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BOGDANOV-TAKENS BIFURCATION OF CODIMENSION 3 IN A PREDATOR-PREY MODEL WITH CONSTANT-YIELD PREDATOR HARVESTING

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Abstract. Recently, we (J. Huang, Y. Gong and S. Ruan, *Discrete Contin. Dynam. Syst. B* **18** (2013), 2101-2121) showed that a Leslie-Gower type predator-prey model with constant-yield predator harvesting has a Bogdanov-Takens singularity (cusp) of codimension 3 for some parameter values. In this paper, we prove analytically that the model undergoes Bogdanov-Takens bifurcation (cusp case) of codimension 3. To confirm the theoretical analysis and results, we also perform numerical simulations for various bifurcation scenarios, including the existence of two limit cycles, the coexistence of a stable homoclinic loop and an unstable limit cycle, supercritical and subcritical Hopf bifurcations, and homoclinic bifurcation of codimension 1.

1. **Introduction.**

Harvesting is commonly practiced in fishery, forestry, and wildlife management (Clark [8]). It is very important to harvest biological resources with maximum sustainable yield while maintain the survival of all interacting populations. Recently, the nonlinear dynamics in predator-prey models with harvesting have been studied extensively and very interesting and complex bifurcation phenomona have been observed depending on various functional responses and different harvesting regimes. We refer to Beddington and Cooke [1], Beddington and May [2], Brauer and Soudack [3, 4, 5], Chen et al. [6], Dai and Tang [9], Etoua and

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Rousseau [12], Gong and Huang [13], Huang et al. [15, 16, 17], Leard et al. [19], May et al. [21], Xiao and Jennings [23], Xiao and Ruan [24], Zhu and Lan [26], and references cited therein.

In order to investigate the interaction between krill (prey) and whale (predator) populations in the Southern Ocean, May et al. [21] proposed the following model subject to various harvesting regimes:

$$\begin{cases} \dot{x} = r_1 x(1 - \dfrac{x}{K}) - axy - H_1, \\ \dot{y} = r_2 y(1 - \dfrac{y}{bx}) - H_2, \end{cases} \tag{1}$$

where $x(t) > 0$ and $y(t) \geq 0$ represent the population densities of the prey and predators at time $t \geq 0$, respectively; r_1 and K describe the intrinsic growth rate and carrying capacity of the prey in the absence of predators, respectively; a is the maximum value at which per capita reduction rate of the prey x can attain; r_2 is the intrinsic growth rate of predators; bx takes on the role of a prey-dependent carrying capacity for predators and b is a measure of the quality of the food for predators. H_1 and H_2 describe the effect of harvesting on the prey and predators, respectively.

(a) When $H_1 = H_2 = 0$, that is, there is no harvesting, system (1) becomes the so-called Leslie-Gower type predator-prey model

$$\begin{cases} \dot{x} = r_1 x(1 - \dfrac{x}{K}) - axy, \\ \dot{y} = r_2 y(1 - \dfrac{y}{bx}), \end{cases} \tag{2}$$

which has been studied extensively, for example, Hsu and Huang [14]. In particular, they showed that the unique positive equilibrium of system (2) is globally asymptotically stable under all biologically admissible parameters.

(b) When $H_1 = h_1$ and $H_2 = 0$, where h_1 is a positive constant, that is, there is constant harvesting on the prey only, Zhu and Lan [26] and Gong and Huang [13] considered system (1) when only the prey population is harvested at a constant-yield rate

$$\begin{cases} \dot{x} = r_1 x(1 - \dfrac{x}{K}) - axy - h_1, \\ \dot{y} = r_2 y(1 - \dfrac{y}{bx}). \end{cases} \tag{3}$$

They obtained various bifurcations including saddle-node bifurcation, supercritical and subcritical Hopf bifurcations of codimension 1, and repelling Bogdanov-Takens bifurcation of codimension 2.

(c) When $H_1 = 0$ and $H_2 = h_2$, where h_2 is a positive constant, that is, there is constant harvesting on the predators only, we (Huang, Gong and Ruan [17]) considered system (1) when only the predator population is harvested at a constant-yield rate

$$\begin{cases} \dfrac{dx}{dt} = r_1 x(1 - \dfrac{x}{K}) - axy, \\ \dfrac{dy}{dt} = r_2 y(1 - \dfrac{y}{bx}) - h_2. \end{cases} \tag{4}$$

By the following scaling

$$t \to r_1 t, x \to \frac{x}{K}, y \to \frac{ay}{r_1},$$

model (4) becomes

$$\begin{cases} \dfrac{dx}{dt} = x(1-x) - xy, \\ \dfrac{dy}{dt} = y(\delta - \beta\dfrac{y}{x}) - h, \end{cases} \tag{5}$$

here $\delta = \frac{r_2}{r_1}$, $\beta = \frac{r_2}{abK}$, $h = \frac{ah_2}{r_1^2}$ are positive constants. The effect of constant-yield predator harvesting on system (5) in the biological-feasible region $\Omega = \{(x, y) : x > 0, y \geq 0\}$ was studied. The saddle-node bifurcation, repelling and attracting Bogdanov-Takens bifurcations of codimension 2, supercritical and subcritical Hopf bifurcations, and degenerate Hopf bifurcation are shown in model (5) as the values of parameters vary. In particular, it was shown that the model has a Bogdanov-Takens singularity (cusp) of codimension 3. However, the existence of Bogdanov-Takens bifurcation (cusp case) of codimension 3 has not been proved analytically, which is the subject of this paper.

This paper is organized as follows. In section 2, we prove analytically the existence of Bogdanov-Takens bifurcation (cusp case) of codimension 3 for model (5) and describe the bifurcation diagram and bifurcation phenomena. Numerical simulations of various bifurcation cases, including the existence of two limit cycles, the coexistence of a stable homoclinic loop and an unstable limit cycle, supercritical and subcritical Hopf bifurcations, and homoclinic bifurcation of codimension 1, are also presented in section 3 to confirm the theoretical analysis. The paper ends with a brief discussion in section 4 about the effect of constant-yield predator harvesting on system (5) and a comparison about different dynamics in systems (2), (3), and (5).

