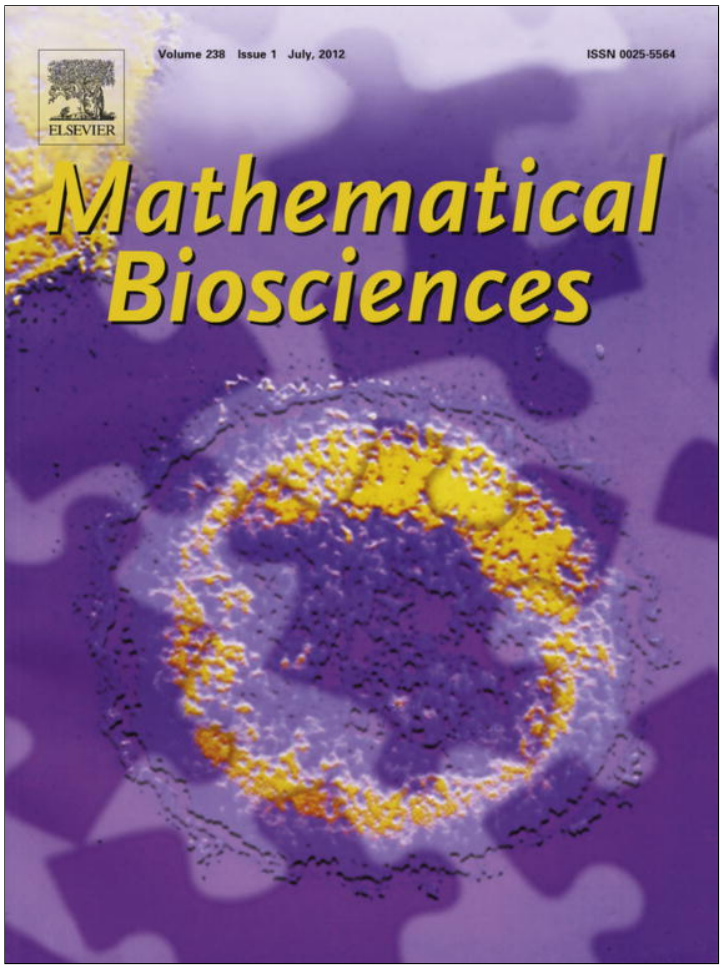

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Analysis of SIR epidemic models with nonlinear incidence rate and treatment

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ABSTRACT

This paper deals with the nonlinear dynamics of a susceptible-infectious-recovered (SIR) epidemic model with nonlinear incidence rate, vertical transmission, vaccination for the newborns of susceptible and recovered individuals, and the capacity of treatment. It is assumed that the treatment rate is proportional to the number of infectives when it is below the capacity and constant when the number of infectives reaches the capacity. Under some conditions, it is shown that there exists a backward bifurcation from an endemic equilibrium, which implies that the disease-free equilibrium coexists with an endemic equilibrium. In such a case, reducing the basic reproduction number less than unity is not enough to control and eradicate the disease, extra measures are needed to ensure that the solutions approach the disease-free equilibrium. When the basic reproduction number is greater than unity, the model can have multiple endemic equilibria due to the effect of treatment, vaccination and other parameters. The existence and stability of the endemic equilibria of the model are analyzed and sufficient conditions on the existence and stability of a limit cycle are obtained. Numerical simulations are presented to illustrate the analytical results.

1. Introduction

One of the focuses of theoretical studies in mathematical epidemiology is to understand the nonlinear dynamics of various epidemic models. Classical susceptible-infectious-recovered (SIR) models with a bilinear incidence rate usually have a disease-free equilibrium and at most one endemic equilibrium (see Capasso and Serio [7] and Hethcote [20]). In such a case, the basic reproduction number is a threshold: when it is less than unity the disease-free equilibrium exists and is stable, which indicates that the disease will die out; when it is greater than unity the disease-free equilibrium becomes unstable and an endemic equilibrium exists, which demonstrates that the disease will persist. The bifurcation leading from a disease-free equilibrium to an endemic equilibrium is called *forward*.

Recently there has been a great interest in investigating the nonlinear dynamics (such as Hopf bifurcation, saddle-node bifurcation, Bogdanov–Takens bifurcation, existence of periodic and homoclinic orbits, coexistence of limit cycles and homoclinic orbits) in epidemic models with multiple endemic equilibria due to social groups with different susceptibilities, nonlinear or nonmonotone incidence rate, stage structure, behavioral change of susceptibles, etc. (see Alexander and Moghadas [1], Derrick and van den Driessche [12], Feng and Thieme [14], Hu et al. [21], Liu et al. [26,27], Ruan and Wang [30], Tang et al. [33], and Xiao and Ruan [39]). For instance, Liu et al. [26] proposed a nonlinear saturated incidence function $\beta SI^{\ell}/(1+\alpha I^{h})$ to model the effect of behavioral changes to certain communicable diseases, where βSI^{ℓ} describes the infection force of the disease, $1/(1+\alpha I^{h})$ measures the inhibition effect from the behavioral change of the susceptible individuals when the number of infectious individuals increases, ℓ, h, and β are all positive constants, and α is a nonnegative constant. The case when $\ell = h = 1$, i.e. the function $\beta SI/(1+\alpha I)$, was used by Capasso and Serio [7] to represent a "crowding effect" or "protection measure" in modeling the cholera epidemics in Bari in 1973. Because of the nonlinearity and saturation property of these incidence functions, it has been shown that SIR epidemic models with such nonlinear incidence rates can possess multiple endemic equilibria. Moreover, such models can exhibit Hopf bifurcation, saddle-node bifurcation and Bogdanov–Takens bifurcation [1,12,26,27,30], the existence of two limit cycles [21], and the coexistence of a limit cycle and a homoclinic cycle [33]. These results demonstrate that the nonlinear dynamics of these epidemic models are very sensitive to the model parameters and various outcomes could occur depending on the initial population sizes. The disease may be eradicated (solutions approach a disease-free equilibrium), persistent (solutions approach an endemic equilibrium), or occurring periodically

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(solutions approach a limit cycle). Understand such nonlinear dynamics and identify the underlie factors is very important in the control and prevention of spread of communicable diseases.

The phenomenon that the disease-free equilibrium coexists with an endemic equilibrium when the basic reproduction number is less than unity was first observed by Castillo-Chavez et al. [8,9] and Huang et al. [23] in multi-group HIV/AIDS models. It was termed as *backward bifurcation* by Hadeler and Castillo-Chavez [17] and Hadeler and van den Driessche [18] and was shown to exist in epidemic models that include behavioral responses (such as via education) to perceived disease risk. The existence of backward bifurcation in epidemic models has important qualitative implications since the disease now cannot be eradicated by simply reducing the basic reproduction number to be less than unity. Dushoff et al. [13] provided a general framework for the mechanisms behind backward bifurcations in simple epidemic models and discussed the biological interpretation of the features of these models that induce backward bifurcations. Therefore, it is also important to study backward bifurcation in epidemic models in order to seek for conditions for the control of diseases (see Arino et al. [3], Blayneh et al. [4], Brauer [5,6], Greenhalgh and Griffiths [15], Kribs-Zaleta and Velasco-Hernandez [24], Regula et al. [29], Safan et al. [31], van den Driessche and Watmough [34], and Wan and Zhu [35]). In this case, the basic reproduction number does not give information on disease elimination; rather disease elimination is determined by the values of critical parameters at the turning points.

