CONVERGENCE AND TRAVELLING FRONTS IN FUNCTIONAL DIFFERENTIAL EQUATIONS WITH NONLOCAL TERMS: A COMPETITION MODEL*

STEPHEN A. GOURLEY† AND SHIGUI RUAN‡

Abstract. In this paper we consider a two-species competition model described by a reaction-diffusion system with nonlocal delays. In the case of a general domain, we study the stability of the equilibria of the system by using the energy function method. When the domain is one-dimensional and infinite, by employing linear chain techniques and geometric singular perturbation theory, we investigate the existence of travelling front solutions of the system.

Key words. competition-diffusion, equilibrium, stability, travelling front, energy function, geometric singular perturbation

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1. Introduction. Let $\mathbf{R} = (-\infty, \infty)$, and let Ω be some open bounded region in $\mathbf{R}^N$, $N \leq 3$, with a smooth boundary $\partial\Omega$. Let $\partial/\partial n$ denote the outward normal derivative on $\partial\Omega$ and let Δ be the Laplacian operator. For $1 \leq p \leq \infty$, let $L^p(\Omega)$ denote the Banach space of measurable functions u on Ω satisfying

$$\|u\|_p = \begin{cases} \left(\int_\Omega |u(x)|^p \, dx \right)^{1/p} < \infty & \text{if} \quad 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{x \in \Omega} |u(x)| < \infty & \text{if} \quad p = \infty. \end{cases}$$

In particular, if $p = 2$, $L^2(\Omega)$ becomes a Hilbert space with the usual inner product $\langle \cdot, \cdot \rangle$ and $\| \cdot \|_2^2 = \langle \cdot, \cdot \rangle$. Also, let $\| \cdot \|_2$ denote the norm in $L^2((0, T); L^2(\Omega; R))$, i.e.,

$$\|u\|_2 = \left(\int_0^T \|u(s)\|_2^2 \, ds \right)^{1/2}.$$

Let $u_1(t, x)$ and $u_2(t, x)$ denote the population densities of two competitors at time t and location x, and let the diffusivities of the two competitors be d_1 and d_2, respectively. This paper is concerned with the following two-species Lotka–Volterra competition-diffusion model with distributed delays:

$$(1.1) \quad \begin{aligned} \frac{\partial u_1}{\partial t} &= d_1 \Delta u_1 + u_1 \left(r_1 - a_1 u_1 - b_1 \int_\Omega \int_{-\infty}^t K_1(x, y, t - s) u_2(s, y) \, ds \, dy \right), \\ \frac{\partial u_2}{\partial t} &= d_2 \Delta u_2 + u_2 \left(r_2 - b_2 \int_\Omega \int_{-\infty}^t K_2(x, y, t - s) u_1(s, y) \, ds \, dy - a_2 u_2 \right) \end{aligned}$$

for $t > 0$, $x \in \Omega$, under the homogeneous Neumann boundary conditions

$$(1.2) \qquad \frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} = 0, \quad x \in \partial\Omega,$$

and initial conditions

$$(1.3) \quad u_1(\theta, x) = \phi_1(\theta, x) \geq 0, \; u_2(\theta, x) = \phi_2(\theta, x) \geq 0, \quad (\theta, x) \in (-\infty, 0] \times \Omega,$$

where ϕ_1 and ϕ_2 are continuous functions. The parameters r_i, a_i, and b_i, $i = 1, 2$, are all positive constants.

The kernels $K_i(x, y, \sigma)$, $i = 1, 2$, are nonnegative functions which are continuous in $(x, y) \in \bar{\Omega} \times \bar{\Omega}$ for each $\sigma \in [0, \infty)$ and measurable in $\sigma \in [0, \infty)$ for each pair $(x, y) \in \bar{\Omega} \times \bar{\Omega}$. We assume that the kernels depend on both the spatial and the temporal variables. The delays in this type of model formulation are called *spatiotemporal delays* or *nonlocal delays*. This is a formulation that aims to account for the fact that, at previous times, individuals have not necessarily been at the same point in space. See Gourley and Britton [8] for a detailed discussion of this modelling issue on an infinite spatial domain and Gourley and So [9], who more recently have treated the finite domain case, explaining in detail why it leads to the type of delay term we are using in (1.1). See also Yamada [16] and the references cited therein. Gourley and So [9] concentrated on the one-dimensional domain $[0, \pi]$ and showed that on this domain a delayed variable $u(t, x)$, representing a population with diffusivity d, should be modelled in the equations by using a term of the form

$$\int_0^\pi \int_{-\infty}^t G(x, y, t - s) k(t - s) u(s, y) \, ds \, dy,$$

where $k(t)$ is the weight given to the population t time units ago and, in the homogeneous Neumann problem,

$$(1.4) \qquad G(x, y, t) = \frac{1}{\pi} + \frac{2}{\pi} \sum_{n=1}^\infty e^{-dn^2 t} \cos nx \cos ny.$$

In our formulation we are, for convenience, absorbing the G and k of each delay term into a single kernel $K_i(x, y, t)$. Regarding these kernels K_i, we shall assume that

$$(1.5) \qquad \int_\Omega K_i(x, y, \sigma) \, dx = \int_\Omega K_i(x, y, \sigma) \, dy = k_i(\sigma), \;\; \sigma \geq 0,$$

and

$$(1.6) \qquad \int_0^\infty k_i(\sigma) \, d\sigma = 1, \quad \sigma \, k_i(\sigma) \in L^1((0, \infty); \mathbf{R}).$$

Assumption (1.5), that integration of $K_i(x, y, s)$ with respect to either x or y removes *both* the x and the y dependence, is easily seen to be reasonable when we have in mind that $K_i(x, y, t)$ is a product of the form $G(x, y, t) k(t)$, with G given by (1.4) or the corresponding expression for whatever domain is under consideration.

The local existence of solutions $(u_1(t, x), u_2(t, x))$ to (1.1)–(1.3) follows from the results in Yamada [17] or Ruan and Wu [11]. The comparison theorem for parabolic differential equations implies that $(u_1(t, x), u_2(t, x))$ exists globally such that

$$(1.7) \qquad \begin{aligned} &0 \leq u_1(t, x) \leq \max\left\{\frac{r_1}{a_1}, \; \sup_{\theta \leq 0} \|\phi_1(\theta, \cdot)\|_{C(\bar{\Omega}; R)}\right\}, \\ &0 \leq u_2(t, x) \leq \max\left\{\frac{r_2}{a_2}, \; \sup_{\theta \leq 0} \|\phi_2(\theta, \cdot)\|_{C(\bar{\Omega}; R)}\right\} \end{aligned}$$

for $x \in \bar{\Omega}$ and $t \in \mathbf{R}$. Also, by the strong maximum principle, if $\phi_1(0, x) \not\equiv 0$ and $\phi_2(0, x) \not\equiv 0$, then we have $u_1(t, x) > 0$, $u_2(t, x) > 0$ for all $x \in \bar{\Omega}$ and $t > 0$.