2. Bogdanov-Takens bifurcation of codimension 3.

Before stating the main results of our paper, we firstly recall the definition of Bogdanov-Takens bifurcation (cusp case) of codimension 3 introduced by Dumortier, Roussarie and Sotomayor [10] (see also Chow, Li and Wang [7], Dumortier et al. [11], Perko [22]) as follows.

Definition 2.1. The bifurcation that results from unfolding the following normal form of a cusp of codimension 3

$$\begin{cases} \dfrac{dx}{dt} = y, \\ \dfrac{dy}{dt} = x^2 \pm x^3 y \end{cases}$$

is called a *Bogdanov-Takens bifurcation (cusp case) of codimension 3*. A universal unfolding of the above normal form is given by

$$\begin{cases} \dfrac{dx}{dt} = y, \\ \dfrac{dy}{dt} = \mu_1 + \mu_2 y + \mu_3 xy + x^2 \pm x^3 y. \end{cases}$$

The following lemma is from Theorem 3.3 (ii) in Huang, Gong and Ruan [17].

Lemma 2.2. *When $\beta = \frac{h^3}{(1-h)^2}$, $\delta = \frac{h+h^2}{1-h}$, and $0 < h < 1$, system (5) has an interior equilibrium $(h, 1-h)$ which is a cusp. Moreover, if $h = 2 - \sqrt{3}$, then $(h, 1-h)$ is a cusp of codimension 3. The phase portrait is given in Figure 1.*

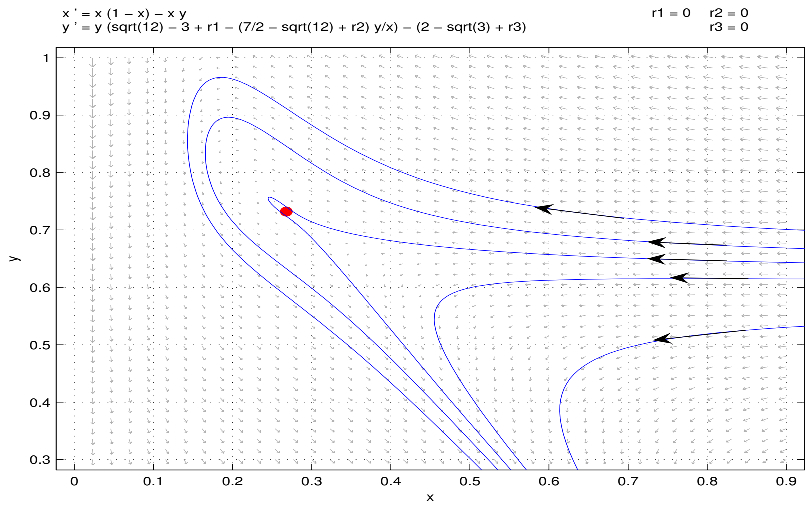

FIGURE 1. A cusp of codimension 3 for system (6).

Substituting $h = 2-\sqrt{3}$ into $\beta = \frac{h^3}{(1-h)^2}$ and $\delta = \frac{h+h^2}{1-h}$, we can find a necessary condition for the existence of higher codimension B-T bifurcations:

$$(\delta_0, \beta_0, h_0) = (-3+2\sqrt{3},\ \frac{7}{2}-2\sqrt{3},\ 2-\sqrt{3}),$$

and the degenerate equilibrium $(h, 1-h)$ is $(2-\sqrt{3}, -1+\sqrt{3})$ under the above conditions.

Lemma 2.2 indicates that system (5) may exibit Bogdanov-Takens bifurcation of codimension 3. In order to make sure if such a bifurcation can be fully unfolded inside the class of system (5) as (δ, β, h) vary in the small neighborhood of (δ_0, β_0, h_0), we let $(\delta, \beta, h) = (\delta_0 + r_1, \beta_0 + r_2, h_0 + r_3)$ in system (5) and obtain the following unfolding system

$$\begin{cases} \dfrac{dx}{dt} = x(1-x) - xy, \\ \dfrac{dy}{dt} = y(\delta_0 + r_1 - (\beta_0 + r_2)\dfrac{y}{x}) - (h_0 + r_3), \end{cases} \tag{6}$$

where $(r_1, r_2, r_3) \sim (0,0,0)$. If we can transform the unfolding system (6) into the following versal unfolding of Bogdanov-Takens singularity (cusp) of codimension 3 by a series of near-identity transformations

$$\begin{cases} \dfrac{dx}{d\tau} = y, \\ \dfrac{dy}{d\tau} = \gamma_1 + \gamma_2 y + \gamma_3 xy + x^2 - x^3 y + R(x,y,r), \end{cases} \tag{7}$$

where

$$R(x,y,r) = y^2 O(|x,y|^2) + O(|x,y|^5) + O(r)(O(y^2) + O(|x,y|^3)) + O(r^2)O(|x,y|), \tag{8}$$

and $\frac{D(\gamma_1,\gamma_2,\gamma_3)}{D(r_1,r_2,r_3)} \neq 0$ for small r, then we can claim that system (5) undergoes Bogdanov-Takens bifurcation (cusp case) of codimension 3 (Dumortier, Roussarie and Sotomayor [10], Chow, Li and Wang [7]). In fact, we have the following main theorem.

