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Traveling wave solutions in delayed lattice differential equations with partial monotonicity

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Abstract

In this paper, we investigate a system of delayed lattice differential equations with partial monotonicity. By using Schauder's fixed point theorem, a new cross-iteration scheme is given to establish the existence of traveling wave solutions. Our main results can deal with the existence of traveling wave solution for a class of delayed reaction diffusion system with partial monotonicity and generalize the results of Wu and Zou (J. Differential Equations 135 (1997) 315–357).

Keywords: Cross-iteration scheme; Schauder's fixed point theorem; Upper and lower solutions; Quasimonotonicity

1. Introduction

Lattice dynamical systems are infinite systems of differential or difference equations indexed by points in a lattice, such as the integer lattice Z which incorporate some aspect of the spatial structure of the lattice. Such systems arise in many applied subjects, such as

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biology, chemical kinetics, imagine processing, material science, neurology, physiology, etc. We refer to the surveys by Chow [6] and Mallet-Paret [15] about some recent results on lattice dynamical systems.

The most studied lattice differential equation is the *single* spatially discrete Fisher or Nagumo equation

$$\frac{\mathrm{d}x_n}{\mathrm{d}t} = d(x_{n-1} - 2x_n + x_{n+1}) + f(x_n), \quad n \in Z, \tag{1.1}$$

where d is a positive constant and f is a Lipschitz continuous function satisfying certain character. Bell and Cosner [3] studied the long time behavior of solutions to (1.1) for some nonlinear function f. The traveling wave solutions were analytically discussed in Britton [4] and numerically computed in Chi et al. [5]. Keener [11] analyzed propagation and its failure for (1.1). By Keener's formation, a traveling wave of Eq. (1.1) is a solution of the form $x_n(t) = \phi(n - ct)$ for each $n \in Z$, for some $\phi : R \to R$ and some $c \in R$. The existence and stability of traveling wave solutions of (1.1) have been extensively studied by Hankerson and Zinner [10], Keener [11], Zinner [24,25], Zinner et al. [26], etc. More recently, Mallet-Paret [13,14] investigated the global structure of the set of all traveling wave solutions within a general framework of dynamical systems and provided existence, uniqueness and monotonicity. See also Chow et al. [7] and the references cited therein.

Recently, researchers have started to study *systems* of lattice differential equations. Renshaw [18] proposed a spatial population dynamics system

$$\begin{cases} \frac{\mathrm{d}u_n}{\mathrm{d}t} = d_1(u_{n+1} - 2u_n + u_{n-1}) + f_1(u_n, v_n), \\ \frac{\mathrm{d}v_n}{\mathrm{d}t} = d_2(v_{n+1} - 2v_n + v_{n-1}) + f_2(u_n, v_n) \end{cases} \tag{1.2}$$

and studied the Turing model for morphogenesis with $f_1(u_n, v_n) = u_n(r_{1f} - b_1 v_n)$, $f_2(u_n, v_n) = v_n(-r_2 + b_2 u_n)$ (see [18, pp. 314–323]). By using the comparison method, Anderson and Sleeman [1] investigated the existence of traveling wave fronts and propagation failure for spatially discretized FitzHugh–Nagumo equations

$$\begin{cases} \frac{\mathrm{d}u_n}{\mathrm{d}t} = u_{n+1} - 2u_n + u_{n-1} + u_n(1 - u_n)(u_n - a) - v_n, \\ \frac{\mathrm{d}v_n}{\mathrm{d}t} = \sigma u_n - \gamma v_n. \end{cases}$$

Nekorkin et al. [16] considered a system of two coupled FitzHugh–Nagumo chains

$$\begin{cases} \frac{\mathrm{d}u_n}{\mathrm{d}t} = d(u_{n+1} - 2u_n + u_{n-1}) + f(u_n) - h_n(u_n - v_n), \\ \frac{\mathrm{d}v_n}{\mathrm{d}t} = d(v_{n+1} - 2v_n + v_{n-1}) + f(v_n) - h_n(v_n - u_n), \end{cases} \tag{1.3}$$

where $f(w) = w(w-1)(a-w)$, $0 < a < 1$; h_n and d are the interchain and intrachain coupling coefficients. See also Erneux and Nicolis [9] for a discrete bistable reaction–diffusion system modeling N coupled Nagumo equations.

Zou and Wu [28] introduced a time delay into the discrete Nagumo-type equation (1.1) and considered the following *delayed* lattice differential equation:

$$\begin{aligned}\frac{\mathrm{d}u_n}{\mathrm{d}t} &= d[u_{n-1}(t) - 2u_n(t) + u_{n+1}(t)] \\ &\quad + f(u_n(t), u_{n-1}(t-\tau), u_n(t-\tau), u_{n+1}(t-\tau)), \end{aligned} \tag{1.4}$$

where $n \in Z, d > 0, \tau \geqslant 0$ are constants and $f : R^4 \to R$ is a sufficiently smooth function. By using the symmetry Hopf bifurcation theory, they studied the existence and stability of the periodic traveling wave solutions. In [20], Wu and Zou considered a more general class of delayed lattice single differential equations and established the existence of wave fronts and slowly oscillatory spatially periodic traveling waves.

In this paper, we consider the following system of two delayed lattice differential equations:

$$\begin{cases} \frac{\mathrm{d}u_n}{\mathrm{d}t} = \sum_{j=1}^{m} a_j[g(u_{n+j}(t)) - 2g(u_n(t)) + g(u_{n-j}(t))] + f_1(u_n(t-\tau), v_n(t-\tau)), \\ \frac{\mathrm{d}v_n}{\mathrm{d}t} = \sum_{j=1}^{m} b_j[g(v_{n+j}(t)) - 2g(v_n(t)) + g(v_{n-j}(t))] \\ \qquad + f_2(u_n(t-\tau), v_n(t-\tau)) \end{cases} \tag{1.5}$$

and study the existence of traveling wave solutions, where $n \in Z, m \geqslant 1$ is an integer, $a_j > 0, b_j > 0, 1 \leqslant j \leqslant m, \tau \geqslant 0, f_i : R^2 \to R$ and $g : R \to R$ are continuous functions.

In the last decade, great attention has been paid to the existence of traveling wave solutions in delayed reaction–diffusion equations, whose discrete versions are the delayed lattice differential equations. A pioneer work by Schaaf [19] was for scalar equations. The monotone iteration and fixed point methods were first used to study the existence of traveling waves (at least in monostable case) by Atkinson and Reuter [2] and Diekmann [8]. Zou and Wu [27] developed a monotone iteration method to establish the existence of traveling wave solutions for systems of delayed reaction–diffusion equations with monotone nonlinearities. By using some nonstandard ordering of the profile set, Wu and Zou [21] showed that the monotone iteration scheme can be employed to cases with both quasimonotone and nonquasimonotone reaction terms. The iteration scheme of Wu and Zou requires that the upper solution of the wave equations converges to two equilibria when $t \to -\infty$ and $t \to \infty$, respectively. Following Wu and Zou [21], Ma [12] relaxed the monotonicity of the iteration scheme by applying the Schauder fixed point theorem to the operator used in Wu and Zou [21] in a properly chosen subset in the Banach space $C(R, R^n)$ equipped with the so-called exponentially decay norm. The subset is constructed in terms of a pair of upper–lower solutions, which is less restrictive than the upper–lower solutions required in Wu and Zou [21]. This makes the searching for the pair of upper–lower solutions easier. For example, the upper solution does not have to be in the profile set. However, Ma [12] only considered systems with quasimonotone reaction terms and did not consider systems with non-quasimonotone reaction terms.

