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Normal Forms for an Age Structured Model

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Abstract In this paper, we apply the recently developed normal form theory for abstract Cauchy problems with non-dense domain in Liu et al. (J Diff Equ 257:921–1011, 2014) to study normal forms for an age structured model. We provide detailed computations for the Taylor's expansion of the reduced system on the center manifold, from which explicit formulae are given to determine the direction of the Hopf bifurcation and the stability and amplitude of the bifurcating periodic solutions.

Keywords Normal form · Non-densely defined Cauchy problem · Age structured model · Hopf bifurcation · Periodic solution

Mathematics Subject Classfication 34K15 · 34C20 · 37L10 · 58F36

1 Introduction

In modeling some biological and epidemiological processes, the age variable (age of the individual, chronological time since infection or time since cell division) plays a key role in determining the birth, growth and death rates of the populations and their interactions with

Dedicated to Professor John Mallet-Paret on the occasion of his 60th birthday.

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each other. Age structured models, described by hyperbolic partial differential equations, have been studied by many researchers (see the monographs of Cushing [6], Diekmann and Heesterbeek [7], Hoppenstead [14], Iannelli [15], Metz and Diekmann [28], Thieme [35], Webb [37], and the references cited therein). Various approaches have been developed to study age structured models, such as (a) characteristics method (Webb [37], Metz and Diekmann [28], Iannelli [15]), (b) variational method (Anita [1]), and (c) integrated semigroups method (Thieme [33,34,36], Magal and Ruan [23,24]).

Classical results on age-structured models focus on the existence, bounded and stability of solutions. Recently, great attention has been paid to the nonlinear dynamics of such models. It has been shown that some age-structured models exhibit non-trivial periodic solutions induced by Hopf bifurcation (see Prüss [30], Cushing [5], Swart [32], Kostava and Li [17], and Bertoni [2]). Since age-structured models can be rewritten as abstract semilinear equations with non-dense domain (Thieme [33,34,36], Magal and Ruan [23,24]), Magal and Ruan [25] developed the center manifold theory for abstract semilinear Cauchy problems with non-dense domain and applied the results to consider Hopf bifurcation of a specific age-structured model of the following form:

$$\begin{cases} \dfrac{\partial u(t,a)}{\partial t} + \dfrac{\partial u(t,a)}{\partial a} = -\mu u(t,a), \ a \in (0,+\infty), \\ u(t,0) = \alpha h\left(\int_0^{+\infty} \gamma(a) u(t,a) da\right), \\ u(0,.) = \varphi_0 \in L^1_+((0,+\infty);\mathbb{R}), \end{cases} \tag{1.1}$$

where $u(t,a)$ denotes the density of a population at time t with age a, $\mu > 0$ is the mortality rate of the population, $\alpha\gamma(a)$ is the fertility rate at a, and the function $h(\cdot)$ describes some limitation for the reproduction. Based on this study, Liu et al. [19] established a Hopf bifurcation theorem for abstract Cauchy problems with non-dense domain and, as an application, obtained a Hopf bifurcation theorem for general age structured models. Center-unstable manifolds for non-densely defined semilinear Cauchy problems were studied in Liu, Magal and Ruan [20].

Normal form theory is very useful in simplifying the forms of equations restricted on the center manifolds when study the nonlinear dynamics, such as the existence of bifurcations and periodic solutions. Normal form theory has been well-developed for various types of equations, including ordinary differential equations (Guckenheimer and Holmes [12], Chow et al. [4]), partial differential equations (Kokubu [16], Eckmann et al. [9]), functional differential equations (Faria and Magalhães [10,11]), etc. More recently, a normal form theory has been developed by Liu et al. [21] for the non-densely defined abstract Cauchy problems.

We already knew (Magal and Ruan [25]) that when $\tau > 0$, system (1.1) undergoes a Hopf bifurcation at the positive equilibrium. The goal of this paper is to apply the normal form theory developed in Liu et al. [21] to the age-structured model (1.1) to determine the direction of the Hopf bifurcation and study the stability and amplitude of the bifurcating periodic solutions. Note that two approaches were developed in Liu et al. [21]: (a) calculating the Taylor expansion of the reduced system of (1.1) on the center manifold, and (b) evaluating the normal forms of (1.1) restricted on the center manifold directly. Equation (1.1) was studied in [21] by using the second approach. In this paper we will use the first approach to study (1.1). Namely, we will calculate the Taylor expansion of the reduced system of (1.1) on the center manifold, determine the direction of the Hopf bifurcation, and study the stability and amplitude of the bifurcating periodic solutions.

