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Existence of traveling wave solutions in a diffusive predator-prey model

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Abstract. We establish the existence of traveling front solutions and small amplitude traveling wave train solutions for a reaction-diffusion system based on a predator-prey model with Holling type-II functional response. The traveling front solutions are equivalent to heteroclinic orbits in R^4 and the small amplitude traveling wave train solutions are equivalent to small amplitude periodic orbits in R^4. The methods used to prove the results are the shooting argument and the Hopf bifurcation theorem.

1. Introduction

The purpose of this paper is to establish the existence of traveling wave solutions and small amplitude traveling wave train solutions for a reaction-diffusion system based on a predator-prey interaction model:

$$\begin{cases} u_t = d_1 u_{xx} + Au(1 - \frac{u}{K}) - B\frac{uw}{1+Eu}, \\ w_t = d_2 w_{xx} - Cw + D\frac{uw}{1+Eu}, \end{cases} \tag{1}$$

where all parameters in (1) are positive. The functions $u(x,t)$ and $w(x,t)$ are the densities of the prey and predator, respectively, d_1 and d_2 are the diffusion coefficients, A is a growth factor for the prey species, C is the death rate for the predator in the absence of prey, K is the carrying capacity, B and D are the interaction rates for the two species, the parameter E measures the "satiation" effect: the consumption of prey by a unit number of predators cannot continue to grow linearly with

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the number of prey available but must "saturate" at the value $1/E$, see Freedman [6] and May [11]. The reaction term is a Holling type-II functional response.

The existence of traveling wave solutions in special cases of the predator-prey system (1) and its variants has been studied by many researchers. Gardner [7] proved the existence of traveling wave solutions connecting stable spatially homogeneous solutions by using a modification of the Conley index called the connection index. See also Mischaikow and Reineck [12]. Dunbar [1,2] investigated the existence of traveling wave solutions for a diffusive Lotka-Volterra model. The traveling wave solutions observed in Dunbar [1,2] are not necessarily monotone but for certain parameter values have damped oscillations at one end of the traveling wave front. Dunbar [3] considered system (1) with $d_1 = 0$ and proved the existence of periodic orbits and traveling wave solutions, that is, heteroclinic orbits connecting a point and a periodic orbit or connecting two points. We refer to Murray [13] and Volpert et al. [16] for more detailed results and references.

Numerical simulation in Owen and Lewis [14] show that system (1) with $d_1 \neq 0$ and $d_2 \neq 0$ possesses traveling wave solutions. They also mentioned that it is an interesting open problem to prove the existence and convergence of the initial data to traveling wave solutions for system (1). In this paper, we consider system (1) when $d_1 \neq 0$ and $d_2 \neq 0$ and establish the existence of traveling wave solutions and small amplitude traveling wave train solutions of system (1). The technique used to establish the existence of the traveling wave is a shooting argument in R^4 together with a Liapunov function and LaSalle's Invariance Principle. The existence of a traveling wave solution is guaranteed by the nonequivalence of a simply connected region and a punctured disk, rather than the nonequivalence of an interval and a disconnected union of two intervals, see Dunbar [1–3].

We should mention that although the techniques we use to show the existence of traveling wave solutions in this paper are similar to that in [2], there are several differences. First, it is a different model. The system considered in [2] is a Lotka-Volterra model while system (1) has a Holling type-II functional response, so it is more difficult to establish the existence of traveling wave solutions. Secondly, we construct a different Wazewski set W which is more complex. Finally, we construct a different Liapunov function to prove our result. Also, the arguments we use to establish the existence of small amplitude traveling wave train solutions are similar to that in [3]. However, in [3], since $d_1 = 0$ the traveling wave equations is a system in R^3 while we study a system in R^4, the geometry in R^4 is more complicated than in R^3.

For further simplification, taking

$$u^* = Eu, \quad w^* = \frac{Bw}{C}, \quad t' = Ct, \quad x' = \sqrt{\frac{C}{d_2}}x, \quad d = \frac{d_1}{d_2}, \quad \alpha = \frac{A}{ECK},$$
$$b = EK, \quad \beta = \frac{d_2}{EC},$$

and dropping the stars on u, w and the primes on x, t for convenience, we obtain

$$\begin{cases} u_t = du_{xx} + u[\alpha(b-u) - \frac{w}{1+u}], \\ w_t = w_{xx} - w(1 - \frac{\beta u}{1+u}). \end{cases} \tag{2}$$

There are several reasonable parameter restrictions. First, we require that $b > 1$ or equivalently that $E > \frac{1}{K}$, so that the satiation effect is great enough. We also require that $\beta > \frac{1+b}{b} > 1$, which ensures that equations (2) has a positive equilibrium point corresponding to constant coexistence of the two species. Finally, $\alpha > 0$ and $0 < d \leq 1$, the latter indicates that the prey population does not disperse faster than the predators. System (2) has three equilibrium points: $(0, 0)$, $(b, 0)$ and (u_0, w_0), which are equilibria of the corresponding ODE system without diffusion, where

$$u_0 = \frac{1}{\beta - 1}, \quad w_0 = \alpha(\frac{1}{\beta - 1} + 1)(b - \frac{1}{\beta - 1}).$$

The equilibrium point $(0, 0)$ corresponding to absence of both species is a saddle point, $(b, 0)$ corresponding to the prey at the environment carrying capacity in the absence of predators is also a saddle point, and (u_0, w_0) corresponds to co-existence of the two species. The traveling wave solution which will be established by the shooting argument is a heteroclinic orbit connecting $(b, 0)$ and (u_0, w_0).

The paper is organized as follows. In section 2 we first recall a lemma which is a variant of the Wazewski's Theorem and is the main tool in proving one of the main theorems. Then we state the main results on the existence of traveling front solutions and small amplitude traveling wave train solutions. Section 3 is devoted to the proofs of the main theorems. Finally, a brief discussion is presented in section 4.

2. Main results

In order to establish the existence of traveling wave solutions of system (2), we assume that the solutions have the special form $u(x, t) = u(x + ct)$, $w(x, t) = w(x + ct)$, where $s = x + ct$, the wave speed parameter c is positive. Then system (2) becomes

$$\begin{cases} cu' = du'' + \alpha u(b - u) - \frac{uw}{1+u}, \\ cw' = w'' - w + \frac{\beta uw}{1+u}. \end{cases} \tag{3}$$

Here $'$ denotes the differentiation with respect to the traveling wave variable s. Recalling the ecological motivation, we require that the traveling wave solutions u and w are nonnegative and satisfy the boundary conditions:

$$u(-\infty) = b, \quad u(+\infty) = u_0, \quad w(-\infty) = 0, \quad w(+\infty) = w_0. \tag{4}$$

Rewrite system (3) as a system of the first order equations in R^4

$$\begin{cases} u' = v, \\ v' = \frac{c}{d}v + \frac{\alpha}{d}u(u - b) + \frac{uw}{d(1+u)}, \\ w' = z, \\ z' = cz + w - \frac{\beta uw}{1+u}. \end{cases} \tag{5}$$

Recall the following result (see [3], pp.1069) which is a variant of the Wazewski's Theorem and is a formation and extension of the shooting method. The proof can be found in [3].

Consider the differential equation:

$$y' = f(y), \quad ' = d/ds, \quad y \in R^n, \tag{$*$}$$

where $f : R^n \to R^n$ is a continuous function and satisfies the Lipschitz condition. Let $y(s, y_0)$ be the unique solution of (*) satisfying $y(0, y_0) = y_0$. For convenience, set $y(s, y_0) = y_0 \cdot s$, let $Y \cdot S$ be the set of points $y_0 \cdot s$, where $y_0 \in Y, s \in S$.

Given $W \subseteq R^n$, define

$$W^- = \{y_0 \in W \big| \forall s > 0, y_0 \cdot [0, s) \nsubseteq W\}.$$

W^- is called *the immediate exit set of* W. Given $\Sigma \subseteq W$, let

$$\Sigma^0 = \{y_0 \in \Sigma \big| \exists s_0 = s_0(y_0) \text{ such that } y_0 \cdot s_0 \notin W\}.$$

For $y_0 \in \Sigma^0$, define

$$T(y_0) = \sup\{s \big| y_0 \cdot [0, s] \subseteq W\}.$$

$T(y_0)$ is called *an exit time*. Note that $y_0 \cdot T(y_0) \in W^-$ and $T(y_0) = 0$ if and only if $y_0 \in W^-$.

Lemma 2.1. *Suppose that*

(i) if $y_0 \in \Sigma$ and $y_0 \cdot [0, s] \subseteq cl(W)$, then $y_0 \cdot [0, s] \subseteq W$;
(ii) if $y_0 \in \Sigma$, $y_0 \cdot s \in W$, $y_0 \cdot s \notin W^-$, then there is an open set V_s about $y_0 \cdot s$ disjoint from W^-;
(iii) $\Sigma = \Sigma^0$, Σ is a compact set and intersects a trajectory of $y' = f(y)$ only once.

