i(x_1, x_2) = O(|x_1, x_2|^3)$.

First, we make an affine transformation $y_1 = x_1$, $y_2 = \alpha_{10}x_1 + \alpha_{01}x_2$. System (3.4) can be rewritten as

$$\begin{aligned}\dot{y}_1 &= y_2 + \left(\alpha_{11} - \frac{2\alpha_{10}\alpha_{12}}{\alpha_{01}}\right)y_1^2 + \frac{2\alpha_{12}}{\alpha_{01}}y_1y_2 + B_1(y_1,y_2),\\ \dot{y}_2 &= \alpha_{01}\beta_{00} + (\alpha_{01}\beta_{10} - \alpha_{10}\beta_{01})y_1 + (\alpha_{10} + \beta_{01})y_2 + \left(\frac{2\alpha_{10}\alpha_{12}}{\alpha_{01}} + 2\beta_{12} - \frac{2\alpha_{10}\beta_{22}}{\alpha_{01}}\right)y_1y_2\\ &\quad + \left(\alpha_{10}\alpha_{11} - \frac{2\alpha_{10}^2\alpha_{12}}{\alpha_{01}} + \alpha_{01}\beta_{11} - 2\alpha_{10}\beta_{12} + \frac{\alpha_{10}^2\beta_{22}}{\alpha_{01}}\right)y_1^2 + \frac{\beta_{22}}{\alpha_{01}}y_2^2 + B_2(y_1,y_2),\end{aligned} \tag{3.5}$$

where B_i $(i = 1, 2)$ is C^∞ in (y_1, y_2) and $B_i(y_1, y_2) = O(|y_1, y_2|^3)$.

Next, we make a C^∞ change of coordinates in a small neighborhood of $(0, 0)$ as

$$z_1 = y_1 - \left(\frac{\alpha_{12}}{\alpha_{01}} + \frac{\beta_{22}}{2\alpha_{01}}\right) y_1^2,$$
$$z_2 = y_2 + \left(\alpha_{11} - \frac{2\alpha_{10}\alpha_{12}}{\alpha_{01}}\right) y_1^2 - \frac{\beta_{22}}{\alpha_{01}} y_1 y_2$$

and obtain from system (3.5) that

$$\begin{aligned} \dot z_1 &= z_2 + C_1(z_1, z_2), \\ \dot z_2 &= \alpha_{01}\beta_{00} + (\alpha_{01}\beta_{10} - \alpha_{10}\beta_{01} - \beta_{00}\beta_{22}) z_1 + (\alpha_{10} + \beta_{01}) z_2 + d_1(\lambda) z_1^2 \\ &\quad + \left(2\alpha_{11} + 2\beta_{12} - \frac{2\alpha_{10}\alpha_{12}}{\alpha_{01}} - \frac{2\alpha_{10}\beta_{22}}{\alpha_{01}}\right) z_1 z_2 + C_2(z_1, z_2), \end{aligned} \tag{3.6}$$

where $d_1(\lambda) = \frac{(\alpha_{01}\beta_{10}+\alpha_{10}\beta_{01})\alpha_{12}}{\alpha_{01}} - \beta_{01}\alpha_{11} + \alpha_{01}\beta_{11} - 2\alpha_{10}\beta_{12} + (\frac{\alpha_{10}^2}{\alpha_{01}} - \frac{\beta_{10}\beta_{01}}{2} + \frac{\alpha_{10}\beta_{01}}{2\alpha_{01}} - \frac{\beta_{00}\alpha_{12}}{\alpha_{01}} - \frac{\beta_{00}\beta_{22}}{2\alpha_{01}})\beta_{22}$, and C_i $(i = 1, 2)$ is C^∞ in (z_1, z_2) and $C_i(z_1, z_2) = O(|z_1, z_2|^3)$.

Now, in a small neighborhood of $(0, 0)$, we make another C^∞ change of coordinates

$$X_1 = z_1, \qquad X_2 = z_2 + C_1(z_1, z_2)$$

and transform system (3.6) into

$$\begin{aligned} \dot X_1 &= X_2, \\ \dot X_2 &= \alpha_{01}\beta_{00} + (\alpha_{01}\beta_{10} - \alpha_{10}\beta_{01} - \beta_{00}\beta_{22}) X_1 + (\alpha_{10} + \beta_{01}) X_2 + d_1(\lambda) X_1^2 + F_1(X_1) \\ &\quad + \left(2\alpha_{11} + 2\beta_{12} - \frac{2\alpha_{10}\alpha_{12}}{\alpha_{01}} - \frac{2\alpha_{10}\beta_{22}}{\alpha_{01}}\right) X_1 X_2 + X_2 F_2(X_1) + X_2^2 F_3(X_1, X_2), \end{aligned} \tag{3.7}$$

where F_1, F_2 are C^∞ in X_1, F_3 is C^∞ in (X_1, X_2), $F_1(X_1) = O(|X_1|^3)$, $F_2(X_1) = O(|X_1|^2)$ and $F_3(X_1, X_2) = O(|X_1, X_2|)$. Substituting values of α_{10}, α_{01}, β_{10}, β_{01}, α_{11}, α_{12}, β_{00}, β_{11}, β_{12} and β_{22} into system (3.7), we obtain the following system

$$\begin{aligned} \dot X_1 &= X_2, \\ \dot X_2 &= \Phi(X_1, \lambda) + \left[\lambda_1 + \frac{2\delta^3(1+\delta)}{(-1+\delta)^3}\lambda_2\right] X_2 \\ &\quad - \frac{3+\delta}{1+\delta} X_1 X_2 + X_2 F_2(X_1) + X_2^2 F_3(X_1, X_2), \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} \Phi(X_1, \lambda) &= \frac{\delta[(-1+\delta)^3\lambda_1 + \delta^3(1+\delta)\lambda_2]}{(1-\delta)^2} - \frac{\lambda_2\delta^3(1+\delta)[(-1+\delta)^3\lambda_1 + \delta^3(1+\delta)\lambda_2]}{(1-\delta)^6} X_1 \\ &\quad + d_1(\lambda) X_1^2 + F_1(X_1), \end{aligned}$$

$$d_1(\lambda) = -\frac{\delta}{1+\delta} + \frac{1+\delta^2}{1-\delta^2}\lambda_1 + \frac{\delta^3(-2+\delta-\delta^2)}{(1-\delta)^4}\lambda_2 + \frac{\delta^3(-1+3\delta+4\delta^2)}{2(1-\delta)^4}\lambda_1\lambda_2$$
$$+ \frac{\delta^5(1+\delta)^2(1-4\delta)}{2(1-\delta)^7}\lambda_2^2 + \frac{\delta^5(1+\delta)^2}{2(1-\delta)^7}\lambda_1\lambda_2^2 - \frac{\delta^8(1+\delta)^3}{2(1-\delta)^10}\lambda_2^3.$$