At first, we make the following assumptions.

Assumption 1.1 Assume that $\mu > 0, \alpha > 0, \gamma \in L^{\infty}(0, +\infty)$ is a map defined by

$$\gamma(a) = (a-\tau)^n e^{-\varsigma(a-\tau)} 1_{[\tau,+\infty)}(a) = \begin{cases} (a-\tau)^n e^{-\varsigma(a-\tau)}, & \text{if } a \geq \tau \\ 0, & \text{otherwise,} \end{cases}$$

where $\tau \geq 0, \varsigma \geq 0, n \in \mathbb{N}$, and assume that $\varsigma > 0$ whenever $n \geq 1$. The map $h : \mathbb{R} \to \mathbb{R}$ is defined by

$$h(x) = x \exp(-\beta x), \ \forall x \in \mathbb{R},$$

where $\beta > 0$.

In order to explore the asymptotic behavior of system (1.1), we observe some basic facts. Set

$$\alpha_0 := \left(\int_0^{+\infty} \gamma(a) e^{-\mu a} da \right)^{-1}.$$

Then for each $\alpha > \alpha_0$, there exists a unique positive equilibrium

$$\overline{u}_\alpha(a) = e^{-\mu a} \bar{C}, \quad \forall a \geq 0,$$

with

$$\bar{C} = \frac{\ln\left(\alpha \int_0^{+\infty} \gamma(a) e^{-\mu a} da\right)}{\beta \int_0^{+\infty} \gamma(a) e^{-\mu a} da}.$$

So α_0 is the first bifurcation point with respect to the parameter α.

When $n = \varsigma = \tau = 0$, the above system can be rewritten as the following simple ordinary differential equation

$$\frac{dU(t)}{dt} = \alpha h(U(t)) - \mu U(t).$$

In this case the asymptotic behavior is fairly simple since the positive equilibrium is globally asymptotic stable (when it exists) and no oscillations occur around the positive equilibrium. This indicates that the oscillations around the positive equilibrium depend strongly on the shape of the function γ.

Set

$$X_k(t) := \int_\tau^{+\infty} (a-\tau)^k e^{-\varsigma a} u(t,a) da, \ \forall k = 0, \ldots, n.$$

Then (by using classical time differentiability results for age structured models), we obtain for $t \geq \tau$ the following system of delay differential equations:

$$\begin{cases} \dfrac{dX_0(t)}{dt} = e^{-(\varsigma+\mu)\tau} \alpha h\left(e^{\varsigma\tau} X_n(t-\tau)\right) - (\varsigma+\mu) X_0(t), \\ \dfrac{dX_k(t)}{dt} = k X_{k-1}(t) - (\varsigma+\mu) X_k(t), \quad \forall k = 1, \ldots, n. \end{cases} \tag{1.2}$$

The observation is interesting itself since we have reduced the scalar age structured model into a class of cyclic feedback systems with delay. We refer to Mallet-Paret and Sell [26,27] for a nice survey and more results on such systems. Here the feedback h is only locally monotone, and theory of monotone cyclic feedback systems can probably be applied locally around the positive equilibrium. We will not study the local oscillating properties of the system (1.2). Nevertheless, this observation can probably be useful in understanding the qualitative properties of system (1.1).

Now consider the map $g(x) := n! \frac{e^{-(\varsigma+\mu)\tau} \alpha h(x)}{(\varsigma+\mu)^{n+1}}$ and define $X_{\max} \in (0, +\infty)$ at which $g(x)$ attains its maximum. Then we can apply the theory of monotone delay differential equations

to (1.2) (see Smith [31] for more results on the subject), or applying the same argument to the age structured model (1.1) we obtain the following result (the proof is left to the reader).

Proposition 1.2 *Let Assumption 1.1 be satisfied and assume that*

$$\overline{X}_n := \int_{\tau}^{+\infty} (a-\tau)^n e^{-\varsigma a} \overline{u}(a) da \leq X_{\max}.$$

Then the positive equilibrium of system (1.1) is globally asymptotically stable in $L^1_+(0,+\infty)\setminus\{0\}$.