Then the mapping $F(y_0) = y_0 \cdot T(y_0)$ is a homeomorphism from Σ to its image on W^-.

A set $W \subseteq R^n$ satisfying the conditions (i) and (ii) is called *a Wazewski set*. Now we state the main results as follows.

Theorem 2.2. *(i) If $0 < c < \sqrt{\frac{4(b\beta-1-b)}{1+b}}$, then there are no nonnegative solutions of system (5) satisfying the boundary conditions (4).*
(ii) If $c > \sqrt{\frac{4(b\beta-1-b)}{1+b}}$, $\frac{b+1}{b} < \beta < \frac{b}{b-1}$, and $(1-\alpha)(\beta-1) \geq \frac{2\beta}{1+b}\sqrt{\frac{b\beta-1-b}{1+b}}$, then there are nonnegative solutions of (5) satisfying the boundary conditions (4), which correspond to traveling wave solutions of system (2).

Theorem 2.3. *If $\frac{b+1}{b} < \beta \leq \frac{1}{1-\sqrt{2/(1+b)}}$, then as the parameter β crosses the bifurcation curve $c^2 = \frac{1}{1+d} - \frac{d(1+d)p}{r}$ at β_0 in the (β, c)-parameter plane, where $r = \frac{\alpha(1+b)}{\beta} - \frac{2\alpha}{\beta-1} < 0$, $p = \frac{\alpha b(\beta-1)-\alpha}{d\beta} < 0$, then system (5) undergoes a Hopf bifurcation at the equilibrium point $(u_0, 0, w_0, 0)$ and there is a small amplitude periodic solution, which corresponds to a small amplitude traveling wave train solution of system (2).*

3. Proofs of the main results

3.1. Proof of Theorem 2.2

The eigenvalues of the linearization of (5) at $(b, 0, 0, 0)$ are

$$\lambda_1 = \frac{\frac{c}{d} - \sqrt{\frac{c^2}{d^2} + \frac{4\alpha b}{d}}}{2}, \qquad \lambda_2 = \frac{c - \sqrt{c^2 - \frac{4(b\beta-1-b)}{1+b}}}{2},$$

$$\lambda_3 = \frac{c + \sqrt{c^2 - \frac{4(b\beta-1-b)}{1+b}}}{2}, \qquad \lambda_4 = \frac{\frac{c}{d} + \sqrt{\frac{c^2}{d^2} + \frac{4\alpha b}{d}}}{2}.$$

If $0 < c < \sqrt{\frac{4(b\beta-1-b)}{1+b}}$, λ_2 and λ_3 are a pair of complex conjugate eigenvalues with positive real part. By Theorems 6.1 and 6.2 in [9], there is a 2-dimensional unstable manifold base at $(b, 0, 0, 0)$, the point is a *spiral* point on this unstable manifold, and the trajectory approaching $(b, 0, 0, 0)$ as $s \to -\infty$ must have $w(s) < 0$ for some s. This violates the requirement that the traveling wave solution must be nonnegative. So the first part of Theorem 2.2 is proved.

We only need to discuss the case $c \geq \sqrt{\frac{4(b\beta-1-b)}{1+b}}$ in the following. It is easy to know that $\lambda_1 < 0 < \lambda_2 < \lambda_3 < \lambda_4$, the eigenvectors e_2, e_3, e_4 associated with $\lambda_2, \lambda_3, \lambda_4$, respectively, are

$$e_2 = (1, \lambda_2, p(\lambda_2), \lambda_2 p(\lambda_2)),\ e_3 = (1, \lambda_3, p(\lambda_3), \lambda_3 p(\lambda_3)),\ e_4 = (1, \lambda_4, 0, 0),$$

where $p(\lambda) = \frac{1+b}{b}[(d-1)\lambda^2 - \frac{\beta b-1-b}{1+b} - \alpha b] < 0$. By Theorems 6.1 and 6.2 in [9], we know that there is a strongest unstable manifold Ω_1 tangent to e_4 at $(b, 0, 0, 0)$, and a parametric representation for the 1-dimension strongest unstable manifold Ω_1 in a small neighborhood of $(b, 0, 0, 0)$ is

$$f_1(m) = (b, 0, 0, 0) + me_4 + O(|m|).$$

There is also a 2-dimension strongly unstable manifold Ω_2 tangent to the span of e_4 and e_3 at $(b, 0, 0, 0)$, and a parametric representation for the 2-dimension strongly unstable manifold Ω_2 in a small neighborhood of $(b, 0, 0, 0)$ is

$$f_2(m, n) = (b, 0, 0, 0) + me_4 + ne_4 + O(|m| + |n|).$$

Finally, there is a 3-dimension unstable manifold Ω_3 tangent to the span of e_4, e_3 and e_2 at $(b, 0, 0, 0)$, and a parametric representation for the 3-dimension unstable manifold Ω_3 in a small neighborhood of $(b, 0, 0, 0)$ is

$$f_2(m, n, l) = (b, 0, 0, 0)^T + me_4 + ne_4 + le_2 + O(|m| + |n| + |l|).$$

The idea of constructing the Wazewski set W is similar to that in Dunbar [2]: it will be the complement of four blocks in R^4, two of which are chosen so that z' has the same sign as z so solutions entering these blocks would not have $z \to 0$ as $s \to \infty$, the other pair of blocks are chosen so that v' has the same sign as v and so solutions entering these blocks will not have $v \to 0$ as $s \to \infty$. Define W as follows

$$W = R^4 \backslash (P \cup Q \cup T \cup S),$$

where

$$\begin{aligned}
P &= \{(u, v, w, z) | u < u_0, w > w_0, z > 0\}, \\
Q &= \{(u, v, w, z) | u > u_0, w < w_0, z < 0\}, \\
S &= \{(u, v, w, z) | u > u_0, \alpha(u - b) + \frac{w}{1+u} > 0, v > 0\}, \\
T &= \{(u, v, w, z) | u < u_0, \alpha(u - b) + \frac{w}{1+u} < 0, v < 0\}.
\end{aligned}$$

Note that $P \cap T \neq \emptyset$, $Q \cap S \neq \emptyset$, while all other pairwise intersections are empty. We have

$$\begin{aligned}
\partial W &= (\partial P \backslash T) \cup (\partial Q \backslash S) \cup (\partial T \backslash P) \cup (\partial S \backslash Q), \\
W^- &= \partial W \backslash (\{(u_0, 0, w_0, 0)\} \cup J_1 \cup J_2), \\
N &= \{(u, v, w, z) | w = z = 0\}, \\
H &= \{(u, v, w, z) | u = v = 0\}.
\end{aligned}$$

Also

$$\begin{aligned}
J_1 = &\{(u, v, w, z) | u > u_0, w \leq 0, z = 0\} \\
&\cup \{(u, v, w, z) | u = b, w \leq w_0, v < 0, z = 0\} \\
&\cup \{(u, v, w, z) | u > u_0, \alpha(u - b) + \frac{w}{1+u} > 0, v \leq 0, z = 0\} \\
&\cup \{(u, v, w, z) | u > u_0, \alpha(u - b) + \frac{w}{1+u} = 0, w \leq w_0, v = 0, z = 0\} \\
&\cup \{(u, v, w, z) | u > u_0, \alpha(u - b) + \frac{w}{1+u} = 0, w > w_0, v < 0, z = 0\} \\
&\cup \{(u, v, w, z) | u > u_0, \alpha(u - b) + \frac{w}{1+u} = 0, w > w_0, z < 0\}, \\
J_2 = &\{(u, v, w, z) | u = 0, w \leq \alpha b, z < 0, v = 0\} \\
&\cup \{(u, v, w, z) | u = 0, w \leq w_0, z \geq 0, v = 0\} \\
&\cup \{(u, v, w, z) | u < 0, \alpha(u - b) + \frac{w}{1+u} = 0, v = 0\} \\
&\cup \{(u, v, w, z) | u < 0, \alpha(u - b) + \frac{w}{1+u} < 0, w \geq w_0, z < 0, v = 0\} \\
&\cup \{(u, v, w, z) | u < u_0, \alpha(u - b) + \frac{w}{1+u} < 0, w < w_0, v = 0\} \\
&\cup \{(u, v, w, z) | u < u_0, \alpha(u - b) + \frac{w}{1+u} = 0, z < 0, v < 0\}.
\end{aligned}$$

J_1 is the set of points on ∂W which do not exit from W into Q, T or S, this can occur in three ways. Some points in the invariant manifold N may not enter T or S immediately, of course, they will remain in N for all time and so do not enter Q.