Theorem 2.3. *System (5) undergoes Bogdanov-Takens bifurcation (cusp case) of codimension 3 in a small neighborhood of the interior equilibrium $(2-\sqrt{3}, -1+\sqrt{3})$ as (δ, β, h) varies near (δ_0, β_0, h_0). Therefore, system (5) can exhibit the coexistence of a stable homoclinic loop and an unstable limit cycle, two limit cycles (the inner one unstable and the outer stable) and semi-stable limit cycle for various parameters values.*

Proof. Firstly, we translate the equilibrium $(2-\sqrt{3}, -1+\sqrt{3})$ of system (6) when $r = 0$ into the origin and expand system (6) in power series around the origin. Let

$$X = x - 2 + \sqrt{3},\ Y = y + 1 - \sqrt{3}.$$

Then system (6) becomes

$$\begin{cases} \dfrac{dX}{dt} = (-2+\sqrt{3})X + (-2+\sqrt{3})Y - XY - X^2, \\ \dfrac{dY}{dt} = Q_0(X,Y) + O(|X,Y|^5), \end{cases} \tag{9}$$

where

$$\begin{aligned} Q_0(X,Y) =& A_{00} + A_{10}X + A_{01}Y + A_{11}XY + A_{20}X^2 + A_{02}Y^2 + A_{30}X^3 + A_{21}X^2Y \\ &+ A_{12}XY^2 + A_{40}X^4 + A_{31}X^3Y + A_{22}X^2Y^2 \end{aligned}$$

in which

$$\begin{aligned} &A_{00} = (-1+\sqrt{3})r_1 - 2r_2 - r_3, && A_{10} = 2-\sqrt{3} + 2(2+\sqrt{3})r_2, \\ &A_{01} = 2-\sqrt{3} + r_1 - 2(1+\sqrt{3})r_2, && A_{11} = -1+\sqrt{3} + 2(5+3\sqrt{3})r_2, \\ &A_{20} = -1 - 2(7+4\sqrt{3})r_2, && A_{02} = \frac{1}{2}(-2+\sqrt{3} - 2(2+\sqrt{3})r_2), \\ &A_{30} = 2+\sqrt{3} + (52+30\sqrt{3})r_2, && A_{21} = -(1+\sqrt{3} + 2(19+11\sqrt{3})r_2), \\ &A_{12} = \frac{1}{2} + (7+4\sqrt{3})r_2, && A_{40} = -7-4\sqrt{3} - 2(97+56\sqrt{3})r_2, \\ &A_{31} = 5+3\sqrt{3} + (142+82\sqrt{3})r_2, && A_{22} = -(26+15\sqrt{3})(\frac{7}{2} - 2\sqrt{3} + r_2). \end{aligned}$$

Next, let

$$x_1 = X,\ y_1 = (-2+\sqrt{3})X + (-2+\sqrt{3})Y - XY - X^2.$$

Then system (9) can be transformed into

$$\begin{cases} \dfrac{dx_1}{dt} = y_1, \\ \dfrac{dy_1}{dt} = Q_1(x_1,y_1) + O(|x_1,y_1|^5), \end{cases} \tag{10}$$

where

$$\begin{aligned} Q_1(x_1,y_1) =& a_{00} + a_{10}x_1 + a_{01}y_1 + a_{11}x_1y_1 + a_{20}x_1^2 + a_{02}y_1^2 + a_{12}x_1y_1^2 \\ &+ a_{21}x_1^2y_1 + a_{22}x_1^2y_1^2 + a_{31}x_1^3y_1 \end{aligned}$$

with

$$\begin{aligned}
a_{00} &= (-2+\sqrt{3})A_{00},\\
a_{10} &= -A_{00} + (2-\sqrt{3})A_{01} - (2-\sqrt{3})A_{10},\\
a_{01} &= -2+\sqrt{3}+A_{01},\\
a_{11} &= -1-2A_{02}+A_{11},\\
a_{20} &= A_{01} + (-2+\sqrt{3})A_{02} - A_{10} + (2-\sqrt{3})A_{11} - (2-\sqrt{3})A_{20},\\
a_{02} &= 2+\sqrt{3} - (2+\sqrt{3})A_{02},\\
a_{12} &= (7+4\sqrt{3})(-1+A_{02}+(-2+\sqrt{3})A_{12}),\\
a_{21} &= -2A_{12}+A_{21},\\
a_{22} &= (-7-4\sqrt{3})(-2-\sqrt{3}+(2+\sqrt{3})A_{02} - A_{12} + (2-\sqrt{3})A_{22}),\\
a_{31} &= -2A_{22}+A_{31}.
\end{aligned}$$

Secondly, following the procedure in Li, Li and Ma [20], we use several steps (I, II, III, IV, V, VI) to transform system (10) to the versal unfolding of Bogdanov-Takens singularity (cusp) of codimension 3, that is system (7).

(I) Removing the y_1^2-term from Q_1 in system (10) when $r = 0$. In order to remove the y_1^2-term from Q_1, we let $x_1 = x_2 + \frac{a_{02}}{2}x_2^2,\ y_1 = y_2 + a_{02}x_2y_2$, which is a near identity transformation for (x_1, y_1) near $(0,0)$, then system (10) is changed into

$$\begin{cases} \dfrac{dx_2}{dt} = y_2, \\ \dfrac{dy_2}{dt} = Q_2(x_2, y_2) + O(|x_2, y_2|^5), \end{cases} \tag{11}$$

where

$$\begin{aligned} Q_2(x_2,y_2) =& b_{00} + b_{10}x_2 + b_{01}y_2 + b_{11}x_2y_2 + b_{20}x_2^2 + b_{12}x_2y_2^2 + b_{21}x_2^2y_2 \\ &+ b_{30}x_2^3 + b_{31}x_2^3y_2 + b_{40}x_2^4 + b_{22}x_2^2y_2^2 \end{aligned}$$

with
$b_{00} = a_{00},\ b_{10} = -a_{00}a_{02} + a_{10},\ b_{01} = a_{01},\ b_{11} = a_{11},\ b_{20} = \dfrac{1}{2}(2a_{00}a_{02}^2 - a_{02}a_{10} + 2a_{20}),\ b_{12} = 2a_{02}^2 + a_{12},\ b_{21} = \dfrac{1}{2}(a_{02}a_{11} + 2a_{21}),\ b_{30} = \dfrac{1}{2}(-2a_{00}a_{02}^3 + a_{02}^2a_{10}),\ b_{31} = a_{02}a_{21} + a_{31},\ b_{40} = \dfrac{1}{4}(4a_{00}a_{02}^4 - 2a_{02}^3a_{10} + a_{02}^2a_{20}),\ b_{22} = \dfrac{1}{2}(-2a_{02}^3 + 3a_{02}a_{12} + 2a_{22})$.