Treatment is an important and effective method to prevent and control the spread of various infectious diseases. In classical epidemic models, the treatment rate of the infectives is assumed to be proportional to the number of the infective individuals [2]. During the SARS outbreaks in 2003, the dramatically increasing of SARS cases in Beijing challenged the normal public-health system and capacity in Beijing City and forced the Chinese government to create the first and only SARS hospital, Beijing Xiaotangshan Hospital, to treat the large number of SARS patients [37]. After this experience, researchers started to consider the capacity of the health-care system from both modeling and analyzing points of view. Wang and Ruan [38] considered a SIR model in which the capacity for the treatment of a disease in a community is a constant. Namely, they used the following function

$$T(I)=\begin{cases} k, & \text{if } I>0,\\ 0, & \text{if } I=0 \end{cases} \tag{1.1}$$

to describe the treatment rate and studied the effect of treatment parameter k on the dynamics of the model. They found that the model undergoes a sequence of bifurcations including saddle-node, Hopf, and homoclinic bifurcations and exhibits homoclinic orbits even though incidence rate is assumed to be bilinear. Moreover, it is shown that it may not be necessary to set k so high to eliminate endemic equilibria since such equilibria may be unstable. Wang [36] proposed the following piecewise linear treatment function

$$T(I)=\begin{cases} kI, & \text{if } 0\leqslant I\leqslant I_0,\\ u=kI_0, & \text{if } I>I_0, \end{cases} \tag{1.2}$$

in a SIR model, where I_0 is the infective level at which the health-care system reaches capacity; that is, treatment increases linearly with I before the capacity is reached and is constant afterward. It was shown that the model has bistable endemic equilibria when I_0 is low and backward bifurcation can occur.

The treatment function (1.2) has been used by some other researchers. For example, Zhang and Liu [41] studied a model with a general incidence rate $\lambda(I+S)^{n-1}SI$ $(0\leqslant n\leqslant 1)$ and the treatment function (1.2). Hu et al. [22] considered an epidemic model with standard incidence rate $\beta SI/N$ and the treatment function (1.2). Li et al. [25] studied an epidemic model with nonlinear incidence rate $\beta I/(1+\alpha I)$ with the treatment function (1.2) and analyzed the stability and bifurcation of the system. Other types of treatment functions have also been proposed. For instance, Zhang and Liu [40] used a saturated treatment function $T(I)=rI/(1+\mu I)$, $r>0$, $\mu\geqslant 0$, and found that the saturated function has the advantage of giving near-linear treatment response when I is low and approaches a constant capacity as I gets large. Eckalbar and Eckalbar [11] introduced a new treatment function, $T(I)=\max\{\gamma I-gI^2,0\}$, $\gamma>0$, $g>0$, into a SIR model with bilinear incidence rate. It was found that the system could have up to four equilibria with possible bi-stability, backward bifurcations, and limit cycles. See also Li et al. [25], Zhou and Fan [42], etc.

In this paper, we consider a SIR epidemic model with the nonlinear incidence rate $\beta SI/(1+\alpha I)$, the treatment rate function (1.2), vertical transmission, and vaccination for the newborns of the susceptible and recovered individuals. To formulate our model, let $S(t), I(t)$ and $R(t)$ be the number of susceptible, infective and recovered individuals at time t, respectively. The basic assumptions are as follows.

(i) The total population size at time t (day) is denoted by $N=S+I+R$. The newborns of S and R are susceptible individuals, and the newborns of I who are not vertically infected are also susceptible individuals.
(ii) The positive constant b (per day) denotes the death rate and birth rate of susceptible and recovered individuals. The positive constant δ (per day) denotes the death rate and birth rate of infective individuals. The positive constant γ (per day) is the natural recovery rate of infective individuals. The positive constant $q(q\leqslant 1)$ (per day) is the vertical transmission rate, and note $p=1-q$ (per day), then $0\leqslant p\leqslant 1$. Fraction m' of all newborns with mothers in the susceptible and recovered classes are vaccinated and appeared in the recovered class, while the remaining fraction, $m=1-m'$, appears in the susceptible class.
(iii) The incidence rate is described by a nonlinear function $\beta SI/(1+\alpha I)$, where β (per day) is a positive constant describing the infection rate and α (per person) is a nonnegative constant represents the half saturation constant.
(iv) The treatment rate of a disease is $T(I)$ given in (1.2).

Under the above assumptions, the SIR epidemic model takes the following form:

$$\begin{cases} \frac{dS}{dt}=bm(S+R)-\beta\frac{SI}{1+\alpha I}-bS+p\delta I,\\ \frac{dI}{dt}=\beta\frac{SI}{1+\alpha I}+q\delta I-\delta I-\gamma I-T(I),\\ \frac{dR}{dt}=\gamma I-bR+bm'(S+R)+T(I). \end{cases} \tag{1.3}$$

Because

$$\frac{dN}{dt}=0,$$

the total number of population N is a constant. For convenience, it is assumed that $N=S+I+R=1$, thus S, I and R are taken as the proportions of susceptible, infective and recovered individuals in the total population. By using $S+R=1-I$, the first two equations of system (1.3) do not contain the variable R. Therefore, system (1.3) is equivalent to the following 2-dimensional system:

$$\begin{cases} \frac{dS}{dt}=-\beta\frac{SI}{1+\alpha I}-bS+bm(1-I)+p\delta I,\\ \frac{dI}{dt}=\beta\frac{SI}{1+\alpha I}-p\delta I-\gamma I-T(I). \end{cases} \tag{1.4}$$

It is easy to verify that the positive invariant set of system (1.4) is

$$D=\{(S,I)|S\geqslant 0,\ I\geqslant 0,\ S+I\leqslant 1\}.$$

When $0 \leqslant I \leqslant I_0$, system (1.4) is

$$\begin{cases} \frac{dS}{dt} = -\beta\frac{SI}{1+\alpha I} - bS + bm(1-I) + p\delta I, \\ \frac{dI}{dt} = \beta\frac{SI}{1+\alpha I} - p\delta I - \gamma I - kI. \end{cases} \tag{1.5}$$

When $I > I_0$, system (1.4) becomes ($u = kI_0$)

$$\begin{cases} \frac{dS}{dt} = -\beta\frac{SI}{1+\alpha I} - bS + bm(1-I) + p\delta I, \\ \frac{dI}{dt} = \beta\frac{SI}{1+\alpha I} - p\delta I - \gamma I - u. \end{cases} \tag{1.6}$$

The purpose of this paper is to study the nonlinear dynamics of system (1.4). Under some conditions, it is shown that there exists a backward bifurcation from an endemic equilibrium, which implies that the disease-free equilibrium coexists with an endemic equilibrium. In such a case, reducing the basic reproduction number less than unity is not enough to control and eradicate the disease, extra measures are needed to ensure that the solutions approach the disease-free equilibrium. When the basic reproduction number is greater than unity, model (1.4) can have multiple endemic equilibria due to the effect of treatment, vaccination and other parameters. The existence and stability of the endemic equilibria of the model are analyzed and sufficient conditions on the existence and stability of a limit cycle are obtained. Numerical simulations are presented to illustrate the analytical results.

