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Uniform Persistence in Functional Differential Equations

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In this paper, we investigate the question of uniform persistence for retarded functional differential equations. By utilizing Liapunov-like functions, Razumikhin techniques, and differential inequalities, we are able to establish criteria for uniform persistence analogous to those obtained by others for ordinary differential equations, difference equations, and reaction–diffusion equations. We apply these criteria to some well known biological models with delay. Our results indicate that the conditions which guarantee the existence of an interior equilibrium are enough to ensure uniform persistence. Moreover, these conditions are equivalent to uniform persistence for the cases without delay as well. © 1995 Academic Press, Inc.

1. INTRODUCTION

In recent years the concept of persistence has played an important role in mathematical ecology. Biologically, when a system of interacting species is persistent in a suitable sense, it means that all the species survive in the long term. Mathematically, persistence of a system means that strictly positive solutions do not have any omega limit points on the boundary of the nonnegative cone.

Various definitions of persistence have been developed in order to analyze mathematical models. (Weak) persistence was considered by Freedman and Waltman [20] and Gard and Hallam [26], and was defined as strong flow-invariance by Gard [25] and Fernandes and

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Zanolin [12]. Persistence was defined and discussed by Freedman and Waltman [21, 22]. Uniform persistence was first introduced by Schuster *et al.* [51]. A dissipative and uniformly persistent system was called cooperative by Hofbauer [33], and permanent by Hofbauer and Sigmund [35] and Hutson [37, 38]. The latter is now often used in the literature. Even through permanence and uniform persistence are equivalent for most purposes, Yang and Ruan [56] have shown that a uniformly persistent system is not necessarily dissipative. The connections of various definitions of persistence have been discussed by Butler *et al.* [5], and recently by Freedman and Moson [17] who also introduced the concept of weakly weak persistence.

Most of the applications of persistence have been to ordinary differential equations (cf. Gard and Hallam [26], Fernandes and Zanolin [14], and the references cited therein), and to difference equations (cf. Hutson and Moran [39]). Dunbar *et al* [11], Hutson and Moran [40], and Cantrell *et al.* [7] investigated persistence for reaction–diffusion models. Burton and Hutson [2] obtained some very interesting results on persistence of autonomous functional differential equations with infinite delay. Criteria for one or more forms of persistence to hold in general dynamical systems were given by Butler and Waltman [6], Fonda [15], Freedman *et al.* [19], Garay [24], Hofbauer [34] and Hofbauer and So [36], among others. Also, Freedman and So [18] dealt with discrete dynamical systems and Hale and Waltman [31] developed persistence criteria for infinite-dimensional systems. For more details and more references on this subject, we refer the reader to the recent survey paper of Hutson and Schmitt [41].

Recently, Freedman and Wu [23] discussed persistence in a delayed system by using the monotone dynamical systems theory developed by Smith [53]. By constructing suitable persistence functionals, Wang and Ma [54] obtained uniform persistence conditions for Lotka–Volterra predator–prey systems with a finite number of discrete delays. Their results suggested that delays are "harmless" for uniform persistence. Similar phenomena were observed by Zanolin [58] in delayed Kolmogorov competing species systems. By utilizing the results in Hale and Waltman [31], Cao *et al.* [8] studied uniform persistence for Kolmogorov-type predator–prey and competition models with per capita net growth rates that are dependent on time-delayed population densities. Kuang and Tang [44] also established sufficient conditions for uniform persistence in non-autonomous Kolmogorov-type delayed population models. See also Cao and Gard [9], Kuang and Tang [45] and Ruan [49]. However, all together there are very few general results on persistence in delay equations which have a significant background in biology (cf. Cushing [10], Gopalsamy [27], MacDonald [48], etc.).

The methods of Liapunov-like functions and differential inequalities are standard techniques in studying stability, flow-invariance, and the existence of periodic solutions (cf. Fernandes and Zanolin [12, 13] and the references cited therein), and have been proposed as techniques in the investigation of persistence (cf. see Hofbauer [33], Hutson [37], etc.). The Razumikhin technique is an important method which is utilized in studying the stability of functional differential equations (see Haddock and Terjéki [28] and Hale [29]). By this technique, one only needs to choose a Liapunov function (instead of a Liapunov functional) and to verify the nonpositivity of the derivative function for some initial data (instead of all initial data) under certain restrictions in order to have stability.

In the present paper, motivated by the work of Fernandes and Zanolin [13, 14] for nonautonomous ordinary differential equations, we investigate retarded functional differential equations. Liapunov-like functions, the Razumikhin technique, and differential inequalities are used to derive uniform persistence criteria for the retarded functional differential equation (RFDE). Our uniform persistence theorems are analogous to that for ordinary differential equations (Hofbauer [33], Hutson [37], and Fernandes and Zanolin [14]), difference equations (Hutson and Moran [39]) and reaction–diffusion equations (Hutson and Moran [40]). Some well known examples are analyzed to illustrate the obtained results. Our result in the first example indicates that the conditions which guarantee the existence of an interior equilibrium are enough to ensure uniform persistence. In Examples 2 and 3, the criteria we obtained are equivalent to uniform persistence for the cases without delay.

2. Preliminaries

Suppose $r \geqslant 0$ is a given real number, $R = (-\infty, \infty)$, $C = C([-r, 0], R^n)$ is the Banach space of continuous functions mapping the interval $[-r, 0]$ into R^n with the topology of uniform convergence. For an element $\varphi \in C$, designate the norm of φ by $\|\varphi\| = \sup_{-r \leqslant \varphi \leqslant 0} |\varphi(\theta)|$. If $t_0 \in R$, $A \geqslant 0$ and $x \in C([t_0 - r, t_0 + A], R^n)$, then for any $t \in [t_0, t_0 + A]$ let $x_t \in C$ be defined by $x_t(\theta) = x(t + \theta)$, $-r \leqslant \theta \leqslant 0$. If $f\colon R \times C \to R^n$ is a given function, we have a RFDE

$$\dot{x}(t) = f(t, x_t). \tag{2.1}$$

For given $(t_0, \varphi) \in R \times C$, by $x_t(t_0, \varphi)$ we denote a solution of RFDE (2.1) with initial value φ at t_0, i.e. $x_{t_0}(t_0, \varphi) = \varphi$. We assume that solutions of RFDE (2.1) globally exist and are unique and continuous.