Points on ∂W with $z = 0$, $w < 0$ will enter W from Q and so will not be immediate exit points. Points on ∂W with $\alpha(u-b) + \frac{w}{1+u} = 0, u > u_0, w > w_0, z < 0$ will not be immediate exit points.

J_2 is the set of points on W which do not exit immediately from W into P, or T, this can also occur in three ways. Some points in the invariant manifold H may not enter P immediately, they will remain in H for all time and so do not enter T. Points on ∂W with $u < 0, \alpha(u-b) + \frac{w}{1+u} < 0, v = 0$ will not be immediately exit points. Points on ∂W with $\alpha(u-b) + \frac{w}{1+u} = 0, u < u_0, z < 0$ will not be immediate exit points.

The details of proving that W^- is the set described above are tedious, we just give the proof of one part of ∂W, $\partial T \backslash P$, and the other proofs are similar.

The boundary of ∂T are $u = u_0, \alpha(u-b) + \frac{w}{1+u} = 0$ or $v = 0$. We consider the following cases to discuss $\partial T \backslash P$.

(1) $u = u_0,\ w = w_0,\ v < 0$.

(i) $z < 0$, then $w < w_0$. $v < 0$ implies that $u < u_0$. Direct calculation shows that

$$\Big[\alpha(u-b)+\frac{w}{1+u}\Big]'\Big|_{(u_0,v,w_0,z)} = \Big[v\Big(\alpha-\frac{w}{(1+u)^2}\Big)+\frac{z}{1+u}\Big]\Big|_{(u_0,v,w_0,z)} < 0.$$

Hence, the trajectory enters T.

(ii) $z = 0$, since $v < 0$, then $u < u_0$, and

$$\Big[\alpha(u-b)+\frac{w}{1+u}\Big]'\Big|_{(u_0,v,w_0,z)} = \Big[v(\alpha-\frac{w}{(1+u)^2}\Big]\Big|_{(u_0,v,w_0,z)} < 0.$$

So the trajectory also enters T.

(iii) $z > 0$, then $w > w_0,\ z > 0,\ u < u_0$, the trajectory enters P.

(2) $u = u_0,\ w < w_0,\ v < 0$.

Since $v < 0$ implies that $u < u_0$, and at the point $u = u_0$, we have

$$\alpha(u-b)+\frac{w}{1+u} = \alpha(u_0-b)+\frac{w}{1+u_0} < \alpha(u_0-b)+\frac{w_0}{1+u_0} = 0.$$

That is, the trajectory enters T.

(3) $u = u_0,\ w = w_0,\ v = 0$.

(i) $z = 0$, this is a singular point and is not in the immediate exit set.

(ii) $z > 0$, then $w > w_0$. We have

$$\begin{aligned} v' &= \frac{1}{d}\Big[cv+\alpha u(u-b)+\frac{wu}{1+u}\Big]\Big|_{(u_0,w_0)} = 0,\\ v'' &= \frac{1}{d}\Big[cv'+v\Big(\alpha(u-b)+\frac{wu}{1+u}\Big)+u[\alpha v+\frac{z}{1+u}-\frac{wv}{(1+u)^2}]\Big]\Big|_{(u_0,w_0)}\\ &= \frac{1}{d}\Big[\frac{uz}{1+u}\Big]_{(u_0,w_0)} > 0. \end{aligned}$$

This implies that u is increasing, and v has a minimum, hence $v > 0$. Since $[\alpha(u-b)+\frac{w}{1+u}]' = \frac{w'}{1+u} > 0$ and $\alpha(u_0-b)+\frac{w_0}{1+u_0} = 0$, we get $\alpha(u-b)+\frac{w}{1+u} > 0$. Therefore, the trajectory enters S.

(iii) $z < 0$, then $w < w_0$. We have

$$v' = \frac{1}{d}\Big[cv + \alpha u(u-b) + \frac{wu}{1+u}\Big]\Big|_{(u_0,w_0)} = 0,$$

$$v'' = \frac{1}{d}\Big[cv' + v\Big(\alpha(u-b) + \frac{wu}{1+u}\Big) + u[\alpha v + \frac{z}{1+u} - \frac{wv}{(1+u)^2}]\Big]\Big|_{(u_0,w_0)}$$
$$= \frac{1}{d}\Big[\frac{uz}{1+u}\Big]_{(u_0,w_0)} < 0,$$

which implies that $\alpha(u-b) + \frac{w}{1+u} < 0$. Hence, u is decreasing, v has a maximum, and $v < 0$, so the trajectory enter T.

(4) $u = u_0,\ w < w_0,\ v = 0.$
Since

$$v'|_{u=u_0} = \frac{1}{d}\Big[cv + u[\alpha(u-b) + \frac{w}{1+u}]\Big]_{u=u_0}$$
$$= \frac{1}{d}\Big[u_0[\alpha(u_0-b) + \frac{w}{1+u_0}]\Big]$$
$$< \frac{1}{d}\Big[u_0[\alpha(u_0-b) + \frac{w_0}{1+u_0}]\Big] = 0,$$

and

$$\alpha(u-b) + \frac{w}{1+u}|_{u=u_0} = \alpha(u_0-b) + \frac{w}{1+u_0}$$
$$< \alpha(u_0-b) + \frac{w_0}{1+u_0} = 0.$$

Hence, the trajectory enters T.

(5) $0 < u < u_0,\ \alpha(u-b) + \frac{w}{1+u} = 0,\ v < 0.$

(i) $z > 0$, since $w = \alpha(b-u)(1+u),\ w_0 = \alpha(b-u_0)(1+u_0)$, and $\beta < \frac{b}{1+b}$, then $u_0 > b-1$, and $w - w_0 = \alpha(u-u_0)(u+u_0+1-b) > 0$. Hence $w > w_0$, the trajectory enters P.

(ii) $z = 0$, since $\beta < \frac{b}{1+b}$, then $1 - \beta + \frac{\beta}{1+u} > 1 - \beta + \frac{\beta}{1+u_0} > 0$, and $z' = cz + w - \frac{\beta uw}{1+u} = w[1 - \beta + \frac{\beta}{1+u}] > 0$, that is, $z > 0$. Similar to that proof of (5i), we have $w > w_0$, so the trajectory enters P.

(iii) $z < 0$, the trajectory does not enter either P or T, and this is included in the portions of J_2.

(6) $0 < u < u_0,\ \alpha(u-b) + \frac{w}{1+u} = 0,\ v = 0.$

(i) $z < 0$, then w is decreasing, and $\Big[\alpha(u-b) + \frac{w}{1+u}\Big]' = \frac{z}{1+u} < 0$, that is, $\alpha(u-b) + \frac{w}{1+u} < 0$. Since $v' = \frac{1}{d}\Big[cv + \alpha u(u-b) + \frac{wu}{1+u}\Big] = 0$ and $v'' = \frac{1}{d}\Big[cv' + v[\alpha(u-b) + \frac{w}{1+u}] + u[\alpha v - \frac{wv}{(1+u)^2 + \frac{z}{1+u}}]\Big] = \frac{1}{d}\frac{uz}{1+u} < 0$, then v is decreasing and has a maximum. Hence $v < 0$, the trajectory enters T.

(ii) $z = 0$, then $z' = cz + w[1 - \frac{\beta u}{1+u}] = w[1 - \beta + \frac{\beta}{1+u}] > w[1 - \beta + \frac{\beta}{1+u_0}] = 0$, which implies that $z > 0$. Similar to (5i), we have $w > w_0$, the trajectory enters P.

(iii) $z > 0$, the trajectory enters P.

(7) $-1 < u < 0,\ \alpha(u-b) + \frac{w}{1+u} = 0,\ v = 0$.

(i) $z < 0$, then $v' = 0,\ v'' = \frac{1}{d}\frac{uz}{1+u} > 0$, so v has a minimum and $v > 0$, but $[\alpha(u-b) + \frac{w}{1+u}]' = \frac{w'}{1+u} < 0$, which implies that $\alpha(u-b) + \frac{w}{1+u} < 0$, the trajectory does not enter either $T,\ S$ or $P,\ Q$ immediately.

(ii) $z = 0$, then $z' = cz + w[1 - \beta + \frac{\beta}{1+u}] > 0$, which implies that $z > 0$, $v'' = 0,\ v''' = \frac{uz'}{1+u} < 0$, so v is decreasing and has a maximum. Hence $v < 0$. Direct calculation shows that $[\alpha(u-b) + \frac{w}{1+u}]' = \frac{z}{1+u} > 0$, that is, $\alpha(u-b) + \frac{w}{1+u} > 0$, the trajectory does not enter T. Similarly, the trajectory does not enter P, Q, S;

(iii) $z > 0$, the trajectory does not enter either T, S or P, Q immediately.