The organization of this paper is as follows. In next section, we analyze the existence and bifurcations of equilibria for (1.4). In Section 3, we study the stability of various equilibria and the existence and stability of a limit cycle in (1.4). In Section 4, we give some numerical simulations to verify our results. A brief discussion is presented in Section 5.

2. Existence of equilibria

In this section, we consider the equilibria of system (1.4). Obviously, $E_0(m,0)$ is the disease-free equilibrium of (1.4).

For the positive equilibrium $E^*(S^*,I^*)$ of system (1.4), when $0 < I^* \leqslant I_0$, let the right side of (1.5) be equal to zero. Then S^* and I^* satisfy the following equations:

$$\begin{cases} -\beta\frac{S^*I^*}{1+\alpha I^*} - bS^* + bm(1-I^*) + p\delta I^* = 0, \\ \beta\frac{S^*I^*}{1+\alpha I^*} - (p\delta+\gamma+k)I^* = 0. \end{cases} \tag{2.1}$$

When $I^* > I_0$, let the right side of (1.6) be equal to zero. Then S^* and I^* satisfy the following equations

$$\begin{cases} -\beta\frac{S^*I^*}{1+\alpha I^*} - bS^* + bm(1-I^*) + p\delta I^* = 0, \\ \beta\frac{S^*I^*}{1+\alpha I^*} - (p\delta+\gamma)I^* - u = 0. \end{cases} \tag{2.2}$$

From (2.1) we obtain that

$$I^* = \frac{b(p\delta+\gamma+k)(R_0-1)}{\beta(bm+\gamma+k)+\alpha b(k+\gamma+\delta p)}, \tag{2.3}$$

where

$$R_0 = \frac{\beta m}{p\delta+\gamma+k} \tag{2.4}$$

is called the *basic reproduction number* of (1.4); if $R_0 > 1, I^*$ in (2.3) is positive. At the same time, I^* in (2.3) must satisfy $I^* \leqslant I_0$, which is equivalent to

$$u \geqslant \frac{kb(p\delta+\gamma+k)(R_0-1)}{\beta(bm+\gamma+k)+\alpha b(k+\gamma+\delta p)} = u_2. \tag{2.5}$$

Therefore, $E_0(m,0)$ is always the disease-free equilibrium of (1.4). $E^*(S^*,I^*)$ is the endemic equilibrium of system (1.4) if and only if $R_0 > 1$ and $u \geqslant u_2$, where

$$S^* = \frac{\alpha bm(p\delta+\gamma+k)+(bm+\gamma+k)(p\delta+\gamma+k)}{\beta(bm+\gamma+k)+\alpha b(p\delta+\gamma+k)}. \tag{2.6}$$

According to (2.2), I^* satisfies the following equation

$$b_0(I^*)^2 + b_1I^* + b_2 = 0, \tag{2.7}$$

where

$$\begin{aligned} b_0 &= \beta(bm+\gamma)+\alpha b(p\delta+\gamma), \\ b_1 &= b(p\delta+\gamma-\beta m)+u(\alpha b+\beta), \\ b_2 &= bu. \end{aligned} \tag{2.8}$$

If $b_1 \geqslant 0$, it is clear that Eq. (2.7) does not have a positive solution. Let us suppose that $b_1 < 0$. Clearly, $b_1 < 0$ is equivalent to

$$\beta m > p\delta+\gamma \quad \text{and} \quad u < \frac{b(\beta m - p\delta - \gamma)}{b\alpha+\beta}. \tag{2.9}$$

From (2.8), we get

$$\begin{aligned} \Delta &= b_1^2 - 4b_0b_2 \\ &= (\alpha b+\beta)^2u^2 - 2b\{(\beta m - p\delta-\gamma)(\alpha b+\beta) + 2[\beta(bm+\gamma) \\ &\quad +\alpha b(p\delta+\gamma)]\}u + b^2(p\delta+\gamma-\beta m)^2. \end{aligned}$$

Note that

$$\begin{aligned} &\{-2b\{(\beta m - p\delta-\gamma)(\alpha b+\beta) + 2[\beta(bm+\gamma)+\alpha b(p\delta+\gamma)]\}\}^2 \\ &\quad - 4(\alpha b+\beta)^2b^2(p\delta+\gamma-\beta m)^2 \\ &= 4[\beta(bm+\gamma)+\alpha b(p\delta+\gamma)][\beta(bm+\gamma)+\alpha b(p\delta+\gamma) \\ &\quad + (\beta m - p\delta-\gamma)(\alpha b+\beta)] > 0. \end{aligned}$$

Denote

$$\begin{aligned} \eta = &[\beta(bm+\gamma)+\alpha b(p\delta+\gamma)][\beta(bm+\gamma)+\alpha b(p\delta+\gamma) \\ &+ (\beta m - p\delta-\gamma)(\alpha b+\beta)]^{\frac{1}{2}}. \end{aligned}$$

It follows that $\Delta \geqslant 0$ is equivalent to

$$u \geqslant \frac{b\{(\beta m - p\delta-\gamma)(\alpha b+\beta) + 2[\beta(bm+\gamma)+\alpha b(p\delta+\gamma)] + 2\eta\}}{(\alpha b+\beta)^2} \tag{2.10}$$

or

$$u \leqslant \frac{b\{(\beta m - p\delta-\gamma)(\alpha b+\beta) + 2[\beta(bm+\gamma)+\alpha b(p\delta+\gamma)] - 2\eta\}}{(\alpha b+\beta)^2} = u_0. \tag{2.11}$$

As $\beta m > p\delta+\gamma$, we have

$$u_0 < \frac{b(\beta m - p\delta-\gamma)}{\alpha b+\beta}.$$

Therefore, $b_1 < 0$, $\Delta \geqslant 0$ if and only if $\beta m > p\delta+\gamma$ and $u \leqslant u_0$.