Let $(X, \|\cdot\|)$ be the metric space with $\varphi \in X$ if $\varphi \in C$, $\|\varphi\|$ exists and each component $\varphi_i(s) \geqslant 0$ for $-r \leqslant s \leqslant 0$. Then X is a subset of C which contains the only biologically meaningful elements. In the following we also use d to denote $\|\cdot\|$ sometimes.

Let $N \subseteq X$ be closed relative to X and suppose that N is a *flow-invariant* set with respect to RFDE (2.1); that is, if $x_t(t_0, \varphi)$ is any (noncontinuable) solution of (2.1) through $(t_0, \varphi) \in R \times N$, then $x_t(t_0, \varphi) \in N$ for all $t \in I_x = [t_0, t_x)$, where I_x is the right maximal interval of existence. For given $M \subset N$, we denote by $\text{int}_N M$, $\text{fr}_N M$ and $\text{cl}_N M$ respectively the interior, boundary (frontier), and closure of M relative to N.

DEFINITION 2.1. Given a set $M \subset N$ with $\text{int}_N M \neq \emptyset$, we say RFDE (2.1) is *persistent* relative to M if for each $(t_0, \varphi) \in R \times \text{int}_N M$ and $x_t(t_0, \varphi)$ the solution of (2.1), we have $x_t(t_0, \varphi) \in \text{int}_N M$ for each $t \in [t_0, t_x)$, such that

$$\liminf_{t \to t_x^-} d(x_t(t_0, \varphi), \text{fr}_N M) > 0.$$

If there is a $\delta > 0$ such that

$$\liminf_{t \to t_x^-} d(x_t(t_0, \varphi), \text{fr}_N M) > \delta$$

for any $x_t(t_0, \varphi) \in \text{int}_N M$, $t \in [t_0, t_x)$, we say system (2.1) is *uniformly persistent* relative to M.

The concept of uniform persistence essentially involves two conditions: flow-invariance of $\text{int}_N M$ and repulsivity of $\text{fr}_N M$ with respect to the solutions of (2.1) with values in $\text{int}_N M$. Flow-invariance was studied by Gard [25] and Fernandes and Zanolin [12, 13] (and the references cited therein) for ODE, and by Seifert [52] for RFDE. Now we suppose that $\emptyset \neq G \subset N$ is a set open relatively to N and $S \subset N$, $S \cap G = \emptyset$. We shall find conditions for the repulsivity of S with respect to the solutions of (2.1) lying in G. Define $S^* = S \cap \text{fr}_N G$.

DEFINITION 2.2. Let $Z \subset N \backslash G$, we say Z is *uniformly repelling* with respect to G if there exist an open neighbourhood $\mathscr{A}$ of Z and $t_1 \in [t_0, t_x)$, such that for any solution $x_t(t_0, \varphi)$ of (2.1) with $x_t(t_0, \varphi) \in G$ for all $t \in [t_0, t_x)$, we have $x_t(t_0, \varphi) \notin \mathscr{A}$ for all $t \in [t_1, t_x)$. In the particular case that $Z = \{u\}$, with $u \notin G$, the point u is said to be *uniformly repulsive* with respect to G.

Following Fernandes and Zanolin [14], we have the following

PROPOSITION 2.3. *If S^* is compact and each point $u \in S^*$ is uniformly repulsive with respect to G, then S is uniformly repelling. If S is* *compact and uniformly repelling with respect to G, then there is* $\delta > 0$ *such that*

$$\liminf_{t \to t_1^-} d(x_t(t_0, \varphi), S) > \delta$$

for each $x_t(t_0, \varphi)$ *solution of* (2.1) *with* $x_t(t_0, \varphi) \in G$, $t \in [t_0, t_1)$.

Choosing $G = \text{int}_N M$ and $S = \text{fr}_N M$ in Proposition 2.3, we get the following.

PROPOSITION 2.4. *Let* $M \subset N$, *such that* $\text{int}_N M \neq \emptyset$ *and* $\text{fr}_N M$ *is compact. Suppose that* $\text{int}_N M$ *is flow-invariant with respect to the RFDE* (2.1) *and each point of* $\text{fr}_N(\text{int}_N M)$ *is uniformly repulsive with respect to* $\text{int}_N M$. *Then the RFDE* (2.1) *is uniformly persistent.*

If $V: R \times R^n \to R$ is a continuous function, then the derivative of V along the solutions of (2.1) is defined as

$$\dot{V}(t, \varphi(0)) = \liminf_{h \to 0^+} \frac{1}{h} [V(t+h, x_{t+h}(t_0, \varphi)) - V(t, \varphi(0))].$$

In the following, we suppose that $t_0 \geqslant 0$ and consider $t \in J = [t_0, b)$, $b \leqslant \infty$.

A special case of (2.1) is the autonomous RFDE

$$\dot{x} = f(x_t). \tag{2.2}$$

Similarly we can define a space X for (2.2). If a map $\pi: X \times J \to X$ satisfies for all $u \in X$ that

(i) $\pi(u, 0) = u$,

(ii) $\pi(\pi(u, s), t) = \pi(u, t+s)$ for $t, s \in J$,

(iii) π is continuous,

then (π, X, J) is a semidynamical system (see Saperstone [50] and Hale [30]). For convenience we often write $\pi(u, t) = u_t$, where u is a unique solution of (2.2). The semi-orbit through u is denoted by

$$\gamma^+(u) = \{u_t : t \in J\},$$

and for a subset $M \subset X$,

$$\gamma^+(M) = \bigcup_{u \in M} \gamma^+(u).$$

The omega limit set of u is defined to be

$$\omega(u) = \bigcap_{s \geqslant 0} \text{cl} \bigcup_{t \geqslant s} u_t,$$

and for a set $M \subset X$, the omega limit set is defined as

$$\omega(M) = \bigcup_{u \in M} \omega(u).$$

It is known that if $\gamma^+(u)$ is relatively compact, then $\omega(u)$ is nonempty, connected, compact, and invariant (see Hale [30]).