If the nonlinear reaction terms in system (1.5) satisfy either the quasimonotonicity condition

(QM) there exist two positive constants β_1 and β_2 such that

$$\begin{cases} f_1(\phi_1(x),\psi_1(x)) - f_1(\phi_2(x),\psi_2(x)) + \beta_1[\phi_1(x)(0) - \phi_2(x)(0)] \\ \quad \geqslant 2A[g(\phi_1(0)) - g(\phi_2(0))], \\ f_2(\phi_1(x),\psi_1(x)) - f_2(\phi_2(x),\psi_2(x)) + \beta_2[\psi_1(x)(0) - \psi_2(x)(0)] \\ \quad \geqslant 2B[g(\psi_1(0)) - g(\psi_2(0))] \end{cases}$$

for $\phi_i(x), \psi_i(x) \in X = C([-\tau, 0]; R)$ with $0 \leqslant \phi_2(x)(s) \leqslant \phi_1(x)(s) \leqslant k_1$, $0 \leqslant \psi_2(x)(s) \leqslant \psi_1(x)(s) \leqslant k_2$ for $s \in [-\tau, 0]$, $i = 1, 2$, where $A = \sum_{j=1}^m a_j$, $B = \sum_{j=1}^m b_j$

or the nonquasimonotonicity condition

(QM*) there exist two positive constants β_1 and β_2 such that

$$\begin{cases} f_1(\phi_1(x),\psi_1(x)) - f_1(\phi_2(x),\psi_2(x)) + \beta_1[\phi_1(x)(0) - \phi_2(x)(0)] \\ \quad \geqslant 2A[g(\phi_1(0)) - g(\phi_2(0))], \\ f_2(\phi_1(x),\psi_1(x)) - f_2(\phi_2(x),\psi_2(x)) + \beta_2[\psi_1(x)(0) - \psi_2(x)(0)] \\ \quad \geqslant 2B[g(\psi_1(0)) - g(\psi_2(0))] \end{cases}$$

for $\phi_i(x), \psi_i(x) \in X = C([-\tau, 0]; R)$, $i=1,2$ with (i) $0 \leqslant \phi_2(x)(s) \leqslant \phi_1(x)(s) \leqslant k_1$, $0 \leqslant \psi_2(x)(s) \leqslant \psi_1(x)(s) \leqslant k_2$ for $s \in [-\tau, 0]$, and (ii) $e^{\beta_1 s}[\phi_1(x)(s) - \phi_2(x)(s)]$ and $e^{\beta_2 s}[\psi_1(x)(s) - \psi_2(x)(s)]$ are nondecreasing in $s \in [-\tau, 0]$, where $A = \sum_{j=1}^m a_j$, $B = \sum_{j=1}^m b_j$,

then the existence of traveling wave front solutions of (1.5) can be obtained by using the results in Wu and Zou [20].

On the other hand, the reaction terms in some models arising from practical problem may not satisfy (QM) or (QM*), such as the following epidemic model with time delays:

$$\begin{cases} \frac{du_n}{dt} = d_1(u_{n+1} - 2u_n + u_{n-1}) + au_n[1 - u_n(t-\tau)] - u_n v_n, \\ \frac{dv_n}{dt} = d_2(v_{n+1} - 2v_n + v_{n-1}) + u_n v_n - bv_n \end{cases} \tag{1.6}$$

to which the main results of Wu and Zou [20] fail to apply.

The purpose of this paper is to generalize the methods of Wu and Zou to the cases in which only one equation in system (1.5) satisfies the condition (QM) or (QM*) while the other equation satisfies neither. We will propose a less restrictive condition on the reaction terms, construct a subset in the Banach space $C(R, R^2)$ equipped with the exponential decay norm and apply the Schauder fixed point theorem to the operator used in Wu and Zou [20] to establish the existence of a traveling wavefront solution of delayed lattice differential equations. The subset is obtained from a pair of upper–lower solutions and the upper solution does not have to satisfy the left-limit condition at $-\infty$. Since the nonlinear functions f_1 and f_2 in (1.5) have different monotonicity with respect to the first and second arguments in the first and second equations, respectively, following Ye and Li [22] and Pao [17], we will introduce definitions of the upper and lower solutions and the new cross iteration scheme, which are different from that defined in Wu and Zou [20].

This paper is organized as follows. In Section 2, some definitions and assumptions are given. Section 3 is devoted to establishing the existence of traveling wave solutions in the case of quasimonotone reactions. The case of nonquasimonotone reaction terms is studied in Section 4.

2. Preliminaries

A traveling wave solution of (1.5) is a pair of solutions of the special form $u_n(t)=\phi(t-nc)$, $v_n(t)=\psi(t-cn)$, where c is a given positive constant. Substituting $u_n(t)=\phi(t-nc)$, $v_n(t)=\psi(t-cn)$ into (1.5), and denoting $\phi_t(s)=\phi(t+s)$, $\psi_t(s)=\psi(t+s)$, we find that (1.5) has a pair of traveling wave solutions if and only if the following wave equations:

$$\begin{cases} \frac{\mathrm{d}\phi}{\mathrm{d}t} = \sum\limits_{j=1}^{m} a_j[g(\phi(t+jc)) - 2g(\phi(t)) + g(\phi(t-jc))] + f_1(\phi_t, \psi_t), \\ \frac{\mathrm{d}\psi}{\mathrm{d}t} = \sum\limits_{j=1}^{m} a_j[g(\psi(t+jc)) - 2g(\psi(t)) + g(\psi(t-jc))] + f_2(\phi_t, \psi_t) \end{cases} \tag{2.1}$$

with asymptotic boundary conditions

$$\begin{aligned} &\lim_{t\to-\infty} \phi(t) = \phi_-, \quad \lim_{t\to+\infty} \phi(t) = \phi_+, \\ &\lim_{t\to-\infty} \psi(t) = \psi_-, \quad \lim_{t\to+\infty} \psi(t) = \psi_+ \end{aligned} \tag{2.2}$$

have a pair of solutions on R, where (ϕ_-, ψ_-) and (ϕ_+, ψ_+) are two equlibria of (2.1).

Without loss of generality, let $\phi_-=0$, $\phi_+=k_1>0$, $\psi_-=0$, $\psi_+=k_2>0$. Then (2.2) can be replaced by

$$\begin{aligned} &\lim_{t\to-\infty} \phi(t) = 0, \quad \lim_{t\to+\infty} \phi(t) = k_1, \\ &\lim_{t\to-\infty} \psi(t) = 0, \quad \lim_{t\to+\infty} \psi(t) = k_2. \end{aligned} \tag{2.3}$$

In this paper, we use the usual notations for the standard ordering in R^2. That is, for $u=(u_1,u_2)^{\mathrm{T}}$ and $v=(v_1,v_2)^{\mathrm{T}}$, we denote $u\leqslant v$ if $u_i\leqslant v_i$, $i=1,2$, and $u<v$ if $u\leqslant v$ but $u\neq v$. In particular, we denote $u\ll v$ if $u\leqslant v$ but $u_i\neq v_i$, $i=1,2$. If $u\leqslant v$, we also denote $(u,v]=\{w\in R^2: u<w\leqslant v\}$, $[u,v)=\{w\in R^2: u\leqslant w<v\}$. Let $|\cdot|$ denote the Euclidean norm in R^2 and $\|\cdot\|$ denote the supremum norm in $C([-\tau,0],R^2)$.

For the convenience of statement, we make the following hypotheses:

(P1) $f_1(\hat{0})=f_1(\hat{K})=0$ and $f_2(\hat{0})=f_2(\hat{K})=0$ with $\hat{0}=(0,0)$, $\hat{K}=(k_1,k_2)$, where $k_1>0$, $k_2>0$.

(P2) $g:[0,K_0]\to R$ is continuously differentiable, monotonically increasing, and $0\leqslant g'(x)\leqslant g'(0)$, $g(0)=0$, where $K_0=\max\{k_1,k_2\}$.

(P3) There exist two positive constants $L_1 > 0$ and $L_2 > 0$ such that

$$|f_1(\phi_{1t}, \psi_{1t}) - f_1(\phi_{2t}, \psi_{2t})| \leqslant L_1 \|\Phi - \Psi\|,$$
$$|f_2(\phi_{1t}, \psi_{1t}) - f_2(\phi_{2t}, \psi_{2t})| \leqslant L_2 \|\Phi - \Psi\|$$

for $\Phi = (\phi_1, \psi_1), \Psi = (\phi_2, \psi_2) \in C([-\tau, 0], R)$ with $0 \leqslant \phi_i(s), \psi_i(s) \leqslant K, s \in [-\tau, 0], i = 1, 2$.

(P4) $f_2(\phi, \psi) = \psi(0)[h(\psi) + a\phi(0)]$, where the functional $h(\phi)$ is continuous and $a > 0$.

Define operators $H_1, H_2 : C(R^2, R) \to C(R^2, R)$ by

$$H_1(\phi, \psi)(t) = f_1(\phi_t, \psi_t) + \beta_1\phi(t) + \sum_{j=1}^{m} a_j[g(\phi(t + jc)) - 2g(\phi(t)) + g(\phi(t - jc))],$$

$$H_2(\phi, \psi)(t) = f_2(\phi_t, \psi_t) + \beta_2\psi(t) + \sum_{j=1}^{m} b_j[g(\psi(t + jc)) - 2g(\psi(t)) + g(\psi(t - jc))].$$

Then, (2.1) can be rewritten as following:

$$\begin{cases} \frac{d\phi}{dt} = -\beta_1\phi + H_1(\phi, \psi), \\ \frac{d\psi}{dt} = -\beta_2\psi + H_2(\phi, \psi). \end{cases} \tag{2.4}$$

Define the operators $F_1, F_2 : C(R^2, R) \to C(R^2, R)$ by

$$F_1(\phi, \psi)(t) = e^{-\beta_1 t} \int_{-\infty}^{t} e^{\beta_1 s} H_1(\phi, \psi)(s)\, ds,$$
$$F_2(\phi, \psi)(t) = e^{-\beta_2 t} \int_{-\infty}^{t} e^{\beta_2 s} H_2(\phi, \psi)(s)\, ds.$$

We can see that F_1 and F_2 are well defined and for any $(\phi(t), \psi(t)) \in C(R, R^2)$, F_1 and F_2 satisfy

$$\begin{aligned} F_1'(\phi, \psi)(t) &= -\beta_1 F_1(\phi, \psi)(t) + H_1(\phi, \psi)(t), \\ F_2'(\phi, \psi)(t) &= -\beta_2 F_2(\phi, \psi)(t) + H_2(\phi, \psi)(t). \end{aligned} \tag{2.5}$$

In the following, we introduce the exponential decay norm. Let $\mu \in (0, \min\{\beta_1, \beta_2\})$ and equip $C(R, R^2)$ with the norm $|\cdot|_\mu$ defined by $|\phi|_\mu = \sup_{t\in R} |\phi(t)|e^{-\mu|t|}$. Define

$$B_\mu(R, R^2) = \left\{ \phi \in C(R, R^2) : \sup_{t \in R} |\phi(t)|e^{-\mu|t|} < \infty \right\}.$$

Then $B_\mu(R, R^2)$ is a Banach space.