Suppose that $u \leqslant u_0$ and $\beta m > p\delta+\gamma$, then (2.7) has two positive solutions I_1 and I_2, where

$$I_1 = \frac{-b_1-\sqrt{\Delta}}{2b_0}, \quad I_2 = \frac{-b_1+\sqrt{\Delta}}{2b_0}. \tag{2.12}$$

Let

$$S_j = \frac{bm - u - (bm+\gamma)I_j}{b}.$$

If $I_j > I_0 (j = 1,2)$, then $E_j(S_j,I_j)(j = 1,2)$ is an endemic equilibrium of (1.4).

By the expression of I_1, we notice that $I_1 > I_0$ is equivalent to

$$-\sqrt{\Delta} > 2b_0I_0 + b_1. \tag{2.13}$$

This implies that $2b_0I_0 + b_1 < 0$. It is easy to prove that $2b_0I_0 + b_1 < 0$ is equivalent to

$$u < \frac{bk(\beta m - p\delta - \gamma)}{2[\beta(bm+\gamma) + b\alpha(p\delta+\gamma)] + k(b\alpha+\beta)} = u_1. \tag{2.14}$$

Furthermore, $I_1 > I_0$ requires that

$$(b_1 + 2b_0 I_0)^2 - \Delta > 0.$$

Following some calculations, we obtain that

$$(b_1 + 2b_0 I_0)^2 - \Delta = \frac{4b_0 I_0}{k}\{kb[p\delta + \gamma + k - \beta m] + [\beta(bm+\gamma+k) + \alpha b(p\delta+\gamma+k)]u\}.$$

(1) If $p\delta + \gamma < \beta m \leqslant p\delta + \gamma + k$, then $(b_1 + 2b_0 I_0)^2 - \Delta > 0$.
(2) If $\beta m > p\delta + \gamma + k$, that is, $R_0 > 1$, then the inequality $(b_1 + 2b_0 I_0)^2 - \Delta > 0$ is equivalent to $u > u_2$.

Therefore, $I_1 > I_0$ holds if and only if either $u < u_1$, $u \leqslant u_0$ and $\beta m > p\delta + \gamma$ or $u < u_1$, $u > u_2$, $u \leqslant u_0$ and $R_0 > 1$. Similarly, $I_2 > I_0$ holds if and only if either $u \leqslant \min\{u_0, u_1\}$ and $\beta m > p\delta + \gamma$, or $u < \min\{u_0, u_2\}$, $u > u_1$ and $R_0 > 1$.

Note that the sign of $u_2 - u_1$ is determined by

$$\Phi = R_0 - R_0^*,$$

where

$$R_0^* = 1 + \frac{k(\beta(bm+\gamma+k) + \alpha b(p\delta+\gamma+k))}{(p\delta+\gamma+k)[\beta(bm+\gamma) + \alpha b(p\delta+\gamma)]}. \tag{2.15}$$

It follows that $u_1 > u_2$ if $1 < R_0 < R_0^*$, and $u_1 < u_2$ if $R_0 > R_0^*$.

Summarizing the above discussions, we have the following conclusions for the existence of equilibria.

Theorem 2.1. *(i) For system (1.4), the disease-free equilibrium $E_0(m, 0)$ always exists. (ii) $E^*(S^*, I^*)$ is an endemic equilibrium of system (1.4) if and only if $R_0 > 1$ and $u \geqslant u_2$. Furthermore, suppose $R_0 > 1$, $u \geqslant u_2$ and one of the following conditions is satisfied:*

(a) $u > u_0$;
(b) $u_1 < u < u_0$.

Then E^* is the unique endemic equilibrium of system (1.4).

Theorem 2.2. *The endemic equilibria $E_1(S_1, I_1)$ and $E_2(S_2, I_2)$ of system (1.4) do not exist if $\beta m \leqslant p\delta + \gamma$ or $u > u_0$. On the other hand, suppose $u \leqslant u_0$ and $\beta m > p\delta + \gamma$, we have the following results.*

(i) *If $R_0 \leqslant 1$ and $u < u_1$, then the equilibria E_1 and E_2 of system (1.4) exist.*
(ii) *If $1 < R_0 < R_0^*$ and $u_2 < u < u_1$, then the equilibria E_1 and E_2 of system (1.4) exist.*
(iii) *If $1 < R_0 < R_0^*$ and $u < u_2$, then the equilibrium E_2 of system (1.4) exists but E_1 does not exist, and the equilibrium E_2 does not exist if $u > u_1$.*
(iv) *If $1 < R_0^* < R_0$, then E_1 does not exist. Furthermore, E_2 exists if $u < u_2$ and does not exist if $u > u_2$.*

We can see that under the conditions in Theorem 2.1 (i), the disease-free equilibrium E_0 coexists with two endemic equilibria E_1 and E_2. In fact, we have the following result.

Corollary 2.3. *If $R_0 < 1$, $\beta m > p\delta + \gamma$, and $u < \min\{u_0, u_1\}$, then system (1.4) has a backward bifurcation of endemic equilibria.*

We now present examples to show that, for various parameter values, system (1.4) has a forward bifurcation from one endemic equilibrium to another endemic equilibrium (see Example 2.4) and a backward bifurcation with a disease-free equilibrium and two endemic equilibria (Example 2.5). Note that by Theorem 2.1 (ii) and Theorem 2.2 (ii), there are conditions that guarantee the existence of all three endemic equilibria E_1, E_2, and E^* (Example 2.6).

Example 2.4. We choose the parameter values as follows: $\beta = 0.34$, $\alpha = 0.4$, $\gamma = 0.01$, $\delta = 0.01$, $b = 0.2$, $p = 0.02$, $k = 0.03$ and $m = 0.3$. By calculations, $R_0 = 2.53731$, $R_0^* = 2.12826$, $R_0 > R_0^*$ and $u_2 = 0.00996346$, case (iv) of Theorem 2.2 holds. When $u \leqslant u_2$, a bifurcation diagram is illustrated in Fig. 1, the bifurcation at $u = u_2$ is forward when the parameter u decreases, and system (1.4) has a unique endemic equilibrium for all $u > 0$.

Example 2.5. We choose the parameter values as follows: $\beta = 0.2, \alpha = 0.4, \gamma = 0.01, \delta = 0.01, b = 0.2, p = 0.02, k = 0.03$ and $m = 0.1$. By calculations, we have $R_0 = 0.497512 < 1, u_0 = 0.000590643$, $u_1 = 0.00266885$, so case (i) of Theorem 2.2 holds. A backward bifurcation diagram is illustrated in Fig. 2, where the horizontal line denotes the disease-free equilibrium E_0. Two endemic equilibria appear simultaneously at $u = u_0$ when the parameter u decreases.