Since $d_1(0) = -\frac{\delta}{1+\delta} < 0$, we make it positive by using a time transform. Let $Y_1 = -X_1$, $Y_2 = X_2$, $\tau = -t$. Then system (3.8) becomes (still denote time by t)

$$\begin{aligned} \dot{Y}_1 &= Y_2, \\ \dot{Y}_2 &= -\Phi(Y_1,\lambda) - \left[\lambda_1 + \frac{2\delta^3(1+\delta)}{(-1+\delta)^3}\lambda_2\right]Y_2 \\ &\quad - \frac{3+\delta}{1+\delta}Y_1Y_2 + Y_2F_2(Y_1) + Y_2^2F_3(Y_1,Y_2). \end{aligned} \tag{3.9}$$

Applying the Malgrange Preparation Theorem (Chow and Hale [9, p. 43]) to $\Phi(Y_1,\lambda)$, we have

$$\Phi(Y_1,\lambda) = \big(\xi_1(\lambda) + \xi_2(\lambda)Y_1 + Y_1^2\big)\Psi(Y_1,\lambda),$$

where $\xi_1(\lambda) = \frac{\delta[(-1+\delta)^3\lambda_1+\delta^3(1+\delta)\lambda_2]}{(1-\delta)^2 d_1(\lambda)}$, $\xi_2(\lambda) = -\frac{\delta^3(1+\delta)\lambda_2[(-1+\delta)^3\lambda_1+\delta^3(1+\delta)\lambda_2]}{(1-\delta)^6 d_1(\lambda)}$, $\Psi(0,\lambda) = -d_1(\lambda)$, and $\Psi(Y_1,\lambda)$ is a power series in Y_1 whose coefficients depend on parameters $\lambda = (\lambda_1,\lambda_2)$.

Now consider $Z_1 = Y_1$, $Z_2 = \frac{Y_2}{\sqrt{\Psi(Y_1,\lambda)}}$, $\tau = \int \sqrt{\Psi(Y_1(s),\lambda)}ds$. Then system (3.9) can be written as

$$\begin{aligned} \dot{Z}_1 &= Z_2, \\ \dot{Z}_2 &= \xi_1(\lambda) + \xi_2(\lambda)Z_1 + Z_1^2 - \frac{\lambda_1 + \frac{2\delta^3(1+\delta)}{(-1+\delta)^3}\lambda_2}{\sqrt{\Psi(Z_1,\lambda)}}Z_2 \\ &\quad - \frac{3+\delta}{1+\delta}\frac{1}{\sqrt{\Psi(Z_1,\lambda)}}Z_1Z_2 + G(Z_1,Z_2,\lambda), \end{aligned} \tag{3.10}$$

where $G(Z_1,Z_2,0)$ is a power series in (Z_1,Z_2) with power $Z_1^iZ_2^j$ satisfying $i+j \geq 3$ and $j \geq 2$. Letting $\eta(\lambda) = -\frac{\lambda_1+\frac{2\delta^3(1+\delta)}{(-1+\delta)^3}\lambda_2}{\sqrt{\Psi(0,\lambda)}}$ and making the parameter dependent affine transformation $x = Z_1 + \frac{1}{2}\xi_2(\lambda)$, $y = Z_2$, we transform system (3.10) to

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \mu_1(\lambda_1,\lambda_2) + \mu_2(\lambda_1,\lambda_2)y + x^2 + \frac{d_2}{\sqrt{-d_1}}xy + R(x,y,\lambda), \end{aligned} \tag{3.11}$$

where $\mu_1(\lambda_1,\lambda_2) = \xi_1(\lambda) - \frac{1}{4}\xi_2^2(\lambda)$, $\mu_2(\lambda_1,\lambda_2) = \eta(\lambda) - \frac{d_2}{2\sqrt{-d_1}}\xi_2(\lambda)$, $d_1(0) = d_1$, $d_2 = -\frac{3+\delta}{1+\delta}$, and $R(x,y,\lambda)$ is a power series in (x,y) with power x^iy^j satisfying $i+j \geq 3$ and $j \geq 2$.

Since $0 < \delta < 1$, we can check that

$$\left|\frac{\partial(\mu_1,\mu_2)}{\partial(\lambda_1,\lambda_2)}\right|_{\lambda=0} = -\frac{\delta^2(1+\delta)^2\sqrt{\delta(1+\delta)}}{(1-\delta)^2} \neq 0.$$

Thus, the parameter transformation is nonsingular and system (3.11) becomes

$$\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= \mu_1(\lambda_1,\lambda_2) + \mu_2(\lambda_1,\lambda_2)y + x^2 - \frac{3+\delta}{1+\delta}\sqrt{\frac{1+\delta}{\delta}}xy + S(x,y,\mu),
\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}
&\mu_1(\lambda_1,\lambda_2) \\
&\quad = \frac{[\delta(\delta-1)^3\lambda_1 + \delta^4(1+\delta)\lambda_2][\delta^5(1-\delta^2)^2(1-\delta)\lambda_1\lambda_2^2 - \delta^8(1+\delta)^3\lambda_2^3 - (1-\delta)^4 Q(\lambda)]}{Q^2(\lambda)}, \\
&\mu_2(\lambda_1,\lambda_2) = \frac{\delta^4(3+\delta)\lambda_2[(-1+\delta)^3\lambda_1 + \delta^3(1+\delta)\lambda_2]}{Q(\lambda)(1-\delta)(\frac{\delta}{1+\delta})^{\frac{2}{3}}} + \frac{\sqrt{2}(-1+\delta)^3\lambda_1 + 2\sqrt{2}\delta^3(1+\delta)\lambda_2}{\sqrt{Q(\lambda)}}, \\
&\quad Q(\lambda) = (1-\delta)^6\left[\frac{2\delta}{1+\delta} - \frac{2(1+\delta^2)}{1-\delta^2}\lambda_1 + \frac{2\delta^3(2-\delta+\delta^2)}{(1-\delta)^4}\lambda_2 - \frac{\delta^2(-1+3\delta+4\delta^2)}{(1-\delta)^4}\lambda_1\lambda_2 \right. \\
&\qquad \left. - \frac{\delta^5(1+\delta)^2(1-4\delta)}{(1-\delta)^7}\lambda_2^2 - \frac{\delta^5(1+\delta)^2}{(1-\delta)^7}\lambda_1\lambda_2^2 + \frac{\delta^8(1+\delta)^3}{(1-\delta)^{10}}\lambda_2^3\right],
\end{aligned}$$

and $S(x,y,\mu)$ is a power series in (x,y,μ_1,μ_2) with powers $x^i y^j \mu_1^k \mu_2^l$ satisfying $i+j+k+l \geq 4$ and $i+j \geq 3$.