3. Partially quasimonotone case

In this section, we study the existence of traveling wave solution of (2.1) when the delayed reaction terms f_1 and f_2 are partially monotonic, i.e. they satisfy the following condition:

(PQM) there exist two constants $\beta_1 > 0$ and $\beta_2 > 0$ such that

$$\begin{cases} f_1(\phi_{1t}, \psi_{1t}) - f_1(\phi_{2t}, \psi_{1t}) + \beta_1[\phi_1(0) - \phi_2(0)] \geqslant 2A[g(\phi_1(0)) - g(\phi_2(0))], \\ f_1(\phi_{1t}, \psi_{1t}) - f_1(\phi_{1t}, \psi_{2t}) \leqslant 0, \\ f_2(\phi_{1t}, \psi_{1t}) - f_2(\phi_{2t}, \psi_{2t}) + \beta_2[\psi_1(0) - \psi_2(0)] \geqslant 2B[g(\psi_1(0)) - g(\psi_2(0))] \end{cases}$$

for $\phi_1, \phi_2, \psi_1, \psi_2 \in C([-\tau, 0], R)$ with $0 \leqslant \phi_2(s) < \phi_1(s) \leqslant k_1$, $0 < \psi_2(s) \leqslant \psi_1(s) \leqslant k_2$, $s \in [-\tau, 0]$, where $A = \sum_{j=1}^{m} a_j$ and $B = \sum_{j=1}^{m} b_j$.

Remark 3.1. Comparing (QM) and (PQM), we can see that f_2 satisfies the same monotone condition while f_1 satisfies a weaker monotone condition in (PQM). Thus, if f_1 and f_2 satisfy (QM), then they satisfy (PQM), but the inverse does not hold.

Definition 3.2. If the continuous functions $(\bar{\phi}(t), \bar{\psi}(t))$ and $(\underline{\phi}(t), \underline{\psi}(t)) : R \to R^2$ are differentiable almost everywhere and satisfy

$$\begin{cases} \bar{\phi}'(t) \geqslant \sum_{j=1}^{m} a_j[g(\bar{\phi}(t + jc)) - 2g(\bar{\phi}(t)) + g(\bar{\phi}(t - jc))] + f_1(\bar{\phi}_t, \underline{\psi}_t), \\ \bar{\psi}'(t) \geqslant \sum_{j=1}^{m} b_j[g(\bar{\psi}(t + jc)) - 2g(\bar{\psi}(t)) + g(\bar{\psi}(t - jc))] + f_2(\bar{\phi}_t, \bar{\psi}_t) \end{cases} \tag{3.1}$$

and

$$\begin{cases} \underline{\phi}'(t) \leqslant \sum_{j=1}^{m} a_j[g(\underline{\phi}(t + jc)) - 2g(\underline{\phi}(t)) + g(\underline{\phi}(t - jc))] + f_1(\underline{\phi}_t, \bar{\psi}_t), \\ \underline{\psi}'(t) \leqslant \sum_{j=1}^{m} b_j[g(\underline{\psi}(t + jc)) - 2g(\underline{\psi}(t)) + g(\underline{\psi}(t - jc))] + f_2(\underline{\phi}_t, \underline{\psi}_t), \end{cases} \tag{3.2}$$

then $(\bar{\phi}(t), \bar{\psi}(t))$ is called *an upper solution* and $(\underline{\phi}(t), \underline{\psi}(t))$ is called a *lower solution* of (2.1).

In what follows, we assume that an upper solution $\bar{\Phi}(t) = (\bar{\phi}(t), \bar{\psi}(t))$ and a lower solution $\underline{\Psi}(t) = (\underline{\phi}(t), \underline{\psi}(t))$ of (2.1) are given so that

(A1) $(0, 0) \leqslant (\underline{\phi}(t), \underline{\psi}(t)) \leqslant (\bar{\phi}(t), \bar{\psi}(t)) \leqslant (k_1, k_2)$;
(A2) $\lim_{t \to -\infty}(\underline{\phi}(t), \underline{\psi}(t)) = (0, 0)$, $\lim_{t \to \infty}(\bar{\phi}(t), \bar{\psi}(t)) = (k_1, k_2)$.

The operator $H = (H_1, H_2)$ defined in Section 2 enjoys the following nice properties.

Lemma 3.3. *Assume that* (P1)–(P3) *and* (PQM) *hold. We have*

(i) $0 \leqslant H_2(\phi, \psi)(t) \leqslant f_2(\tilde{k_1}, \tilde{k_2}) + \beta_2 k_2$ *for* $(\phi, \psi) \in C_{[0,K]}(R, R^2)$;
(ii) $H_2(\phi, \psi)(t)$ *is nondecreasing in* $t \in R$ *if* $(\phi, \psi) \in C_{[0,K]}(R, R^2)$ *is nondecreasing in* $t \in R$;
(iii) $H_1(\phi_2, \psi_1)(t) \leqslant H_1(\phi_1, \psi_1)(t)$ *and* $H_2(\phi_2, \psi_2)(t) \leqslant H_2(\phi_1, \psi_1)(t)$ *for* $t \in R$ *if* $(\phi_1, \psi_1), (\phi_2, \psi_2) \in C_{[0,K]}(R, R^2)$ *satisfy* $\phi_2(t) \leqslant \phi_1(t), \psi_2(t) \leqslant \psi_1(t)$ *for* $t \in R$.

Proof. By (P2) and (PQM), direct calculation shows that

$$\begin{aligned}
&H_1(\phi_1, \psi_1)(t) - H_1(\phi_2, \psi_1)(t)\\
&\quad = f_1(\phi_{1t}, \psi_{1t}) - f_1(\phi_{2t}, \psi_{1t}) + \beta_1[\phi_1(t) - \phi_2(t)]\\
&\qquad + \sum_{j=1}^{m} a_j\{[g(\phi_1(t+jc)) - g(\phi_2(t+jc))] - 2[g(\phi_1(t)) - g(\phi_2(t))]\\
&\qquad + [g(\phi_1(t-jc)) - g(\phi_2(t-jc))]\}\\
&\quad \geqslant 2A[g(\phi_1(t)) - g(\phi_2(t))]\\
&\qquad + \sum_{j=1}^{m} \{a_j[g(\phi_1(t+jc)) - g(\phi_2(t+jc))] - 2[g(\phi_1(t)) - g(\phi_2(t))]\\
&\qquad + [g(\phi_1(t-jc)) - g(\phi_2(t-jc))]\}\\
&\quad \geqslant 0,
\end{aligned}$$

which implies that $H_1(\phi_1, \psi_1)(t) \geqslant H_1(\phi_2, \psi_1)(t)$.

Following a similar argument as in Wu and Zou [20, Proposition 3.1], we can verify the proposition for $H_2(\phi, \psi)$. This completes the proof. □

We have the following lemma for $F = (F_1, F_2)$ defined in Section 2, which is a direct consequence of Lemma 3.3.

Lemma 3.4. *Assume that* (P1)–(P3) *and* (PQM) *hold. Then*

(i) $F_2(\phi, \psi)(t)$ *is nondecreasing in* R *if* $(\phi, \psi)(t) \in C_{[0,K]}(R, R^2)$ *is nondecreasing in* $t \in R$;
(ii) $F_1(\phi_2, \psi_1)(t) \leqslant F_1(\phi_1, \psi_1)(t)$ *and* $F_2(\phi_2, \psi_2)(t) \leqslant F_2(\phi_1, \psi_1)(t)$ *for* $t \in R$ *if* $(\phi_1, \psi_1), (\phi_2, \psi_2) \in C_{[0,K]}(R, R^2)$ *satisfy* $\phi_2(t) \leqslant \phi_1(t), \psi_2(t) \leqslant \psi_1(t)$ *for* $t \in R$.