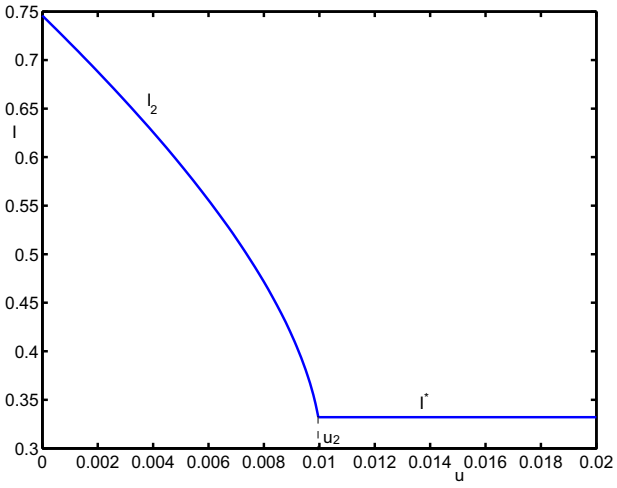

Fig. 1. The forward bifurcation diagram from I^* to I_2 versus u for (1.4).

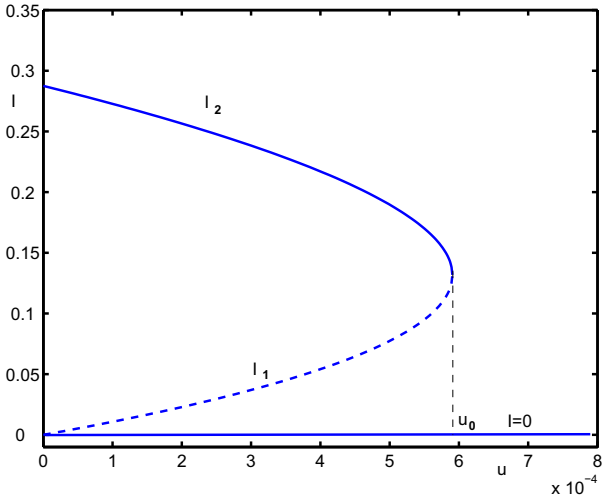

Fig. 2. The backward bifurcation diagram of I_1 and I_2 versus u.

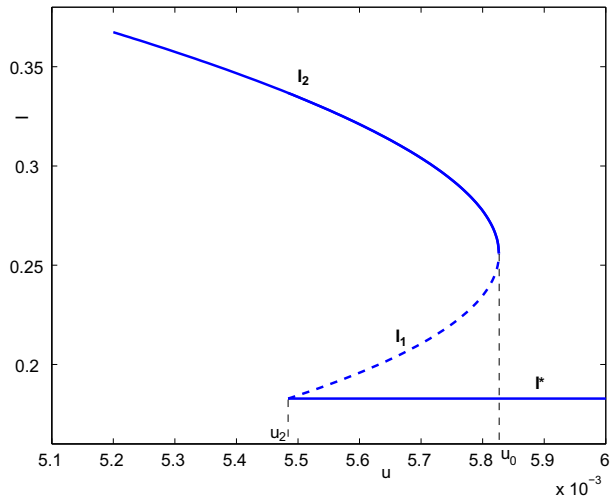

Fig. 3. The bifurcation diagram with multiple endemic equilibria versus u.

Example 2.6. Choose $\beta = 0.34,\ \alpha = 0.4,\ \gamma = 0.01,\ \delta = 0.01,\ b = 0.2,\ p = 0.02,\ k = 0.03$ and $m = 0.2$. We have $R_0 = 1.69154$, $R_0^* = 2.274\ 05$, $u_0 = 0.00582652$, $u_1 = 0.00719025$ and $u_2 = 0.00548396$, so case (ii) of Theorem 2.2 holds. A bifurcation diagram is illustrated in Fig. 3, where the horizontal line denotes the endemic equilibrium E^*. It shows that there is a bifurcation at $u = u_0$ when the parameter u decreases, which gives rise to the existence of multiple endemic equilibria E^*, E_1, and E_2.

3. Stability of equilibria

For the stability of the disease-free equilibrium $E_0(m, 0)$, we have the following theorem.

Theorem 3.1. *The disease-free equilibrium $E_0(m, 0)$ is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$. Moreover, $E_0(m, 0)$ is globally asymptotically stable in D if $R_0 < 1$ and $u > u_0$.*

Proof. It is easy to obtain that the characteristic roots to the linearized equation of system (1.4) at $E_0(m, 0)$ are $\lambda_1 = -b < 0$ and $\lambda_2 = (p\delta + \gamma + k)(R_0 - 1)$. Thus, E_0 is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.

Next, if $R_0 < 1, E^*$ does not exist by Theorem 2.1. By Theorem 2.2, E_1 and E_2 do not exist if $u > u_0$. Therefore, E_0 is the unique equilibrium of system (1.4). Since D is the invariant set of system (1.4) and E_0 is locally asymptotically stable, it follows from Bendixson Theorem that every solution of system (1.4) in D approaches E_0 when t tends to positive infinity. □

For the endemic equilibrium $E^*(S^*, I^*)$, we have the following theorem.

Theorem 3.2. *If the endemic equilibrium $E^*(S^*, I^*)$ of system (1.4) exists, then it is locally asymptotically stable.*

Proof. According to Theorem 2.1, the endemic equilibrium $E^*(S^*, I^*)$ exists if and only if $R_0 > 1$ and $u \geqslant u_2$. The Jacobian matrix of system (1.4) at $E^*(S^*, I^*)$ is

$$J^* = \begin{pmatrix} -b - \beta\frac{I^*}{1+\alpha I^*} & -\beta\frac{S^*}{(1+\alpha I^*)^2} - bm + p\delta \\ \beta\frac{I^*}{1+\alpha I^*} & \beta\frac{S^*}{(1+\alpha I^*)^2} - p\delta - \gamma - k \end{pmatrix}. \tag{3.1}$$

Because S^* and I^* satisfy Eq. (2.1), by means of (2.1), the trace and determinant of J^* are simplified into

$$\mathrm{tr}(J^*) = -\frac{b + [\beta + \alpha(b + p\delta + \gamma + k)]I^*}{1 + \alpha I^*} < 0,$$
$$\det(J^*) = \frac{b\alpha(p\delta + \gamma + k) + \beta(bm + \gamma + k)}{1 + \alpha I^*} I^* > 0.$$

Therefore, all eigenvalues of matrix J^* have negative real parts when $R_0 > 1,\ u \geqslant u_2$. It follows that $E^*(S^*, I^*)$ is locally asymptotically stable. □

Now we discuss the stability of endemic equilibria $E_i(S_j, I_j)(j = 1, 2)$. The Jacobian matrix of system (1.4) at $E_j(S_j, I_j)$ is

$$J_j = \begin{pmatrix} -b - \beta\frac{I_j}{1+\alpha I_j} & -\beta\frac{S_j}{(1+\alpha I_j)^2} - bm + p\delta \\ \beta\frac{I_j}{1+\alpha I_j} & \beta\frac{S_j}{(1+\alpha I_j)^2} - p\delta - \gamma \end{pmatrix}, \quad j = 1, 2. \tag{3.2}$$

Theorem 3.3. *If the endemic equilibrium $E_1(S_1, I_1)$ of system (1.4) exists, then it is unstable.*

Proof. For the endemic equilibrium $E_1(S_1, I_1), S_1$ and I_1 satisfy (2.2). By (2.2), (2.7) and (2.12), after some calculations, the determinant of the matrix J_1 is

$$\det(J_1) = \frac{-\sqrt{\Delta}}{1 + \alpha I_1} < 0. \tag{3.3}$$

Therefore, $E_1(S_1, I_1)$ is a saddle and unstable. □

For the endemic equilibrium $E_2(S_2, I_2)$, similarly, the determinant of J_2 is

$$\det(J_2) = \frac{\sqrt{\Delta}}{1 + \alpha I_2} > 0. \tag{3.4}$$

Thus, E_2 may be a node, focus, or center.