(II) Eliminating the $x_2y_2^2$-term from Q_2 in system (11) when $r = 0$. Let $x_2 = x_3 + \frac{b_{12}}{6}x_3^3,\ y_2 = y_3 + \frac{b_{12}}{2}x_3^2y_3$, then we obtaint the following system

$$\begin{cases} \dfrac{dx_3}{dt} = y_3, \\ \dfrac{dy_3}{dt} = Q_3(x_3, y_3) + O(|x_3, y_3|^5), \end{cases} \tag{12}$$

where

$$\begin{aligned} Q_3(x_3,y_3) =& c_{00} + c_{10}x_3 + c_{01}y_3 + c_{11}x_3y_3 + c_{20}x_3^2 + c_{21}x_3^2y_3 + c_{30}x_3^3 \\ &+ c_{31}x_3^3y_3 + c_{40}x_3^4 \end{aligned}$$

in which
$c_{00} = b_{00},\ c_{10} = b_{10},\ c_{01} = b_{01},\ c_{11} = b_{11},\ c_{20} = b_{20} - \dfrac{b_{12}b_{00}}{2},\ c_{21} = b_{21},\ c_{30} = b_{30} - \dfrac{b_{12}b_{10}}{3},\ c_{31} = b_{31} + \dfrac{b_{11}b_{12}}{6},\ c_{40} = b_{40} + \dfrac{b_{00}b_{12}^2}{4} - \dfrac{b_{12}b_{20}}{6}.$

(III) Removing the x_3^3 and x_3^4-terms from Q_3 in system (12) when $r = 0$. Note that $c_{20} = \frac{1}{2} + O(r)$, $c_{20} \neq 0$ for small r. We let $x_3 = x_4 - \frac{c_{30}}{4c_{20}}x_4^2 + \frac{15c_{30}^2 - 16c_{20}c_{40}}{80c_{20}^2}x_4^3, y_3 = y_4$, and obtain the following system from system (12):

$$\begin{cases} \dfrac{dx_4}{dt} = y_4\left(1 + \dfrac{c_{30}}{2c_{20}}x_4 + \dfrac{-25c_{30}^2 + 48c_{20}c_{40}}{80c_{20}^2}x_4^2 + \dfrac{-35c_{30}^3 + 48c_{20}c_{30}c_{40}}{80c_{20}^3}x_4^3\right), \\ \dfrac{dy_4}{dt} = Q_4^*(x_4, y_4) + O(|x_4, y_4|^5), \end{cases} \tag{13}$$

where

$$\begin{aligned} Q_4^*(x_4, y_4) =& D_{00} + D_{10}x_4 + D_{01}y_4 + D_{11}x_4y_4 + D_{20}x_4^2 + D_{21}x_4^2y_4 + D_{30}x_4^3 \\ &+ D_{40}x_4^4 + D_{31}x_4^3y_4 \end{aligned}$$

and
$D_{00} = c_{00},\ D_{10} = c_{10},\ D_{01} = c_{01},\ D_{11} = c_{11},\ D_{20} = c_{20} - \dfrac{c_{10}c_{30}}{4c_{20}},\ D_{21} = c_{21} - \dfrac{c_{11}c_{30}}{4c_{20}},\ D_{30} = \dfrac{c_{30}}{2} + \dfrac{3c_{10}c_{30}^2}{16c_{20}^2} - \dfrac{c_{40}c_{10}}{5c_{20}},\ D_{31} = c_{31} + \dfrac{3c_{11}c_{30}^2}{16c_{20}^2} - \dfrac{5c_{21}c_{30} + 2c_{11}c_{40}}{10c_{20}},\ D_{40} = \dfrac{3c_{40}}{5} - \dfrac{5c_{30}^2}{16c_{20}}.$

Next, introducing a new time variable τ by

$$d\tau = \left(1 + \frac{c_{30}}{2c_{20}}x_4 + \frac{-25c_{30}^2 + 48c_{20}c_{40}}{80c_{20}^2}x_4^2 + \frac{-35c_{30}^3 + 48c_{20}c_{30}c_{40}}{80c_{20}^3}x_4^3\right)dt,$$

we can obtain (still denote τ by t)

$$\begin{cases} \dfrac{dx_4}{dt} = y_4, \\ \dfrac{dy_4}{dt} = Q_4(x_4, y_4) + O(|x_4, y_4|^5), \end{cases} \tag{14}$$

where

$$\begin{aligned} Q_4(x_4, y_4) =& d_{00} + d_{10}x_4 + d_{01}y_4 + d_{11}x_4y_4 + d_{20}x_4^2 + d_{21}x_4^2y_4 \\ &+ d_{30}x_4^3 + d_{40}x_4^4 + d_{31}x_4^3y_4 \end{aligned}$$

and
$d_{00} = c_{00},\ d_{10} = c_{10} - \dfrac{c_{00}c_{30}}{2c_{20}},\ d_{01} = c_{01},\ d_{11} = c_{11} - \dfrac{c_{01}c_{30}}{2c_{20}},\ d_{20} = c_{20} - \dfrac{60c_{10}c_{20}c_{30} - 45c_{00}c_{30}^2 + 48c_{00}c_{20}c_{40}}{80c_{20}^2},\ d_{21} = c_{21} - \dfrac{60c_{11}c_{20}c_{30} - 45c_{01}c_{30}^2 + 48c_{01}c_{20}c_{40}}{80c_{20}^2},$
$d_{30} = \dfrac{c_{10}(35c_{30}^2 - 32c_{20}c_{40})}{40c_{20}^2},\ d_{31} = c_{31} + \dfrac{7c_{11}c_{30}^2}{8c_{20}^2} - \dfrac{5c_{21}c_{30} + 4c_{11}c_{40}}{5c_{20}},$
$d_{40} = \dfrac{1}{6400c_{20}^4}(100c_{10}c_{20}c_{30}(-15c_{30}^2 + 16c_{20}c_{40}) + c_{00}(-275c_{30}^4 - 1440c_{20}c_{30}^2c_{40} + 2304c_{20}^2c_{40}^2))$. $d_{30} = 0$ and $d_{40} = 0$ when $r = 0$.