Now, we assume that there exist an upper solution $(\bar{\phi}(t), \bar{\psi}(t))$ and a lower solution $(\underline{\phi}(t), \underline{\psi}(t))$ of (2.1) satisfying

(A3) $\sup_{s \leqslant t} \underline{\phi}(s) < \bar{\phi}(t), \sup_{s \leqslant t} \underline{\psi}(s) < \bar{\psi}(t)$.

Define

$$\Gamma([\underline{\phi}, \underline{\psi}], [\bar{\phi}, \bar{\psi}]) := \left\{ (\phi, \psi) \in C(R, R^2) \begin{array}{ll} \text{(i)} & \psi(t) \text{ is nondecreasing in } R, \\ \text{(ii)} & \underline{\phi}(t) \leqslant \phi(t) \leqslant \bar{\phi}(t) \text{ and } \underline{\psi}(t) \leqslant \psi(t) \leqslant \bar{\psi}(t) \end{array} \right\}.$$

We can see that $\Gamma([\underline{\phi}, \underline{\psi}], [\bar{\phi}, \bar{\psi}])$ is nonempty. In fact, let $\phi_0(t) = \sup_{s \leqslant t} \underline{\phi}(s)$, $\psi_0(t) = \sup_{s \leqslant t} \underline{\psi}(s)$. Then $(\phi_0(t), \psi_0(t)) \in \Gamma([\underline{\phi}, \underline{\psi}], [\bar{\phi}, \bar{\psi}])$.

Lemma 3.5. *Assume that* (P1)–(P3) *hold, then* $F = (F_1, F_2) : B_\mu(R, R^2) \to B_\mu(R, R^2)$ *is continuous with respect to the norm* $|\cdot|_\mu$.

Proof. We first prove that $H_1 : B_\mu(R, R^2) \to B_\mu(R, R^2)$ is continuous. If $\Phi = (\phi_1, \psi_1)$, $\Psi = (\phi_2, \psi_2) \in B_\mu(R, R^2)$ satisfy

$$|\Phi - \Psi|_\mu = \sup_{t \in R} |\Phi(t) - \Psi(t)| e^{-\mu|t|} < \delta,$$

then

$$\begin{aligned}
&|H_1(\phi_1, \psi_1)(t) - H_1(\phi_2, \psi_2)(t)| \\
&\quad = \Big| f_1(\phi_{1t}, \psi_{1t}) - f_1(\phi_{2t}, \psi_{2t}) + \beta_1(\phi_1(t) - \phi_2(t)) \\
&\qquad + \sum_{j=1}^m a_j \{[g(\phi_1(t + r_j)) - g(\phi_2(t + r_j))] - 2[g(\phi_2(t)) - g(\phi_1(t))] \\
&\qquad + [g(\phi_2(t - r_j)) - g(\phi_1(t - r_j))]\} \Big| \\
&\quad \leqslant |f_1(\phi_{1t}, \psi_{1t}) - f_1(\phi_{2t}, \psi_{2t})| + \beta_1 |\phi_1(t) - \phi_2(t)| \\
&\qquad + \sum_{j=1}^m a_j |g(\phi_1(t + r_j)) - g(\phi_2(t + r_j))| + 2 \sum_{j=1}^m a_j |g(\phi_2(t)) - g(\phi_1(t))| \\
&\qquad + \sum_{j=1}^m a_j |g(\phi_1(t - r_j)) - g(\phi_2(t - r_j))| \\
&\quad \leqslant |f_1(\phi_{1t}, \psi_{2t}) - f_1(\phi_{2t}, \psi_{2t})| + (\beta_1 + 2Ag'(0)) |\phi(t) - \psi(t)| \\
&\qquad + \sum_{j=1}^m a_j g'(0) [|\phi_1(t + r_j) - \phi_2(t + r_j)| + |\phi_1(t - r_j) - \phi_2(t - r_j)|].
\end{aligned}$$

We can check that

$$\begin{aligned}
&|\phi_1(t - r_j) - \phi_2(t - r_j)| e^{-\mu|t|} + |\phi_1(t + r_j) - \phi_2(t + r_j)| e^{-\mu|t|} \\
&\quad \leqslant 2e^{\mu r_j} |\phi_1(s) - \phi_2(s)|_\mu.
\end{aligned} \tag{3.3}$$

Thus, it follows that

$$
\begin{aligned}
&|H_1(\phi_1,\psi_1)(t)-H_1(\phi_2,\psi_2)(t)|\mathrm{e}^{-\mu|t|}\\
&\quad\leqslant |f_1(\phi_{1t},\psi_{1t})-f_1(\phi_{2t},\psi_{2t})|\mathrm{e}^{-\mu|t|}+(\beta_1+2Ag'(0))|\phi_1(t)-\phi_2(t)|\mathrm{e}^{-\mu|t|}\\
&\qquad+\sum_{j=1}^m a_j g'(0)[|\phi_1(t+r_j)-\phi_2(t+r_j)|+|\phi_1(t-r_j)-\phi_2(t-r_j)|]\mathrm{e}^{-\mu|t|}\\
&\quad\leqslant |f_1(\phi_{1t},\psi_{1t})-f_1(\phi_{1t},\psi_{2t})|\mathrm{e}^{-\mu|t|}\\
&\qquad+\left(\beta_1+2Ag'(0)+2\sum_{j=1}^m(a_j\mathrm{e}^{\mu r_j})g'(0)\right)|\phi_1(t)-\phi_2(t)|_\mu\\
&\quad\leqslant L_1\|\Phi_t-\Psi_t\|_{X_c}\mathrm{e}^{-\mu|t|}\\
&\qquad+\left(\beta_1+2Ag'(0)+2\sum_{j=1}^m(a_j\mathrm{e}^{\mu r_j})g'(0)\right)|\phi_1(t)-\phi_2(t)|_\mu\\
&\quad\leqslant L_1\sup_{s\in[-c\tau,0]}|\phi_1(t+s)-\phi_1(t+s)|\mathrm{e}^{-\mu|t|}\\
&\qquad+\left(\beta_1+2Ag'(0)+2\sum_{j=1}^m(a_j\mathrm{e}^{\mu r_j})g'(0)\right)|\phi_1(t)-\phi_2(t)|_\mu\\
&\quad\leqslant L_1\sup_{\theta\in R}|\Phi(\theta)-\Psi(\theta)|\mathrm{e}^{-\mu|\theta|}\mathrm{e}^{\mu|s|}\\
&\qquad+\left(\beta_1+2Ag'(0)+2\sum_{j=1}^m(a_j\mathrm{e}^{\mu r_j})g'(0)\right)|\phi_1(t)-\phi_2(t)|_\mu\\
&\quad\leqslant\left(L_1\mathrm{e}^{\mu c\tau}+\beta_1+2Ag'(0)+2\sum_{j=1}^m(a_j\mathrm{e}^{\mu r_j})g'(0)\right)|\Phi(t)-\Psi(t)|_\mu.
\end{aligned}
$$

Denote $N=L_1\mathrm{e}^{\mu c\tau}+\beta_1+2Ag'(0)+2g'(0)\sum_{j=1}^m(a_j\mathrm{e}^{\mu r_j})$. For any fixed $\varepsilon>0$, let δ be such that $\delta<\varepsilon/N$. If $\Phi,\Psi\in B_\mu(R,R^2)$ satisfy $|\Phi-\Psi|_\mu=\sup_{t\in R}|\Phi(t)-\Psi(t)|\mathrm{e}^{-\mu|t|}<\delta$, then

$$
\begin{aligned}
&|H_1(\phi_1,\psi_1)(t)-H_1(\phi_2,\psi_2)(t)|_\mu\\
&\quad\leqslant\left(L_1\mathrm{e}^{\mu c\tau}+\beta_1+2Ag'(0)+2g'(0)\sum_{j=1}^m(a_j\mathrm{e}^{\mu r_j})\right)|\Phi(t)-\Psi(t)|_\mu\\
&\quad=N|\Phi(t)-\Psi(t)|_\mu\\
&\quad\leqslant N\delta\leqslant\varepsilon,
\end{aligned}
$$

which implies that $H_1:\ B_\mu(R,R^2)\to B_\mu(R,R^2)$ is continuous.