Hence, regardless of the sign of z, this is a part of J_2.

(8) $0 < u < u_0,\ \alpha(u-b) + \frac{w}{1+u} < 0,\ v = 0,\ w > w_0$.

(i) $z < 0$, then $v' = \frac{1}{d}\Big[cv + u[\alpha(u-b) + \frac{w}{1+u}]\Big] < 0$, this implies that v is decreasing and $v < 0$, so the trajectory enters T.

(ii) $z = 0$, then $z' = cz + w[1 - \beta + \frac{\beta}{1+u}] > w[1 - \beta + \frac{\beta}{1+u_0}] = 0$, that is, z is increasing, and $z > 0$. The trajectory enters P.

(iii) $z < 0$, these points are in P and not considered.

(9) $0 < u < u_0,\ \alpha(u-b) + \frac{w}{1+u} < 0,\ w < w_0,\ v = 0$.

We have $v' = \frac{1}{d}\Big[cv + u[\alpha(u-b) + \frac{w}{1+u}]\Big] < 0$, which implies that v is decreasing and $v < 0$. Hence, the trajectory enters T.

(10) $u < 0,\ \alpha(u-b) + \frac{w}{1+u} < 0,\ w \geq w_0,\ v = 0$.

(i) $z > 0$, these points are in P and will not be considered.

(ii) $z = 0$, since $z' = cz + w[1 - \frac{\beta u}{1+u}] = w[1 - \beta + \frac{\beta}{1+u}] > 0$, which implies that z is increasing, we have $z > 0$, so the trajectory enters P.

(iii) $z < 0$, then $v' = \frac{1}{d}\Big[cv + u[\alpha(u-b) + \frac{w}{1+u}]\Big] > 0$, that is, v is increasing and $v > 0$, the trajectory does not enter either T, Q or P, S immediately, this is included in J_2.

(11) $u < 0,\ \alpha(u-b) + \frac{w}{1+u} < 0,\ w < w_0,\ v = 0$.

We have $v' = \frac{1}{d}\Big[cv + u[\alpha(u-b) + \frac{w}{1+u}]\Big] > 0$, and $v > 0$, the trajectory does not enter either T, Q or P, S immediately, this is included in J_2.

(12) $u = 0,\ v = 0,\ \alpha(u-b) + \frac{w}{1+u} < 0$.

The points are on the invariant manifold H. The trajectory are the solution of the equations

$$\begin{cases} w' = z, \\ z' = cz + w. \end{cases}$$

Then $w < \alpha b$. Since $\beta < \frac{b}{b-1} < \frac{b+1}{b-1}$, we have $\alpha b - w_0 = \alpha u_0[\frac{1}{\beta-1} + 1 - b] > 0$, so $\alpha b > w_0$.

(i) $z > 0$, if $w > w_0$, then these points are in P and will not be considered; if $w \leq w_0$, then $v' = \frac{1}{d}\Big[cv + u[\alpha(u-b) + \frac{w}{1+u}]\Big] = 0,\ v'' = 0, \cdots$, $v^{(n)} = 0$. The trajectory does not enter either $T,\ Q$ or $P,\ T$ immediately, this is included in J_2.

(ii) $z = 0$, then $z' = cz + w[1 - \frac{\beta u}{1+u}] = w > 0$, z is increasing and $z > 0$. Similar to (12i), if $w > w_0$, these points are in P and will not be considered; if $w \leq w_0$, then the trajectory does not enter either T, Q or P, S immediately, this is included in J_2.

(iii) $z < 0$, the trajectory does not enter either T, Q or P, S immediately, this is included in J_2.

(13) $u = 0,\ v = 0,\ \alpha(u - b) + \frac{w}{1+u} = 0$, that is $w = \alpha b > w_0$.

(i) $z > 0$, these points are in P and will not be considered.

(ii) $z = 0$, then $z' = w > 0$, the trajectory enters P.

(iii) $z < 0$, the trajectory does not enter either T, Q or P, S immediately, this is included in J_2.

In order to use Lemma 2.1, we construct the set Σ by a series of lemmas (Lemma 3.1 to Lemma 3.6). Then we prove that there must be a trajectory through Σ which does not leave W by Lemmas 3.7 and 3.8. Finally, we choose a Liapunov function and use LaSalle's Invariance Principle to show that the trajectory approaches $(u_0, 0, w_0, 0)$.

Lemma 3.1. *Consider a solution $y(s, y_0)$ with $y_0 \in \Omega_1$, and $u_0 < b$, then there is a finite s_0 such that $u(s_0, y_0) < u_0, v(s_0, y_0) < 0$, that is, if choose m_1, m_2 such that $m_1 < \frac{c}{d} < \lambda_4 < m_2$, $m_2[u(0) - b] < v(0) < m_1[u(0) - b]$, then $m_2[u(s) - b] < v(s) < m_1[u(s) - b]$.*

Proof. Consider the system

$$\begin{cases} u' = v, \\ v' = \frac{c}{d}v + \frac{u}{d}[\alpha(u - b)]. \end{cases} \tag{6}$$

The solution of (5) in N is given by $(u(s), v(s), 0, 0)$, where $(u(s), v(s))$ is the solution of (6), so the strongest unstable manifold Ω_1 is contained in the invariant manifold N. Similarly, the strongest unstable manifold Ω_1 is contained in the invariant manifold W.

We first consider the solution of (6), it is easy to know the solution in Ω_1 must approach $(b, 0)$ tangent to the eigenvector $(-1, -\lambda_4)$ in the region $u < b, v < 0$. If the initial condition y_0 satisfies $m_2[u(0) - b] < v(0) < m_1[u(0) - b]$, take $m_1 < c/d < \lambda_4 < m_2$, then in the region $0 < u < b, v < 0$, the trajectory of a solution starting from Ω_1 satisfies

$$m_2[u(s) - b] < v(s) < m_1[u(s) - b].$$

In fact, if there exists some $s > 0$ such that $m_2[u(s) - b] \geq v(s)$, let $s_1 = \inf\{s | m_2[u(s) - b] \geq v(s)\}$. For $s \in [0, s_1)$, we have $v(s) > m_2[u(s) - b]$, $v(s_1) = m_2[u(s_1) - b]$ and $u(s) > 0$, so $v'(s_1) < m_2 u'(s_1)$. Substituting (6) into $v'(s_1) < m_2 u'(s_1)$, we obtain

$$(\frac{c}{d} - m_2)v(s_1) + \frac{\alpha}{d}u(s_1)[u(s_1) - b] < 0.$$

Using $v(s_1) = m_2[u(s_1) - b]$, we have

$$(\frac{c}{d} - m_2)m_2[u(s_1) - b] + \frac{\alpha}{d}u(s_1)[u(s_1) - b] < 0.$$

If there exists some s_2 such that $u(s_2) - b = 0, u'(s_2) \geq 0, 0 < s_2 \leq s_1$, $0 \leq u'(s_2) < 0$, then $u(s_2) < b$. Thus,

$$m_2(m_2 - \frac{c}{d}) - \frac{\alpha}{d}u(s_1) < 0.$$

Since $0 \leq u(s_1) \leq b, m_2(m_2 - \frac{c}{d}) - \frac{\alpha b}{d} < 0$, we have $m_2 < \frac{c/d+\sqrt{(c/d)^2+4b\alpha/d}}{2} = \lambda_4$, this is a contradiction to the choice of $m_2 > \lambda_4$. So we have $m_2[u(s)-b] < v(s)$. Similarly, we can prove that

$$v(s) < m_1[u(s) - b].$$

Since $u' = v$, integrating yields

$$b - c_1e^{m_2s} < u(s) < b - c_2e^{m_1s}$$

for s such that $u(s)$ satisfies $0 < u(s) < b$. Then for s_0 large enough, $u(s_0) < u_0, v(s_0) < 0$.

Using the vector field, the outward direction of the line $m_2[u(s)-b]-v(s) = 0$ is denoted as $n_t = \{m_2, -1\}$, then $n_t \cdot F' = [b - u(s)][-m_2 + \frac{c}{d}m_2 - \frac{\alpha b}{d}] < 0$ in the region $u(s) < b, v < 0$. So the trajectory enters transversally into this region. Similarly, we can prove the trajectory transversally intersects the line $v(s) = m_1[u(s) - b]$.

Lemma 3.2.

(i) A solution $y(s, y_0)$ on Ω_1 which approaches $(b, 0, 0, 0)$ as $s \to -\infty$ in the region $u > b, v > 0$ will remain in that region for all s.
(ii) Any trajectory which has a point such that $w(0) > 0, z(0) > \frac{c}{2}w(0)$ will have $w(s) > 0$ and $z(s) > \frac{c}{2}w(s)$ for all $s > 0$ such that $u \leq b$.