By (2.2), (2.7) and (2.12), the trace $\mathrm{tr}(J_2)$ of matrix J_2 is very complicated, but the sign of $\mathrm{tr}(J_2)$ is determined by

$$\varphi = -2bb_0b_2 + b_1u[b(\beta + b\alpha) - \beta(bm + \gamma)] - [b_2(\beta + \alpha(b + p\delta + \gamma)) + b_0u]\sqrt{\Delta}. \tag{3.5}$$

(1) According to $b_1 < 0$ and (3.5), if $b(\beta + b\alpha) \geqslant \beta(bm + \gamma)$, then $\varphi < 0$.
(2) If $b(\beta + b\alpha) < \beta(bm + \gamma)$, because

$$\begin{aligned} &-2bb_0b_2 + b_1u[b(\beta + b\alpha) - \beta(bm + \gamma)] \\ &= u\Big\{-2b^2[\beta(bm + \gamma) + b\alpha(p\delta + \gamma)] + b(p\delta + \gamma - m\beta)[b(\beta \\ &+b\alpha) - \beta(bm + \gamma)] + (b\alpha + \beta)[b(\beta + b\alpha) - \beta(bm + \gamma)]u\Big\}, \end{aligned} \tag{3.6}$$

it is known from (3.6) that if

$$0 < \beta m - p\delta - \gamma \leqslant \frac{2b[\beta(bm + \gamma) + b\alpha(p\delta + \gamma)]}{\beta(bm + \gamma) - b(\beta + b\alpha)} = \eta_1, \tag{3.7}$$

$$u \geqslant \frac{-2b^2[\beta(bm + \gamma) + b\alpha(p\delta + \gamma)] + b(p\delta + \gamma - m\beta)[b(\beta + b\alpha) - \beta(bm + \gamma)]}{(b\alpha + \beta)[\beta(bm + \gamma) - b(\beta + b\alpha)]} = u_3, \tag{3.8}$$

then $\varphi < 0$.

(3) It follows from (3.6) that if $b(\beta + b\alpha) < \beta(bm + \gamma)$, $\beta m - p\delta - \gamma > \eta_1$ and then $\varphi < 0$.

(4) It is known from (3.5) and (3.6) that if $b(\beta + b\alpha) < \beta(bm + \gamma)$, $\beta m - p\delta - \gamma > \eta_1$, $u < u_3$ and

$$\chi = \Delta - \frac{\{-2bb_0b_2 + b_1u[b(\beta + b\alpha) - \beta(bm + \gamma)]\}^2}{[b_2(\beta + \alpha(b + p\delta + \gamma)) + b_0u]^2} > 0, \tag{3.9}$$

then $\varphi < 0$.

Summarizing the above discussions, we have the following results on the stability of the equilibrium $E_2(S_2, I_2)$.

Theorem 3.4. *Suppose that the endemic equilibrium $E_2(S_2, I_2)$ exists, if one of following conditions is satisfied:*

(i) $b(\beta + b\alpha) \geqslant \beta(bm + \gamma)$;
(ii) $b(\beta + b\alpha) < \beta(bm + \gamma)$ *and* $\beta m - p\delta - \gamma \leqslant \eta_1$;
(iii) $b(\beta + b\alpha) < \beta(bm + \gamma)$, $\beta m - p\delta - \gamma > \eta_1$ *and* $u \geqslant u_3$;
(iv) $b(\beta + b\alpha) < \beta(bm + \gamma)$, $\beta m - p\delta - \gamma > \eta_1$, $u < u_3$ and $\chi > 0$,

then $E_2(S_2, I_2)$ is locally asymptotically stable. It is unstable if $b(\beta + b\alpha) < \beta(bm + \gamma)$, $\beta m - p\delta - \gamma > \eta_1$, $u < u_3$ and $\chi < 0$.

The existence of limit cycles plays an important role in determining the dynamical behaviors of the system. For example, if there is no limit cycle in system (1.4) and its endemic equilibrium is unique and locally asymptotically stable, then it must be globally stable. Now, we consider the existence of limit cycles in system (1.4).

Theorem 3.5. *Suppose $R_0 > R_0^*$ and $u < \min\{u_0, u_2\}$. If $\varphi > 0$, then system (1.4) has at least a stable limit cycle which encircles E_2.*

Proof. As $R_0 > R_0^* > 1$ and $u < u_2$, it is known from Theorem 2.1 that the equilibrium E^* of system (1.4) does not exist. Again, because $R_0 > R_0^*$, $u < u_0$ and $u < u_2$, it follows from Theorem 2.2 that the equilibrium E_1 of system (1.4) does not exist, but the equilibrium E_2 exists.

It is known from $\varphi > 0$ that E_2 is an unstable focus or node. It is easy to check that the unstable manifold at $E_0(m, 0)$ which is a saddle point, is in the first quadrant. As the set D is positively invariant for system (1.4), and system (1.4) does not have any equilibrium in the interior of $D \setminus \{E_2\}$. It follows from Poincaré–Bendixson theorem that system (1.4) has at least a stable limit cycle which encircles E_2. □

4. Numerical simulations

In this section, we present some numerical simulations of system (1.4) to illustrate our results.

Example 4.1 (Example 2.4 continued). Choose $\beta = 0.34, \alpha = 0.4, \gamma = 0.01, \delta = 0.01, b = 0.2, p = 0.02, k = 0.03$ and $m = 0.3$. We have R_0=2.53731, $R_0^* = 2.12826, R_0 > R_0^*, u_0 = 0.0101133$ and $u_2 = 0.00996346$. A forward bifurcation diagram was given in Fig. 1.

If we select $I_0 = 0.4$, then $u = 0.012 > u_2$, the equilibrium $E^*(0.133942, 0.332115)$ exists, but E_1 and E_2 do not exist (see Theorem 2.1(ii)). Its phase portrait is given in Fig. 4, which shows that the unique equilibrium E^* is globally asymptotically stable.

If we choose $I_0 = 0.2$, then $u = 0.006 < u_2$, the equilibrium $E_2(0.0754805, 0.55577)$ exists, but E_1 and E^* do not exist (see Theorem 2.2(iv)). Its phase portrait is illustrated in Fig. 5. The unique equilibrium E_2 is globally asymptotically stable in D.