The results in Bogdanov [3,4] and Takens [35] or Perko [31] imply that system (3.12) is strongly topologically equivalent to

$$\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= \mu_1 + \mu_2 y + x^2 - xy.
\end{aligned} \tag{3.13}$$

Choosing μ_1 and μ_2 as bifurcation parameters we know that system (3.13) undergoes a Bogdanov–Takens bifurcation when the parameters (λ_1,λ_2) vary in a small neighborhood of the origin. The local representations of the bifurcation curves in a small neighborhood of the origin are given as follows:

(a) The saddle-node bifurcation curve

$$SN = \left\{(\lambda_1,\lambda_2) : \mu_1(\lambda_1,\lambda_2) = 0,\ \text{i.e., } \lambda_2 = \frac{(1-\delta)^3}{\delta^3(1+\delta)}\lambda_1 + O(\lambda_1^2)\right\},$$

which consists of $SN^+ = \{(\lambda_1,\lambda_2) : \mu_1(\lambda_1,\lambda_2) = 0,\ \mu_2(\lambda_1,\lambda_2) > 0\}$ and $SN^- = \{(\lambda_1,\lambda_2) : \mu_1(\lambda_1,\lambda_2) = 0,\ \mu_2(\lambda_1,\lambda_2) < 0\}$;

(b) The Hopf bifurcation curve

$$
\begin{aligned}
H = \Bigg\{ & (\lambda_1,\lambda_2) : \mu_2(\lambda_1,\lambda_2) = -\frac{3+\delta}{1+\delta}\sqrt{\frac{1+\delta}{\delta}}\sqrt{-\mu_1(\lambda_1,\lambda_2)},\ \mu_1(\lambda_1,\lambda_2) < 0 \Bigg\} \\
= \Bigg\{ & (\lambda_1,\lambda_2) : -\frac{(-1+\delta)(3+\delta)^2}{\delta}\lambda_1 - \frac{\delta^2(1+\delta)(3+\delta)^2}{(-1+\delta)^2}\lambda_2 \\
& + \frac{\delta(27+40\delta+32\delta^2+30\delta^3+13\delta^4+25\delta^5)}{(-1+\delta)^3}\lambda_1\lambda_2 \\
& + \frac{9+7\delta+11\delta^2+6\delta^3+\delta^4}{\delta^2}\lambda_1^2 + \frac{\delta^4(1+\delta)^2(18+7\delta+9\delta^2+5\delta^3+\delta^4)}{(-1+\delta)^6}\lambda_2^2 \\
& + O\big(|\lambda_1,\lambda_2|^3\big) = 0,\ \mu_1(\lambda_1,\lambda_2) < 0 \Bigg\};
\end{aligned}
$$

(c) The homoclinic bifurcation curve

$$
\begin{aligned}
HL = \Bigg\{ & (\lambda_1,\lambda_2) : \mu_2(\lambda_1,\lambda_2) = -\frac{5(3+\delta)}{7(1+\delta)}\sqrt{\frac{1+\delta}{\delta}}\sqrt{-\mu_1(\lambda_1,\lambda_2)},\ \mu_1(\lambda_1,\lambda_2) < 0 \Bigg\} \\
= \Bigg\{ & (\lambda_1,\lambda_2) : -\frac{25(-1+\delta)(3+\delta)^2}{49\delta}\lambda_1 - \frac{25\delta^2(1+\delta)(3+\delta)^2}{49(-1+\delta)^2}\lambda_2 \\
& + \frac{\delta(675+1096\delta+992\delta^2+846\delta^3+325\delta^4+50\delta^5)}{49(-1+\delta)^3}\lambda_1\lambda_2 \\
& + \frac{225+199\delta+299\delta^2+150\delta^3+25\delta^4}{49\delta^2}\lambda_1^2 \\
& + \frac{\delta^4(1+\delta)^2(450+271\delta+321\delta^2+125\delta^3+25\delta^4)}{49(-1+\delta)^6}\lambda_2^2 \\
& + O\big(|\lambda_1,\lambda_2|^3\big) = 0,\ \mu_1(\lambda_1,\lambda_2) < 0 \Bigg\};
\end{aligned}
$$

(d) Recall that the Hopf bifurcation curve of system (3.1) at E_2^* is given by

$$
\begin{aligned}
H_2 &= \big\{(\lambda_1,\lambda_2) : (\lambda_1,\lambda_2) \text{ satisfy (3.3) and } w_1 < w < w_2\big\} \\
&= \Bigg\{(\lambda_1,\lambda_2) : \lambda_2 = \frac{(1-\delta)^3(3+4\delta+3\delta^2+2\delta^3+2\delta^4)}{\delta^3(1+\delta)^2(3+\delta+\delta^2+\delta^3)}\lambda_1\Bigg\}.
\end{aligned}
$$

The bifurcation diagram of system (3.1) is sketched in Fig. 3.1. □

These bifurcation curves H_2, SN^+, H, HL, and SN^- divide a small neighborhood of the origin in the (λ_1,λ_2) parameter plane into six regions. In each region, the dynamics of system (3.1) can be described as follows.

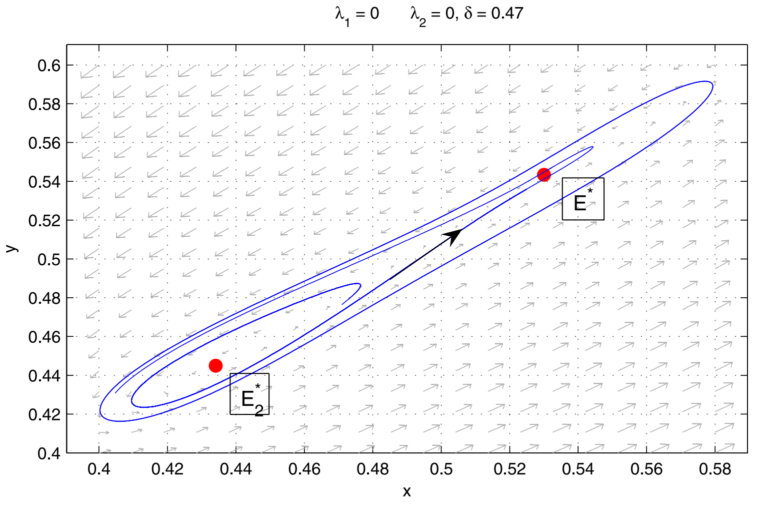

Fig. 3.2. When $(\lambda_1, \lambda_2) = (0, 0)$, E^* is a cusp of codimension 2, E_2^* is an unstable multiple focus with multiplicity one, there exists a large stable limit cycle surrounding the two non-hyperbolic positive equilibria.