Proof. It is easy to check that the outward direction of the line $v = 0 (u > b)$ is $n_t = \{0, 1\}$, so $n_t \cdot F' = \frac{\alpha}{d}u(u - b] > 0$. The outward direction of line $u = b, (v > 0)$ is $n_t = \{1, 0\}$, so $n_t \cdot F' = \frac{c}{d}v > 0$. Thus, the region $u > b, v > 0$ is an invariant region.

Suppose to the contrary that there exists an s such that $u(s) < b$, but $z(s) \leq \frac{c}{2} w(s)$, let $s_1 = inf\{s|z(s) \leq \frac{c}{2}, u(s) < b\}$. Since $w(0) > 0$, for $s \in [0, s_1)$, $w'(s) = z(s) > \frac{c}{2}$, we have $w(s_1) > 0, z'(s_1) - \frac{c}{2}w'(s) \leq 0$. Using that $z(s_1) = (c/2)w(s_1)$, we obtain $\frac{c^2}{4}+1-\beta+\frac{\beta}{1+u(s)} \leq 0$. Since $u(s) \leq b$, it follows that $\frac{c^2}{4}+1-\beta+\frac{\beta}{1+b} < 0$, that is, $c^2 \leq \frac{4(b\beta-b-1)}{1+b}$, a contradiction with $c^2 > \frac{4(b\beta-b-1)}{1+b}$. This completes the proof of Lemma 3.2.

Lemma 3.3. *Let $y(s)$ be a solution approaching $(b,0,0,0)$ and tangent to e_3 in the region where $u<b$ as $s\to-\infty$. Suppose that $(1-\alpha)(\beta-1)\geq\frac{2\beta}{1+b}\sqrt{\frac{b\beta-1-b}{1+b}}$ and $u(s)$ is decreasing until $y(s)$ enters the region*

$$T=\{(u,v,w,z)|u>u_0,w>0,\alpha(u-b)+\frac{w}{1+u}<0\}.$$

Then the solution must satisfy $v(s)<-\frac{c}{2\alpha(1+b)}w(s)$.

Proof. The solution $y(s)$ approaches $(b,0,0,0)$ and is tangent to e_3. The eigenvector e_3 at $(b,0,0,0)$ has components $v=\lambda_3(u-b)$, $w=p(\lambda_3)(u-b)$, where $p(\lambda_3)<0$, then $v=u'<0$. Thus u is decreasing in the region $u<b$. Because $0<d\leq 1$, we have

$$\begin{aligned}\alpha(u-b)+\frac{w}{1+u}&=\alpha(u-b)+\frac{p(\lambda_3)(u-b)}{1+u}=(b-u)[-\frac{p(\lambda_3)}{1+u}-\alpha]\\&\geq(b-u)[-\frac{p(\lambda_3)}{1+u}-\alpha]\\&\geq\frac{b-u}{b}[(1-d)\lambda_3^2+\frac{\beta b-1-b}{1+b}]>0.\end{aligned}$$

Then in the region $u<b$, the eigenvector e_3 at $(b,0,0,0)$ lies in the region where $\alpha(u-b)+\frac{w}{1+u}>0$.

Therefore, asymptotically as $s\to-\infty$, the solution $y(s)$ satisfies

$$u_0<u<b,\ \ v<0,\ \ w>0,\alpha(u-b)+\frac{w}{1+u}>0.$$

Now we suppose $y(s)\in T$, $v(s_1)\geq-\frac{c}{2\alpha(1+b)}w(s)$, and s_1 is the first value such that $u(s)\leq b$ for $s<s_1$. Lemma 3.2 implies that $z(s_1)>\frac{c}{2}w(s_1)$, thus, $\alpha(1+b)v(s_1)+z(s_1)>0$. Let

$$s_2=\sup\{s<s_1|\alpha(u-b)+\frac{w}{1+u}\geq 0\}.$$

Since as $s\to-\infty$, $\alpha(u-b)+\frac{w}{1+u}>0$, the value s_2 is finite. Since $\alpha(u-b)+\frac{w}{1+u}<0$ for $s>s_2$ and $\alpha(u(s_2)-b)+\frac{w(s_2)}{1+u(s_2)}=0$, we have $[\alpha(u(s)-b)+\frac{w(s)}{1+u(s)}]'_{s=s_2}\leq 0$.

Rewrite the last inequality as $v(s_2)[\alpha(1-b)+2\alpha u(s_2)]+z(s_2)\leq 0$. But as $s=s_1$, we have

$$\begin{aligned}&v(s_1)[\alpha-\frac{w(s_1)}{1+u(s_1)^2}]+\frac{z(s_1)}{1+u(s_1)}\\&=\frac{1}{1+u(s_1)}[v(s_1)(\alpha+\alpha u(s_1)-\frac{w(s_1)}{1+u(s_1)}]+z(s_1)\\&\geq\frac{1}{1+u(s_1)}[v(s_1)(\alpha(1+b)-\frac{v(s_1)w(s_1)}{1+u(s_1)}+z(s_1)]\\&\geq\frac{1}{1+u(s_1)}[\alpha(1+b)v(s_1)+z(s_1)]>0.\end{aligned}$$

By the continuity, $\alpha(u-b)+\frac{w}{1+u}$ has a positive minimum for some $s_3 \in (s_2, s_1)$, and $y(s_3) \in T$. Then

$$[\alpha(u-b)+\frac{w}{1+u}]'_{s=s_3}=0, \quad [\alpha(u-b)+\frac{w}{1+u}]''_{s=s_3} \geq 0,$$

which can be written as

$$\alpha v'(s_3)+\frac{z'(s_3)}{1+u(s_3)}+\frac{2\alpha v(s_3)^2}{1+u(s_3)}-\frac{w(s_3)v(s_3)}{(1+u(s_3))^2} \geq 0.$$

Since $s_2 < s_3 < s_1$, $\alpha(u(s_3)-b)+\frac{w(s_3)}{1+u(s_3)} < 0$, by the assumption that $(1-\alpha)(\beta-1) \geq \frac{2\beta}{1+b}\sqrt{\frac{b\beta-1-b}{1+b}}$, it follows that $\frac{(1-\alpha)w(s_3)}{1+u(s_3)}+2\alpha v(s_3) \geq 0$. Since $y(s_3) \in T$, $\beta > \frac{b+1}{b}$, we can see that

$$\frac{w(s_3)}{1+u(s_3)}[1-\frac{\alpha^2 u(s_3)^2}{1+u(s_3)}+\frac{(\alpha^2 b-\beta)u(u_3)}{1+u(s_3)}-\frac{2\alpha w(s_3)u(s_3)}{(1+u(s_3)^2}] < 0.$$

Therefore

$$\alpha v'(s_3)+\frac{z'(s_3)}{1+u(s_3)}+\frac{2\alpha v(s_3)^2}{1+u(s_3)}-\frac{w(s_3)v(s_3)}{(1+u(s_3))^2} < 0,$$

a contradiction. Thus, the inequality $v(s_1) \geq -\frac{c}{2\alpha(1+b)}w(s)$ cannot hold. This proves Lemma 3.3.

Consider a small circle on Ω_2 parametrically given by

$$g(\theta)=\begin{pmatrix} b+\varepsilon\cos(\theta+\varphi)+\varepsilon\sin(\theta+\varphi)+O(\varepsilon) \\ \lambda_4\varepsilon\cos(\theta+\varphi)+\lambda_3\varepsilon\sin(\theta+\varphi)+O(\varepsilon) \\ p(\lambda_3)\varepsilon\sin(\theta+\varphi)+O(\varepsilon) \\ \lambda_3 p(\lambda_3)\varepsilon\sin(\theta+\varphi)+O(\varepsilon) \end{pmatrix},$$

The phase φ is fixed so that $g(0)$ is on Ω in the region $u < b$, and the parameter $\theta \in [0, 2\pi]$. Choose g so that as θ increases from 0, $b+\varepsilon\cos(\theta+\varphi)+\varepsilon\sin(\theta+\varphi)+O(\varepsilon)$ decreases and $p(\lambda_3)\varepsilon\sin(\theta+\varphi)+O(\varepsilon)$ increases from 0. Let A be the component of the set $\{\theta \in [0, 2\pi]$, there exists s_0 such that $u(s_0, g(\theta)) = u_0, v(s, g(\theta)) \leq 0, s \leq s_0\}$. Then A contains 0 from Lemmas 3.1 and 3.2, A is nonempty and bounded. Let $\theta_1 = \sup A$ and $y_1 = g(\theta_1)$.

Lemma 3.4. *There exists an s_0 such that*

$$u(s_0, y_1) = u_0, \quad w(s_0, y_1) > w_0, \quad v(s_0, y_1) = 0.$$

Proof. We prove the lemma in the following several steps.