(IV) Removing the $x_4^2y_4$-term from Q_4 in system (14) when $r = 0$. Since $d_{20} = \frac{1}{2} + O(r)$, $d_{20} \neq 0$ for small r, we can make the parameter dependent affine transformation $x_4 = x_5,\ y_4 = y_5 + \frac{d_{21}}{3d_{20}} y_5^2 + \frac{d_{21}^2}{36d_{20}^2} y_5^3$, then system (14) becomes

$$\begin{cases} \dfrac{dx_5}{dt} = y_5(1 + \dfrac{d_{21}}{3d_{20}} y_5 + \dfrac{d_{21}^2}{36d_{20}^2} y_5^2), \\ \dfrac{dy_5}{dt} = Q_5^*(x_5, y_5) + O(|x_5, y_5|^5), \end{cases} \tag{15}$$

where

$$\begin{aligned} Q_5^*(x_5, y_5) =& F_{00} + F_{10}x_5 + F_{01}y_5 + F_{20}x_5^2 + F_{11}x_5y_5 + F_{02}y_5^2 + F_{30}x_5^3 + F_{21}x_5^2y_5 \\ &+ F_{12}x_5y_5^2 + F_{03}y_5^3 + F_{40}x_5^4 + F_{31}x_5^3y_5 + F_{22}x_5^2y_5^2 + F_{13}x_5y_5^3 + F_{04}y_5^4 \end{aligned}$$

and
$F_{00} = d_{00},\ F_{10} = d_{10},\ F_{01} = d_{01} - \dfrac{2d_{21}d_{00}}{3d_{20}},\ F_{20} = d_{20},\ F_{11} = d_{11} - \dfrac{2d_{10}d_{21}}{3d_{20}},\ F_{02} = \dfrac{-12d_{01}d_{20}d_{21} + 13d_{00}d_{21}^2}{36d_{20}^2},\ F_{30} = d_{30},\ F_{21} = \dfrac{d_{21}}{3},\ F_{12} = \dfrac{-12d_{11}d_{20}d_{21} + 13d_{10}d_{21}^2}{36d_{20}^2},$
$F_{03} = \dfrac{9d_{01}d_{20}d_{21}^2 - 10d_{00}d_{21}^3}{54d_{20}^3},\ F_{40} = d_{40},\ F_{31} = d_{31} - \dfrac{2d_{21}d_{30}}{3d_{20}},\ F_{22} = \dfrac{d_{21}^2}{36d_{20}},$
$F_{13} = \dfrac{9d_{11}d_{20}d_{21}^2 - 10d_{10}d_{21}^3}{54d_{20}^3},\ F_{04} = \dfrac{-108d_{01}d_{20}d_{21}^3 + 121d_{00}d_{21}^4}{1296d_{20}^4}.$

Once again, introducing a new time variable τ by

$$d\tau = (1 + \frac{d_{21}}{3d_{20}} y_5 + \frac{d_{21}^2}{36d_{20}^2} y_5^2)dt,$$

we obtain (still denote τ by t)

$$\begin{cases} \dfrac{dx_5}{dt} = y_5, \\ \dfrac{dy_5}{dt} = Q_5(x_5, y_5) + R_5(x_5, y_5, r), \end{cases} \tag{16}$$

where

$$Q_5(x_5, y_5) = e_{00} + e_{10}x_5 + e_{01}y_5 + e_{11}x_5y_5 + e_{20}x_5^2 + e_{31}x_5^3y_5$$

and
$e_{00} = d_{00},\ e_{10} = d_{10},\ e_{01} = d_{01} - \frac{d_{00}d_{21}}{d_{20}},\ e_{11} = d_{11} - \frac{d_{10}d_{21}}{d_{20}},\ e_{20} = d_{20},\ e_{31} = d_{31},$ $R_5(x_5, y_5, r)$ has the property (8).

(V) Changing e_{20} to 1 and e_{31} to -1 in Q_5 in system (16). We can see that $e_{20} = \frac{1}{2} + O(r) > 0$ and $e_{31} = -\frac{1}{2}(2 + \sqrt{3}) + O(r) < 0$ for small r. By making the following changes of variables and time:

$$x_5 = e_{20}^{\frac{1}{5}} e_{31}^{\frac{-2}{5}} x_6,\ y_5 = -e_{20}^{\frac{4}{5}} e_{31}^{\frac{-3}{5}} y_6,\ t = -e_{20}^{\frac{-3}{5}} e_{31}^{\frac{1}{5}} \tau,$$

system (16) becomes (still denote τ by t)

$$\begin{cases} \dfrac{dx_6}{dt} = y_6, \\ \dfrac{dy_6}{dt} = Q_6(x_6, y_6) + R_6(x_6, y_6, r), \end{cases} \tag{17}$$

where

$$Q_6(x_6, y_6) = f_{00} + f_{10}x_6 + f_{01}y_6 + f_{11}x_6y_6 + x_6^2 - x_6^3y_6$$

and
$f_{00} = e_{00}e_{31}^{\frac{4}{5}}e_{20}^{-\frac{7}{5}}$, $f_{10} = e_{10}e_{31}^{\frac{2}{5}}e_{20}^{-\frac{6}{5}}$, $f_{01} = -e_{01}e_{31}^{\frac{1}{5}}e_{20}^{-\frac{3}{5}}$, $f_{11} = -e_{11}e_{31}^{-\frac{1}{5}}e_{20}^{-\frac{2}{5}}$, $R_6(x_6, y_6, r)$ has the property (8).

(VI) Removing the x_6-term from Q_6 in system (17) when $r = 0$. Let $x_7 = x_6 + \frac{f_{10}}{2}$, $y_7 = y_6$, then system (17) becomes

$$\begin{cases} \dfrac{dx_7}{d\tau} = y_7, \\ \dfrac{dy_7}{dt} = \gamma_1 + \gamma_2 y_7 + \gamma_3 x_7 y_7 + x_7^2 - x_7^3 y_7 + R_7(x_7, y_7, r), \end{cases} \tag{18}$$

where $\gamma_1 = f_{00} - \frac{1}{4}f_{10}^2$, $\gamma_2 = f_{01} + \frac{1}{8}(f_{10}^3 - 4f_{10}f_{11})$, $\gamma_3 = f_{11} - \frac{3}{4}f_{10}^2$, and $R_7(x_7, y_7, r)$ has the property (8).