Next, we need to estimate $|F_1(\phi_1,\psi_1)(t)-F_1(\phi_2,\psi_2)(t)|$. By the definition of $F_1(\phi,\psi)(t)$, we have

$$
\begin{aligned}
&|F_1(\phi_1,\psi_1)(t)-F_1(\phi_2,\psi_2)(t)|\\
&\quad=\left|\mathrm{e}^{-\beta_1 t}\int_{-\infty}^{t}\mathrm{e}^{\beta_1 s}H_1(\phi_1,\psi_1)(s)\,\mathrm{d}s-\mathrm{e}^{-\beta_1 t}\int_{-\infty}^{t}\mathrm{e}^{\beta_1 s}H_1(\phi_2,\psi_2)(s)\,\mathrm{d}s\right|\\
&\quad\leqslant \mathrm{e}^{-\beta_1 t}\int_{-\infty}^{t}\mathrm{e}^{\beta_1 s}|H_1(\phi_1,\psi_1)(s)-H_1(\phi_2,\psi_2)(s)|\,\mathrm{d}s\\
&\quad\leqslant |H_1(\phi_1,\psi_1)(s)-H_1(\phi_2,\psi_2)(s)|_\mu \mathrm{e}^{-\beta_1 t}\int_{-\infty}^{t}\mathrm{e}^{\beta_1 s+\mu|s|}\,\mathrm{d}s.
\end{aligned}
$$

(a) If $t<0$, we have

$$
|F_1(\phi_1,\psi_1)(t)-F_1(\phi_2,\psi_2)(t)|\leqslant\frac{1}{\beta_1-\mu}\mathrm{e}^{-\mu t}|H_1(\phi_1,\psi_1)(s)-H_1(\phi_2,\psi_2)(s)|_\mu.
$$

It follows that

$$
|F_1(\phi_1,\psi_1)(t)-F_1(\phi_2,\psi_2)(t)|\mathrm{e}^{-\mu|t|}\leqslant\frac{1}{\beta_1-\mu}|H_1(\phi_1,\psi_1)(t)-H_1(\phi_2,\psi_2)(t)|_\mu.
$$

(b) If $t>0$, it follows that

$$
\begin{aligned}
&|F_1(\phi_1,\psi_1)(t)-F_1(\phi_2,\psi_2)(t)|\\
&\quad\leqslant \mathrm{e}^{-\beta_1 t}\left[\frac{1}{\beta_1-\mu}-\frac{1}{\beta_1+\mu}+\frac{1}{\beta_1+\mu}\mathrm{e}^{(\beta_1+\mu)t}\right||H_1(\phi_1,\psi_1)(t)-H_1(\phi_2,\psi_2)(t)|_\mu.
\end{aligned}
$$

Therefore, we have that

$$
\begin{aligned}
&|F_1(\phi_1,\psi_1)(t)-F_2(\phi_2,\psi_2)(t)|\mathrm{e}^{-\mu|t|}\\
&\quad\leqslant\left\{\left[\frac{1}{\beta_1-\mu}-\frac{1}{\beta_1+\mu}\right]\mathrm{e}^{-(\beta_1+\mu)t}+\frac{1}{\beta_1+\mu}\right\}|H_1(\phi_1,\psi_1)(t)-H_1(\phi_2,\psi_2)(t)|_\mu\\
&\quad\leqslant\frac{1}{\beta_1-\mu}|H_1(\phi_1,\psi_1)(t)-H_1(\phi_2,\psi_2)(t)|_\mu.
\end{aligned}
$$

Thus, by using the fact that $H_1(\Phi)(t)-H_1(\Psi)(t)$ is continuous in $B_\rho(R,R^2)$, it follows that $F_1(\Phi)(t)-F_1(\Psi)(t)$ is continuous with respect to the norm $|\cdot|_\mu$.

By using a similar argument as above, we can also prove that $F_2(\Phi)(t)-F_2(\Psi)(t)$ is continuous with respect to the norm $|\cdot|_\mu$. This completes the proof. □

Lemma 3.6. *If* (P1)–(P3) *and* (PQM) *hold, then* $F(\Gamma([\underline{\phi},\underline{\psi}],[\bar{\phi},\bar{\psi}]))\subset\Gamma([\underline{\phi},\underline{\psi}],[\bar{\phi},\bar{\psi}])$.

Proof. We first claim that

$$
F_1(\underline{\phi},\bar{\psi})\leqslant F_1(\phi,\psi)\leqslant F_1(\bar{\phi},\underline{\psi}),\qquad F_2(\underline{\phi},\underline{\psi})\leqslant F_2(\phi,\psi)\leqslant F_2(\bar{\phi},\bar{\psi}). \tag{3.4}
$$

In fact, for any $(\phi, \psi) \in [\underline{\phi}, \bar{\phi}] \times [\underline{\psi}, \bar{\psi}]$, by (A2) and (P2) and Lemma 3.3(ii), it follows that

$$\begin{aligned}
&H_1(\bar{\phi}, \underline{\psi})(t) - H_1(\phi, \psi)(t)\\
&\quad = f_1(\bar{\phi}_t, \underline{\psi}_t) - f_1(\phi_t, \psi_t) + \beta_1[\bar{\phi}(t) - \phi(t)]\\
&\qquad + \sum_{j=1}^{m} a_j\{[g(\bar{\phi}(t+jc)) - g(\phi(t+jc))] - 2[g(\bar{\phi}(t)) - g(\phi(t))]\\
&\qquad + [g(\bar{\phi}(t-jc)) - g(\phi(t-jc))] - 2[g(\bar{\phi}(t)) - g(\phi(t))]\}\\
&\quad = [f_1(\bar{\phi}_t, \underline{\psi}_t) - f_1(\phi_t, \underline{\psi}_t)] + \beta_1[\bar{\phi}(t) - \phi(t)] + [f_1(\phi_t, \underline{\psi}_t) - f_1(\phi_t, \psi_t)]\\
&\qquad + \sum_{j=1}^{m} a_j\{[g(\bar{\phi}(t+jc)) - g(\phi(t+jc))] - 2[g(\bar{\phi}(t)) - g(\phi(t))]\\
&\qquad + [g(\bar{\phi}(t-jc)) - g(\phi(t-jc))]\}\\
&\quad \geqslant 2A[g(\bar{\phi}(t)) - g(\phi(t))] + f_1(\phi_t, \underline{\psi}_t) - f_1(\phi_t, \psi_t)\\
&\qquad + \sum_{j=1}^{m} a_j\{[g(\bar{\phi}(t+jc)) - g(\phi(t+jc))] - 2[g(\bar{\phi}(t)) - g(\phi(t))]\\
&\qquad + [g(\bar{\phi}(t-jc)) - g(\phi(t-jc))]\}\\
&\quad \geqslant 0,
\end{aligned}$$

which implies that $H_1(\bar{\phi}, \underline{\psi}) \geqslant H_1(\phi, \psi)$. Similarly, $H_1(\underline{\phi}, \bar{\psi}) \leqslant H_1(\phi, \psi)$. Repeating the above argument shows that $F_2(\underline{\phi}, \underline{\psi}) \leqslant F_2(\phi, \psi) \leqslant F_2(\bar{\phi}, \bar{\psi})$. Thus, (3.4) is proved.

Since $(\bar{\phi}(t), \bar{\psi}(t))$ is an upper solution, we have

$$\bar{\phi}'(t) + \beta_1\bar{\phi}(t) - H_1(\bar{\phi}, \underline{\psi})(t) \geqslant 0. \tag{3.5}$$

Recalling (2.5) for $(\bar{\phi}(t), \underline{\psi}(t))$ implies that

$$F_1'(\bar{\phi}, \underline{\psi})(t) + \beta_1 F_1(\bar{\phi}, \underline{\psi})(t) - H_1(\bar{\phi}, \underline{\psi})(t) = 0. \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$[F_1(\bar{\phi}, \underline{\psi})(t) - \bar{\phi}(t)]' + \beta_1[F_1(\bar{\phi}, \underline{\psi})(t) - \bar{\phi}(t)] \leqslant 0. \tag{3.7}$$

Let $w(t) = F_1(\bar{\phi}, \underline{\psi})(t) - \bar{\phi}(t)$ and denote $r(t) = w'(t) + \beta_1 w(t)$. Then, it follows from (3.7) that $r(t) \leqslant 0$. Now, by using the fundamental theorem of first-order ordinary differential equation, we obtain that

$$w(t) = c_1 \mathrm{e}^{-\beta_1 t} + \int_{-\infty}^{t} \mathrm{e}^{-\beta_1(t-s)} r(s)\,\mathrm{d}s.$$

Since $w(t)$ is bounded on $(-\infty, \infty)$, we have $c_1=0$. Thus, $w(t)=\int_{-\infty}^{t} \mathrm{e}^{-\beta_1(t-s)} r(s)\, \mathrm{d}s \leqslant 0$, which means that

$$F_1(\bar{\phi}, \underline{\psi})(t) \leqslant \bar{\phi}(t). \tag{3.8}$$

Similarly, we can prove that

$$F_1(\underline{\phi}, \bar{\psi})(t) \geqslant \underline{\phi}(t). \tag{3.9}$$

Combining (3.4), (3.8) and (3.9) yields

$$\underline{\phi}(t) \leqslant F_1(\phi, \psi)(t) \leqslant \bar{\phi}(t).$$

For any $(\phi, \psi) \in [\underline{\phi}, \bar{\phi}] \times [\underline{\psi}, \bar{\psi}]$. Repeating the above argument, we have

$$\underline{\psi}(t) \leqslant F_2(\underline{\phi}, \underline{\psi})(t) \leqslant F_2(\phi, \psi)(t) \leqslant F_2(\bar{\phi}, \bar{\psi})(t) \leqslant \bar{\psi}(t).$$