3. Main Results

THEOREM 3.1. *Assume that* $f: J \times C \to R^n$ *is completely continuous,* $u \in \text{fr}_N G$, *and there is an open neighbourhood* $X_1 \subset X$ *of* u *and two continuous functions* $V(t, x)$ *and* $g(t, z)$ *with* $V: J \times R^n \to R$ *locally Lipschitzian in* x *and* $g: J \times (-\infty, 0) \to R$ *such that*

(i) $K = \{\psi \in X_1 \cap \text{fr}_N G: \limsup_{t \to b^-, \varphi \to \psi} V(t, \varphi(0)) = 0\}$ *is compact and* $u \in K \subset X_1$;

(ii) $\lim_{t \to b^-} V(t, \varphi(0)) = 0$ *uniformly for* $\varphi \in G$ *and* $d(\varphi, K) \to 0$;

(iii) $\limsup_{t \to \sigma^-, \varphi \to \psi} V(t, \varphi(0)) < 0$ *for all* $(\sigma, \psi) \in J \times (G \cap X_1)$;

(iv) $\dot{V}_{(2.1)}(t, \varphi(0)) \leqslant g(t, V(t, \varphi(0)))$ *if* $V(t+\theta, \varphi(\theta)) \leqslant V(t, \varphi(0))$, $\theta \in [-r, 0]$ *for all* $(t, \varphi) \in J \times (G \cap X_1)$;

(v) *for every* $k > 0$, *there exist* $\eta_k > 0$ *and* $\delta > 0$ *such that for each* $t_0 \leqslant \sigma < b$, *the problem*

$$\dot{z} = g(t, z), \qquad z(\sigma) = -k \tag{3.1}$$

has a maximial solution $z(t)$ *on* $[t_0, t_z)$ *with*

$$\sup_{t \geqslant \sigma} z(t) < -\eta_k \qquad \textit{and} \qquad \liminf_{t \to t_z^-} z(t) < -\delta.$$

Then u *is uniformly repulsive with respect to* G.

Proof. For $\rho > 0$, denote

$$B[K, \rho] = \{\psi \in X: \exists \varphi \in K, \|\psi - \varphi\| \leqslant \rho\}.$$

Let ε_1 be fixed such that $0 < \varepsilon_1 < \delta$. Since $K \subset X_1$ is compact, there is a $\rho_0 > 0$ such that $B[K, \rho_0] \subset X_1$. Condition (ii) implies that there are β_1 and ρ_1 with $t_0 \leqslant \beta_1 < b$, $0 < \rho_1 \leqslant \rho_0$ such that

$$\inf\{V(t, \varphi(0)): t \in [\beta_1, b), \varphi \in G \cap B[K, \rho_1]\} \geqslant -\varepsilon_1. \tag{3.2}$$

Claim I. There is β_2 with $\beta_1 \leqslant \beta_2 < b$ such that

$$\sup\{V(t, \varphi(0)): t \in [\beta_2, b), \varphi \in G \cap \text{fr}\, B[K, \rho_1]\} = -\varepsilon_2 < 0.$$

Indeed, condition (iii) implies that there is a β_2 such that $-\varepsilon_2 \leqslant 0$. If $\varepsilon_2 = 0$, then we can find a sequence $(t_n, \varphi_n) \in [\beta_1, b) \times (G \cap \text{fr}\, B[K, \rho_1])$ with $t_n \to b$, such that $V(t_n, \varphi_n(0)) \to 0$. Without loss of generality, we assume that $\varphi_n \to \psi \in \text{cl}_N G \cap \text{fr}\, B[K, \rho_1] \subset \text{cl}_N G \cap X_1$. Since $\psi \notin K$, by condition (i), we have $\psi \in G$, which contradicts condition (iii). Hence Claim I is proved.

According to condition (v), let $\eta_{\varepsilon_2} > 0$ be chosen. Condition (ii) again implies that there are β_3 and ρ with $\beta_3 \leqslant \beta_2 < b$, $0 < \rho < \rho_1$ such that

$$\inf\{V(t, \varphi(0)) : t \in [\beta_3, b), \varphi \in G \cap B[K, \rho]\} = -\varepsilon_3 > -\eta_{\varepsilon_2}. \tag{3.3}$$

Define $\mathscr{A} = B(K, \rho) = \{\psi \in X : \exists \varphi \in K, \|\psi - \varphi\| < \rho\}$, an open neighbourhood of u. Since $\text{cl}\, \mathscr{A} = B[K, \rho] \subset X$ is a closed set and f is completely continuous, if $t_x < b$, then by the continuation of solutions (see Hale [29], Theorem 3.2 in Chapter 2), there is $t_1 \geqslant t_0$ such that $x_t(t_0, \varphi) \notin \mathscr{A}$ for $t \in [t_1, t_x)$. Hence we assume $t_x = b$ and set $\gamma = \max\{t_0, \beta_3\}$, $x(t) = x_t(t_0, \varphi)$.

If $x(t) \notin B[K, \rho_1]$ for all $t \in [\gamma, b)$, then for $t_1 = \gamma$, $x(t) \notin \mathscr{A}$ for all $t \in [t_1, b)$ and we prove the theorem. Now suppose that there is $\gamma_1 \geqslant \gamma$ such that $x(\gamma_1) \in B[K, \rho_1]$.

Claim II. There is $\gamma_2 > \gamma_1$ such that $x(\gamma_2) \notin B[K, \rho_1]$.