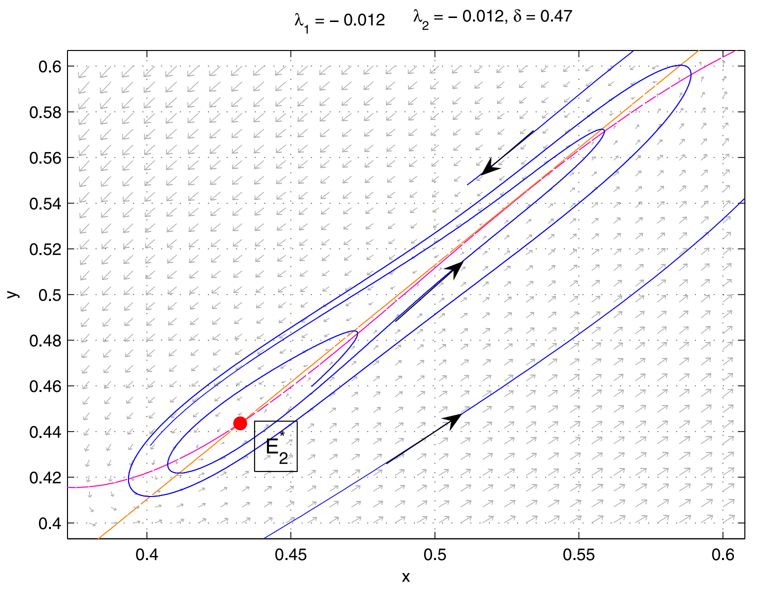

Fig. 3.3. When $(\lambda_1, \lambda_2) = (-0.012, -0.012)$ lies in region I, E_2^* is an unstable focus and there exists a large stable limit cycle.

(i) When $(\lambda_1, \lambda_2) = (0, 0)$, system (3.1) has two positive equilibria: E_2^* is an unstable multiple focus with multiplicity one and E^* is a cusp of codimension 2. From the Poincaré–Bendixon theorem, there exists at least one limit cycle enclosing these two positive equilibria (see Fig. 3.2).

(ii) When the parameters lie in region I (i.e., the region between the curves SN^+ and H_2), system (3.1) has a unique positive equilibrium E_2^* which is an unstable focus. This implies that system (3.1) has at least a stable limit cycle by Poincaré–Bendixon theorem (see Fig. 3.3).

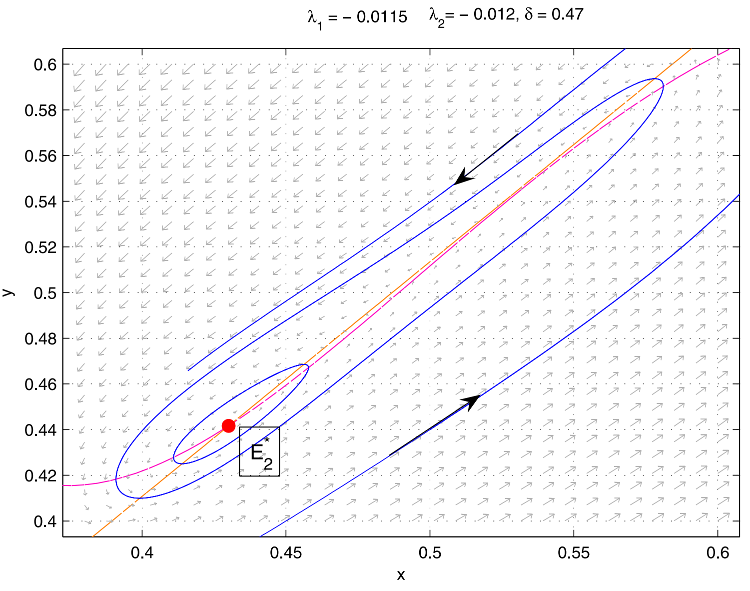

Fig. 3.4. When $(\lambda_1, \lambda_2) = (-0.0115, -0.012)$ lies in region VI, E_2^* is a stable focus, there exist a small unstable limit cycle and a large stable limit cycle.

(iii) When parameters lie on the Hopf bifurcation curve H_2 at E_2^*, the unique positive equilibrium E_2^* is an unstable multiple focus with multiplicity one and the stable limit cycle still exists.

(iv) When parameters cross H_2 into region VI (i.e., the region between H_2 and SN^-), E_2^* becomes a stable focus and a new unstable limit cycle bifurcates from E_2^* following the subcritical Hopf bifurcation. Thus system (3.1) has at least two limit cycles, the outer is stable and the inner is unstable (see Fig. 3.4).

(v) When parameters lie on the saddle-node bifurcation curve SN^+, system (3.1) has two positive equilibria, one is a saddle-node E^* and the other is an unstable focus E_2^* (see also case (xiii)).

(vi) When parameters cross the curve SN^+ into region II (i.e., the region between SN^+ and H), system (3.1) has three positive equilibria E_1^*, E_2^* and E_3^*, in which E_1^* and E_3^* bifurcate from the saddle-node. When $(\lambda_1, \lambda_2) = (-0.01232, -0.012)$, which lie in region II, we can see that E_1^* and E_2^* are unstable foci and E_3^* is a saddle. Poincaré–Bendixon theorem implies that there exists at least one stable limit cycle enclosing these three hyperbolic positive equilibria (see Fig. 3.5).

(vii) When parameters lie on the Hopf bifurcation curve H, system (3.1) has three positive equilibria E_1^*, E_2^*, and E_3^*, in which E_1^* is an unstable multiple focus with multiplicity one, E_3^* is a hyperbolic saddle, and E_2^* is an unstable hyperbolic focus. There exists at least one stable limit cycle enclosing these three positive equilibria.

(viii) When the parameters cross the curve H into region III (i.e., the region between H and HL), E_1^* becomes a stable focus and an unstable limit cycle bifurcates from E_1^* according to the subcritical Hopf bifurcation. Taking $(\lambda_1, \lambda_2) = (-0.01235, -0.012)$, system (3.1) has three positive equilibria, a small limit cycle enclosing the stable focus E_1^* and a large stable limit cycle enclosing all three positive equilibria (see Fig. 3.6).

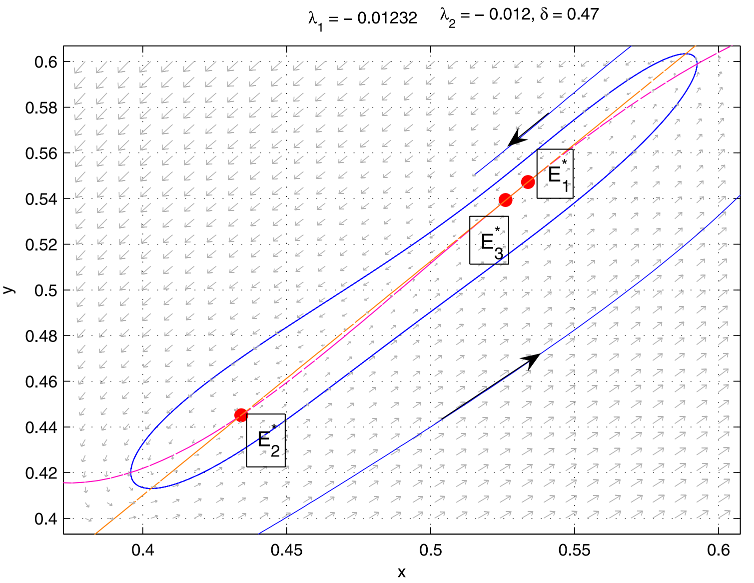

Fig. 3.5. When $(\lambda_1, \lambda_2) = (-0.01232, -0.012)$ lies in region II, all three equilibria E_1^*, E_2^*, E_3^* are unstable and there exists a large stable limit cycle surrounding these three hyperbolic positive equilibria.