Combining this with Lemma 3.4(ii), we can see that $F_2(\phi, \psi)$ is nondecreasing in $t \in R$. This completes the proof. □

Lemma 3.7. *If* (P2)–(P3) *and* (PQM) *hold, then operator* $F : \Gamma([\underline{\phi}, \underline{\psi}], [\bar{\phi}, \bar{\psi}]) \to \Gamma([\underline{\phi}, \underline{\psi}], [\bar{\phi}, \bar{\psi}])$ *is compact.*

Proof. We first established an estimate for F. For any $(\phi(t), \psi(t) \in \Gamma([\underline{\phi}, \underline{\psi}], [\bar{\phi}, \bar{\psi}])$, direct calculation shows

$$\begin{aligned}
0 &\leqslant F_2'(\phi, \psi)(t) \\
&= -\mu \mathrm{e}^{-\beta_1 t} \int_{-\infty}^{t} \mathrm{e}^{\beta_2 s} H_2(\phi, \psi)(s)\, \mathrm{d}s + H_2(\phi, \psi)(t) \\
&= -\beta_1 F_2(\phi, \psi)(t) + H_2(\phi, \psi)(t) \\
&\leqslant -\beta_1 F_2(0, 0) + H_2(k_1, k_2) \\
&\leqslant H_2(k_1, k_2) - H_2(0, 0) \\
&= \beta_1 k_2,
\end{aligned}$$

which implies that $F_2'(\phi, \psi)(t)$ is uniformly bounded.

From Lemma 3.6, we have $0 \leqslant \underline{\phi}(t) \leqslant F_1(\phi, \psi)(t) \leqslant \bar{\phi}(t) \leqslant k_1$. It follows that

$$\begin{aligned}
F_1'(\phi, \psi)(t) &= -\beta_1 \mathrm{e}^{-\beta_1 t} \int_{-\infty}^{t} \mathrm{e}^{\beta_1 s} H_1(\phi, \psi)(s)\, \mathrm{d}s + H_1(\phi, \psi)(t) \\
&\leqslant \bar{\phi}(t) - \beta_1 \mathrm{e}^{-\beta_1 t} \int_{-\infty}^{t} \mathrm{e}^{\beta_1 s} \underline{\phi}(s)\, \mathrm{d}s \\
&\leqslant \bar{\phi}(t) \leqslant k_1
\end{aligned}$$

and

$$
\begin{aligned}
F_1'(\phi,\psi)(t) &\geqslant \underline{\phi}(t) - \beta_1 \mathrm{e}^{-\beta_1 t}\int_{-\infty}^{t} \mathrm{e}^{\beta_1 s}\bar{\phi}(s)\,\mathrm{d}s \\
&\geqslant -\beta_1 \mathrm{e}^{-\beta_1 t}\int_{-\infty}^{t} \mathrm{e}^{\beta_1 s}\bar{\phi}(s)\,\mathrm{d}s \\
&\geqslant -\beta_1 \mathrm{e}^{-\beta_1 t}\int_{-\infty}^{t} \mathrm{e}^{\beta_1 s}k_1\,\mathrm{d}s \\
&= -k_1.
\end{aligned}
$$

It implies that $|F_1'(\phi,\psi)(t)|\leqslant k_1$. Hence, $F\Gamma([\underline{\phi},\underline{\psi}],[\bar{\phi},\bar{\psi}])$ is equicontinuous on $\Gamma([\underline{\phi},\underline{\psi}],[\bar{\phi},\bar{\psi}])$.

Define $F^n(\phi,\psi)=(F_1^n(\phi,\psi),F_2^n(\phi,\psi))$, where

$$
F_i^n(\phi,\psi)(t)=\begin{cases} F_i(\phi,\psi)(t), & t\in[-n,n],\\ F_i(\phi,\psi)(-n), & t\in(-\infty,-n),\\ F_i(\phi,\psi)(n), & t\in(n,+\infty).\end{cases}\qquad i=1,2.
$$

Thus, $F^n(\phi,\psi)(t)$ is equicontinuous and uniformly bounded. Ascoli–Arzela lemma implies that $F_n(\phi,\psi)(t)$ is compact.

Since $\{F_n(\phi,\psi)(t)\}_0^\infty$ is a compact series, and

$$
\begin{aligned}
&\sup_{t\in R}|F_n(\phi,\psi)(t)-F(\phi,\psi)(t)|\mathrm{e}^{-\rho|t|}\\
&\quad=\sup_{t\in(-\infty,-n)\cup(n.\infty)}|F^n(\phi,\psi)(t)-F(\phi,\psi)(t)|\mathrm{e}^{-\mu|t|}\\
&\quad\leqslant 2K\mathrm{e}^{-\rho n}\to 0,\quad n\to\infty.
\end{aligned}
$$

By using Proposition 2.12 in [23], we know that $\{G_n(\phi,\psi)(t)\}_0^\infty$ is convergent to $F(\phi,\psi)(t)$ in $\Gamma([\underline{\phi},\underline{\psi}],[\bar{\phi},\bar{\psi}])$ with respect to the norm $|\cdot|_\mu$. Therefore, $F(\phi,\psi)(t)$ is compact. □

Theorem 3.8. *Assume that* (P1)–(P3) *and* (PQM) *hold. In addition to* (A1)–(A3), *we assume that an upper solution* $(\bar{\phi},\bar{\psi})$ *and a lower solution* $(\underline{\phi},\underline{\psi})$ *satisfy*

(A4) $f=(f_1,f_2)\neq 0$ *for any* $(\phi,\psi)\in(0,\inf_{t\in R}\bar{\phi}]\times(0,\inf_{t\in R}\bar{\psi}]\cup[\sup_{t\in R}\underline{\phi},k_1)\times[\sup_{t\in R}\underline{\psi},k_2)$.

Then (2.1) *has a traveling wave solution connecting* $(0,0)$ *and* (k_1,k_2). *Moreover, the second component of traveling wave solution is monotonically nondecreasing in* $t\in R$.

Proof. The boundedness of the set $\Gamma([\underline{\phi},\underline{\psi}],[\bar{\phi},\bar{\psi}])$ is obvious. We can see that $\Gamma([\underline{\phi},\underline{\psi}],[\bar{\phi},\bar{\psi}])$ is closed, convex and nonempty.

Since $F:C_{[0,K]}(R,R^2)\to C_{[0,K]}(R,R^2)$ is continuous with respect to the norm $|\cdot|_\mu$ in $B_\mu(R,R^2)$ from Lemma 3.5. Lemmas 3.6 and 3.7 imply that F is compact and

$F(\Gamma[\underline{\phi},\underline{\psi}],[\bar{\phi},\bar{\psi}])\subset\Gamma([\underline{\phi},\underline{\psi}],[\bar{\phi},\bar{\psi}])$. Schauder's fixed point theorem implies that there exists a fixed point $(\phi^*(t),\psi^*(t))\in\Gamma([\underline{\phi},\underline{\psi}][\bar{\phi},\bar{\psi}])$,which is a solution of (2.1).

In order to prove that this solution is a traveling wave solution, we need to verify the asymptotic boundary condition (2.3). First of all, by (P2) and the fact that $0\leqslant(\underline{\phi},\underline{\psi})(t)\leqslant(\phi^*(t),\psi^*(t))\leqslant(\bar{\phi},\bar{\psi})(t)\leqslant(k_1,k_2)$, we know

$$\lim_{t\to-\infty}(\phi^*,\psi^*)(t)=(0,0).$$

Secondly, $(\phi^*(t),\psi^*(t))\in\Gamma((\underline{\phi},\underline{\psi}),(\bar{\phi},\bar{\psi}))$ implies that $\psi^*(t)$ is monotone nondecreasing in $t\in R$, and hence, $\lim_{t\to\infty}\psi^*(t)$ exists and satisfies $k_2^*:=\lim_{t\to\infty}\psi^*(t)=\sup_{t\in R}\psi^*(t)\geqslant\sup_{t\in R}\underline{\psi}(t)>0$. Now, employing the 1' Hospital's rule to $\psi^*(t)=F_2(\phi^*(t),\psi^*(t))$, we have