(a) Since $g(0) \in \Omega_1$ with $u < b$, if $(u(s_0), g(0)) = u_0$, then $v(s_0, g(0)) = (d/ds)(u(s_0),\ g(0)) < 0$, so $(u(s_0(\theta)), g(0)) = u_0$ for θ in a small neighborhood of $\theta = 0$. Therefore, $\theta_1 \neq 0$. By Lemma 2, if $g(\theta^*)$ is in the branch of Ω_1 with $u > b$, then $\theta_1 < \theta^*$.

(b) $y(s, y_1) \notin \{(u, v, w, z)|u_0 < u < b, 0 < w < w_0, \forall s > 0\}$. Otherwise, $w' = z > 0, \forall s > 0$, so w could not be bounded.

(c) There is no s_1 such that $v(s_1, y_1) = 0, u_0 < u(s_1, y_1) < b, w(s_1, y_1) > 0$. Otherwise, if there exists such an s_1, it is easy to know from Lemma 3.3 that $\alpha(u - b) + \frac{w}{1+u} \geq 0, u_0 < u(s) < b, w(s) > 0$. If $\alpha(u - b) + \frac{w}{1+u} = 0$, then from $z(s) > 0$, it follows that w is increasing. Since $v(s_1, y_1) = 0$, we have $[\alpha(u-b)+\frac{w}{1+u}]' = \frac{z(s_1)}{1+u(s_1)} > 0$ and $\alpha(u(s)-b)+\frac{w(s)}{1+u(s)} > 0, u(s) > 0$ for $s > s_1$. There exists $\delta > 0$ such that $\alpha(u(s) - b) + \frac{w(s)}{1+u(s)} < 0$ for $s \in (s_1 - \delta, s_1)$, then the trajectory $y(s)$ enters T, as in Lemma 7 of [2], it follows that $v(s) > -\frac{c}{4}w(s)$. This is a contradiction to Lemma 3.3. If there exists s_1 such that $u(s_1) > 0$, $\alpha(u-b)+\frac{w}{1+u} > 0$, then $v'(s_1) = cv+u[\alpha(u-b)+\frac{w}{1+u}] > 0$, so v increases, based on the assumption that $v(s_1, y_1) = 0, u(s_1, y_1) > u_0$. The Implicit Function Theorem and the continuity of solutions on the initial conditions imply that there exists an $s_1 = s_1(\theta)$ such that $v(s_1(\theta), y_1(\theta)) = 0, u(s_1(\theta), \ y_1(\theta)) > u_0$ for all θ in a small neighborhood of θ_1. Since it does not occur in T, $\alpha(u(s_1(\theta)) - b) + \frac{w(s(\theta))}{1+u(s_1(\theta))} > 0$, then $v'(s_1(\theta), g(\theta)) > 0$ for $\theta < \theta_1$. This contradicts the definition of θ_1 and proves the case (c).

(d) It is impossible that $u(s, y_1) > u_0$ holds for all s. Since $u(s)$ is decreasing and $w(s)$ is increasing, $w(s)$ cannot be bounded. If it is true, then $\alpha(u(s) - B) + \frac{w}{1+u} > \alpha(u(s) - b) + \frac{w}{1+b}$. Thus, $\alpha(u(s) - B) + \frac{w}{1+u}$ cannot be bounded. Argued as in the proof of Lemma 3.1, we have $v(s) > m_1(u(s) - b)$, where $m_1 > \lambda_4$. This means that $v(s)$ is bounded from low by $m_1(u_0 - b)$. Then $v' = \frac{c}{d}v + \frac{u}{d}[\alpha(u - b) + \frac{w}{1+u}]$ is increasing, this means that v does not remain negative, so u is not decreasing, and this contradicts with part (c).

(e) It follows from part (c) that $v(s, g(\theta)) < 0$ as long as $u(s, g(\theta)) \geq u_0$, and if there exists an s_0 such that $u(s_0, y_1) = u_0, w(s_0, y_1) < w_0$, then

$$\alpha(u_0 - b) + \frac{w}{1 + u_0} < \alpha(u_0 - b) + \frac{w_0}{1 + u_0} = 0.$$

This means that the trajectory will enter T, then we have $v(s_0, g(\theta)) < 0$. This contradicts with the definition of θ_1. Thus there is no s_0 such that $u(s_0, y_1) = u_0, w(s_0, y_1) < w_0$.

(f) There is no s_0 such that $u(s_0, y_1) = u_0, w(s_0, y_1) = w_0$. Otherwise, from part (e), if $v(s_0, y_1) < 0$, it contradicts with the definition of θ_1. If $v(s_0, y_1) = 0$, the trajectory will enter T, by using the similar argument in part (c), a contradiction.

Summarizing (a) to (f), we complete the proof of Lemma 3.4.

Lemma 3.5. *There exists a value θ_2 such that the v coordinate of $g(\theta_2)$ is zero and $\theta_2 > \theta_1$.*

Proof. The proof is similar to that of Lemma 8 in [2] and is omitted.

We know from Lemmas 3.4 and 3.5 that solutions starting from the arc $g(\theta)$, $0 < \theta < \theta_1$, on Ω_2 enter either the region T or P. Arg $g(\theta)$ provides one side of the quadrilateral Σ, the second side of the quadrilateral is composed of two portions, one portion is an arg $g(\theta), \theta_1 < \theta < \theta_2$, where θ_2 satisfies $\lambda_4\varepsilon\cos(\theta + \varphi) +$

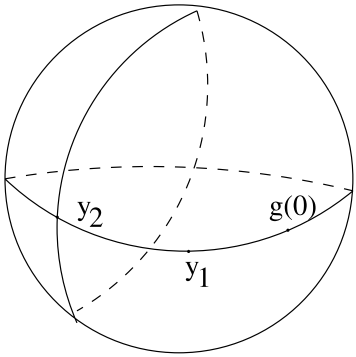

Fig. 1. The second side of the quadrilateral Σ.

$\lambda_3\varepsilon\sin(\theta+\varphi)+O(\varepsilon)=0$, let $y_2=g(\theta_2)$, the second is an arc of the circle of intersection of a small sphere surrounding $(b,0,0,0)$ in Ω and the hyperplane $v=0$ (see Figure 1).

Now, we construct the other sides of the quadrilateral Σ.

Lemma 3.6. *The sphere intersects the hyperplane defined by $v=0$ and $z=0$ in a smooth closed curve, and there exists a point, say y_3, on the sphere such that the v and z coordinates of y_3 are both zero.*

Proof. Denote

$$g_1(\theta,\varphi)=\begin{pmatrix} b+\varepsilon\cos\theta\sin\varphi+\varepsilon\sin\theta\sin\varphi+\varepsilon\cos\varphi+O(\varepsilon) \\ \lambda_4\varepsilon\cos\theta\sin\varphi+\lambda_3\varepsilon\sin\theta\sin\varphi+\lambda_2\varepsilon\cos\varphi+O(\varepsilon) \\ p(\lambda_3)\varepsilon\sin\theta\sin\varphi+p(\lambda_2)\varepsilon\cos\varphi+O(\varepsilon) \\ \lambda_3 p(\lambda_3)\varepsilon\sin\theta\sin\varphi+\lambda_2 p(\lambda_2)\varepsilon\cos\varphi+O(\varepsilon) \end{pmatrix}.$$

We will show that there exists a C^1 function $\varphi(\theta)$ for $\theta\in[0,2\pi]$ and $\varphi(\theta)\in[0,\pi]$ such that the v, w coordinates of $g_1(\theta,\varphi)=0$ satisfy

$$\lambda_4\varepsilon\cos\theta\sin\varphi+\lambda_3\varepsilon\sin\theta\sin\varphi+\lambda_2\varepsilon\cos\varphi+O(\varepsilon)=0, \tag{7}$$

$$\lambda_3 p(\lambda_3)\varepsilon\sin\theta\sin\varphi+\lambda_2 p(\lambda_2)\varepsilon\cos\varphi+O(\varepsilon)=0. \tag{8}$$

Divide (7), (8) by ε on both sides respectively and denote

$$G(\theta,\varphi)=\lambda_4\cos\theta\sin\varphi+\lambda_3\sin\theta\sin\varphi+\lambda_2\cos\varphi+O(1), \tag{9}$$

$$H(\theta,\varphi)=\lambda_3 p(\lambda_3)\sin\theta\sin\varphi+\lambda_2 p(\lambda_2)\cos\varphi+O(1). \tag{10}$$

At $(\theta_2,\pi/2)$, $\partial G/\partial\varphi=-\lambda_2\neq 0$, $\partial H/\partial\varphi\neq 0$, so the Implicit Function Theorem implies that, for ε sufficiently small, there exists a curve defined by

$$\cot\varphi=(\lambda_4\cos\theta+\lambda_3\sin\theta)/(-\lambda_2)$$

such that θ and φ lie in a neighborhood of the curve.