Notice that system (1.1) has a trivial equilibrium $E_0 = (0, 0)$, two semitrivial spatially homogeneous equilibria

$$E_1 = \left(\frac{r_1}{a_1}, 0 \right), \quad E_2 = \left(0, \frac{r_2}{a_2} \right),$$

and a positive spatially homogeneous equilibrium

(1.8)
$$E^* = \left(\frac{r_1 a_2 - r_2 b_1}{a_1 a_2 - b_1 b_2}, \ \frac{r_2 a_1 - r_1 b_2}{a_1 a_2 - b_1 b_2} \right),$$

provided that $a_1 a_2 \neq b_1 b_2$ and either (i) $r_2 b_1 < r_1 a_2$ and $r_1 b_2 < r_2 a_1$ or (ii) $r_2 b_1 > r_1 a_2$ and $r_1 b_2 > r_2 a_1$. The trivial equilibrium E_0 is of no interest here. The stability of the semitrivial equilibrium E_i means that the ith competitor ($i = 1, 2$) wins the competition. These semitrivial equilibria are of considerable interest ecologically because of the possibility of a transition between the two. In fact we shall prove in this paper that, when the coexistence equilibrium E^* is absent, a transition can occur between E_1 and E_2 in the form of a travelling wave-front solution.

Various special cases of system (1.1) have been studied by many researchers. When the delay kernels are independent of the spatial variable (i.e., when the delays are local), Ruan and Wu [11] studied the stability of the equilibria. See also Ruan and Zhao [12] for competition models with finite delays and Schiaffino and Tesei [13] for a nonlinear competition system. When there are no delays, the stability of the competition-diffusion model was investigated by Zhou and Pao [18]. Delayed competition models without diffusion have been studied by Cushing [2] and by Gopalsamy [6], and the monograph by Wu [15] provides a very comprehensive description of current research into delay-diffusion equations. When the domain $\Omega = (-\infty, \infty)$ and there are no delays, Conley and Gardner [1], Gardner [4], Kan-on [10], and Tang and Fife [14] have shown that the competition-diffusion model has travelling front solutions connecting the boundary equilibria

(1.9)
$$\left(\frac{r_1}{a_1}, \ 0 \right) \quad \text{and} \quad \left(0, \ \frac{r_2}{a_2} \right).$$

The existence of such solutions even for the nondelay problem is a highly nontrivial matter because one is seeking a heteroclinic connection between equilibria in a four-dimensional phase space. The introduction of delays increases the dimension to eight (for the particular delays we consider). However, when the delays are small, considerable progress can be achieved by the use of geometric singular perturbation theory.

In this paper we shall first discuss the stability of the equilibria E_1, E_2, and E^* by using the energy function method (see Yamada [16, 17]). Then, for the case when $\Omega = (-\infty, \infty)$, we will study the existence of travelling front solutions of system (1.1) connecting the two boundary equilibria E_1 and E_2.

2. Convergence. The main result of this section is a theorem on the global stability of each of the equilibria. First, we shall derive an inequality that will be needed in the proof of the main theorem. The hypotheses of this lemma are not restricted to this application (see, in particular, Gourley and So [9]).

LEMMA 2.1. *Let* $K(x, y, t) = G(x, y, t)k(t)$, $x, y \in \Omega \subset \mathbf{R}^N$, *where* $k(t) \geq 0$ *and* $G(x, y, t)$ *is the solution of*

$$(2.1) \qquad \frac{\partial G}{\partial t} = d\,\nabla^2 G, \qquad \frac{\partial G}{\partial n} = 0 \ \ on \ \partial\Omega, \qquad G(x, y, 0) = \delta(x - y).$$

Then

$$\left\| \int_\Omega \int_{-\infty}^t K(x, y, t - s)u(s, y)\, ds\, dy \right\|_2 \leq \int_{-\infty}^t k(t - s)\|u(s)\|_2\, ds$$

for any function $u(t, x)$ *such that* $\partial u/\partial n = 0$ *on* $\partial\Omega$.

Remark 2.2. Before we prove this lemma let us stress that x and y are both vectors in $\mathbf{R}^N$ here. For the purposes of computing G, ∇^2 is calculated with respect to either of these vectors (say x for definiteness) with the other one, y, held fixed. In the case considered in detail in [9], Ω is one-dimensional, $\nabla^2 = \partial^2/\partial x^2$, and $G(x, y, t)$ is given by (1.4).

Proof of Lemma 2.1. We have

$$\left\| \int_\Omega \int_{-\infty}^t K(x, y, t - s)u(s, y)\, ds\, dy \right\|_2$$

$$= \left\| \int_{-\infty}^t \int_\Omega K(x, y, t - s)u(s, y)\, dy\, ds \right\|_2$$

$$\leq \int_{-\infty}^t \left\| \int_\Omega K(x, y, t - s)u(s, y)\, dy \right\|_2 ds$$

$$= \int_{-\infty}^t k(t - s) \left\| \int_\Omega G(x, y, t - s)u(s, y)\, dy \right\|_2 ds.$$

Therefore, we want to show that

$$\left\| \int_\Omega G(x, y, t - s)u(s, y)\, dy \right\|_2 \leq \|u(s)\|_2$$

for $s \leq t$. Let λ_k, $k = 0, 1, 2, \ldots$, be the eigenvalues of $-d\nabla^2$ under homogeneous Neumann boundary conditions, with corresponding normalized (in L^2) eigenfunctions $\phi_k(x)$ so that

$$-d\nabla^2\phi_k = \lambda_k\phi_k, \qquad \frac{\partial\phi_k}{\partial n} = 0 \ \ on \ \partial\Omega.$$

Then $\lambda_0 = 0$ with $\phi_0 = $ constant, and $\lambda_k > 0$ for all other k. The solution $G(x, y, t)$ of (2.1) will be given by a Fourier series expansion in terms of these functions $\phi(x)$ with coefficients depending on y. In fact,

$$G(x, y, t) = \sum_{n=0}^\infty e^{-\lambda_n t}\phi_n(x)\phi_n(y).$$

Also, $u(t, x)$ satisfies the boundary conditions and therefore can be expanded in terms of the ϕ_n:

$$u(t, x) = \sum_{n=0}^\infty a_n(t)\phi_n(x).$$

Therefore, since the ϕ_k are orthonormal,

$$\int_\Omega G(x,y,t-s)u(s,y)\,dy = \sum_{n=0}^\infty a_n(s)e^{-\lambda_n(t-s)}\phi_n(x),$$

and hence, by Parseval's identity,

$$\left\|\int_\Omega G(x,y,t-s)u(s,y)\,dy\right\|_2 = \left(\sum_{n=0}^\infty a_n^2(s)e^{-2\lambda_n(t-s)}\right)^{1/2}$$

$$\leq \left(\sum_{n=0}^\infty a_n^2(s)\right)^{1/2} = \|u(s)\|_2$$

as desired. The proof is complete.