In fact, suppose by contradiction that $x(t) \in B[K, \rho_1]$ for all $t \in [\gamma_1, b)$, define a function

$$v(t) = V(t, x(t)).$$

For all $t \in [\gamma_1, b)$, condition (iii) implies

$$v(t) < 0, \tag{3.4}$$

inequality (3.2) implies

$$v(t) \geqslant -\varepsilon_1, \tag{3.5}$$

and by condition (iv) we have

$$\dot{v}(t) \leqslant g(t, v(t)) \tag{3.6}$$

if $v(\xi) \leqslant v(t)$ for $t - \tau \leqslant \xi \leqslant t$. Let $z(t) = z(t, \gamma_1, z^0)$ be the maximal solution of

$$\dot{z} = g(t, z), \qquad z(\gamma_1) = v(\gamma_1)$$

on $[\gamma_1, t_z)$ according to condition (v). Since $v(\gamma_1) = z(\gamma_1, \gamma_1, z^0) = z^0$, $v(\xi) \leqslant z(\xi, \gamma_1, z^0) \leqslant z(\gamma_1, \gamma_1, v(\gamma_1)) = v(\gamma_1)$ for $\xi \in [\gamma_1 - r, \gamma_1]$, we have

$$\dot{v}(\gamma_1) \leqslant g(\gamma_1, v(\gamma_1)) = \dot{z}(\gamma_1, \gamma_1, z^0),$$

which means that at $t=\gamma_1$, the derivative of $v(t)-z(t,\gamma_1,z^0)$ is negative, but

$$v(\gamma_1)-z(\gamma_1,\gamma_1,z^0)=0.$$

Hence, by continuity, there exists $l>0$ sufficiently small such that

$$v(t)\leqslant z(t,\gamma_1,z^0) \qquad \text{for} \quad \gamma_1\leqslant t\leqslant \gamma_1+l.$$

We claim that

$$v(t)\leqslant z(t) \qquad \text{for all} \quad \gamma_1\leqslant t<t_z.$$

Suppose that there is $t_1>\gamma_1+l$ such that

$$v(t)>z(t,\gamma_1,z^0) \qquad \text{for} \quad t\geqslant t_1.$$

Let $z_m(t)$ be any solution of

$$\dot z=g(t,z)+\frac{1}{m}, \qquad z(\gamma_1)=z^0=v(\gamma_1), \qquad m=1,2,....$$

We know that the maximal solution $z(t)$ can be expressed as (see Lakshmikantham and Leela [46])

$$z(t)=\lim_{m\to\infty} z_m(t).$$

Then there is a sufficiently large $m>0$ such that

$$v(t)>z_m(t) \qquad \text{for some} \quad t\geqslant t_1.$$

Since $z_m(t)$ is nondecreasing and $v(\gamma_1)\leqslant z_m(\gamma_1)$, by continuity, there is $t_2\in[\gamma_1,t_1]$ such that $v(t_2)=z_m(t_2)$, $v(t)\leqslant z_m(t)\leqslant z_m(t_2)=v(t_2)$ for all $t\in[\gamma_1,t_2]$, and moreover

$$\dot v(t_2)\geqslant \dot z_m(t_2)=g(t_2,z_m(t_2))+\frac{1}{m}=g(t_2,v(t_2))+\frac{1}{m}.$$

On the other hand, since $v(t_2)=z_m(t_2)\geqslant z_m(\gamma_1)\geqslant v(\gamma_1)$, i.e. $v(\xi)\leqslant v(t_2)$ for $t_2-r\leqslant\xi\leqslant t_2$, by condition (iv) we have

$$\dot v(t_2)\leqslant g(t_2,v(t_2)),$$

a contradiction. Therefore

$$v(t)\leqslant z(t,\gamma_1,z^0)=z(t) \qquad \text{for} \quad \gamma_1\leqslant t<t_z.$$

By (3.2) and (3.5) it follows that

$$\liminf_{t \to t_z^-} z(t) = \liminf_{t \to t_z^-} v(t) \geqslant -\varepsilon_1 > -\delta,$$

which contradicts condition (v). Claim II is proved.

Thus $x(\gamma_2) \notin B[K, \rho_1]$ for some $\gamma_2 > \gamma_1$. If $x(t) \notin B(K, \rho_1)$ for all $t \geqslant \gamma_2$, then for $t_1 = \gamma_2$, $x(t) \notin \mathscr{A}$ for all $t \geqslant t_1$ and we prove the theorem. So, suppose there is $\bar{t} > \gamma_2$ such that $x(\bar{t}) \in B(K, \rho_1)$; define

$$\bar{\gamma} = \sup\{t \in [\gamma_2, \bar{t}] : x(t) \notin B[K, \rho_1]\},$$

then $\gamma_2 < \bar{\gamma} < \bar{t}$, $x(\bar{\gamma}) \in G \cap \operatorname{fr} B[K, \rho_1]$, and $x(t) \in B[K, \rho_1]$ for $t \in [\bar{\gamma}, \bar{t}]$. Similarly, $v(t)$ satisfies (3.4), (3.5), (3.6), and, by Claim I, $v(\bar{\gamma}) \leqslant -\varepsilon_2$.

Let $\bar{z}(t)$ be the maximal solution of

$$\dot{z} = g(t, z), \qquad z(\bar{\gamma}) = -\varepsilon_2$$

on $[\bar{\gamma}, t_{\bar{z}})$ according to condition (v), then $\bar{z}(t) < -\eta_{\varepsilon_2}$ for all $t \in [\bar{\gamma}, t_{\bar{z}})$. Again by the comparison technique used above, we have $v(t) \leqslant \bar{z}(t)$ for $t \geqslant \bar{\gamma}$. Now by the continuity of $v(t)$ we have $\bar{t} < t_{\bar{z}}$. Hence $v(t) \leqslant \bar{z}(t)$ for all $t \in [\bar{\gamma}, \bar{t}]$ and $v(\bar{t}) < -\eta_\varepsilon{}^2$.