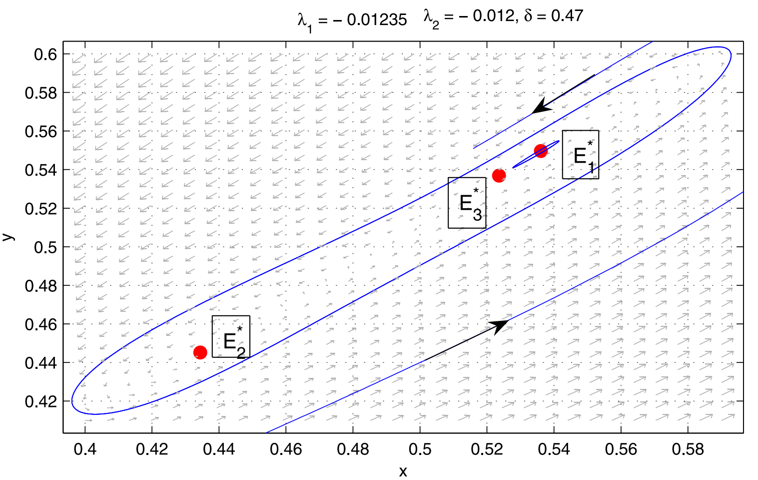

Fig. 3.6. When $(\lambda_1, \lambda_2) = (-0.01235, -0.012)$ lies in region III, E_1^* is a stable focus and E_2^*, E_3^* are unstable, there exist a small unstable limit cycle surrounding E_1^* and a large stable limit cycle enclosing the small limit cycle and equilibria E_2^*, E_3^*.

(ix) When parameters lie on the homoclinic bifurcation curve HL, system (3.1) has three positive equilibria E_1^*, E_2^*, E_3^* and an unstable homoclinic loop enclosing the stable focus E_1^* (see Fig. 3.7).

(x) When the parameters cross the curve HL into region IV (i.e., the region between HL and H_2), the above homoclinic loop is broken. There exist a stable focus E_1^*, a hyperbolic

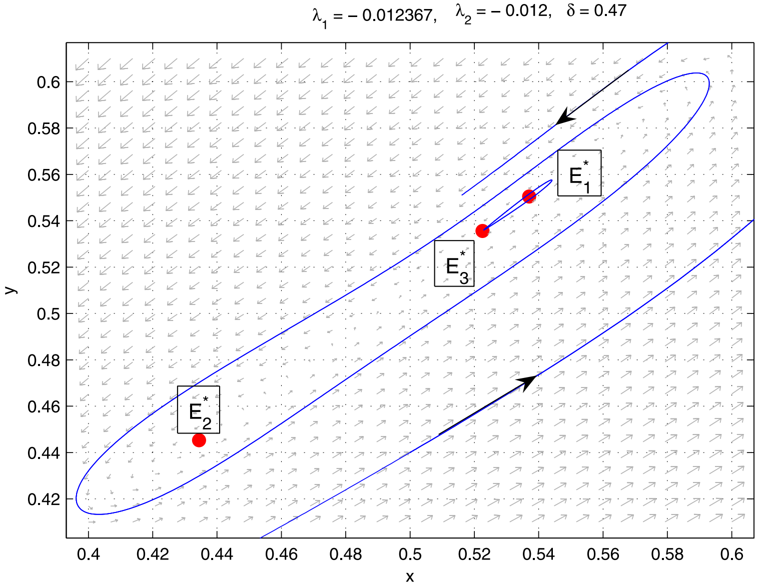

Fig. 3.7. When $(\lambda_1, \lambda_2) = (-0.012367, -0.012)$ lies on the *HL* curve, E_1^* is a stable focus, E_3^* is a saddle, E_2^* is an unstable focus, there exist a homoclinic loop surrounding E_1^* and a large stable limit cycle enclosing the homoclinic loop and equilibria E_2^*, E_3^*.

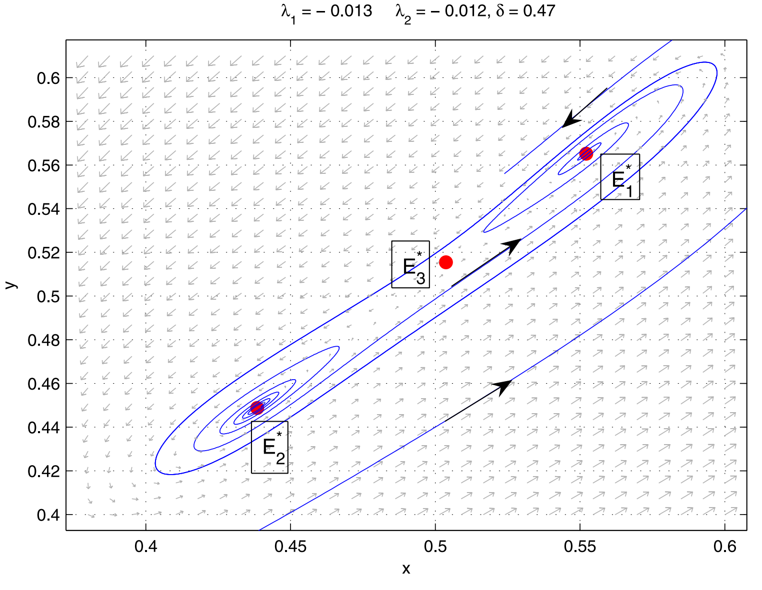

Fig. 3.8. When $(\lambda_1, \lambda_2) = (-0.013, -0.012)$ lies in region IV, E_2^* is an unstable focus, E_1^* is a stable focus, and E_3^* is a saddle, and there exists a large stable limit cycle enclosing all three equilibria.

saddle E_3^*, an unstable focus E_2^*, and a stable limit cycle enclosing these three hyperbolic positive equilibria (see Fig. 3.8).