Next, we state our main theorem of this section.

THEOREM 2.3. *Let $(u_1(t,x),u_2(t,x))$ satisfy (1.1) with boundary conditions (1.2) and initial conditions (1.3), with $\phi_1(0,x) \not\equiv 0$ and $\phi_2(0,x) \not\equiv 0$.*

(i) *If $r_1/r_2 > a_1/b_2 > b_1/a_2$, then $\lim_{t\to\infty}(u_1(t,x),u_2(t,x)) = (r_1/a_1,0)$ uniformly for $x \in \bar\Omega$.*

(ii) *If $r_1/r_2 < b_1/a_2 < a_1/b_2$, then $\lim_{t\to\infty}(u_1(t,x),u_2(t,x)) = (0,r_2/a_2)$ uniformly for $x \in \bar\Omega$.*

(iii) *If $b_1/a_2 < r_1/r_2 < a_1/b_2$, then $\lim_{t\to\infty}(u_1(t,x),u_2(t,x)) = (u_1^*,u_2^*)$ uniformly for $x \in \bar\Omega$, where u_1^* and u_2^* are the components of the equilibrium E^* given by (1.8).*

Proof. We prove only (i); the proofs of (ii) and (iii) are similar. To study the stability of the semitrivial equilibrium $E_1 = (r_1/a_1,0)$, define

(2.2) $$E(u_1) = \int_\Omega \left[u_1 - \frac{r_1}{a_1} - \frac{r_1}{a_1}\log\frac{u_1}{r_1/a_1}\right]dx, \quad F(u_2) = \int_\Omega u_2\,dx.$$

Then $E(u_1) \geq 0$ and $F(u_2) \geq 0$. For some constant $\alpha > 0$ to be found later, we have

$$\frac{d}{dt}[\alpha E(u_1) + F(u_2)]$$

$$= \alpha\int_\Omega \frac{\partial u_1}{\partial t}\left(1 - \frac{r_1/a_1}{u_1}\right)dx + \int_\Omega \frac{\partial u_2}{\partial t}\,dx$$

$$= -\alpha d_1 \frac{r_1}{a_1}\int_\Omega \frac{|\nabla u_1|^2}{u_1^2}\,dx - \alpha a_1\int_\Omega \left(u_1 - \frac{r_1}{a_1}\right)^2 dx$$

$$\quad - \alpha b_1\int_\Omega \left(\int_\Omega\int_{-\infty}^t K_1(x,y,t-s)u_2(s,y)\,ds\,dy\right)\left(u_1 - \frac{r_1}{a_1}\right)dx - a_2\int_\Omega u_2^2(t,x)\,dx$$

$$\quad + r_2\int_\Omega u_2(t,x)\,dx - b_2\int_\Omega \left(\int_\Omega\int_{-\infty}^t K_2(x,y,t-s)u_1(s,y)\,ds\,dy\right)u_2(t,x)\,dx$$

$$= -\alpha d_1\frac{r_1}{a_1}\int_\Omega \frac{|\nabla u_1|^2}{u_1^2}\,dx - \alpha a_1\int_\Omega \left(u_1 - \frac{r_1}{a_1}\right)^2 dx$$

$$\quad - \alpha b_1\int_\Omega \left(\int_\Omega\int_{-\infty}^t K_1(x,y,t-s)u_2(s,y)\,ds\,dy\right)\left(u_1 - \frac{r_1}{a_1}\right)dx - a_2\int_\Omega u_2^2(t,x)\,dx$$

$$\quad - b_2\int_\Omega \left(\int_\Omega\int_{-\infty}^t K_2(x,y,t-s)\left(u_1(s,y) - \frac{r_2}{b_2}\right)ds\,dy\right)u_2(t,x)\,dx,$$

where we have used (1.5) and (1.6). By hypothesis, $r_2/b_2 < r_1/a_1$, so

$$\frac{d}{dt}[\alpha E(u_1) + F(u_2)]$$

$$\leq -\alpha d_1 \frac{r_1}{a_1} \int_\Omega \frac{|\nabla u_1|^2}{u_1^2}\, dx - \alpha a_1 \int_\Omega \left(u_1 - \frac{r_1}{a_1}\right)^2 dx$$

$$- \alpha b_1 \int_\Omega \left(\int_\Omega \int_{-\infty}^t K_1(x,y,t-s)u_2(s,y)\, ds\, dy\right)\left(u_1 - \frac{r_1}{a_1}\right) dx - a_2 \int_\Omega u_2^2(t,x)\, dx$$

$$- b_2 \int_\Omega \left(\int_\Omega \int_{-\infty}^t K_2(x,y,t-s)\left(u_1(s,y) - \frac{r_1}{a_1}\right) ds\, dy\right) u_2(t,x)\, dx.$$

(2.3)

Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $L^2(\Omega; \mathbf{R})$, $\|\cdot\|_2^2 = \langle \cdot, \cdot \rangle$. Then we have the following inequality:

(2.4)
$$\frac{d}{dt}[\alpha E(u_1) + F(u_2)] + \alpha d_1 \frac{r_1}{a_1} \int_\Omega \frac{|\nabla u_1|^2}{u_1^2}\, dx + \alpha a_1 \left\|u_1 - \frac{r_1}{a_1}\right\|_2^2 + a_2 \|u_2\|_2^2$$

$$\leq -\alpha b_1 \left\langle \int_\Omega \int_{-\infty}^t K_1(x,y,t-s)u_2(s,y)\, ds\, dy,\ u_1(t) - \frac{r_1}{a_1}\right\rangle$$

$$- b_2 \left\langle \int_\Omega \int_{-\infty}^t K_2(x,y,t-s)\left(u_1(s,y) - \frac{r_1}{a_1}\right) ds\, dy,\ u_2(t)\right\rangle.$$