Finally by (3.3) we get $x(\bar{t}) \notin \mathscr{A}$. Since $\bar{t}$ is arbitrary we have proved that

$$x(t) \notin \mathscr{A} \qquad \text{for all} \quad t \geqslant t_1 = \gamma_2.$$

This completes the proof. ∎

Remark 3.2. The assumptions in Theorem 3.1 also guarantee that the point u is not reachable through G; that is, there is no solution of RFDE (2.1) with $\varphi \in G$ such that $x_{t_1}(t_0, \varphi) = u$ for some $t_1 \in I_x$ and $x_t(t_0, \varphi) \in G$ for all $t \in [t_0, t_1)$.

COROLLARY 3.3. *In Theorem* 3.1 *if*

$$g(t, z) = -\rho(t)\, q(|z|) \tag{3.7}$$

with $q: (0, +\infty) \to (0, +\infty)$ and $\rho: J \to R$ continuous functions such that

$$\lim_{\mu \to 0} \int_\mu^{\mu_0} \frac{1}{q(\xi)}\, d\xi = +\infty \qquad (\mu_0 > 0) \tag{3.8}$$

and

$$\int_{t_0}^{b} \rho(s)\, ds = +\infty, \qquad \int_{s_1}^{s_2} \rho(s)\, ds > -m = \text{constant} \tag{3.9}$$

for all $s_1, s_2 \in J$ with $s_1 < s_2$, then the conclusion of Theorem 3.1 *holds.*

Proof. Let $z(t)$ be a maximal solution of (3.1) with (3.7) and let $t_z \leqslant b$ be such that $z(t) < 0$ for all $t \in [\sigma, t_z)$, with t_z maximal. From (3.7) and (3.9), for $t \in [\sigma, t_z)$ and $k = |z(\sigma)|$ we have

$$\int_{|z(t)|}^{k} \frac{1}{q(\xi)}\, d\xi = -\int_{t}^{\sigma} \frac{z'(y)}{q(|z(y)|)}\, dy = -\int_{\sigma}^{t} \rho(s)\, ds < m,$$

that is,

$$\int_{|z(t)|}^{\mu_0} \frac{1}{q(\xi)}\, d\xi \leqslant -\int_{\sigma}^{t} \rho(s)\, ds + \left| \int_{k}^{\mu_0} \frac{1}{q(\xi)}\, d\xi \right|. \tag{3.10}$$

Define

$$\Phi(\mu) = \int_{\mu}^{\mu_0} \frac{1}{q(\xi)}\, d\xi.$$

Observe that $\Phi \colon R^+ \to (-L, +\infty)$ is decreasing and

$$\liminf_{\mu \to 0^+} \Phi(\mu) = +\infty$$

by condition (3.8), where $L = \int_{\mu_0}^{\infty} (1/q(\xi))\, d\xi$. Now the second inequality in (3.9) implies that

$$\Phi(|z(t)|) \leqslant m + \left| \int_{k}^{\mu_0} \frac{1}{q(\xi)}\, d\xi \right|.$$

Hence there is $\eta_k = \Phi^{-1}(m + |\int_{k}^{\mu_0} (1/q(\xi))\, d\xi|) > 0$, such that

$$z(t) < -\eta_k < 0 \qquad \text{for all} \quad t \in [\sigma, t_z^-). \tag{3.11}$$

If $t_z < b$, then $\lim_{t \to t_z^-} z(t) = -\infty$. If $t_z = b$, taking limits as $t \to b^-$ in both sides of (3.10) and using the first equality in (3.9), we get

$$\lim_{t \to b^-} \int_{\mu_0}^{|z(t)|} \frac{1}{q(\xi)}\, d\xi = +\infty,$$

and by (3.11), we have $L = +\infty$ and

$$\lim_{t \to b^-} |z(t)| = +\infty.$$

Hence assumption (v) follows. ∎

Suppose that V can be chosen independent of t, i.e., $V = V(x) \colon R^n \to R$, set $V_0 = \{\varphi \in X \colon V(\varphi(0)) = 0\}$.

THEOREM 3.4. *Assume that* $f: J\times C\to R^n$ *is completely continuous,* $\mathrm{fr}_N M$ *is a compact set, and, for each* $u\in \mathrm{fr}_N(\mathrm{int}_N M)$, *there is a differentiable function* $V: R^n\to R$ *such that*

(i) $V(u(0))=0$ *for* $u\in \mathrm{fr}_N M$ *and* $V(\varphi(0))<0$ *for* $\varphi\in \mathrm{int}_N M$.

Assume that there exist a continuous function $g: J\times(-\infty, 0)\to R$ *and an open neighbourhood* X_2 *of* $V_0\cap \mathrm{fr}_N(\mathrm{int}_N M)$, *such that*

(ii) $\dot V_{(2.1)}(\varphi(0))\leqslant g(t, V(\varphi(0)))$ *if* $V(\varphi(\theta))\leqslant V(\varphi(0))$, $\theta\in[-r, 0]$ *for all* $(t, \varphi)\in J\times(X_2\cap \mathrm{int}_N M)$;

(iii) *condition* (v) *of Theorem* 3.1 *holds.*

Then the RFDE (2.1) *is uniformly persistent.*

Proof. Define the open set $G=\mathrm{int}_N M$, the closed sets $S=\mathrm{fr}_N M$ and $S^*=\mathrm{fr}_N G$. First we note that G is flow-invariant for the RFDE (2.1). Indeed, by Remark 3.2, no point of S^* is reachable through G. Since, for each $u\in S^*$, the set $V_0\cap \mathrm{fr}_N G\subset X$ is compact, Theorem 3.1 ensures that each point $u\in S^*$ is uniformly repulsive with respect to $\mathrm{int}_N M$. Thus Proposition 2.2 implies the conclusion. ∎

In Theorem 3.4, we suppose that $\mathrm{fr}_N M$, a subset of C, is compact. Here we introduce the following definition.