$$\begin{aligned}\lim_{t\to\infty}\psi^*(t)&=\lim_{t\to\infty}F_2(\phi^*(t),\psi^*(t))\\&=\lim_{t\to\infty}\frac{\int_{-\infty}^{t}\mathrm{e}^{\beta_2 s}H_2(\phi^*,\psi^*)(s)\,\mathrm{d}s}{\mathrm{e}^{\beta_2 t}}\\&=\lim_{t\to\infty}\frac{1}{\beta_2}\left[f_2(\phi_t^*,\psi_t^*)(s)+\beta_2\psi(t)\right.\\&\qquad\left.+\sum_{j=1}^{m}b_j[g(\psi(t+jc))-2g(\psi(t))+g(\psi(t+jc))]\right]\\&=\lim_{t\to\infty}\left[\frac{f_{2c}(\phi_t^*,\psi_t^*)}{\beta_2}+\psi^*(t)\right]\\&=\lim_{t\to\infty}\frac{f_{2c}(\phi_t^*,\psi_t^*)}{\beta_2}+\lim_{t\to\infty}\psi^*(t),\end{aligned}$$

which implies that

$$\lim_{t\to\infty}f_{2c}(\phi_t^*,\psi_t^*)=0.$$

By $f_{2c}(\phi_t^*,\psi_t^*)=\psi^*(t)[h_c(\phi_t^*)+a\psi^*(t))$, we know that

$$\phi^*(t)=\frac{1}{a}\left[\frac{f_{2c}(\phi_t^*,\psi_t^*)}{\psi^*(t)}-h_c(\psi_t^*)\right]$$

which shows that $k_1^*:=\lim_{t\to\infty}\phi^*(t)$ also exists. By Proposition 2.1 in [20], we must have $(f_{1c}(\tilde{k}_1^*,\tilde{k}_2^*),\ f_{2c}(\tilde{k}_1^*,\tilde{k}_2^*))=(0,0)$. Note that (P4) implies that

$$0<\sup_{t\in R}\underline{\psi}(t)\leqslant k_2^*\leqslant k_2,$$

$$0<\sup_{t\in R}\underline{\phi}(t)=\lim_{t\to\infty}\underline{\phi}(t)\leqslant\lim_{t\to\infty}\phi^*(t)=k_1^*\leqslant k_1.$$

Again by (P4), we conclude that $k_1^* = k_1$ and $k_2^* = k_2$. Therefore, the fixed point does satisfy the boundary condition (2.4), giving a traveling wave front of (1.3). The proof is completed. □

4. Partially exponential quasimonotone case

In this section, we consider the case that (PQM) is not satisfied. We replace (PQM) by the following assumption.

(PQM*) There exist two constants $\beta_1 > 0$ and $\beta_2 > 0$ such that

$$\begin{cases} f_1(\phi_{1t}, \psi_{1t}) - f_2(\phi_{2t}, \psi_{1t}) + \beta_1[\phi_1(0) - \phi_2(0)] \geqslant 2A[g(\phi_1(0)) - g(\phi_2(0))], \\ f_1(\phi_{1t}, \psi_{1t}) - f_1(\phi_{1t}, \psi_{2t}) \leqslant 0, \\ f_2(\phi_{1t}, \psi_{1t}) - f_2(\phi_{2t}, \psi_{2t}) + \beta_2[\psi_1(0) - \psi_2(0)] \geqslant 2B[g(\psi_1(0)) - g(\psi_2(0))] \end{cases}$$

for $(\phi_1, \psi_1), (\phi_2, \psi_2) \in C([-\tau, 0], R)$ with (i) $0 \leqslant \phi_2(s) < \phi_1(s) \leqslant k_1$, $0 < \psi_2(s) \leqslant \psi_1(s) \leqslant k_2$, $s \in [-\tau, 0]$ and (ii) $e^{\beta_1 s}[\phi_1(s) - \phi_2(s)]$ and $e^{\beta_2 s}[\psi_1(s) - \psi_2(s)]$ are nondecreasing in $s \in [-\tau, 0]$, where $A = \sum_{j=1}^m a_j$ and $B = \sum_{j=1}^m b_j$.

Remark 4.1. Similarly, we can see that (QM*) implies (PQM*).

In what follows, we assume that an upper solution $(\bar{\phi}(t), \bar{\psi}(t))$ and a lower solution $(\underline{\phi}(t), \underline{\psi}(t))$ satisfy (A1)–(A2) and the following additional assumption:

(A5) The set $\Gamma([\underline{\phi}, \underline{\psi}], [\bar{\phi}, \bar{\psi}])^*$ is non-empty, where

$$\Gamma([\underline{\phi}, \underline{\psi}], [\bar{\phi}, \bar{\psi}])^* = \left\{ (\phi, \psi) \in C(R, R^2); \begin{array}{ll} \text{(i)} & \psi(t) \text{ is nondecreasing in } R; \\ \text{(ii)} & (\underline{\phi}(t), \underline{\psi}(t)) \leqslant (\phi(t), \psi(t)) \leqslant (\bar{\phi}(t), \bar{\psi}(t)); \\ \text{(iii)} & e^{\beta_1 t}[\bar{\phi}(t) - \phi(t)], e^{\beta_2 t}[\bar{\psi}(t) - \psi(t)], \\ & e^{\beta_1 t}[\phi(t) - \underline{\phi}(t)] \text{ and } e^{\beta_2 t}[\phi(t) - \underline{\phi}(t)] \\ & \text{are nondecreasing in } t \in R; \\ \text{(iv)} & e^{\beta_2 t}[\psi(t+s) - \psi(t)] \quad \text{is nondecreasing in} \\ & t \in R \text{ for every } s > 0 \end{array} \right\}.$$

The proofs of the following lemmas are similar to that of Lemmas 3.3–3.5 and are omitted here.

Lemma 4.2. *Assume that* (P1)–(P3) *and* (PQM*) *hold, then for any* $(\phi_1, \psi_1), (\phi_2, \psi_2) \in C_{[0,K]}(R, R^2)$ *with* (α) $0 \leqslant \phi_2(t) \leqslant \phi_1(t) \leqslant k_1$, $0 \leqslant \psi_2(t) \leqslant \psi_1(t) \leqslant k_2$ *for* $t \in R$; (β) $e^{\beta_1 t}[\phi_1(t) - \phi_2(t)]$ *and* $e^{\beta_2 t}[\psi_1(t) - \psi_2(t)]$ *are nondecreasing in* $t \in R$, *we have*

(i) $0 \leqslant H_2(\phi, \psi)(t) \leqslant f_2(\tilde{k_1}, \tilde{k_2}) + \beta_2 k_2$ *for* $(\phi, \psi) \in C_{[0,K]}(R, R^2)$;

(ii) $H_2(\phi,\psi)(t)$ *is nondecreasing in* $t\in R$ *if* $(\phi,\psi)\in C_{[0,K]}(R,R^2)$ *is nondecreasing in* $t\in R$;

(iii) $H_1(\phi_2,\psi_1)(t)\leqslant H_1(\phi_1,\psi_1)(t)$ *and* $H_2(\phi_2,\psi_2)(t)\leqslant H_2(\phi_1,\psi_1)(t)$ *if* (ϕ_1,ψ_1), $(\phi_2,\psi_2)\in C_{[0,K]}(R,R^2)$ *satisfy* (a) $0\leqslant\phi_2(t)\leqslant\phi_1(t)\leqslant k_1$, $0\leqslant\psi_2(t)\leqslant\psi_1(t)\leqslant k_2$ *for* $t\in R$; (b) $\mathrm{e}^{\beta_1 t}[\phi_1(t)-\phi_2(t)]$ *and* $\mathrm{e}^{\beta_2 t}[\psi_1(t)-\psi_2(t)]$ *are nondecreasing in* $t\in R$.

Lemma 4.3. *Assume that* (P1)–(P3) *and* (PQM*) *hold, then*

(i) $F_2(\phi,\psi)(t)$ *is nondecreasing in* R *if* $(\phi,\psi)(t)\in C_{[0,K]}(R,R^2)$ *is nondecreasing in* $t\in R$;

(ii) $F_1(\phi_2,\psi_1)(t)\leqslant F_1(\phi_1,\psi_1)(t)$ *and* $F_2(\phi_2,\psi_2)(t)\leqslant F_2(\phi_1,\psi_1)(t)$ *for* $t\in R$ *if* (ϕ_1,ψ_1), $(\phi_2,\psi_2)\in C_{[0,K]}(R,R^2)$ *satisfy* (a) $0\leqslant\phi_2(t)\leqslant\phi_1(t)\leqslant k_2$, $0\leqslant\psi_2(t)\leqslant\psi_1(t)\leqslant k_2$ *for* $t\in R$; (b) $\mathrm{e}^{\beta_1 t}[\phi_1(t)-\phi_2(t)]$ *and* $\mathrm{e}^{\beta_2 t}[\psi_1(t)-\psi_2(t)]$ *are non-decreasing in* $t\in R$.

Lemma 4.4. *Assume that* (P1)–(P3) *hold, then* $F=(F_1,F_2): B_\mu(R,R^2)\to B_\mu(R,R^2)$ *is continuous with respect to the norm* $|\cdot|_\mu$.