By Lemma 2.1, we have

(2.5)
$$\left\|\int_\Omega \int_{-\infty}^t K_1(x,y,t-s)u_2(s,y)\, ds\, dy\right\|_2 \leq \int_{-\infty}^t k_1(t-s)\|u_2(s)\|_2\, ds$$

$$\leq \sup_{s\leq 0}\|u_2(s)\|_2 \int_t^\infty k_1(s)\, ds + \int_0^t k_1(t-s)\|u_2(s)\|_2\, ds$$

and

$$\left\|\int_\Omega \int_{-\infty}^t K_2(x,y,t-s)\left(u_1(s,y) - \frac{r_1}{a_1}\right) ds\, dy\right\|_2 \leq \int_{-\infty}^t k_2(t-s)\left\|u_1(s) - \frac{r_1}{a_1}\right\|_2 ds$$

$$\leq \sup_{s\leq 0}\left\|u_1(s) - \frac{r_1}{a_1}\right\|_2 \int_t^\infty k_2(s)\, ds + \int_0^t k_2(t-s)\left\|u_1(s) - \frac{r_1}{a_1}\right\|_2 ds.$$

(2.6)

Thus, for any $T > 0$,

$$\left|\int_0^T \left\langle \int_\Omega \int_{-\infty}^t K_1(x,y,t-s)u_2(s,y)\, ds\, dy, u_1(t) - \frac{r_1}{a_1}\right\rangle dt\right|$$

$$\leq \int_0^T \left\|u_1(t) - \frac{r_1}{a_1}\right\|_2 \left\|\int_\Omega \int_{-\infty}^t K_1(x,y,t-s)u_2(s,y)\, ds\, dy\right\|_2 dt$$

$$\leq \sup_{s\leq 0}\|u_2(s)\|_2 \sup_{0\leq t\leq T}\left\|u_1(t) - \frac{r_1}{a_1}\right\|_2 \int_0^\infty s\, k_1(s)\, ds$$

$$+ \int_0^T \left\|u_1(t) - \frac{r_1}{a_1}\right\|_2 \int_0^t k_1(t-s)\|u_2(s)\|_2\, ds\, dt.$$

We now estimate the second term in the above as follows:

$$\int_0^T \left\| u_1(t) - \frac{r_1}{a_1} \right\|_2 \int_0^t k_1(t-s) \|u_2(s)\|_2 \, ds \, dt$$

$$\leq \left\| u_1 - \frac{r_1}{a_1} \right\|_2 \left(\int_0^T \left(\int_0^t k_1(t-s) \|u_2(s)\|_2 \, ds \right)^2 dt \right)^{1/2}$$

$$\leq \left\| u_1 - \frac{r_1}{a_1} \right\|_2 \left(\int_0^T \left(\underbrace{\int_0^t k_1(t-s) \, ds}_{\leq 1} \right) \left(\int_0^t k_1(t-s) \|u_2(s)\|_2^2 \, ds \right) dt \right)^{1/2}$$

$$\leq \left\| u_1 - \frac{r_1}{a_1} \right\|_2 \left(\int_0^T \int_0^t k_1(t-s) \|u_2(s)\|_2^2 \, ds \, dt \right)^{1/2}$$

$$= \left\| u_1 - \frac{r_1}{a_1} \right\|_2 \left(\int_0^T \|u_2(s)\|_2^2 \underbrace{\int_s^T k_1(t-s) \, dt}_{\leq 1} \, ds \right)^{1/2}$$

$$\leq \left\| u_1 - \frac{r_1}{a_1} \right\|_2 \|u_2\|_2,$$

where $\| \cdot \|_2$ denotes the norm in $L^2((0,T); L^2(\Omega; R))$, i.e.,

$$\|u\|_2 = \left(\int_0^T \|u(s)\|_2^2 \, ds \right)^{1/2}.$$

Therefore, for any $T > 0$,

$$\left| \int_0^T \left\langle \int_\Omega \int_{-\infty}^t K_1(x,y,t-s) u_2(s,y) \, ds \, dy, \, u_1(t) - \frac{r_1}{a_1} \right\rangle dt \right|$$

$$\leq \sup_{s \leq 0} \|u_2(s)\|_2 \sup_{0 \leq t \leq T} \|u_1(t) - \frac{r_1}{a_1}\|_2 \int_0^\infty s \, k_1(s) \, ds + \left\| u_1 - \frac{r_1}{a_1} \right\|_2 \|u_2\|_2.$$

In a similar way, we have

$$\left| \int_0^T \left\langle \int_\Omega \int_{-\infty}^t K_2(x,y,t-s) \left(u_1(s,y) - \frac{r_1}{a_1} \right) ds \, dy, \, u_2(t) \right\rangle dt \right|$$

$$\leq \sup_{s \leq 0} \left\| u_1(s) - \frac{r_1}{a_1} \right\|_2 \sup_{0 \leq t \leq T} \|u_2(t)\|_2 \int_0^\infty s \, k_2(s) \, ds + \left\| u_1 - \frac{r_1}{a_1} \right\|_2 \|u_2\|_2.$$

(2.7)

Integrating (2.4) over $[0,T]$ and noting that $\sup_{0 \leq t \leq T} \|u_2(t)\|_2$ and $\sup_{0 \leq t \leq T} \|u_1(t) - \frac{r_1}{a_1}\|_2$ can be bounded independently of T (by (1.7)), we obtain that there exists a positive constant C independent of T such that

$$\frac{\alpha d_1 r_1}{a_1} \left\| \frac{\nabla u_1}{u_1} \right\|_2^2 + \alpha a_1 \left\| u_1 - \frac{r_1}{a_1} \right\|_2^2 + a_2 \|u_2\|_2^2 \leq C + (\alpha b_1 + b_2) \left\| u_1 - \frac{r_1}{a_1} \right\|_2 \|u_2\|_2$$

(2.8)

or, by using Young's inequality,

$$
(2.9) \quad
\begin{aligned}
&\frac{\alpha d_1 r_1}{a_1} \left\| \frac{\nabla u_1}{u_1} \right\|_2^2 + \alpha a_1 \left\| u_1 - \frac{r_1}{a_1} \right\|_2^2 + a_2 \|u_2\|_2^2 \\
&\qquad \leq C + (\alpha b_1 + b_2) \left(\frac{1}{2} \lambda \left\| u_1 - \frac{r_1}{a_1} \right\|_2^2 + \frac{1}{2\lambda} \|u_2\|_2^2 \right)
\end{aligned}
$$

for any $\lambda > 0$. If we choose

$$
\lambda = \frac{\alpha b_1 + b_2}{2 a_2},
$$

then (2.9) reads as

$$
(2.10) \quad \frac{\alpha d_1 r_1}{a_1} \left\| \frac{\nabla u_1}{u_1} \right\|_2^2 + \alpha a_1 \left\| u_1 - \frac{r_1}{a_1} \right\|_2^2 \leq C + \frac{(\alpha b_1 + b_2)^2}{4 a_2} \left\| u_1 - \frac{r_1}{a_1} \right\|_2^2.
$$

From (2.10) we can conclude that

$$
(2.11) \quad \left\| \frac{\nabla u_1}{u_1} \right\|_2 \leq C_1
$$

and

$$
(2.12) \quad \left\| u_1 - \frac{r_1}{a_1} \right\|_2 \leq C_2
$$

for some constants C_1, C_2 independent of T, provided that $\alpha > 0$ can be chosen such that

$$
2 \sqrt{\alpha a_1 a_2} > \alpha b_1 + b_2,
$$

which is possible by the assumption $a_1 a_2 > b_1 b_2$.