Lengthy computations by *Mathematica* software show that

$$\frac{D(\gamma_1, \gamma_2, \gamma_3)}{D(r_1, r_2, r_3)} = 4\sqrt{3}\ 2^{\frac{3}{5}}\ (2 + \sqrt{3})^{\frac{4}{5}} + O(r) > 0$$

for small r, it is obvious that system (18) is exactly in the form of system (7). By the results in Dumortier, Roussarie and Sotomayor [10] and Chow, Li and Wang [7], system (18) is the versal unfolding of the Bogdanov-Takens singularity (cusp) of codimension 3, the remainder term $R_7(x_7, y_7, r)$ with the property (8) has no influence on the bifurcation phenomena, and the dynamics of system (5) in a small neighborhood of the interior equilibrium $(2 - \sqrt{3}, -1 + \sqrt{3})$ as (δ, β, h) varing near (δ_0, β_0, h_0) are equivalent to that of system (18) in a small neighborhood of $(0, 0)$ as $(\gamma_1, \gamma_2, \gamma_3)$ varing near $(0, 0, 0)$. □

3. **Bifurcation diagram and numerical simulations.** We describe the bifurcation diagram of system (18) following the bifurcation diagram given in Figure 3 of Dumortier, Roussarie and Sotomayor [10] (see also Zhu, Campbell and Wolkowicz [25], Lamontagne, Coutu and Rousseau [18]) based on a time reversal transformation. System (18) has no equilibria for $\gamma_1 > 0$. $\gamma_1 = 0$ is a saddle-node bifurcation plane in a neighborhood of the origin, crossing the plane in the direction of decreasing γ_1, two equilibria are created: a saddle, and a node or focus. The other surfaces of bifurcation are located in the half space $\gamma_1 < 0$. The bifurcation diagram has the conical structure in $\mathbb{R}^3$ starting from $(\gamma_1, \gamma_2, \gamma_3) = (0, 0, 0)$. It can best be shown by drawing its intersection with the half sphere

$$S = \{(\gamma_1, \gamma_2, \gamma_3) | \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = \lambda^2, \gamma_1 \le 0, \lambda > 0 \text{ sufficiently small}\}.$$

To see the trace of intersection clearly, we draw the projection of the trace onto the (γ_2, γ_3)-plane, see Figure 2.

Now we summarize the bifurcation phenomena of system (18), which is equivalent to the original system (5). There are three bifurcation curves on S as shown in Figure 2:

C : homoclinic bifurcation curve;
H : Hopf bifurcation curve;
L : saddle-node bifurcation curve of limit cycles.

The curve L is tangent to H at a point h_2 and tangent to C at a point c_2. The curves H and C have first order contact with the boundary of S at the points b_1 and b_2. In the neighborhood of b_1 and b_2, system (18) is an unfolding of the cusp singularity of codimension 2.

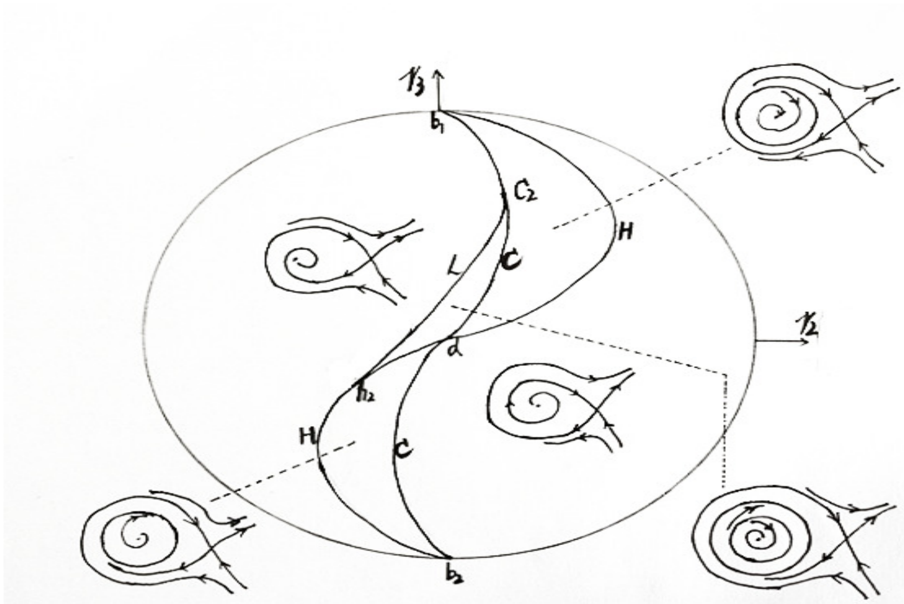

FIGURE 2. Bifurcation diagram for system (18) on S.

(a) Along the curve C, except at the point c_2, a homoclinic bifurcation of codimension 1 occurs. When crossing the arc b_1c_2 of C from left to right, the two separatrices of the saddle point coincide and an unstable limit cycle appears. The same phenomenon gives rise to a stable limit cycle when crossing the arc c_2b_2 of C from right to left. The point c_2 corresponds to a homoclinic bifurcation of codimension 2.

(b) Along the arc b_1h_2 of the curve H, a subcritical Hopf bifurcation occurs with an unstable limit cycle appearing when crossing the arc b_1h_2 of H from right to left. Along the arc h_2b_2 of the curve H, a supercritical Hopf bifurcation occurs with a stable limit cycle appearing when crossing the arc h_2b_2 of H from left to right. The point h_2 is a degenerate Hopf bifurcation point, i.e., a Hopf bifurcation point of codimension 2.

(c) The curves H and C intersect transversally at a unique point d representing a parameter value of simultaneous Hopf and homoclinic bifurcation.