Example 4.2 (Example 2.5 continued). Choose $\beta = 0.2, \alpha = 0.4, \gamma = 0.01, \delta = 0.01, b = 0.2, p = 0.02, k = 0.03$ and $m = 0.1$. By calculations, $R_0 = 0.497512 < 1, u_0 = 0.000590643, u_1 = 0.00266885$ and $R_0 < 1$. A backward bifurcation diagram was given in Fig. 2. If we choose $I_0 = 0.01$, then $u = 0.0003 < u_0$, the equilibria $E_1(0.0929588, 0.036941)$ and $E_2(0.060531, 0.238294)$ exist, but the equilibrium E^* does not exist. Its phase portrait is illustrated in Fig. 6. It shows that the equilibria E_0 and E_2 are asymptotically stable.

Example 4.3 (Example 2.6 continued). For system (1.4), we choose parameter values as follows: $\beta = 0.34, \alpha = 0.4, \gamma = 0.01, \delta = 0.01, b = 0.2, p = 0.02, k = 0.03, m = 0.2$ and I_0=0.1867. Then we have R_0= 1.69154, R_0^*=2.274 05. $b(\beta + b\alpha) - \beta(bm + \gamma) = 0.067 > 0, u_0 = 0.00582652$, $u_1 = 0.00719025$, $u_2 = 0.00548396$ and $u = 0.0056$. The parameter values satisfy the conditions of Theorem 3.2, the condition (ii) of Theorem 2.2 and the condition (i) of Theorem 3.4. Therefore, all four equilibria $E_0(0.2, 0), E^*(0.126881, 0.182799), E_1(0.123039, 0.195843)$ and

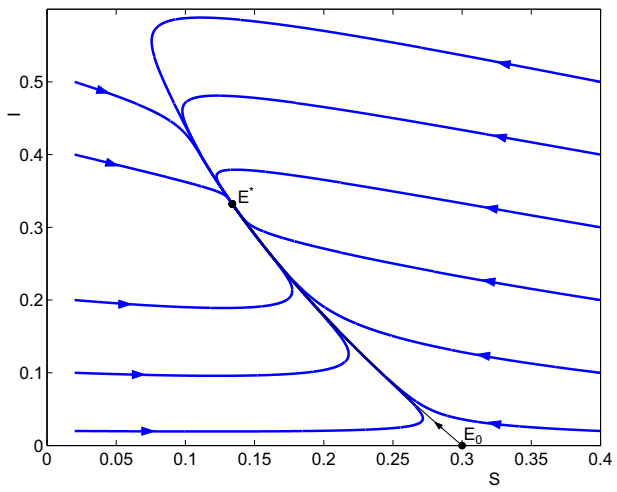

Fig. 4. The phase portrait of system (1.4) when E^* is globally asymptotically stable.

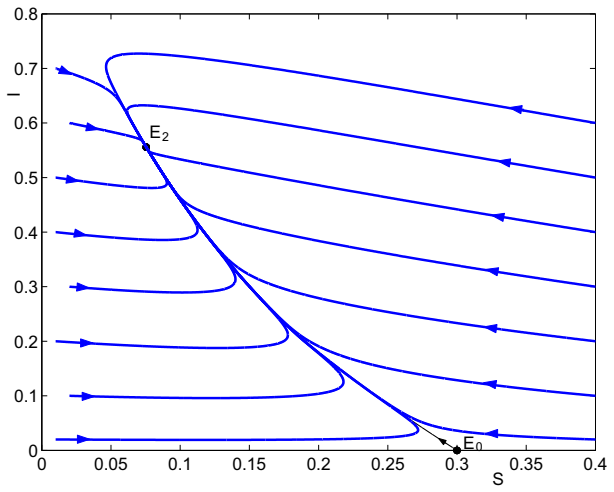

Fig. 5. The phase portrait of system (1.4) when E_2 is globally asymptotically stable.

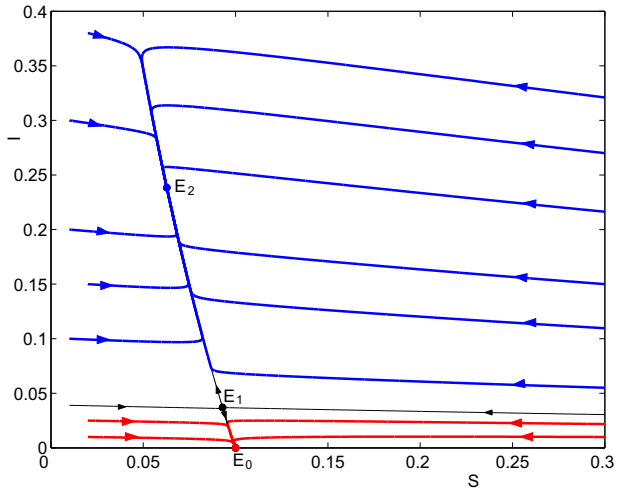

Fig. 6. The phase portrait of system (1.4) when E_2 and E_0 are stable and E_1 is unstable.

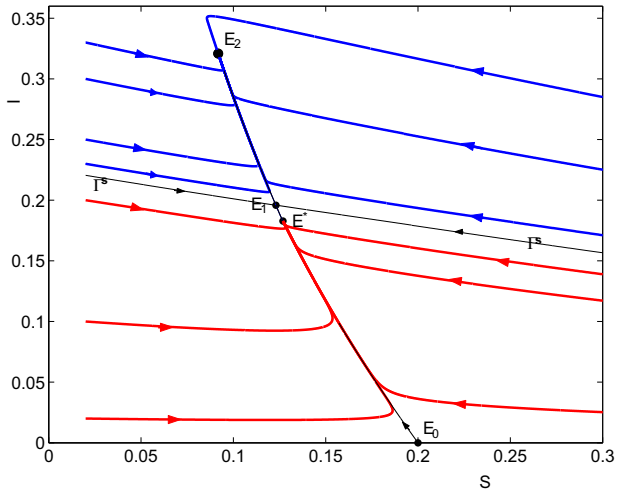

Fig. 7. The phase portrait of system (1.4) with bistable endemic equilibria.

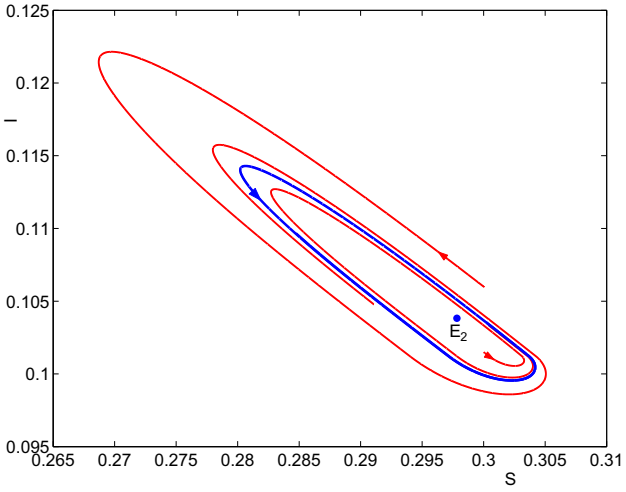

Fig. 8. A stable limit cycle of system (1.4) encircling the unstable equilibrium E_2.