Because of (1.7) we may deduce from (2.11) that, for some constant C_3 independent of T,

$$
(2.13) \quad \|\nabla u_1\|_2 \leq C_3.
$$

Since all this is for any $T > 0$, (2.13) and (2.12) imply that $u_1 - r_1/a_1 \in L^2((0, \infty); W^{1,2}(\Omega; R))$ and thus

$$
(2.14) \quad \lim_{t \to \infty} \left\| u_1(t) - \frac{r_1}{a_1} \right\|_{W^{1,2}} = 0.
$$

Therefore,

$$
\lim_{t \to \infty} \left\| u_1(t) - \frac{r_1}{a_1} \right\|_{C(\bar{\Omega}; R)} = 0.
$$

We deduce $\lim_{t \to \infty} \|u_2(t)\|_{C(\bar{\Omega}; R)} = 0$ in a similar way (for example, λ in (2.9) would be chosen differently). This completes the proof.

Remark 2.4.   Theorem 2.3 indicates that if $r_1/r_2 > a_1/b_2 > b_1/a_2$, then the competitor with density u_1 wins the competition; if $r_1/r_2 < b_1/a_2 < a_1/b_2$, then the

competitor with density u_2 overcompetes the one with density u_1; and if $b_1/a_2 < r_1/r_2 < a_1/b_2$, then the two competing species coexist in the sense of existence and stability of a positive steady state. Theorem 2.3 extends Propositions 7.5–7.7 in Ruan and Wu [11] on competition-diffusion systems with infinite time delays, Theorem 3.1 in Zhou and Pao [18] on competition-diffusion systems, and the results in Gopalsamy [6] on competition systems with finite delays.

Remark 2.5. It is known (see Yamada [16]) that in the case of the single-species delay equation

$$\frac{\partial u}{\partial t} = \Delta u + u \left(a - bu - \int_{-\infty}^{t} f(t-s)u(x,s)\,ds \right)$$

on homogeneous Neumann boundary conditions, where a and b are nonnegative constants, bifurcations can occur from the nonzero homogeneous equilibrium state for certain kernels and for suitable values of the parameters a and b, which include the requirement that b be sufficiently small. However, in the competition model (1.1), bifurcations to spatially patterned or to spatiotemporal structures are not expected to occur from the equilibrium E^*, given by (1.8). Let us explain why this is so. A standard linearized analysis about the boundary equilibrium $(r_1/a_1, 0)$ shows that, regardless of the delay kernels, this equilibrium is unstable to perturbations in which $u_2 > 0$ if $r_1/r_2 < a_1/b_2$. Similarly, the equilibrium $(0, r_2/a_2)$ is unstable to perturbations in which $u_1 > 0$ if $r_1/r_2 > b_1/a_2$. Now, if the interior equilibrium E^* were to lose stability and bifurcate to a spatial or spatiotemporal structure, we would expect that both boundary equilibria would remain unstable throughout this process so that they act as repellers. Yet the conditions for both boundary equilibria to be linearly unstable can be summarized as

$$\frac{b_1}{a_2} < \frac{r_1}{r_2} < \frac{a_1}{b_2},$$

which is precisely the condition for global convergence to E^* given in (iii) of Theorem 2.3. Hence, bifurcations from E^* cannot occur if the boundary equilibria are to remain unstable.

Remark 2.6. If we assume that, in the absence of the other competitor, each competitor's growth is governed by a Volterra integrodifferential equation with both instantaneous and delay self-regulatory terms (see Cushing [2], Schiaffino and Tesei [13], and Yamada [17]), then we have a more general model of the following form:

$$\frac{\partial u_1}{\partial t} = d_1 \Delta u_1 + u_1 \left(r_1 - a_1 u_1 - c_1 \int_{\Omega}\int_{-\infty}^{t} H_1(x,y,t-s)u_2(s,y)\,ds\,dy \right.$$

$$\left. -b_1 \int_{\Omega}\int_{-\infty}^{t} K_1(x,y,t-s)u_2(s,y)\,ds\,dy \right),$$

(2.15)

$$\frac{\partial u_2}{\partial t} = d_2 \Delta u_2 + u_2 \left(r_2 - b_2 \int_{\Omega}\int_{-\infty}^{t} K_2(x,y,t-s)u_1(s,y)\,ds\,dy \right.$$

$$\left. -a_2 u_2 - c_2 \int_{\Omega}\int_{-\infty}^{t} H_2(x,y,t-s)u_1(s,y)\,ds\,dy \right),$$

where $a_i \geq 0$, $b_i > 0$, $c_i \geq 0$, $i = 1, 2$, are constants and the kernels H_i, $i = 1, 2$, satisfy similar properties as the K_i of (1.1). Notice that system (1.1) is a special case of (2.15) with $c_i = 0$. When $a_i = 0$, even when there is no diffusion (i.e., the ordinary delay competition model), both stability (see Gomatam and MacDonald [5])

and bifurcation (see Gopalsamy and Aggarwala [7]) are possible. We anticipate that system (2.15) will exhibit more complex dynamics, such as Hopf bifurcations, and we leave this for future consideration.