The points which satisfy $\partial G/\partial\varphi \neq 0$ are in a neighborhood of the curve defined by

$$\cot\varphi = \lambda_2/(\lambda_4\cos\theta + \lambda_3\sin\theta).$$

Thus, θ on the sphere given by $\varphi(\theta)$ may be extended to a smooth closed curve $h(\theta,\varphi)$ on the sphere. We can get the similar result about (10). The proof of the rest is similar to the first part of this lemma.

Now, we choose a small enough neighborhood of $(b, 0, 0, 0)$ such that the conditions required in Lemma 3.1 to Lemma 3.6 are satisfied.

Let ε be small enough to ensure the conditions required in Lemmas 3.5 and 3.6. We know from Lemma 3.1 to Lemma 3.6 that there is a topological triangle defined on the sphere, the three corners are y_0 which is determined by Lemma 3.1, $y_2 = g(\theta_2)$, and y_3 which is determined by Lemma 3.6. But this triangle does not satisfy the requirement (ii) of Lemma 2.1 because there has a small neighborhood in R^4 around the point $(b, 0, 0, 0)$ which contains points of W^-. So we need to modify the corner y_0. As in [2], from Lemma 3.6, let U be a small neighborhood in R^4 around the point y_0. Let U be small enough so that it does not contain y_1 or y_3. Also let U be small enough so that if $y^* \in U$, then there is an $s_0(y^*)$ such that $u(s_0(y^*), y^*) = u_0$. Recall that $y(s, y_0)$ crosses $u = u_0$ transversally at $s = s_0$. Let E be a small ball in R^4 centered at y_0 contained in U. Consider the curve of the intersection of the sphere ∂E with the sphere in Ω_3 defined by $h(\theta,\varphi)$. This curve is the fourth side of Σ. Let y_4 be the intersection of ∂W with the arc of intersection of the sphere defined by $h(\theta,\varphi)$ and the hyperplane $z = 0$, let y_5 be intersection of the sphere defined by $g(\theta)$. Thus, we have determined the third and fourth sides (see Figure 2).

Lemma 3.7. *There exists a* $y^* \in \Sigma$ *such that the solution* $y(s, y^*) = (u(s), (v_1(s), w_1(s), z_1(s))$ *remains in the region* W, *and* $0 < u_1(s) < b$, $0 < w_1(s) < L$, *where* L *is a some positive real for all* s.

Proof. It is easy to see that the set W is closed. In order to use Lemma 2.1 to prove this lemma, we need to check the conditions (ii) and (iii) of Lemma 2.1. Suppose

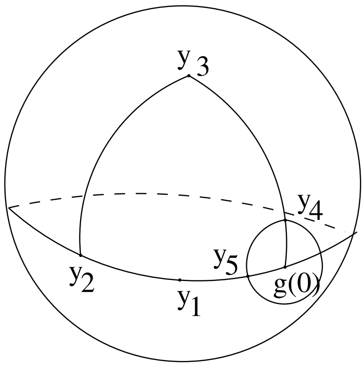

Fig. 2. The four sides of the quadrilateral Σ.

$y_0 \in \Sigma, s < T(y_0), Y(s, y_0) \in W$ and $Y(s, y_0) \notin W^-$. Since $W^- \subseteq \text{int}W$ or $W^- \subseteq \partial W \backslash W^-$, if $Y(s, y_0) \in \text{int}W$, then there is an open set U around $Y(s, y_0)$ disjoint from ∂W.

If $Y(s, y_0) \in \partial W \backslash W^-$, because N and H are invariant manifolds, $Y(s, y_0) \notin N$ or H. There are several cases need to be eliminated.

If $Y(s, y_0)$ is in the portion of $\partial W \backslash W^-$ with $u > u_0, w < 0, z = 0$, then $z' = cz + w(1 - \beta + \frac{\beta}{1+u}) > w(1 - \beta + \frac{\beta}{1+u_0}) = 0$ and the trajectory was previously in the set Q, this contradicts with the assumption that $s < T(y_0)$.

If $Y(s, y_0)$ is in the portion of $\partial W \backslash W^-$ with $u < 0, \alpha(u-b) + \frac{w}{1+u} < 0, v = 0$, then $v' = \frac{c}{d}v + \frac{u}{d}[\alpha(u - b) + \frac{w}{1+u}] > 0$ and the trajectory was previously in T, this contradicts with the assumption that $s < T(y_0)$.

By construction it follows that Σ is compact, intersects each trajectory only once and is simply connected. If $\Sigma = \Sigma^0$, since W^- is not simply connected, by Lemma 2.1, this is impossible. So $\Sigma \neq \Sigma^0$, that is, there exists some point y^* such that $Y(s, y^*) \in W$ for all s.

Suppose s_1 is the first time such that $w_1(s_1) > L$ with $u(s_1) \leq u_0$, since $Y(s, y^*)$ is in W, $Y(s, y^*) \notin P$, so $z_1(s_1) < 0$. This implies that $w_1(s)$ is decreasing and must have exceeded L at some other time $s_2 \neq s_1$. This contradicts with the first time s_1. Therefore, $w_1(s)$ is bounded. Similarly we can prove that $u_1(s)$ is bounded.

Lemma 3.8. *The solution $y(s, y^*)$ remains in Ω for all s, where*

$$q > \frac{1}{2}\left(c + \sqrt{c^2 - 4(\beta - 1 - \frac{\beta}{1+b}}\right).$$

$$\Omega = \{(u, v, w, z) | 0 < u < b, 0 < w < L, -\frac{1}{c}w < z < qw, -\frac{L+1}{c}u < v < \frac{b\alpha}{c}u\}.$$

Proof. Suppose there exists an s_1 such that $z_1(s_1) < -\frac{1}{c}w_1(s_1)$. If there exists an s_2 such that $z_1(s_2) = -\frac{1}{c}w_1(s_2)$, then $z_1'(s_2) + \frac{1}{c}w_1'(s_2) \geq 0$. Substitution from $w' = z$ and $z' = cz + w - \frac{\beta uw}{1+u}$ yields

$$-\frac{1}{c}(\frac{1}{c} + c)w_1(s_2) + w_1(s_2)[1 - \beta + \frac{\beta}{1 + u_1(s_1)}] \geq 0,$$
$$-\frac{1}{c}(\frac{1}{c} + c)w_1(s_2) + w_1(s_2)[1 - \beta + \frac{\beta}{1 + b}] \geq 0.$$

These imply that $-\frac{1}{c^2} \geq 0$, a contradiction, so it follows that $z_1(s_1) < -\frac{1}{c}w_1(s_1)$. The inequality continues to hold for $s > s_1$, so $z_1'(s) = cz_1(s) + w_1(s)[1 - \beta + \frac{\beta}{1+u_1(s)}] \leq cz_1(s) + w_1(s) < 0$ and $z_1(s) < z_1(s_1)$ for $s > s_1$. Therefore $w_1'(s)$ is strictly negative and bounded away from zero by $z_1(s_1)$, and $w_1(s_1) < 0$ for some finite s. This is a contraction. The proof of the rest of the lemma is similar.

Lemma 3.9. *The trajectory $y(s, y^*) \to (u_0, 0, w_0, 0)$ as $s \to +\infty$.*

Proof. The characteristic equation of systems (5) linearized at $(u_0, 0, w_0, 0)$ is given by

$$\lambda^4 - (c + \frac{c}{d})\lambda^3 + \frac{c^2 - r}{d}\lambda^2 + \frac{cr}{d}\lambda + \frac{\alpha b(\beta - 1) - \alpha}{d\beta} = 0. \tag{11}$$

where $r(\beta) = \frac{\alpha(1+b)}{\beta} - \frac{2\alpha}{\beta-1}$. Since $\frac{b+1}{b} < \beta < \frac{b+1}{b-1}$, we have $r < 0$, $\frac{\alpha b(\beta-1)-\alpha}{d\beta} > 0$. By using the Routh-Hurwitz criteria, we can see that the characteristic equation has two eigenvalues with positive real part and two eigenvalues with negative real part. By Theorem 6.2 of [9], there is a 2-dimensional stable manifold at $(u_0, 0, w_0, 0)$. In order to show the trajectory will approach the point $(u_0, 0, w_0, 0)$, we construct a Liapunov function as follows

$$V = [c(u-u_0)-dv]+u_0[d\frac{v}{u}-c\log\frac{w}{w_0}]+u_0[c(w-w_0)-z]+u_0w_0[\frac{z}{w}-c\log\frac{w}{w_0}].$$