(d) For parameter values in the triangle dh_2c_2, there exist exactly two limit cycles: the inner one is unstable and the outer one is stable. These two limit cycles coalesce in a generic way in a saddle-node bifurcation of limit cycles when the curve L is crossed from right to left. On the arc L itself, there exists a unique semistable limit cycle.

In the following, we give some numerical simulations for system (6) to confirm the existence of Bogdanov-Takens bifurcation (cusp case) of codimension 3. In Figure 3, we fix $r_1 = 0, r_3 = -0.01$. An unstable hyperbolic focus A for $r_2 = -0.012$ is shown in Figure 3(a); when r_2 increases to $r_2 = -0.011$, an unstable limit cycle arrounding a stable hyperbolic focus A appears by subcritical Hopf bifurcation (see Figure 3(b)); when $r_2 = -0.009999$, the coexistence of a stable homoclinic loop and an unstable limit cycle is shown in Figure 3(c), the homoclinic orbit breaks for larger $r_2 = -0.0095$ (see Figure 3(d)). Comparing with Figure 3(b) and (d), we can see that the relative locations of the stable manifold and unstable manifold for the saddle B are reversed, which implies the occurence of a homoclinic bifurcation when r_2 is between $r_2 = -0.011$ and $r_2 = -0.0095$.

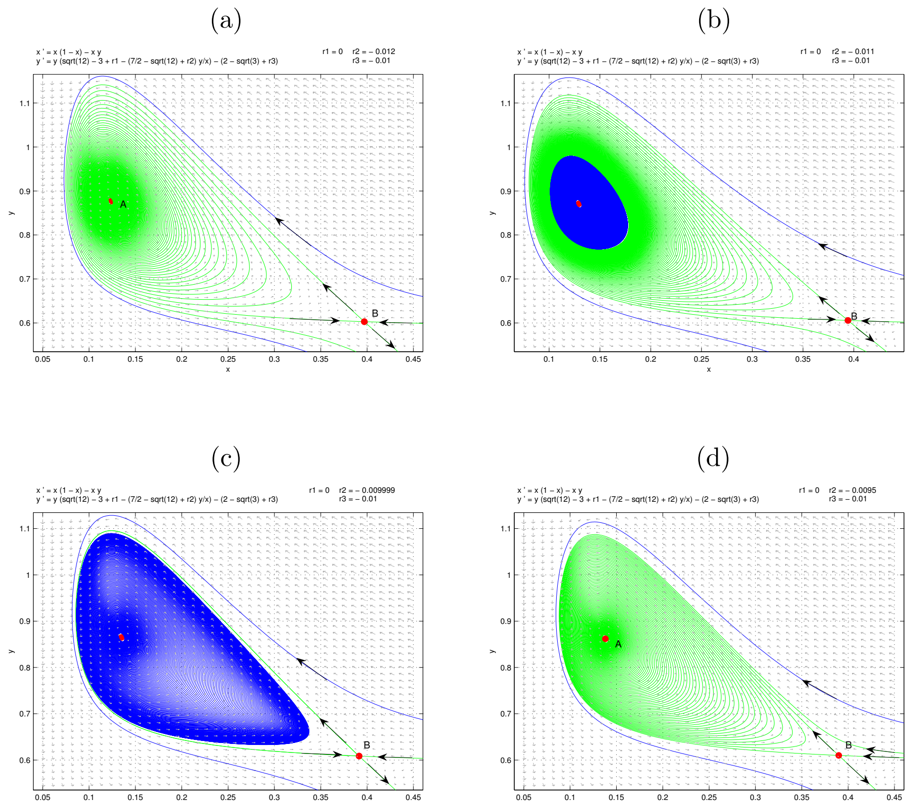

FIGURE 3. The coexistence of a stable homoclinic loop and an unstable limit cycle in system (6) with $r_1 = 0, r_3 = -0.01$. (a) An unstable hyperbolic focus A for $r_2 = -0.012$; (b) An unstable limit cycle arrounding a stable focus A for $r_2 = -0.011$; (c) The coexistence of a stable homoclinic loop and an unstable limit cycle for $r_2 = -0.009999$; (d) A stable hyperbolic focus A for $r_2 = -0.0095$.

In Figure 4, we fix $r_1 = -0.1541, r_2 = -0.0234$. A stable hyperbolic focus A for $r_3 = -0.081$ is shown in Figure 4(a); when r_3 increases to $r_3 = -0.0799$, a stable limit cycle arrounding an unstable hyperbolic focus A appears by supercritical Hopf bifurcation (see Figure 4(b)); when $r_3 = -0.07443$, a stable homoclinic loop is shown in Figure 4(c), the homoclinic orbit breaks for larger $r_3 = -0.073$ (see Figure 4(d)). Figure 4(b) and (d) show that the relative locations of the stable manifold and unstable manifold for the saddle B are reversed, which implies the occurence of homoclinic bifurcation when r_3 is between $r_3 = -0.0799$ and $r_3 = -0.073$.