DEFINITION 3.5. A system of the RFDE (2.1) is *dissipative* (or equivalently *uniformly ultimately bounded*) if for any solution $x_t(t_0, \varphi)$ of (2.1) there is a constant B such that, for any $(t_0, \varphi)\in R\times C$, there is a constant $T(t_0, \varphi)>0$ such that $|x_t(t_0, \varphi)|\leqslant B$ for $t\geqslant t_0+T(t_0, \varphi)$.

Remark 3.6. The above definition of dissipativeness is equivalent to *bounded dissipativeness* in Hale [30].

Suppose that $x_t(t_0, \varphi)$ inherits the nonnegative property; that is, if $\varphi\in X$, then any solution $x_t(t_0, \varphi)$ defined for $t\geqslant 0$ satisfies $x_t^i(t_0, \varphi)\geqslant 0$ for $0\leqslant t<\infty$ and $i=1, 2, ..., n$ if $\varphi_i(s)\geqslant 0$. Let B be defined as in Definition 3.5, define $X_0=\{\varphi\in X: \|\varphi\|\leqslant B\}$, and let $N=\{\psi: \exists t\geqslant 0, \exists\varphi\in X_0, \psi=x_t(t_0, \varphi)\}$. Similar to Lemmas 3.4, 3.5 and 3.6 of Burton and Hutson [2], we know that N is flow-invariant and is compact in the $\|\cdot\|$ norm.

Let S be a closed subset of N consisting of those φ such that $\varphi_j(0)=0$ for at least one j. For the sake of biological relevance, assume that RFDE (2.1) is such that S and $M=N\backslash S$ are flow-invariant. Since $S=\mathrm{fr}_N M$ is a closed subset of compact set N, it is compact. Hence we have the following result.

THEOREM 3.7. *Suppose that* $f: J \times C \to R^n$ *is completely continuous and system* (2.1) *is dissipative. Let* $P: R^n \to R$ *be a differentiable function satisfying*

(i) $P(\varphi(0)) = 0$ *for* $\varphi \in \mathrm{fr}_N M$, $P(\varphi(0)) > 0$ *for* $\varphi \in \mathrm{int}_N M$.

Assume further that there exist a continuous function $\rho: J \to R$ *and a neighborhood* X_3 *of* $\mathrm{fr}_N M$ *such that for every* $\varphi \in X_3 \cap \mathrm{int}_N M$ *and* $t \in J$,

(ii) $\dot{P}_{(2.1)}(\varphi(0)) \geqslant P(\varphi(0))\, \rho(t)$ *if* $P(x(\xi)) \geqslant P(x(t))$ *for* $t - r \leqslant \xi \leqslant t$;

(iii) $\int_{t_0}^{b} \rho(t)\, dt = +\infty$ *and* $\int_{t_1}^{t_2} \rho(s)\, ds > -m$ *for every* $t_0 \leqslant t_1 < t_2 < b$.

Then the RFDE (2.1) *is uniformly persitent.*

Proof. Setting

$$V(\varphi(0)) = -P(\varphi(0)),$$

$$g(t, z) = -\rho(t)\, |z|,$$

then by Corollary 3.3 and Theorem 3.5 the theorem follows. ▮

Remark 2.8. Varieties of Theorem 3.7 have been proved by Hofbauer [33] and Hutson [37] for autonomous ODE, by Fernades and Zanolin [14] for nonautonomous ODE, by Hutson and Moran [39] for difference equations, by Hutson and Moran [40] for reaction–diffusion equations, and by Burton and Hutson [2] for autonomous equations with infinite delay.

Remark 3.9. If $f: J \times C \to R^n$ is T-periodic in the time variable, then the hypothesis on the function $\rho(t)$ in Theorem 3.7 is satisfied provided that

$$\langle \rho \rangle = \frac{1}{T} \int_0^T \rho(s)\, ds > 0.$$

Under minor modifications, the above results hold for the autonomous retarded functional differential equation (2.2). Especially, as in Hutson [37], Burton and Hutson [2], and Fonda [15], the differential inequalities in Theorems 3.1 and 3.4 only need to hold in $\omega(u)$ and in the omega limit set of $u \in S$, and in Theorem 3.7 assumption (ii) only needs to be true for $\varphi \in \omega(S)$, the omega limit set of the boundary S. In fact, we have the following.

THEOREM 3.10. *Suppose that* $f: C \to R^n$ *is completely continuous and system* (2.2) *is dissipative. Let* $P: R^n \to R$ *be a differentiable function satisfying*

(i) $P(\varphi(0)) = 0$ *for* $\varphi \in S$, $P(\varphi(0)) > 0$ *for* $\varphi \in \mathrm{int}_N M$.

Assume further that there exists a continuous function $\rho: J \to R$ such that for every $\varphi \in \omega(S)$

(ii) $\dot{P}_{(2.2)}(\varphi(0)) \geqslant P(\varphi(0))\, \rho(t)$ *if* $P(x(\xi)) \geqslant P(x(t))$ *for* $t-r \leqslant \xi \leqslant t$;

(iii) $\int_{t_0}^{b} \rho(t)\, dt = +\infty$ *and* $\int_{t_1}^{t_2} \rho(s)\, ds > -m$ *for every* $t_0 \leqslant t_1 < t_2 < b$.

Then RFDE (2.2) is uniformly persistent.

Remark 3.11. As in the classical Razumikhin-type theorems in stability theory (see Hale [29] and Haddock and Terjéki [28]), we do not require that the differential inequalities in the above persistence theorems hold for all initial values, but for some initial values under the restriction $P(x(\xi)) \geqslant P(x(t))$ for $t-r \leqslant \xi \leqslant t$. This kind of condition is usually called *Razumikhin condition* (see Haddock and Terjéki [28]).

If $P(\cdot)$ is a Liapunov-like functional, as in Burton and Hutson [2], we have the folowing.