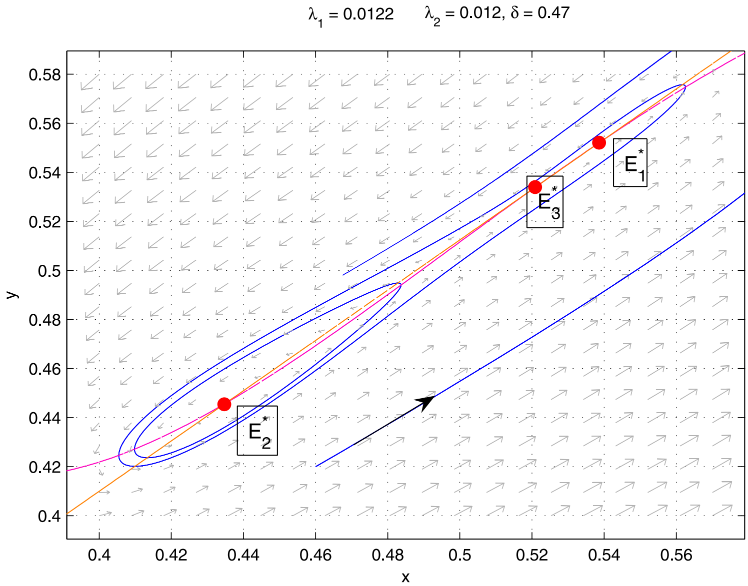

Fig. 3.9. When $(\lambda_1, \lambda_2) = (0.0122, 0.012)$ lies in region V, E_1^*, E_2^* are stable foci, and E_3^* is a saddle, there exists a small unstable limit cycle surrounding E_2^* with a larger stable limit cycle.

(xi) When parameters lie on the Hopf bifurcation curve H_2 at E_2^*, system (3.1) has three positive equilibria E_1^*, E_2^*, and E_3^*, in which E_2^* is an unstable multiple focus with multiplicity one (see also case (iii)).

(xii) When the parameters cross the curve H_2 into region V (i.e., the region between H_2 and SN^-), E_2^* becomes stable and an unstable limit cycle bifurcates from E_2^* via the Hopf bifurcation surrounded by a larger stable limit cycle (see Fig. 3.9).

(xiii) When parameters lie on the saddle-node bifurcation curve SN^-, system (3.1) has two positive equilibria, one is a stable focus E_2^* and the other is a saddle-node E^* (see also case (v)).

(xiv) When parameters cross the curve SN^- into region VI, it returns to case (iv), that is, system (3.1) has at least two limit cycles enclosing the unique positive equilibrium E_2^*, the outer is stable and the inner is unstable (see Fig. 3.4).

Remark 3.2. When $(\lambda_1, \lambda_2) = (0.0122, 0.012)$ lies in region V, there exists a tri-stability phenomenon for system (3.1): stable foci E_1^* and E_2^*, and a large stable limit cycle surrounding these three positive equilibria (see Fig. 3.9).

3.2. Degenerate focus type Bogdanov–Takens bifurcation of codimension 3

In this subsection, we consider case (b2) of Lemma 2.1 when system (1.6) has a unique degenerate equilibrium E^*. From Theorem 2.3, we know that if $(a, \delta, \beta) = (\frac{(1-b)^3}{27}, \frac{2+b}{b-1}, \frac{27}{(1-b)(2+b)^2})$, then the unique degenerate positive equilibrium $E^*(\frac{3}{1-b}, -\frac{(2+b)^3}{9(1-b)})$ of system (1.6) is a codimension 3 Bogdanov–Takens singularity (focus or center case). If we choose a, δ and β as bifurcation parameters, system (1.6) may exhibit degenerate focus type Bogdanov–Takens bifurcation of codimension 3. Let

$$a = \frac{(1-b)^3}{27} + \alpha_1, \qquad \delta = \frac{2+b}{b-1} + \alpha_2, \qquad \beta = \frac{27}{(1-b)(2+b)^2} + \alpha_3, \qquad \alpha = (\alpha_1, \alpha_2, \alpha_3)$$

and consider the unfolding system of system (1.6) as follows

$$\begin{aligned} \dot{x} &= x(1-x) - \frac{x^2 y}{(\frac{(1-b)^3}{27} + \alpha_1)x^2 - bx + 1}, \\ \dot{y} &= y\left(\frac{2+b}{b-1} + \alpha_2 - \left(\frac{27}{(1-b)(2+b)^2} + \alpha_3\right)\frac{y}{x}\right), \end{aligned} \tag{3.14}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a parameter vector in a small neighborhood of $(0, 0, 0)$.

Theorem 3.3. *When parameters $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ vary in a small neighborhood of the origin, system (3.14) undergoes degenerate focus type Bogdanov–Takens bifurcation of codimension* 3 *in a small neighborhood of $E^*(\frac{3}{1-b}, -\frac{(2+b)^3}{9(1-b)})$. Moreover, in a small neighborhood of the point $(a, \delta, \beta) = (\frac{(1-b)^3}{27}, \frac{2+b}{b-1}, \frac{27}{(1-b)(2+b)^2})$ of the parameter space, there exist a Hopf bifurcation surface, two homoclinic bifurcation surfaces, two saddle-node loop bifurcation surfaces, a multiple limit cycle bifurcation surface, and two saddle-node bifurcation surfaces for system (1.6). When parameters (a, δ, β) cross these surfaces, system (1.6) undergoes Hopf bifurcation, homoclinic bifurcation, saddle-node loop bifurcation, multiple limit cycle bifurcation, and saddle-node bifurcation, respectively.*

Proof. Firstly, we make a sequence of smooth coordinate transformations (I), (II) and (III), which were used in the proof of Theorem 2.3, to obtain the following system from system (3.14)

$$\begin{aligned} \dot{X} &= Y + c_{00}(\alpha) + c_{10}(\alpha)X + c_{01}(\alpha)Y + c_{20}(\alpha)X^2 + c_{11}(\alpha)XY + c_{02}(\alpha)Y^2 + c_{30}(\alpha)X^3 \\ &\quad + c_{12}(\alpha)XY^2 + c_{21}(\alpha)X^2Y + c_{03}(\alpha)Y^3 + O\big(|X, Y|^4\big), \\ \dot{Y} &= d_{00}(\alpha) + d_{10}(\alpha)X + d_{01}(\alpha)Y + d_{20}(\alpha)X^2 + d_{11}(\alpha)XY + d_{02}(\alpha)Y^2 + d_{30}(\alpha)X^3 \\ &\quad + d_{12}(\alpha)XY^2 + d_{21}(\alpha)X^2Y + d_{03}(\alpha)Y^3 + O\big(|X, Y|^4\big), \end{aligned} \tag{3.15}$$

where $c_{ij}(\alpha)$ and $d_{ij}(\alpha)$ are smooth functions whose long expressions are omitted here for the sake of brevity, $c_{00}(0) = c_{10}(0) = c_{01}(0) = c_{20}(0) = c_{11}(0) = c_{02}(0) = c_{30}(0) = c_{21}(0) = d_{00}(0) = d_{10}(0) = d_{01}(0) = d_{20}(0) = d_{02}(0) = 0$, $c_{03}(0) = c_{03}$, $c_{12}(0) = c_{12}$, $d_{11}(0) = d_{11}$, $d_{30}(0) = d_{30}$, $d_{12}(0) = d_{12}$, $d_{21}(0) = d_{21}$, $d_{03}(0) = d_{03}$, and c_{03}, c_{12}, d_{11}, d_{30}, d_{12}, d_{21}, d_{03} are given in system (2.12).