It is easy to see that $V(u, v, w, z)$ is continuous and bounded below on Ω, and

$$\begin{aligned}
\frac{dV}{ds} &= \frac{\partial V}{\partial u}\cdot u_t + \frac{\partial V}{\partial v}\cdot v_t + \frac{\partial V}{\partial w}\cdot w_t + \frac{\partial V}{\partial z}\cdot z_t \\
&= -\frac{u_0v^2}{u^2} + \alpha(u-u_0)(b-u) - \frac{uw}{1+u} - \frac{u_0w_0z^2}{w^2} + \left[\frac{u_0w}{1+u} - u_0w\right] \\
&\quad + \frac{\beta u_0uw}{1+u} + w_0u_0 - \frac{\beta u_0w_0u}{1+u} \\
&= -\frac{u_0v^2}{u^2} + \alpha(u-u_0)(b-u) - \frac{uw}{1+u} - \frac{u_0w_0z^2}{w^2} - \frac{u_0uw}{1+u} \\
&\quad + \frac{\beta u_0uw}{1+u} + w_0u_0 - \frac{\beta u_0w_0u}{1+u} \\
&= -\frac{u_0v^2}{u^2} + \alpha(u-u_0)(b-u) + \left[\frac{\beta u_0uw}{1+u} - \frac{uw}{1+u} - \frac{u_0uw}{1+u}\right] - \frac{u_0w_0z^2}{w^2} \\
&\quad + w_0u_0 - \frac{\beta u_0w_0u}{1+u} \\
&= -\frac{u_0v^2}{u^2} + \alpha(u-u_0)(b-u) - \frac{u_0w_0z^2}{w^2} + w_0u_0 - \frac{\beta u_0w_0u}{1+u} \\
&= -\frac{u_0v^2}{u^2} - \frac{u_0w_0z^2}{w^2} + \alpha(b-u)(u-u_0) - w_0u_0[1 - \frac{\beta u}{1+u}] \\
&= -\frac{u_0v^2}{u^2} - \frac{u_0w_0z^2}{w^2} + \frac{\alpha(u-u_0)^2}{1+u}[b-1-u_0-u].
\end{aligned}$$

Since $\frac{b+1}{b} < \beta < \frac{b}{b-1}$, $\frac{dV}{ds}$ is always non-positive in Ω. $\frac{dV}{ds} = 0$ if and only if $u = u_0, z = 0$, the largest invariant subset of this segment is the single point $(u_0, 0, w_0, 0)$. By the LaSalle's Invariance Principle, it follows that $y(s) \to (u_0, 0, w_0, 0)$ as $s \to +\infty$.

3.2. Proof of Theorem 2.3

In order to prove Theorem 2.3, we take α, d, and b as fixed, β and c as parameters. This parameter choice amounts to fixing the values of the growth rate and the carrying capacity of the prey and allowing the predator effectiveness to vary. We search for purely imaginary roots of the characteristic equation (11). Substituting $\lambda = ki$ into (11) and simplifying, we have

$$\begin{cases} k^4 - \frac{c^2 - r(\beta)}{d}k^2 + p(\beta) = 0, \\ k^2 = -\frac{r(\beta)}{1+d}, \end{cases}$$

where $r(\beta) = \frac{\alpha(1+b)}{\beta} - \frac{2\alpha}{\beta - 1}$, $p(\beta) = \frac{\alpha b(\beta-1)-\alpha}{d\beta}$. Since $\beta < \frac{b+1}{b-1}$, we obtain that $r(\beta) < 0$, $p(\beta) < 0$. Thus, a pair of imaginary eigenvalues exists if the parameters β and c satisfy the condition $c^2 = \frac{1}{d+1} - \frac{d(1+d)}{r}p$.

Considering λ as a function of β, and differentiating the characteristic equation (11) with respect to β, we obtain

$$\frac{d\lambda(\beta)}{d\beta} = \frac{\frac{r'}{d}\lambda^2(\beta) - \frac{cr'}{d}\lambda(\beta) - p'}{4\lambda^3(\beta) - 3\lambda^2(\beta)(c + \frac{c}{d}) + \frac{2(c^2-r)}{d}\lambda(\beta) + \frac{cr}{d}}. \tag{12}$$

Substituting $\lambda = ki$ into (12), we obtain

$$\frac{d\lambda(\beta)}{d\beta} = -\frac{(r'k^2 + p'd) + cr'ki}{3(k^2c(1+d) + cr) + (2k(c^2 - r) - 4dk^3)i}.$$

After some calculation, we have

$$\begin{aligned} \mathrm{Re}\Big(\frac{d\lambda(\beta)}{d\beta}\Big) &= -cr\Big[\frac{rr'(3-d)}{(1+d)^2} - 2dp' + \frac{(r-2c^2)r'}{1+d}\Big] \\ &= \frac{-2c\alpha r}{1+d}\Big\{\Big(\frac{2}{\beta - 1} - \frac{1+b}{\beta}\Big)\frac{2\alpha}{1+d}\Big(\frac{1+b}{\beta^2} - \frac{2}{(\beta-1)^2}\Big) \\ &\qquad + \frac{1+b}{\beta^2}(c^2 - 1 - d) - \frac{2c^2}{(\beta-1)^2}\Big\}. \end{aligned}$$

Let $m = (\frac{2}{\beta-1} - \frac{1+b}{\beta})\frac{2\alpha}{1+d}$, and rewrite $\mathrm{Re}(\frac{d\lambda(\beta)}{d\beta})$ as

$$\mathrm{Re}\Big(\frac{d\lambda(\beta)}{d\beta}\Big) = \frac{-2\alpha cr}{1+d}\ \Big\{(m + c^2)\Big(\frac{1+b}{\beta^2} - \frac{2}{(\beta-1)^2}\Big) - \frac{(1+b)(1+d)}{\beta^2}\Big\}.$$

Since $\frac{b+1}{b} < \beta \leq \frac{1}{1-\sqrt{2/(1+b)}}$, we have $\mathrm{Re}(\frac{d\lambda(\beta)}{d\beta}) < 0$. This implies that the transversal condition is satisfied. Thus, we have proved Theorem 2.3.

4. Discussion

Since the pioneering work of Fisher [5] and Kolmogorov et al. [10], many researchers have been paying attention to the existence of traveling wave solutions in biological systems, see the monographs Fife [4], Murray [13], Volpert et al. [16], and the references cited therein. The basic idea is that a reaction-diffusion system can give rise to a moving zone of transition from absence to an equilibrium state, that is, a traveling wave front.

In this paper we have studied the existence of traveling wave solutions and the small amplitude traveling wave train solutions for a reaction-diffusion system based on a predator-prey model with Holling type-II functional response. By constructing the Wazewski set, using a shooting argument and LaSalle's Invariance Principle, we showed the existence of a heteroclinic orbit connecting two equilibrium points in R^4 which corresponds to a traveling wave solution for the reaction-diffusion system. By using the Hopf Bifurcation theorem, we proved that there is a small amplitude periodic solution in R^4 which corresponds to a small amplitude traveling wave train solution to the reaction-diffusion system. In comparison, Dunbar [2] investigated a Lotka-Volterra type predator-prey model while we studied a predator-prey model with Holling type-II functional response. Dunbar [3] studied system (1) with $d_1 = 0$ and considered the traveling wave equations in R^3, we considered system with $d_1 \neq 0$ and $d_2 \neq 0$ and established the existence of traveling wave solutions in R^4.

In Theorem 2.2, sufficient conditions were given to ensure the existence of traveling wave solutions connecting two steady states $(b, 0)$ and (u^0, w^0). Returning to the original parameters in system (1) we know that the traveling wave solutions connect the prey carrying capacity steady state $(K, 0)$ and the predator-prey coexistence steady state $(\bar{u}, \bar{w})$; that is, there is a zone of transition from the state $(K, 0)$ with saturation of prey and none or few predator to the state $(\bar{u}, \bar{w})$ with decreased prey level and increased predator level. Biologically, if we consider a one dimensional habitat such as a coastline or river and if the linear habitat is initially uniformly saturated with prey at its carrying capacity, introducing a few predators at one end of the habitat may result in a "wave of invasion" of predators. We refer to Owen and Lewis [14], Sherratt et al. [15] and the references cited therein for further study on traveling waves and predator-prey invasion.

One of the generalizations of the basic idea of Fisher and Kolmogorov et al. is that there could be a traveling wave train solution connecting an equilibrium state and a periodic solution (see Dunbar [3] and Sherratt et al. [15]). It would be interesting to investigate the existence of such traveling wave solutions for system (1). Also, it would be very interesting to study the existence of traveling wave solutions in system (1) with nonlocal effect (see Gourley and Britton [8]). We leave these for future consideration.

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