THEOREM 3.12. *Suppose that $f: C \to R^n$ is completely continuous and system (2.2) is dissipative. Let $P: C \to R$ be a differentiable functional satisfying*

(i) $P(\varphi(0)) = 0$ *for* $\varphi \in S$, $P(\varphi(0)) > 0$ *for* $\varphi \in \operatorname{int}_N M$;

(ii) $\psi(u) = \dot{P}(u)/P(u) > 0$ *for* $u \in \omega(S)$.

Then the autonomous RFDE(2.2) is uniformly persistent.

4. APPLICATIONS

1. Consider the predator–prey model with delay proposed by Leung [47] (see also Kuang [42])

$$\begin{aligned} \dot{x} &= x(t)[a - bx(t) - cy(t)] \\ \dot{y} &= \alpha y(t)[x(t-\tau) - \beta], \end{aligned} \tag{4.1}$$

where a, b, c, α, β, and τ are positive constants. The initial population sizes are provided in the form

$$x(s) = \varphi(s) \geqslant 0, \qquad s \in [-\tau, 0], \qquad y(0) = y_0 \geqslant 0.$$

It is known that if

$$a - b\beta > 0 \tag{4.2}$$

then system (4.1) has a unique equilibrium $E^* = (x^*, y^*)$ where

$$x^* = \beta, \qquad y^* = \frac{a - b\beta}{c}.$$

In [46], Leung showed that, under (4.2) and other conditions, there exist periodic solutions, hence coexistence could occur. We shall see that actually (4.2) implies uniform persistence of system (4.1).

Let

$$x(t) = X(t) + x^*, \qquad y(t) = Y(t) + y^*.$$

System (4.1) can be transformed into

$$\begin{aligned} \dot{X} &= (X(t) + x^*)[-bX(t) - cY(t)] \\ \dot{Y} &= \alpha(Y(t) + y^*)\, X(t-\tau) \end{aligned} \tag{4.3}$$

Choose a Liapunov function as follows

$$V(X(t), Y(t)) = X(t) - x^* \ln\left(1 + \frac{X(t)}{x^*}\right) + \frac{c}{\alpha q}\left[\, Y(t) - y^* \ln\left(1 + \frac{Y(t)}{y^*}\right)\right],$$

where $q > 1$ is a given constant. If $X(t) = Y(t) = 0$, then $V = 0$, and V is positive definite for bounded $X(t) > \beta$, $Y(t) > (a - b\beta)/c$. We have

$$\begin{aligned} \dot{V}_{(4.3)}(X(t), Y(t)) &= \frac{X(t)}{X(t) + x^*} \cdot \dot{X} + \frac{c}{\alpha q} \frac{Y(t)}{Y(t) + y^*} \cdot \dot{Y} \\ &= -bX^2(t) - cX(t)\, Y(t) + \frac{c}{q} X(t-\tau)\, Y(t) \\ &\leqslant -bX^2(t) \end{aligned}$$

if $|X(t-\tau)| \leqslant q|X(t)|$ and $|X(t)| \geqslant \beta$, $|Y(t)| \geqslant (a - b\beta)/c$ (see Hale [29, Chapter 5]). Thus by the classical Liapunov–Razumikhin theorem for boundedness (see Hale [29], Theorem 4.3 in Chapter 5), system (4.3) and hence system (4.1) is dissipative.

THEOREM 4.1. *If* (4.2) *holds, then system* (4.1) *is uniformly persistent.*

Proof. System (4.1) has two boundary equilibria, $E_0 = (0, 0)$ and $E_1 = (a/b, 0)$. For $(x, y) \in \omega(S)$, the omega limit set of the boundary S, if $y = 0$, from the first equation of (4.1) it follows that $\pi(x, t) = x_t \to (a/b)$ as $t \to \infty$. If $x(0) = 0$ then $\pi(x, t) \equiv 0$ and $\pi(y, t) = y_t \to 0$ as $t \to \infty$. Thus the omega limit set of S is the union of $E_0 = (0, 0)$ and $E_1 = (a/b, 0)$.

Now choose $P(x, y) = x^{\alpha_1} y^{\alpha_2}$, where α_1 and α_2 are positive undetermined constants. Defining

$$\rho(t) = \alpha_1[a - bx(t) - cy(t)] + \alpha_2 \alpha[x(t) - \beta],$$

we have

$$\begin{aligned}\dot{P}(x, y) &= P(x, y) \cdot \{\alpha_1[a - bx(t) - cy(t)] + \alpha_2 \alpha[x(t-\tau) - \beta]\} \\ &\geqslant P(x, y)\, \rho(t)\end{aligned}$$

if $x(\xi) \geqslant x(t)$ for $t - \tau \leqslant \xi \leqslant t$. The choice $\alpha_1 = 1$ ensures that $\rho(t) > 0$ at E_0. If (4.2) holds, the second term in $\rho(t)$ is positive at E_1. Hence there is always a choice of α_2 to ensure $\rho(t) > 0$. The result follows from Theorem 3.10. ∎

2. For the Lotka–Volterra competition model with delay

$$\begin{aligned}\dot{x} &= x(t)(r_1 - a_1 x(t-\tau) - b_1 y(t-\tau)) \\ \dot{y}(t) &= y(t)(r_2 - a_2 x(t-\tau) - b_2 y(t-\tau))\end{aligned} \tag{4.4}$$

under initial conditions

$$\begin{aligned}x(s) &= \varphi_1(s) \geqslant 0, \qquad s \in [-\tau, 0], \qquad \varphi_1(0) > 0 \\ y(s) &= \varphi_2(s) \geqslant 0, \qquad s \in [-\tau, 0], \qquad \varphi_2(0) > 0,\end{aligned}$$

where r_i, a_i, b_i $(i = 1, 2)$ and τ are positive constants, both $\varphi_1(s)$ and $\varphi_2(s)$ are continuous on $[-\tau, 0]$. System (4.4) has three boundary equilibria $E_0 = (0, 0)$, $E_1 = (r_1/a_1, 0)$ and $E_2 = (0, r_2/b_2)$.