Therefore, in order to obtain some undamped oscillations for system (1.1), we need the two following ingredients

$$\tau > 0 \text{ and } h'\left(\overline{X}_n\right) < 0.$$

From the results in Magal and Ruan [25, Chapter 5], we know that when $\tau > 0$, there exists $\alpha_1 > \alpha_0$, such that the positive equilibrium $\overline{u}_\alpha$ undergoes a Hopf bifurcation (see also [3] for more detailed computations). Now assume that we can compute the normal form for the FDE (1.2) by using one of the above mentioned methods. Then the stability of the bifurcating periodic orbits of system (1.1) for the L^1 topology will not follow from such a study. This simple remark shows that we need to compute the normal form for system (1.1) in order to derive some qualitative properties related to the original topology in L^1.

The paper is organized as follows. In Sect. 2 we reformulate system (1.1) as an abstract non-densely defined Cauchy problem and present the Hopf bifurcation results obtained in [3,25]. In Sect. 3, we apply the normal form theory to system (1.1) and compute the third Taylor's expansion of the reduced system on the center manifold, from which explicit formulae are given to determine the direction of the Hopf bifurcation and the stability and amplitude of the bifurcated periodic solutions.

2 Existence of Hopf Bifurcation

In this section, we reformulate the PDE (1.1) as a non-densely defined Cauchy problem. Following the approach introduced by Thieme [34], we consider

$$X := \mathbb{R} \times L^1((0,+\infty), \mathbb{R})$$

endowed with the product norm

$$\left\| \begin{pmatrix} \chi \\ \varphi \end{pmatrix} \right\| = |\chi| + \|\varphi\|_{L^1}.$$

Let $A : D(A) \subset X \to X$ be the linear operator on X defined by

$$A \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi(0) \\ -\varphi' - \mu\varphi \end{pmatrix}$$

with

$$D(A) = \{0_{\mathbb{R}}\} \times W^{1,1}((0,+\infty), \mathbb{R}).$$

Then A is non-densely defined and

$$\overline{D(A)} = \{0_{\mathbb{R}}\} \times L^1((0,+\infty), \mathbb{R}) := X_0.$$

Let $H : X_0 \to X$ be the map defined by

$$H\left(\begin{pmatrix} 0 \\ \varphi \end{pmatrix}\right) = \begin{pmatrix} h\left(\int_0^{+\infty} \gamma(a)\varphi(a)da\right) \\ 0_{L^1} \end{pmatrix}.$$

Then by identifying $u(t,.)$ to $v(t) = \begin{pmatrix} 0_{\mathbb{R}} \\ u(t,.) \end{pmatrix} \in X_0$, the system (1.1) can be reformulated as the following non-densely defined abstract Cauchy problem

$$\frac{dv(t)}{dt} = Av(t) + \alpha H(v(t)), \text{ for } t \geq 0, \ v(0) = \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in \overline{D(A)}. \tag{2.1}$$

Here the solution of system (2.1) is understood as an integrated solution, that is, $v \in C\left([0,\tau], \overline{D(A)}\right)$ and satisfies

$$\int_0^t \begin{pmatrix} 0 \\ u(s,.) \end{pmatrix} ds \in D(A),$$

and for each $t \in [0,\tau]$,

$$\begin{pmatrix} 0 \\ u(t,.) \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi(.) \end{pmatrix} + A\int_0^t \begin{pmatrix} 0 \\ u(s,.) \end{pmatrix} ds + \int_0^t \alpha H\left(\begin{pmatrix} 0 \\ u(s,.) \end{pmatrix}\right) ds.$$

We have

$$\rho(A) = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > -\mu\},$$

and for each $\lambda \in \rho(A)$,

$$\begin{aligned} &(\lambda I - A)^{-1}\begin{pmatrix} \chi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \\ &\Leftrightarrow \varphi(a) = e^{-(\lambda+\mu)a}\chi + \int_0^a e^{-(\lambda+\mu)(a-s)}\psi(s)ds. \end{aligned}$$

It is readily checked that

$$\left\|(\lambda I - A)^{-1}\right\| \leq \frac{1}{\lambda+\mu}, \forall \lambda > -\mu,$$

so A is a Hille-Yosida operator. Now we consider A_0, the part of A in X_0, which is defined by

$$A_0\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = A\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ -\varphi' - \mu\varphi \end{pmatrix}, \forall \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in D(A_0),$$

and

$$D(A_0) = \left\{\begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in \{0_{\mathbb{R}}\} \times W^{1,1}((0,+\infty), \mathbb{R}) : \varphi(0) = 0\right\}.$$