3. Travelling front solutions. In this section we discuss the modifications necessary to system (1.1) for the case of an infinite one-dimensional domain $\Omega = (-\infty, \infty)$, and travelling front solutions of the resulting system. The infinite domain case is in some respects slightly simpler from a modelling point of view since there are no boundaries for individuals to interact with as they drift from their past to their present positions. Because of this, in contrast to the finite domain case, the nonlocal averaging associated with the delay takes the form of a spatial convolution, so that the model assumes the form

$$\frac{\partial u_1}{\partial t} = d_1 \frac{\partial^2 u_1}{\partial x^2} + u_1 \left(r_1 - a_1 u_1 - b_1 \int_{-\infty}^{\infty} \int_{-\infty}^{t} G_1(x - y, t - s) k_1(t - s) u_2(s, y) \, ds \, dy \right),$$

$$\frac{\partial u_2}{\partial t} = d_2 \frac{\partial^2 u_2}{\partial x^2} + u_2 \left(r_2 - b_2 \int_{-\infty}^{\infty} \int_{-\infty}^{t} G_2(x - y, t - s) k_2(t - s) u_1(s, y) \, ds \, dy - a_2 u_2 \right),$$

(3.1)

where the k_i satisfy $\int_0^{\infty} k_i(s) \, ds = 1$, $i = 1, 2$, and the G_i satisfy diffusion equations as in Lemma 2.1 but without the boundary conditions. To be more precise, G_1 is a weighting function describing the distribution at past times of the individuals of the species u_2 who are at position x at time t. The u_2 individuals diffuse at diffusivity d_2; thus G_1 must satisfy

$$\frac{\partial G_1}{\partial t} = d_2 \frac{\partial^2 G_1}{\partial x^2}, \qquad G_1(x, 0) = \delta(x),$$

and similarly, G_2 satisfies

$$\frac{\partial G_2}{\partial t} = d_1 \frac{\partial^2 G_2}{\partial x^2}, \qquad G_2(x, 0) = \delta(x),$$

so that G_1, G_2 are both fundamental solutions of heat equations. With these assumptions, system (3.1) still preserves the same equilibria E_0, E_1, E_2, and (possibly) E^* enumerated earlier.

Our interest in this section is in the possibility of a transition between the boundary equilibria E_1 and E_2 in the form of a travelling wave-front solution. This is of ecological interest since it corresponds to a situation where an environment is initially inhabited only by the weaker of the two competitors at its carrying capacity, and some of the stronger competitor are introduced and then invade the domain, dominate, and drive the weaker to extinction so that the end result is that only the stronger species is present, at its carrying capacity.

In this section the assumptions we shall make on the parameters are those which ensure that the corresponding system without diffusion and without delay (removal of delay can be effected by setting each $k_i(t) = \delta(t)$ in (3.1)) has E_1 unstable and E_2 asymptotically stable. Elementary analysis shows that the conditions for this to happen are

(3.2) $$r_1 b_2 < r_2 a_1 \quad \text{and} \quad r_1 a_2 < r_2 b_1.$$

Note that if (3.2) is satisfied, then the coexistence equilibrium E^* is absent. In the two-dimensional (u_1, u_2) phase plane, the diffusionless undelayed ODEs possess a heteroclinic connection from E_1 to E_2. It is known from the papers referred to in the introduction that under these circumstances the (undelayed) reaction-diffusion system has travelling-front solutions connecting these equilibria. Our intention now is to prove, for certain choices of the kernels k_i, that these travelling fronts persist under the introduction of delay, at least for small delays.

We shall consider the situation when the kernels k_i are given by

$$(3.3) \qquad k_1(t) = \frac{1}{\tau_1} e^{-t/\tau_1}, \qquad k_2(t) = \frac{1}{\tau_2} e^{-t/\tau_2},$$

where the delays $\tau_1, \tau_2 > 0$, and we shall prove the following.

THEOREM 3.1. *Let k_1 and k_2 be given by (3.3) and assume that (3.2) holds. Then, for sufficiently small delays τ_1, τ_2, system (3.1) possesses travelling front solutions connecting the semitrivial equilibria $E_1 = (r_1/a_1, 0)$ and $E_2 = (0, r_2/a_2)$.*

Proof. With the kernels given by (3.3), it is straightforward to see that system (3.1) is equivalent to

$$
\begin{aligned}
\frac{\partial u_1}{\partial t} &= d_1 \frac{\partial^2 u_1}{\partial x^2} + u_1(r_1 - a_1 u_1 - b_1 w_1), \\[2mm]
\frac{\partial u_2}{\partial t} &= d_2 \frac{\partial^2 u_2}{\partial x^2} + u_2(r_2 - b_2 w_2 - a_2 u_2), \\[2mm]
\frac{\partial w_1}{\partial t} &= d_2 \frac{\partial^2 w_1}{\partial x^2} + \frac{1}{\tau_1} u_2 - \frac{1}{\tau_1} w_1, \\[2mm]
\frac{\partial w_2}{\partial t} &= d_1 \frac{\partial^2 w_2}{\partial x^2} + \frac{1}{\tau_2} u_1 - \frac{1}{\tau_2} w_2.
\end{aligned}
$$

(3.4)

Converting to travelling wave form, by writing

$$u_1(t, x) = u_1(z), \qquad z = x + ct,$$

and similarly for the other state variables, gives

$$
\begin{aligned}
cu_1' &= d_1 u_1'' + u_1(r_1 - a_1 u_1 - b_1 w_1), \\[1mm]
cu_2' &= d_2 u_2'' + u_2(r_2 - b_2 w_2 - a_2 u_2), \\[1mm]
cw_1' &= d_2 w_1'' + \frac{1}{\tau_1} u_2 - \frac{1}{\tau_1} w_1, \\[1mm]
cw_2' &= d_1 w_2'' + \frac{1}{\tau_2} u_1 - \frac{1}{\tau_2} w_2,
\end{aligned}
$$

where prime denotes differentiation with respect to z. Let us introduce

$$v_1 = d_1 u_1', \qquad v_2 = d_2 u_2', \qquad v_3 = d_2 w_1', \qquad v_4 = d_1 w_2'.$$

Also, we shall replace τ_1 and τ_2 with $\varepsilon^2 \tau_1$ and $\varepsilon^2 \tau_2$, respectively, since we are interested

in the situation when the delays are small. The system becomes

$$u_1' = \frac{1}{d_1}v_1,$$

$$v_1' = \frac{c}{d_1}v_1 - u_1(r_1 - a_1u_1 - b_1w_1),$$

$$u_2' = \frac{1}{d_2}v_2,$$

$$v_2' = \frac{c}{d_2}v_2 - u_2(r_2 - b_2w_2 - a_2u_2),$$

(3.5)

$$w_1' = \frac{1}{d_2}v_3,$$

$$\varepsilon^2 v_3' = \frac{\varepsilon^2 c}{d_2}v_3 - \frac{1}{\tau_1}u_2 + \frac{1}{\tau_1}w_1,$$

$$w_2' = \frac{1}{d_1}v_4,$$

$$\varepsilon^2 v_4' = \frac{\varepsilon^2 c}{d_1}v_4 - \frac{1}{\tau_2}u_1 + \frac{1}{\tau_2}w_2.$$