$E_2(0.0917512, 0.320995)$ exist. The phase portrait of system (1.4) is illustrated in Fig. 7, where the black thin lines are the separatrices of saddle points E_0 and E_1. It is known from Fig. 7 that the equilibria E_2 and E^* are asymptotically stable, and E_0 and E_1 are unstable. Thus, system (1.4) has bistable endemic equilibria E^* and E_2.

We can see from Fig. 7 that the stable separatrices Γ^s of saddle point E_1 separate the positive invariant set D into two regions, the basin of attraction for the stable equilibrium E_2 is the region above Γ^s and the basin of attraction for the stable equilibrium E^* is the region below Γ^s.

Example 4.4. Choose $\beta = 0.2, \alpha = 0.8, \gamma = 0.01, \delta = 0.03, b = 0.01, p = 0.2, k = 0.04$ and $m = 0.9$. We have $R_0 = 3.21429, R_0^* = 3.22723, u_0 = 0.00404965, u_1 = 0.00405539$ and $u_2 = 0.00404964$. If we choose $I_0 = 0.101225$, then $u = 0.004049 < u_0, \mathrm{tr}(J_2) = 0.00407523 > 0$, the equilibria $E_2(0.29783, 0.103826)$ exists but is unstable, and the equilibria E^* and E_1 do not exist. The parameter values satisfy the conditions of Theorem 3.5. Its phase portrait is given in Fig. 8, which shows that system (1.4) has a stable limit cycle which encircles E_2. Therefore, under some conditions, system (1.4) has a stable periodic orbit which encircles the equilibrium E_2.

5. Discussion

In this paper, we have analyzed a SIR epidemic model to study the effect of limited resources for the treatment of patients in the public-health system, which could occur when there is a very large number of patients but the medical facilities are insufficient, the number of beds is limited, or the number of health-care workers is short-handed. We also considered nonlinear incidence rate, vertical transmission and vaccination for the newborns of the susceptible and recovered individuals in the model. Theorem 2.2 and Corollary 2.3 imply that a backward bifurcation occurs when $R_0 < 1$, that is, the disease-free equilibrium coexists with an endemic equilibrium. Theorems 2.2 and 3.4 indicate that system (1.4) has multiple endemic equilibrium when $R_0 > 1$ where a bifurcation diagram displays forward bifurcations. When there are two endemic equilibria, one of them is always unstable and the other one is stable under certain conditions. When there are three endemic equilibria, bistable endemic equilibria can occur. Numerical simulation confirmed that system (1.4) has a stable periodic orbit which encircles an endemic equilibrium under some conditions. The existence and stability of equilibria of system (1.4) can be summarized in Table 1 (DNE = does not exist).

We can see that when the basic reproduction number $R_0 < 1$ and the treatment term $u < \min\{u_0, u_1\}$, a backward bifurcation occurs with a disease-free equilibrium and two endemic equilibria. Note that the disease-free equilibrium is always stable since $R_0 < 1$, the stability of the endemic equilibria depends on other conditions. Recall that $u = kI_0$, where k is the treatment parameter and I_0 represents the infective level at which the health-care system reaches capacity,

$$R_0 = \frac{\beta m}{p\delta + \gamma + k},$$

$$u_0 = \frac{b\{(\beta m - p\delta - \gamma)(\alpha b + \beta) + 2[\beta(bm + \gamma) + \alpha b(p\delta + \gamma)] - 2\eta\}}{(\alpha b + \beta)^2},$$

$$u_1 = \frac{bk(\beta m - p\delta - \gamma)}{2[\beta(bm + \gamma) + b\alpha(p\delta + \gamma)] + k(b\alpha + \beta)},$$

Table 1
Existence and stability of equilibria for system (1.4).

Conditions			E_0	E^*	E_1	E_2	Figure
$R_0 < 1$			Stable	DNE	DNE	DNE	
	$u < \min\{u_0, u_1\}$		Stable	DNE	Unstable	Stable	Fig. 6
$R_0 > 1$		$u_2 \leqslant u$	Unstable	Stable	DNE	DNE	Fig. 4
	$R_0 < R_0^*$	$u_2 \leqslant u \leqslant \min\{u_0, u_1\}$	Unstable	Stable	Unstable	Stable	Fig. 7
	$R_0^* < R_0$	$u \leqslant \min\{u_0, u_2\}$	Unstable	DNE	DNE	Stable	Fig. 5
						Unstable	Fig. 8

and

$$u_2 = \frac{kb(p\delta + \gamma + k)(R_0 - 1)}{\beta(bm + \gamma + k) + \alpha b(k + \gamma + \delta p)}.$$

Therefore, in order to eradicate the disease, the basic reproduction number R_0 must be lowered than a threshold or the treatment $u > \max\{u_0, u_1\}$ by other control measures, possibly a combination of different types of measures. Epidemic models with two different types of interventions have been studied by some other researchers, see Brauer [6] and Sun and Yang [32]. For example, reduce the rate of vertical transmission rate $q(= 1 - p)$, increase the vaccination rate m', or increase the treatment rate k, so that the disease approaches a lower endemic steady state for a range of parameters. If the capacity for treatment increases further to a certain level, then the disease may be eradicated.

We would like to mention that related models have been studied and similar results have been obtained by other researchers, for example, Cui et al. [10], Li et al. [25], and Hu et al. [22]. However, we have considered more factors and components in our model and obtained different and new results. For example, we assumed that the disease can be transmitted vertically and vaccination applies to the newborns of susceptible and recovered individuals. Moreover, we established the existence and stability of a limit cycle when there is a unique unstable endemic equilibrium.

As pointed out by Greenhalgh and Griffiths [15], the study of backward bifurcation in epidemic modeling is relatively new and there are many issues deserving further investigation for this new and interesting phenomenon. So far most studies focus on the theoretical aspects of backward bifurcation and very few relate to real communicable diseases (see Greenhalgh and Griffiths [15] on a Bovine Respiratory Syncytial virus epidemic model and Blayneh et al. [4] and Wan and Zhu [35] on West Nile virus epidemic models). As mentioned in the Introduction, we started the project with modeling the SARS outbreaks in 2003 [38]. The second SARS outbreak in Toronto, Canada in 2003 (see Gumel et al. [16]) may be explained as a result of backward bifurcation since the basic reproduction number was certainly less than unity after the first outbreak when restrictive and suitable infection-control procedures were being taken [19]. We expect that our results in this paper might be helpful to study the endemic of hepatitis B virus in China [43] and the outbreaks of cholera in Haiti or Zimbabwe [28]. We propose to study these in the future.

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