The linear operator A_0 is the infinitesimal generator of a strongly continuous semigroup $\left\{T_{A_0}(t)\right\}_{t\geq 0}$ of bounded linear operators on X_0, which is defined by

$$T_{A_0}(t)\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \widehat{T}_{\widehat{A}}(t)\varphi \end{pmatrix}$$

with

$$\widehat{T}_{\widehat{A}}(t)(\varphi)(a) = \begin{cases} e^{-\mu t}\varphi(a-t), & \text{if } a - t \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then we consider $\{S_A(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ the integrated semigroup generated by A. That is the family of bounded linear operators on X, such that for each $x = \begin{pmatrix} \chi \\ \psi \end{pmatrix} \in X$, the map $t \to S_A(t)x$ is an integrated solution of the Cauchy problem

$$\frac{dS_A(t)x}{dt} = AS_A(t)x + x, \text{ for } t \geq 0, \text{ and } S_A(0)x = 0.$$

Thus, we deduce that

$$S_A(t)\begin{pmatrix} \chi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ L(t)\chi \end{pmatrix} + \int_0^t T_{A_0}(l)\begin{pmatrix} 0 \\ \psi \end{pmatrix} dl$$

with

$$L(t)(\chi)(a) = \begin{cases} 0, & \text{if } a - t \geq 0, \\ e^{-\mu a}\chi, & \text{if } a - t \leq 0. \end{cases}$$

Finally define a convolution

$$(S_A * f)(t) = \int_0^t S_A(t-s)f(s)ds$$

for $f \in L^1(0,\tau;X)$. Then for each $f \in L^1(0,\tau;X)$, the map $t \to (S_A * f)(t)$ belongs to $C^1([0,\tau], X_0) \cap C([0,\tau], D(A))$, and

$$(S_A \diamond f)(t) := \frac{d}{dt}(S_A * f)(t)$$

satisfies

$$(S_A \diamond f)(t) = A\int_0^t (S_A \diamond f)(l)dl + \int_0^t f(l)dl, \ \forall t \in [0,\tau].$$

Then the integrated solution of system (2.1) is unique and is given by

$$v(t) = T_{A_0}(t)\begin{pmatrix} 0 \\ \varphi \end{pmatrix} + (S_A \diamond \alpha H(v(.)))(t), \ \forall t \geq 0.$$

Set

$$X_{0+} := \{0\} \times L^1_+(0,+\infty).$$

Since $h : [0,+\infty) \to [0,+\infty)$ is Lipschitz continuous, we have the following results (see Thieme [34] or Magal [22]).

Proposition 2.1 *Let Assumption 1.1 be satisfied. Then for each $\alpha \geq 0$, there exists a family of continuous maps $\{U_\alpha(t)\}_{t\geq 0}$ on X_{0+} such that for each $x = \begin{pmatrix} 0 \\ \varphi \end{pmatrix} \in X_{0+}$, the map $t \to U_\alpha(t)x$ is the unique integrated solution of (2.1), that is,*

$$U_\alpha(t)x = x + A\int_0^t U_\alpha(s)xds + \int_0^t \alpha H(U_\alpha(l)x)dl, \ \forall t \geq 0,$$

or equivalently,

$$U_\alpha(t)x = T_{A_0}(t)x + \frac{d}{dt}(S_A * \alpha H(U_\alpha(.)x))(t), \ \forall t \geq 0.$$

Moreover, $\{U_\alpha(t)\}_{t\geq 0}$ is a continuous semiflow, that is,

$$U_\alpha(t)U_\alpha(s) = U_\alpha(t+s), \ \forall t,s \geq 0, \quad U_\alpha(0) = Id,$$

and the map $(t,x) \to U_\alpha(t)x$ is continuous from $[0,+\infty) \times X_{0+}$ into X_{0+}.

The positive equilibrium solution of (2.1) is given for each $\alpha > \alpha_0$ by

$$\bar{v}_\alpha = \begin{pmatrix} 0 \\ \bar{u}_\alpha \end{pmatrix}.$$

The linearized system of (2.1) around $\bar{v}_\alpha$ is

$$\frac{dw(t)}{dt} = Aw(t) + \alpha DH(\bar{v}_\alpha)w(t) \text{ for } t \geq 0,\ \ v(t) \in X_0,$$

where

$$\alpha DH(\bar{v}_\alpha)\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} \eta(\alpha)\int_0^{+\infty}\gamma(a)\varphi(a)da \\ 0 \end{pmatrix}$$

and

$$\eta(\alpha) = \alpha h'\left(\int_0^{+\infty}\gamma(a)\bar{u}_\alpha(a)da\right) = \frac{1 - \ln\left(\alpha\int_0^{+\infty}\gamma(a)\,e^{-\mu a}da\right)}{\int_0^{+\infty}\gamma(a)\,e^{-\mu a}da}.$$

To simplify the notation, we set

$$B_\alpha x = Ax + \alpha DH(\bar{v}_\alpha)x \text{ with } D(B_\alpha) = D(A).$$

Let $\{T_{B_\alpha}(t)\}_{t\geq 0}$ be the linear C_0-semigroup on X generated by B_α.