If we introduce the new state variables

$$\tilde{u}_1 = u_1, \quad \tilde{v}_1 = v_1, \quad \tilde{u}_2 = u_2, \quad \tilde{v}_2 = v_2, \quad \tilde{w}_1 = w_1, \quad \tilde{v}_3 = \varepsilon v_3, \quad \tilde{w}_2 = w_2, \quad \tilde{v}_4 = \varepsilon v_4$$

and then drop the tildes, we have

$$u_1' = \frac{1}{d_1}v_1,$$

$$v_1' = \frac{c}{d_1}v_1 - u_1(r_1 - a_1u_1 - b_1w_1),$$

$$u_2' = \frac{1}{d_2}v_2,$$

$$v_2' = \frac{c}{d_2}v_2 - u_2(r_2 - b_2w_2 - a_2u_2),$$

(3.6)

$$\varepsilon w_1' = \frac{1}{d_2}v_3,$$

$$\varepsilon v_3' = \frac{\varepsilon c}{d_2}v_3 - \frac{1}{\tau_1}u_2 + \frac{1}{\tau_1}w_1,$$

$$\varepsilon w_2' = \frac{1}{d_1}v_4,$$

$$\varepsilon v_4' = \frac{\varepsilon c}{d_1}v_4 - \frac{1}{\tau_2}u_1 + \frac{1}{\tau_2}w_2.$$

When $\varepsilon = 0$, system (3.6) reduces to the equations satisfied by travelling wave solutions of the undelayed problem studied by previous investigators [1, 4, 10, 14]. In this degenerate case the system is four-dimensional, but for $\varepsilon > 0$ (i.e., delay is present), existence of a travelling front solution of (3.1) between E_1 and E_2 is equivalent to existence of a heteroclinic connection between the equilibrium points of the eight-dimensional system (3.6) that correspond to E_1 and E_2 of (3.1). We shall still denote

these equilibria by E_1 and E_2; for system (3.6) they are given by

$$(3.7) \qquad E_1 = \left(\frac{r_1}{a_1}, 0, 0, 0, 0, 0, \frac{r_1}{a_1}, 0 \right), \qquad E_2 = \left(0, 0, \frac{r_2}{a_2}, 0, \frac{r_2}{a_2}, 0, 0, 0 \right).$$

Our intention is to apply the geometric singular perturbation theory described in [3], in particular, Theorem 9.1 of that paper. System (3.6) above will henceforth be referred to as the *slow system*. By introducing a new independent variable η defined by

$$z = \varepsilon \eta,$$

system (3.6) transforms into

$$\dot{u}_1 = \frac{\varepsilon}{d_1} v_1,$$

$$\dot{v}_1 = \varepsilon \left(\frac{c}{d_1} v_1 - u_1 (r_1 - a_1 u_1 - b_1 w_1) \right),$$

$$\dot{u}_2 = \frac{\varepsilon}{d_2} v_2,$$

$$\dot{v}_2 = \varepsilon \left(\frac{c}{d_2} v_2 - u_2 (r_2 - b_2 w_2 - a_2 u_2) \right),$$

(3.8)

$$\dot{w}_1 = \frac{1}{d_2} v_3,$$

$$\dot{v}_3 = \frac{\varepsilon c}{d_2} v_3 - \frac{1}{\tau_1} u_2 + \frac{1}{\tau_1} w_1,$$

$$\dot{w}_2 = \frac{1}{d_1} v_4,$$

$$\dot{v}_4 = \frac{\varepsilon c}{d_1} v_4 - \frac{1}{\tau_2} u_1 + \frac{1}{\tau_2} w_2,$$

where dots denote differentiation with respect to η. System (3.8) is called the *fast system*. Geometric singular perturbation theory uses both the slow and the fast systems. The two are equivalent when $\varepsilon > 0$, but when $\varepsilon = 0$, the slow system (3.6) does not define a dynamical system in the whole of $\mathbf{R}^8$ but rather the dynamics takes place only on

$$M_0 = \{(u_1, v_1, u_2, v_2, w_1, v_3, w_2, v_4) \in \mathbf{R}^8 : \quad v_3 = 0, \quad v_4 = 0, \quad w_1 = u_2, \quad w_2 = u_1\},$$
(3.9)

which is a four-dimensional submanifold of $\mathbf{R}^8$. Note that M_0 consists of the equilibria of the fast system when $\varepsilon = 0$. If M_0 is normally hyperbolic then, for sufficiently small $\varepsilon > 0$, Theorem 9.1 in [3] provides us with a four-dimensional invariant manifold M_ε for the system (3.6). It will be shown that the equilibrium points E_1 and E_2 lie on M_ε. By studying the system (3.6) reduced to this manifold, the dimensionality is reduced back to four and the existence of the heteroclinic connection we are seeking can be established.

To verify normal hyperbolicity it is necessary to use the fast system (3.8). We need to verify that the linearization of (3.8), restricted to M_0, has exactly four ($= \dim M_0$)

eigenvalues on the imaginary axis with the remainder of the spectrum hyperbolic. The linearization of the fast system, when $\varepsilon = 0$, is given by

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \dfrac{1}{d_2} & 0 & 0 \\
0 & 0 & -\dfrac{1}{\tau_1} & 0 & \dfrac{1}{\tau_1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dfrac{1}{d_1} \\
-\dfrac{1}{\tau_2} & 0 & 0 & 0 & 0 & 0 & \dfrac{1}{\tau_2} & 0
\end{pmatrix},
$$

which has eigenvalues $\{0, 0, 0, 0, \pm 1/\sqrt{\tau_2 d_1}, \pm 1/\sqrt{\tau_1 d_2}\}$. Thus, normal hyperbolicity is verified and there exists an invariant manifold $M_{\mathcal{E}}$, close to M_0, for the perturbed system (3.6) for $\varepsilon > 0$ sufficiently small. In fact, $M_{\mathcal{E}}$ can be expressed in the form

$$
\begin{aligned}
(3.10) \quad M_{\mathcal{E}} = \big\{ (u_1, v_1, u_2, v_2, w_1, v_3, w_2, v_4) \in \mathbf{R}^8 : \quad & v_3 = h_1(u_1, v_1, u_2, v_2; \varepsilon), \\
v_4 = h_2(u_1, v_1, u_2, v_2; \varepsilon), \quad w_1 = u_2 + & h_3(u_1, v_1, u_2, v_2; \varepsilon), \\
w_2 = u_1 + & h_4(u_1, v_1, u_2, v_2; \varepsilon) \big\}
\end{aligned}
$$

with $h_i(u_1, v_1, u_2, v_2; 0) = 0$ for $i = 1, 2, 3, 4$. The h_i can be computed by substitution into (3.6).