Secondly, to simplify the third order terms when $\alpha = 0$, we make the following coordinate transformation

$$(IV): \quad X = x_1 + \frac{d_{12}}{6}x_1^3 + \frac{c_{12} + d_{03}}{2}x_1^2 y_1 + c_{03}x_1 y_1^2, \qquad Y = y_1 + \frac{d_{12}}{2}x_1^2 y_1 + d_{03}x_1 y_1^2$$

and rewrite system (3.15) as follows

$$\begin{aligned}
\dot{x}_1 &= y_1 + e_{00}(\alpha) + e_{10}(\alpha)x_1 + e_{01}(\alpha)y_1 + e_{20}(\alpha)x_1^2 + e_{11}(\alpha)x_1y_1 + e_{02}(\alpha)y_1^2 + e_{30}(\alpha)x_1^3 \\
&\quad + e_{12}(\alpha)x_1y_1^2 + e_{21}(\alpha)x_1^2y_1 + e_{03}(\alpha)y_1^3 + O\big(|x_1, y_1|^4\big), \\
\dot{y}_1 &= f_{00}(\alpha) + f_{10}(\alpha)x_1 + f_{01}(\alpha)y_1 + f_{20}(\alpha)x_1^2 + f_{11}(\alpha)x_1y_1 + f_{02}(\alpha)y_1^2 + f_{30}(\alpha)x_1^3 \\
&\quad + f_{12}(\alpha)x_1y_1^2 + f_{21}(\alpha)x_1^2y_1 + f_{03}(\alpha)y_1^3 + O\big(|x_1, y_1|^4\big),
\end{aligned} \tag{3.16}$$

where $e_{ij}(\alpha)$ and $f_{ij}(\alpha)$ can be expressed by $c_{ij}(\alpha)$, $d_{ij}(\alpha)$, d_{12}, c_{12}, d_{03}, and c_{03}, we also omit their expressions here to save spaces.

Thirdly, introduce the following transformation

$$\begin{aligned}
(V):\quad x_2 &= x_1, \\
y_2 &= y_1 + e_{00}(\alpha) + e_{10}(\alpha)x_1 + e_{01}(\alpha)y_1 + e_{20}(\alpha)x_1^2 + e_{11}(\alpha)x_1y_1 + e_{02}(\alpha)y_1^2 \\
&\quad + e_{30}(\alpha)x_1^3 + e_{12}(\alpha)x_1y_1^2 + e_{21}(\alpha)x_1^2y_1 + e_{03}(\alpha)y_1^3 + O\big(|x_1, y_1|^4\big)
\end{aligned}$$

and rewrite system (3.16) as

$$\begin{aligned}
\dot{x}_2 &= y_2, \\
\dot{y}_2 &= g_{00}(\alpha) + g_{10}(\alpha)x_2 + g_{01}(\alpha)y_2 + g_{20}(\alpha)x_2^2 + g_{11}(\alpha)x_2y_2 + g_{02}(\alpha)y_2^2 + g_{30}(\alpha)x_2^3 \\
&\quad + g_{12}(\alpha)x_2y_2^2 + g_{21}(\alpha)x_2^2y_1 + g_{03}(\alpha)y_2^3 + O\big(|x_2, y_2|^4\big),
\end{aligned} \tag{3.17}$$

where $g_{ij}(\alpha)$ can be expressed by $e_{ij}(\alpha)$ and $f_{ij}(\alpha)$, we also omit their expressions here.

Finally, following the steps in Xiao and Zhang [42], we can rewrite system (3.17) as

$$\begin{aligned}
\dot{u} &= \frac{\sigma(\alpha)}{\nu(\alpha)}v, \\
\dot{v} &= \frac{-g_{30}(\alpha)}{\sigma(\alpha)}\big[\lambda_1(\alpha) + \lambda_2(\alpha)\nu(\alpha)u - \nu^3(\alpha)u^3\big] + g_{21}(\alpha)v\big[\lambda_3(\alpha) + A(\alpha)\nu(\alpha)u + \nu^2(\alpha)u^2\big] \\
&\quad + v^2Q_1(u, v, \alpha) + O\big(|u, v|^4\big),
\end{aligned} \tag{3.18}$$

where $\lambda_1(\alpha) = -\frac{g_{00}(\alpha)}{g_{30}(\alpha)} + \frac{g_{10}(\alpha)g_{20}(\alpha)}{3g_{30}^2(\alpha)} - \frac{g_{20}^3(\alpha)}{9g_{30}^3(\alpha)} + \frac{g_{20}^3(\alpha)}{27g_{30}^3(\alpha)}$, $\lambda_2(\alpha) = -\frac{g_{10}(\alpha)}{g_{30}(\alpha)} + \frac{g_{20}^2(\alpha)}{3g_{30}^2(\alpha)}$, $\lambda_3(\alpha) = \frac{g_{01}(\alpha)}{g_{21}(\alpha)} - \frac{g_{11}(\alpha)g_{20}(\alpha)}{3g_{21}(\alpha)g_{30}(\alpha)} + \frac{g_{21}(\alpha)g_{20}^2(\alpha)}{9g_{21}(\alpha)g_{30}^2(\alpha)}$, $A(\alpha) = \frac{g_{11}(\alpha)}{g_{21}(\alpha)} + \frac{2g_{20}(\alpha)}{3g_{30}(\alpha)}$, $Q_1(u, v, \alpha) = \sigma(\alpha)[g_{02}(\alpha) + \frac{g_{12}(\alpha)g_{20}^2(\alpha)}{9g_{30}^2(\alpha)} + \sigma(\alpha)g_{03}(\alpha)v + \nu(\alpha)g_{12}(\alpha)u]$.

By using the computer software *Mathematica* we can calculate that $g_{30}(0) = \frac{243(b-1)^2}{(2+b)^7} < 0$ and $g_{21}(0) = \frac{243(b-1)(16-5b+7b^2)}{2(2+b)^8} < 0$ (because $b < -2$), so we can choose

$$\sigma(\alpha) = -\frac{g_{30}(\alpha)}{g_{21}(\alpha)}\nu(\alpha), \qquad \nu(\alpha) = \sqrt{-\frac{g_{30}(\alpha)}{g_{21}^2(\alpha)}}$$

in the small neighborhood of $\alpha = (0, 0, 0)$. In order to obtain the canonical unfolding of the focus type Bogdanov–Takens singularity of codimension 3, we make the last time transformation $\tau = -\frac{g_{30}(\alpha)}{g_{21}(\alpha)} t$, and still denote τ by t, system (3.18) becomes