The essential growth bound $\omega_{0,\text{ess}}(B_\alpha) \in [-\infty, +\infty)$ of B_α is defined by

$$\omega_{0,\text{ess}}(B_\alpha) := \lim_{t\to+\infty}\frac{\ln\left(\left\|T_{B_\alpha}(t)\right\|_{\text{ess}}\right)}{t}.$$

To conclude this section we summarize some results obtained in Magal and Ruan [25] and Chu et al. [3].

Lemma 2.2 *Let Assumption 1.1 be satisfied. Then the linear operator $B_\alpha : D(A) \subset X \to X$ is a Hille-Yosida operator and*

$$\omega_{0,\text{ess}}(B_\alpha) \leq -\mu.$$

Set

$$\Omega := \{\lambda \in \mathbb{C} : \text{Re}(\lambda) > -\mu\}.$$

Recall that the esolvent set of B_α is defined by $\rho(B_\alpha) = \{\lambda \in \mathbb{C} : \lambda I - B_\alpha \text{ is invertible}\}$. Denote by $\sigma(B_\alpha) := \mathbb{C}\backslash\rho(B_\alpha)$ the spectrum of B_α. By using the above lemma, we know that for each $\lambda \in \Omega$,

$$\lambda \in \sigma(B_\alpha) \Leftrightarrow \Delta(\alpha, \lambda) = 0,$$

where the characteristic function is

$$\Delta(\alpha, \lambda) := 1 - \eta(\alpha)\int_0^{+\infty}\gamma(a)\,e^{-(\lambda+\mu)a}da \text{ for each } \lambda \in \Omega.$$

Moreover, by using the fact that $\gamma(a) = (a-\tau)^n e^{-\varsigma(a-\tau)}1_{[\tau,+\infty)}(a)$, for each $\lambda \in \Omega$, the characteristic equation

$$\Delta(\alpha, \lambda) = 0$$

is equivalent to

$$1 = n!\eta(\alpha)\frac{e^{-(\lambda+\mu)\tau}}{(\varsigma + \lambda + \mu)^{n+1}}. \tag{2.2}$$

In the following, we regard α as the bifurcation parameter in considering the Hopf bifurcation of system (2.1).

Proposition 2.3 *Let Assumption 1.1 be satisfied and assume that $\tau > 0$. Then the characteristic equation (2.2) with $\alpha = \alpha_k$, $k \in \mathbb{N} \setminus \{0\}$, has a unique pair of purely imaginary roots $\pm i\omega_k$, where*

$$\alpha_k = \frac{(\varsigma + \mu)^{n+1}}{n! e^{-\mu\tau}} \exp\left(1 + \left(\frac{\sqrt{(\varsigma + \mu)^2 + \omega_k^2}}{\beta + \mu}\right)^{n+1}\right)$$

and $\omega_k > 0$ is the unique solution of

$$-\left(\omega\tau + (n+1)\arctan\frac{\omega}{\varsigma + \mu}\right) = \pi - 2k\pi.$$

By proving in addition the transversality condition, we obtain the following result on Hopf bifurcation (Magal and Ruan [25]).

Theorem 2.4 (Hopf Bifurcation) *Let Assumption 1.1 be satisfied and assume that $\tau > 0$. Then there exists a positive sequence $\{\alpha_k\}$, $k = 1, 2, \ldots$, where α_k is defined in Proposition 2.3, such that the age structured model (1.1) undergoes a Hopf bifurcation at the equilibrium $u = \overline{u}_{\alpha_k}$. In particular, a non-trivial periodic solution bifurcates from the equilibrium $u = \overline{u}_{\alpha_k}$ when $\alpha = \alpha_k$.*

3 Direction and Stability of Hopf Bifurcation

In this section we study the direction and stability of the Hopf bifurcation by applying the normal form theory developed in Liu, Magal and Ruan [21] to the Cauchy problem (2.1).