Lemma 4.5. *Assume that* (P1)–(P3) *and* (PQM*) *hold, then* $\Gamma([\underline{\psi},\underline{\psi}],[\bar{\phi},\bar{\psi}])^*$ *is a closed, bounded, and convex subset of* $B_\mu(R,R^2)$.

Proof. We firstly show that $\Gamma([\underline{\psi},\underline{\psi}],[\bar{\phi},\bar{\psi}])^*$ is closed. Let $(\phi_n,\psi_n)\in\Gamma([\underline{\phi},\underline{\psi}],[\bar{\phi},\bar{\psi}])^*$. Assume that $\{\phi_n(t)\}_1^\infty$ and $\{\psi_n(t)\}_1^\infty$ are convergent, then there exists a continuous function $(\phi(t),\psi(t))$ such that

$$\lim_{n\to+\infty}\sup_{t\in R}|\phi_n(t)-\phi(t)|\mathrm{e}^{-\mu|t|}=0,\qquad \lim_{n\to+\infty}\sup_{t\in R}|\psi_n(t)-\psi(t)|\mathrm{e}^{-\mu|t|}=0.$$

Therefore, $(\phi_n(t),\psi_n(t))$ converges to $(\phi(t),\psi(t))$ pointwisely for every $t\in R$ as $n\to+\infty$. For any $t_1,t_2\in R$, let $t_1\geqslant t_2$. Since $\phi_n(t)\in\Gamma([\underline{\psi},\underline{\psi}],[\bar{\phi},\bar{\psi}])^*$, it follows that

$$\begin{aligned}
&\mathrm{e}^{\mu t_1}[\phi_n(t_1+s)-\phi_n(t_1)]\geqslant\mathrm{e}^{\mu t_2}[\phi_n(t_2+s)-\phi_n(t_2)],\\
&\mathrm{e}^{\mu t_1}[\bar{\phi}(t_1)-\phi_n(t_1)]\geqslant\mathrm{e}^{\mu t_2}[\bar{\phi}(t_2)-\phi_n(t_2)],\\
&\mathrm{e}^{\mu t_1}[\phi_n(t_1)-\underline{\phi}(t_1)]\geqslant\mathrm{e}^{\mu t_2}[\phi_n(t_2)-\underline{\phi}(t_2)].
\end{aligned}$$

Taking limit as $n\to\infty$, we have

$$\begin{aligned}
&\mathrm{e}^{\mu t_1}[\phi(t_1+s)-\phi(t_1)]\geqslant\mathrm{e}^{\mu t_2}[\phi(t_2+s)-\phi(t_2)],\\
&\mathrm{e}^{\mu t_1}[\bar{\phi}(t_1)-\phi(t_1)]\geqslant\mathrm{e}^{\mu t_2}[\bar{\phi}(t_2)-\phi(t_2)],\\
&\mathrm{e}^{\mu t_1}[\phi(t_1)-\underline{\phi}(t_1)]\geqslant\mathrm{e}^{\mu t_2}[\phi(t_2)-\underline{\phi}(t_2)],
\end{aligned}$$

which imply that $\phi(t)$ satisfies the condition (ii) of $\Gamma([\underline{\psi},\underline{\psi}],[\bar{\phi},\bar{\psi}])^*$. It is easy to show that $\phi(t)$ satisfies the condition (i) of $\Gamma([\underline{\psi},\underline{\psi}],[\bar{\phi},\bar{\psi}])^*$. Hence $\phi(t)\in\Gamma([\underline{\psi},\underline{\psi}],[\bar{\phi},\bar{\psi}])^*$. Similarly, $\psi(t)\in\Gamma([\underline{\psi},\underline{\psi}],[\bar{\phi},\bar{\psi}])^*$.Therefore, $\Gamma([\underline{\phi},\underline{\psi}],[\bar{\phi},\bar{\psi}])^*$ is closed.

It is easy to know that $\Gamma([\underline{\phi},\underline{\psi}],[\bar{\phi},\bar{\psi}])^*$ is convex by the definition, and the boundedness is obvious. This completes the proof. $\square$

Lemma 4.6. *If* (P1)–(P3) *and* (PQM*) *hold, then* $F(\Gamma([\underline{\phi},\underline{\psi}],[\bar{\phi},\bar{\psi}])^*) \subset \Gamma([\underline{\phi},\underline{\psi}],[\bar{\phi},\bar{\psi}])^*$.

Proof. For any $(\phi(t),\psi(t)) \in \Gamma([\underline{\phi},\underline{\psi}],[\bar{\phi},\bar{\psi}])^*$, by Lemma 4.3(i), we know that $F_2(\phi,\psi)(t)$ is nondecreasing in $t \in R$. Repeating the proof of Lemma 3.6, we have

$$\underline{\phi} \leqslant F_1(\phi,\psi) \leqslant \bar{\phi}, \quad \underline{\psi} \leqslant F_2(\phi,\psi) \leqslant \bar{\psi}.$$

By a similar argument to that of Proposition 4.1 in Wu and Zou [20], we know that $e^{\beta_1 t}[\bar{\phi}(t)-F_1(\phi,\psi)(t)]$, $e^{\beta_1 t}[F_1(\phi,\psi)(t)-\underline{\phi}(t)]$, $e^{\beta_2 t}[\bar{\psi}(t)-F_2(\phi,\psi)(t)]$ and $e^{\beta_2 t}[F_2(\phi,\psi)(t)-\underline{\psi}(t)]$ are nondecreasing in $t \in R$.

Next, we verify the condition (iv) of $\Gamma([\underline{\phi},\underline{\psi}],[\bar{\phi},\bar{\psi}])^*$. For any $(\phi(t),\psi(t)) \in \Gamma([\underline{\phi},\underline{\psi}],[\bar{\phi},\bar{\psi}])^*$, $t \in R$, and $s > 0$, we have

$$\begin{aligned}
&[F_1(\phi,\psi)(t+s) - F_1(\phi,\psi)(t)]e^{\beta_1 t}\\
&\quad = e^{-\beta_1 s}\int_{-\infty}^{t+s} e^{\beta_1\theta} H_1(\phi,\psi)(\theta)\,d\theta - \int_{-\infty}^{t} e^{\beta_1\theta} H_1(\phi,\psi)(\theta)\,d\theta\\
&\quad = e^{-\beta_1 s}\int_{-\infty}^{t} e^{\beta_1(\theta+s)} H_1(\phi,\psi)(\theta+s)\,d\theta - \int_{-\infty}^{t} e^{\beta_1\theta} H_1(\phi,\psi)(\theta)\,d\theta\\
&\quad = \int_{-\infty}^{t} e^{\beta_1\theta}[H_1(\phi,\psi)(\theta+s) - H_1(\phi,\psi)(\theta)]\,d\theta.
\end{aligned}$$

By Lemma 4.2, it follows that

$$\begin{aligned}
&\frac{d}{dt}\{[F_1(\phi,\psi)(t+s) - F_1(\phi,\psi)(t)]e^{\beta_1 t}\}\\
&\quad = e^{\beta_1 t}[H_1(\phi,\psi)(t+s) - H_1(\phi,\psi)(t)] \geqslant 0,
\end{aligned}$$

which implies that $[F_1(\phi,\psi)(t+s) - F_1(\phi,\psi)(t)]e^{\beta_1 t}$ is nondecreasing in $t \in R$ for all $s > 0$. Similarly, $[F_2(\phi,\psi)(t+s) - F_2(\phi,\psi)(t)]e^{\beta_2 t}$ is also nondecreasing in $t \in R$ for all $s > 0$. Therefore, $F(\phi,\psi)(t) \in \Gamma([\underline{\phi},\underline{\psi}],[\bar{\phi},\bar{\psi}])^*$. It completes the proof. □

Lemma 4.7. *If* (PQM*) *holds, then* $F : \Gamma([\underline{\phi},\underline{\psi}],[\bar{\phi},\bar{\psi}])^* \to \Gamma([\underline{\phi},\underline{\psi}],[\bar{\phi},\bar{\psi}])^*$ *is compact.*

Proof. The proof is similar to that of Lemma 3.7 and is omitted here. □

By Lemmas 4.4–4.7 and Theorem 3.8, we have

Theorem 4.8. *Assume that* (P1)–(P4) *and* (PQM*) *hold. If there exist an upper solution* $(\bar{\phi}(t),\bar{\psi}(t))$ *and a lower solution* $(\underline{\phi}(t),\underline{\psi}(t))$ *satisfying* (A1)–(A2), (A4) *and* (A5), then (2.1) *has a traveling wave front solution with nondecreasing second component, which connects* $(0,0)$ *and* (k_1,k_2).

Remark 4.9. For some discrete predator–prey model with time delays or epidemic models with time delays, such as the model (1.6) mentioned in the introduction, we may find a suitable pair of upper and lower solutions, which satisfies the assumptions in Theorem 3.8 or Theorem 4.8, and obtain the existence of traveling wave solutions. We leave these for future consideration.

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