The slow system (3.6), restricted to $M_{\mathcal{E}}$, is

$$
\begin{aligned}
(3.11) \qquad
u_1' &= \frac{1}{d_1} v_1, \\
v_1' &= \frac{c}{d_1} v_1 - u_1(r_1 - a_1 u_1 - b_1 u_2 - b_1\, h_3(u_1, v_1, u_2, v_2; \varepsilon)), \\
u_2' &= \frac{1}{d_2} v_2, \\
v_2' &= \frac{c}{d_2} v_2 - u_2(r_2 - a_2 u_2 - b_2 u_1 - b_2\, h_4(u_1, v_1, u_2, v_2; \varepsilon)).
\end{aligned}
$$

When $\varepsilon = 0$ (i.e., no delay), system (3.11) again reduces to the system satisfied by travelling wave solutions of the undelayed equations, which has been studied previously. What we now claim is that, for $\varepsilon > 0$ sufficiently small, system (3.11) still possesses as equilibrium points

$$
(3.12) \qquad E_1 = \left(\frac{r_1}{a_1}, 0, 0, 0 \right), \qquad E_2 = \left(0, 0, \frac{r_2}{a_2}, 0 \right)
$$

and that it falls within the class of systems studied by Gardner [4]. Neither is immediately clear. Indeed, Gardner studied competition systems of the form

$$
\begin{aligned}
\partial u_1/\partial t &= d_1 \partial^2 u_1/\partial x^2 + u_1 M(u_1, u_2), \\
\partial u_2/\partial t &= d_2 \partial^2 u_2/\partial x^2 + u_2 N(u_1, u_2),
\end{aligned}
$$

which, in travelling wave form, read as

(3.13)
$$u_1' = \frac{1}{d_1} v_1,$$
$$v_1' = \frac{c}{d_1} v_1 - u_1 M(u_1, u_2),$$
$$u_2' = \frac{1}{d_2} v_2,$$
$$v_2' = \frac{c}{d_2} v_2 - u_2 N(u_1, u_2).$$

Comparing (3.11) with (3.13) we see that, for Gardner's results to be applicable, the functions h_3 and h_4 in (3.11) would need to involve u_1 and u_2 only. We shall now show that this is indeed the case, up to order ε^2.

Indeed, straightforward but tedious calculations, utilizing the fact that M_ε is an invariant manifold for (3.6), yield that the h_i satisfy

$$\varepsilon \left[\frac{1}{d_2} v_2 + \frac{1}{d_1} v_1 \frac{\partial h_3}{\partial u_1} + \frac{\partial h_3}{\partial v_1} \left(\frac{c}{d_1} v_1 - u_1 (r_1 - a_1 u_1 - b_1 u_2 - b_1 h_3) \right) \right.$$
$$\left. + \frac{1}{d_2} v_2 \frac{\partial h_3}{\partial u_2} + \frac{\partial h_3}{\partial v_2} \left(\frac{c}{d_2} v_2 - u_2 (r_2 - a_2 u_2 - b_2 u_1 - b_2 h_4) \right) \right] = \frac{1}{d_2} h_1$$

together with three other similar equations. Attempting solutions of the equations in the form

$$h_1(u_1, v_1, u_2, v_2; \varepsilon) = \varepsilon h_1^{(1)}(u_1, v_1, u_2, v_2) + \varepsilon^2 h_1^{(2)}(u_1, v_1, u_2, v_2) + \cdots,$$

and similarly for the other h_i, yields, after some further algebra, that

$$h_1^{(1)} = v_2, \quad h_2^{(1)} = v_1, \quad h_3^{(1)} = 0, \quad h_4^{(1)} = 0$$

and

$$h_1^{(2)} = 0, \quad h_2^{(2)} = 0,$$
$$h_3^{(2)} = -\tau_1 u_2 (r_2 - a_2 u_2 - b_2 u_1),$$
$$h_4^{(2)} = -\tau_2 u_1 (r_1 - a_1 u_1 - b_1 u_2).$$

Thus, system (3.11) becomes, to order ε^2,

(3.14)
$$u_1' = \frac{1}{d_1} v_1,$$
$$v_1' = \frac{c}{d_1} v_1 - u_1 (r_1 - a_1 u_1 - b_1 u_2 + \varepsilon^2 b_1 \tau_1 u_2 (r_2 - a_2 u_2 - b_2 u_1)),$$
$$u_2' = \frac{1}{d_2} v_2,$$
$$v_2' = \frac{c}{d_2} v_2 - u_2 (r_2 - a_2 u_2 - b_2 u_1 + \varepsilon^2 b_2 \tau_2 u_1 (r_1 - a_1 u_1 - b_1 u_2)),$$

which has the structure of the system (3.13). Also, note that E_1 and E_2, given by (3.12), are indeed equilibria of (3.14). Therefore, the results in [4] are applicable,

yielding a heteroclinic connection between the equilibria E_1 and E_2 of (3.14). We have shown that travelling fronts exist for system (3.1) when the kernels k_1 and k_2 are given by (3.3), so the proof of Theorem 3.1 is complete.

Remark 3.2. Let us briefly discuss the role of the terms of order ε^2. If the system (3.14) is linearized about the equilibrium E_2, we find that the eigenvalues λ of the linearization satisfy an equation that does not involve ε, namely

$$(3.15) \qquad (d_2\lambda^2 - c\lambda - r_2)(d_1a_2\lambda^2 - ca_2\lambda + r_1a_2 - r_2b_1) = 0.$$

About the equilibrium E_1, the eigenvalue equation becomes

$$(3.16) \qquad (d_1\lambda^2 - c\lambda - r_1)(d_2a_1\lambda^2 - ca_1\lambda + r_2a_1 - r_1b_2) = 0,$$

which again does not involve ε. These observations suggest that the manner in which the front approaches the equilibria E_1 and E_2 as $z \to -\infty$ and $z \to \infty$, respectively, is independent of ε for $\varepsilon > 0$ sufficiently small and therefore that the front's qualitative profile is not sensitive to the delays, provided they are both sufficiently small. Of course, system (3.14) is itself the result of a perturbation analysis for small ε, and therefore no conclusions can be drawn for larger ε. In conclusion, we may state that the travelling front solutions of the corresponding undelayed competition model appear to be very *robust*, not only in the sense that they persist under the introduction of delays, but also in that they are not sensitive to small delays in the sense that, if the delays are small, they look qualitatively the same as they do with no delay at all.

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