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= \mu_1(\alpha) + \mu_2(\alpha)u - u^3 + v\left[\mu_3(\alpha) + A_1(\alpha)u + u^2\right] \\ &\quad + v^2 Q_2(u, v, \alpha) + O\left(|u, v|^4\right), \end{aligned} \tag{3.19}$$

where $A_1(\alpha) = \frac{g_{21}(\alpha)\sqrt{-g_{30}(\alpha)}}{g_{30}(\alpha)} A(\alpha)$, $Q_2(u, v, \alpha) = -\frac{g_{21}(\alpha)}{g_{30}(\alpha)} Q_1(u, v, \alpha)$,

$$\begin{gathered} \mu_1(\alpha) = \frac{g_{21}^3(\alpha)}{g_{30}(\alpha)\sqrt{-g_{30}(\alpha)}}\lambda_1(\alpha), \qquad \mu_2(\alpha) = -\frac{g_{21}^2(\alpha)}{g_{30}(\alpha)}\lambda_2(\alpha), \\ \mu_3(\alpha) = -\frac{g_{21}^2(\alpha)}{g_{30}(\alpha)}\lambda_3(\alpha). \end{gathered} \tag{3.20}$$

Since

$$\left|\frac{\partial(\mu_1(\alpha), \mu_2(\alpha), \mu_3(\alpha))}{\partial(\alpha_1, \alpha_2, \alpha_3)}\right|_{\alpha=0} = \frac{3\sqrt{3}(16 - 5b + 7b^2)^6}{64(b-1)^{10}(2+b)^3\sqrt{-(2+b)}} \neq 0$$

when $b < -2$, the parameter transformation (3.20) is a homeomorphism in a small neighborhood of the origin, and μ_1, μ_2 and μ_3 are independent parameters. Furthermore, for system (3.19), the coefficients of u^3 and u^2v are -1 and 1, respectively, the coefficient of uv is $A_1(\alpha)$, which can be calculated as follows when $\alpha = 0$:

$$A_1(0) = \frac{\sqrt{-3(2+b)}}{1-b} < 2\sqrt{2}.$$

By the results in [12] or [42], we know that system (3.19) is a generic 3-parameter family or standard family of Bogdanov–Takens singularity of codimension 3 (focus case). Thus, system (1.6) will undergo a degenerate focus type Bogdanov–Takens bifurcation of codimension 3 by choosing a, δ and β as bifurcation parameters in a small neighborhood of $(\frac{(1-b)^3}{27}, \frac{2+b}{b-1}, \frac{27}{(1-b)(2+b)^2})$. $\square$

Remark 3.4. Note that the maximum number of limit cycles for the versal unfolding of a focus type Bogdanov–Takens singularity of codimension 3 is still an open problem (see [12] for general theory or [42] for a predator–prey system with generalist and specialist predators).

4. Discussion

It is well-known that predation can induce oscillations in interacting species and many field and experimental data demonstrate periodic fluctuations in both predator and prey populations (May [29,30]). The existence of limit cycles in predator–prey models has been extensively studied, see for example, Bazykin [2], Kuang and Freedman [22], May [29], etc.

Wollkind, Collings and Logan [39] employed the predator–prey system (1.4) of Leslie type with the Holling type II functional response to model the interaction between *Metaseiulus occidentalis* and the phytophagous spider mite *Tetranychus mcdanieli* on apple trees, the existence of periodic solutions in the model mimics the observed biological data on population oscillations and outbreaks. The existence and uniqueness of limit cycles in this model has been studied by many researchers, see Hsu and Huang [18–20], Collings [10], Gasull, Kooij and Torregrosa [16], Braza [5], Sáez and González-Olivares [33], etc.

As Collings [10] suggested, the Holling type II functional response in system (1.4) can be replaced by other functions, including the Holling types III and IV function responses. The case with Holling type IV function response was considered by Li and Xiao [27]. It is shown that the model has two non-hyperbolic positive equilibria for some values of parameters, one is a cusp of codimension 2 and the other is a multiple focus of multiplicity one. Via Bogdanov–Takens bifurcation and subcritical Hopf bifurcation in the small neighborhoods of these two equilibria, it was shown that the model can have a stable limit cycle enclosing two equilibria, or an unstable limit cycle enclosing a hyperbolic equilibrium, or two limit cycles enclosing a hyperbolic equilibrium by choosing different values of parameters.

In this paper, we studied the dynamical behavior of a predator–prey model (1.6) of Leslie type with the generalized Holling type III functional response. We have shown that the model has very rich and complicated dynamics such as the existence of a stable limit cycle enclosing two non-hyperbolic positive equilibria, a stable limit cycle enclosing an unstable homoclinic loop, two limit cycles enclosing a hyperbolic positive equilibrium, or one stable limit cycle enclosing three hyperbolic positive equilibria. In particular, we have shown that the model undergoes degenerate focus type Bogdanov–Takens bifurcation of codimension 3, the coexistence of three stable states (two stable equilibria and a stable limit cycle) are also shown by numerical simulations, these new dynamical behaviors were not observed in [27]. Since the generalized Holling type III functional response has similar properties as the Holling type IV functional response, our model with $b > -2\sqrt{a}$ exhibits much more complex and far richer dynamics not only than the case when $b > 0$ but also than the model with Holling type IV functional response considered by Li and Xiao [27]. Moreover, the analytical results and the numerical simulations predict population oscillations and outbreaks in the predator–prey model of Leslie type with the generalized Holling type III functional response with various parameter values. Furthermore, these results provide new bifurcation phenomena, such as the degenerate focus type Bogdanov–Takens bifurcation of codimension 3, the coexistence of three stable states (two stable equilibria and a stable limit cycle), the existence of two limit cycles enclosing a hyperbolic positive equilibrium or one stable limit cycle enclosing three hyperbolic positive equilibria, that have not been observed in classical Gause type predator–prey systems.

The generic bifurcation of three-parameter families of planar vector fields around nilpotent singular points is a very important and challenging problem in the bifurcation theory of dynamical systems (Dumortier et al. [11,12], Xiao [40]). It was shown that there are three topological types of degenerate Bogdanov–Takens bifurcation of codimension 3: saddle, elliptic, and focus. It is very interesting to apply the theory to realistic biological and physical systems that exhibit such degenerate Bogdanov–Takens bifurcation of codimension 3. Recently, Cai, Chen, and Xiao [7] demonstrated that an epidemiological model with strong Allee effect undergoes degenerate *elliptic* type Bogdanov–Takens bifurcation of codimension 3 among other types of bifurcations. In this paper we have shown that a predator–prey model of Leslie type with the generalized Holling type III functional response exhibits degenerate *focus* type Bogdanov–Takens bifurcation of codimension 3. These results indicate that the nonlinear dynamics of such biological and epidemiological models not only depend on more bifurcation parameters but also are very sensitive to parameter perturbations, which are important for the control of biological species or infectious diseases. It will be interesting to see if the degenerate *saddle* type Bogdanov–Takens bifurcation of codimension 3 occurs in realistic biological or epidemiological models.

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