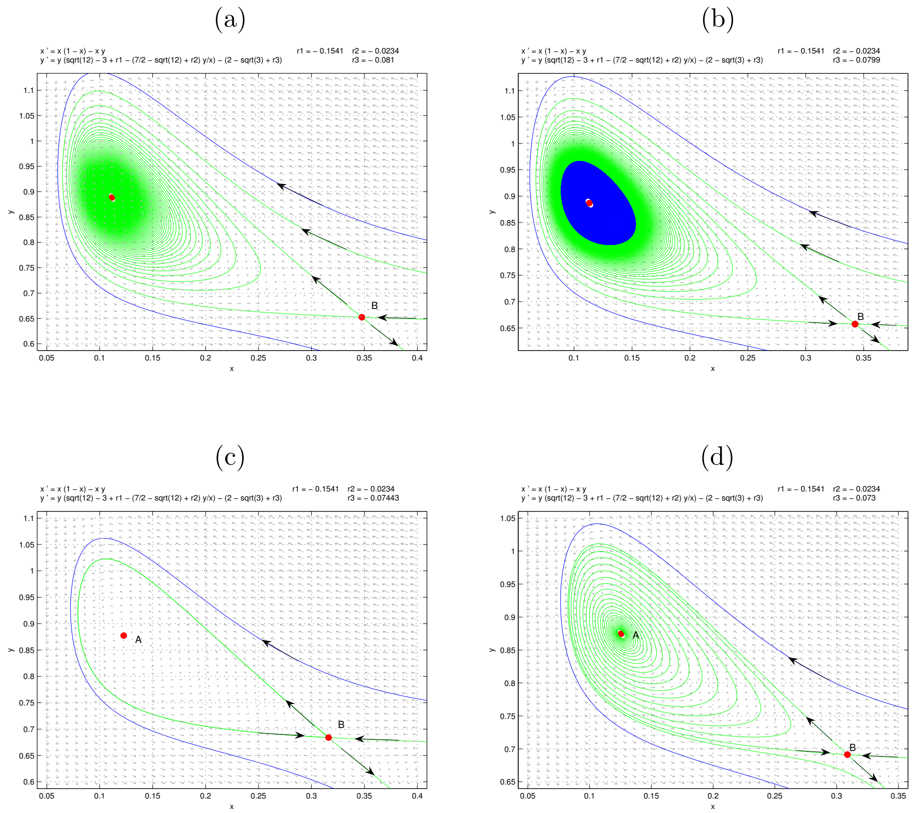

Figure 4. Numerical simulations for the supercritical Hopf bifurcation and homoclinic bifurcation in system (6) with $r_1 = -0.1541, r_2 = -0.0234$. (a) A stable hyperbolic focus A for $r_3 = -0.081$; (b) A stable limit cycle arrounding an unstable hyperbolic focus A for $r_3 = -0.0799$; (c) A stable homoclinic loop for $r_3 = -0.07443$; (d) An unstable hyperbolic focus A for $r_3 = -0.073$.

In Figure 5, we fix $r_1 = 0.38 + 3 - \sqrt{12}, r_2 = \frac{1}{80} - \frac{7}{2} + \sqrt{12}$ and $r_3 = \frac{21}{100} - 2 + \sqrt{3}$, the existence of two limit cycles is shown, in which the repelling cycle is surrounded by an attracting cycle.

4. **Discussion.** Our analytical results confirmed the conjecture in Huang, Gong and Ruan [17] about the existence of Bogdanov-Takens bifurcation of codimension 3 in system (5), so there exist some new dynamics in system (5), such as the coexistence of a stable homoclinic loop and an unstable limit cycle, two limit cycles (the inner one unstable and the outer stable) and a semi-stable limit cycle for various parameters values, which were only numerically simulated in [17].

Notice that these complex dynamics cannot occur in the unharvested systems (2) and the case (3) with only constant-yield prey harvesting. The unharvested model (2) has only one positive equilibrium which is globally stable under all admissible

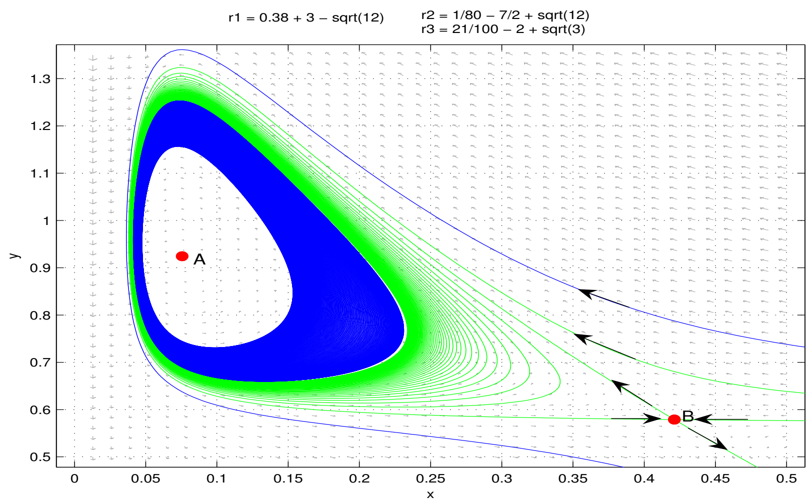

FIGURE 5. The existence of two limit cycles for system (6).

parameters (Hsu and Huang [14]), while model (3) with only constant-yield prey harvesting has at most two positive equilibria and exhibits Hopf bifurcation of codimension 1 (Zhu and Lan [26]) and Bogdanov-Takens bifurcation of codimension 2 (Gong and Huang [13]). Combining the results in Huang, Gong and Ruan [17] and in this paper, we can see that the model (5) with constant-yield predator harvesting has a Bogdanov-Takens singularity (cusp) of codimension 3 or a weak focus of multiplicity two for some parameter values, respectively, and exhibits saddle-node bifurcation, repelling and attracting Bogdanov-Takens bifurcations, supercritical and subcritical Hopf bifurcations, degenerate Hopf bifurcation, and Bogdanov-Takens (cusp) bifurcation of codimension 3 as the values of parameters vary. Thus the constant-yield predator harvesting in system (5) can cause more complex dynamical behaviors and bifurcation phenomena compared with the unharvested system (2) or system (3) with only constant-yield prey harvesting. In Huang, Gong and Ruan [17], we have shown that the constant-yield predator harvesting h can affect the number and type of equilibria, and the type of bifurcations of the model (5) (see Lemma 2.1 and Theorems 2.2, 2.3, 3.3 and 3.4 in [17]), from Figure 4 in this paper we can also see that the dynamics of the model (5) change dramatically as h changes even slightly. Therefore, our results demonstrate that the dynamical behaviors of model (5) are sensitive to the constant-yield predator harvesting and this suggests careful management of resource and harvesting policies in the applied conservation and renewable resource contexts.

Recall that in the original model (1) proposed by May et al. [21], both the prey and predators are subject to harvesting. It will be very interesting (and challenging) to investigate the bifurcations in the model with constant-yield harvesting on both the prey and predators (Beddington and Cooke [1]).

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