THEOREM 4.2. *If*

$$r_2 a_1 - r_1 a_2 > 0, \tag{4.5}$$

$$r_1 b_2 - r_2 b_1 > 0, \tag{4.6}$$

then system (4.4) *is uniformly persistent.*

Proof. It is not difficult to prove that system (4.3) is dissipative. Now for $u \in \omega(S)$, if $u_1(0) = 0$, then $\pi(u_1, t) = 0$ and $\pi(u_2, t) \to r_2/b_2$ as $t \to \infty$. If $u_2(0) = 0$, then $\pi(u_2, t) = 0$ and $\pi(u_1, t) \to r_1/a_1$ as $t \to \infty$. Thus the omega limit set of S is the union of E_0, E_1 and E_2.

Choosing $P(u) = u_1^{\alpha_1} u_2^{\alpha_2}$, where $u = (u_1, u_2)$, α_1 and α_2 are positive undetermined constants, we have

$$\psi(u) = \frac{\dot{P}(u)}{P(u)} = \alpha_1[r_1 - a_1 u_1(\theta) - b_1 u_2(\theta)] + \alpha_2[r_2 - a_2 u_1(\theta) - b_2 u_2(\theta)].$$

For any positive α_1 and α_2, ψ is always positive at E_0. If (4.5) holds, the second term of ψ is positive at E_1; if (4.6) holds the first term of ψ is positive at E_2, hence we can choose α_1 and α_2 to ensure $\psi > 0$. By Theorem 3.12, system (4.4) is uniformly persistent. ▌

Remark 4.3 It is well known that (4.5) and (4.6) are persistence conditions for system (4.4) without delay (see Hallam [32]). Recently, Cao, *et al.* [8] showed that

$$r_2 a_1 - r_1 a_2 e^{r_1 \tau} > 0, \tag{4.7}$$

$$r_1 b_2 - r_2 b_1 e^{r_2 \tau} > 0 \tag{4.8}$$

are sufficient for uniform persistence in system (4.4). Obviously (4.5) and (4.6) are sharper than (4.7) and (4.8).

3. Now consider the following delayed Gause-type predator–prey model with Michaelis–Menten functional response

$$\begin{aligned} \dot{x} &= x(t)\left[\gamma - ax(t) - \frac{by(t)}{1+cx(t)}\right] \\ \dot{y} &= y(t)\left[-v + \frac{dx(t-\tau)}{1+cx(t-\tau)}\right], \end{aligned} \tag{4.9}$$

where a, b, γ, v, and τ are positive constants, $x(t) = \varphi(t) \geqslant 0$ on $[-\tau, 0]$, $y(0) = y_0 \geqslant 0$. Zhao *et al.* [59] showed that system (4.9) is dissipative and has a Hopf bifurcation under certain restrictions on the parameters. Here we have the following result.

THEOREM 4.4 *If*

$$\frac{d\gamma}{a + c\gamma} > v, \tag{4.10}$$

then system (4.9) *is uniformly persistent.*

Proof. Similar to the proof of Theorem 4.1, we know that the omega limit set of the boundary $S = \text{fr}_N M$ is the union of the boudary equilibria $E_0 = (0, 0)$ and $E_1 = (\gamma/a, 0)$. We choose $P(x(t), y(t)) = x(t)^{\alpha_1} y(t)^{\alpha_2}$, where α_1 and α_2 are undetermined positive constants, then

$$\begin{aligned} \dot{P}(x(t), y(t)) = P(x(t), y(t)) &\left[\alpha_1\left(\gamma - ax(t) - \frac{by(t)}{1+cx(t)}\right)\right. \\ &\left. + \alpha_2\left(-v + \frac{dx(t-\tau)}{1+cx(t-\tau)}\right)\right] \\ \geqslant P(x(t), y(t))\, \rho(t) & \end{aligned}$$

if $x(\xi) \geqslant x(t)$ for $t-\tau \leqslant \xi \leqslant t$, where

$$\rho(t)=\alpha_1\left(\gamma-ax(t)-\frac{by(t)}{1+cx(t)}\right)+\alpha_2\left(-v+\frac{dx(t)}{1+cx(t)}\right).$$

The choice $\alpha_1=1$ ensures that $\rho(t)$ is positive at E_0. If (4.10) holds, the second term of $\rho(t)$ is positive at E_1. Hence there is always a choice of α_2 such that $\rho(t)>0$. The result again follows from Theorem 3.10. ▌

Note that condition (4.10) is equivalent to the uniform persistent criterion for system (4.9) without delay; that is the delay in system (4.9) is also "harmless" for uniform persistence.

5. Discussion

By using Liapunov-like functions, Razumikhin technique, and differential inequalities, we have obtained criteria for the uniform persistence of retarded functional differential equations. These criteria are quite general in their applicability to ecological systems. The conditions for uniform persistence do not require any special assumptions on the interaction terms and the conclusions are very precise in specific cases.

The examples described in Section 4 show that for some models uniform persistence criteria are exactly the conditions used previously to establish the existence of an interior equilibrium. Similar results were obtained by Cantrell *et al.* [7] for reaction–diffusion systems. It is well known (see Butler *et al.* [5] and Hutson [38]) that uniform persistence actually implies the presence of an interior equilibrium. Generally, however, possessing an interior equilibrium is neither necessary nor sufficient for persistence. We see that in a rather wide range of problems the conditions which guarantee the existence of an interior equilibrium are in fact enough to ensure uniform persistence.

For the Lotka–Volterra competition model with delay and the delayed Gause-type predator–prey model, our results indicate that the criteria for uniform persistence in the delay models are equivalent to criteria for uniform persistence in the ODE cases. Hence, as shown by Wang and Ma [52], the delay is "harmless" for uniform persistence.

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