3.1 Spectral Decomposition

We first include the parameter α into the state variable. Consider the system

$$\begin{cases} \dfrac{d\alpha(t)}{dt} = 0, \\ \dfrac{dv(t)}{dt} = Av(t) + \alpha(t)\, H(v(t)), \\ \alpha(0) = \alpha_0 \in \mathbb{R}, \ v(0) = v_0 \in X_0. \end{cases} \tag{3.1}$$

Making a change of variables

$$v(t) = \widehat{v}(t) + \overline{v}_\alpha,$$

we obtain the system

$$\begin{cases} \dfrac{d\alpha(t)}{dt} = 0, \\ \dfrac{d\widehat{v}(t)}{dt} = A\widehat{v}(t) + \alpha(t)H(\widehat{v}(t) + \overline{v}_\alpha) - \alpha(t)H(\overline{v}_\alpha). \end{cases}$$

Now set

$$\alpha = \widehat{\alpha} + \alpha_k,$$

we obtain

$$\begin{cases} \dfrac{d\widehat{\alpha}(t)}{dt} = 0, \\ \dfrac{d\widehat{v}(t)}{dt} = A\widehat{v}(t) + \widehat{H}(\widehat{\alpha}, \widehat{v}), \end{cases} \tag{3.2}$$

where

$$\widehat{H}\left(\widehat{\alpha},\widehat{v}\right):=\left(\widehat{\alpha}+\alpha_k\right)\left[H(\widehat{v}(t)+\overline{v}_{(\widehat{\alpha}+\alpha_k)})-H(\overline{v}_{(\widehat{\alpha}+\alpha_k)})\right].$$

We have

$$\partial_{\widehat{v}}\widehat{H}(\widehat{\alpha},\widehat{v})\,(w)=\left(\widehat{\alpha}+\alpha_k\right)DH(\widehat{v}+\overline{v}_{(\widehat{\alpha}+\alpha_k)})\,(w)$$

and

$$\begin{aligned}\partial_{\widehat{\alpha}}\widehat{H}(\widehat{\alpha},\widehat{v})\,(\widetilde{\alpha})=\widetilde{\alpha}\Big\{&H(\widehat{v}+\overline{v}_{(\widehat{\alpha}+\alpha_k)})-H(\overline{v}_{(\widehat{\alpha}+\alpha_k)})\\&+\left(\widehat{\alpha}+\alpha_k\right)\Bigg[DH(\widehat{v}+\overline{v}_{(\widehat{\alpha}+\alpha_k)})\left(\frac{d\overline{v}_{(\widehat{\alpha}+\alpha_k)}}{d\widehat{\alpha}}\right)\\&-DH(\overline{v}_{(\widehat{\alpha}+\alpha_k)})\left(\frac{d\overline{v}_{(\widehat{\alpha}+\alpha_k)}}{d\widehat{\alpha}}\right)\Bigg]\Big\}.\end{aligned}$$

So

$$\partial_{\widehat{v}}\widehat{H}(0,0)=\alpha_k DH(\overline{v}_{\alpha_k})\text{ and }\partial_{\widehat{\alpha}}\widehat{H}(0,0)=0.$$

Set

$$\mathcal{X}=\mathbb{R}\times X,\,\mathcal{X}_0=\mathbb{R}\times\overline{D(A)}.$$

Consider the linear operator $\mathcal{A}:D(\mathcal{A})\subset\mathcal{X}\to\mathcal{X}$ defined by

$$\mathcal{A}\begin{pmatrix}\widehat{\alpha}\\\widehat{v}\end{pmatrix}=\begin{pmatrix}0\\\left(A+\alpha_k DH\left(\overline{v}_{\alpha_k}\right)\right)\widehat{v}\end{pmatrix}=\begin{pmatrix}0\\B_{\alpha_k}\widehat{v}\end{pmatrix}$$

with

$$D(\mathcal{A})=\mathbb{R}\times D(A),$$

and the map $F:\overline{D(A)}\to\mathcal{X}$ defined by

$$F\begin{pmatrix}\widehat{\alpha}\\\widehat{v}\end{pmatrix}=\begin{pmatrix}0\\W\begin{pmatrix}\widehat{\alpha}\\\widehat{v}\end{pmatrix}\end{pmatrix},$$

where $W:\overline{D(A)}\to\mathcal{X}$ is defined by

$$W\begin{pmatrix}\widehat{\alpha}\\\widehat{v}\end{pmatrix}:=\left(\widehat{\alpha}+\alpha_k\right)\left[H(\widehat{v}+\overline{v}_{(\widehat{\alpha}+\alpha_k)})-H(\overline{v}_{(\widehat{\alpha}+\alpha_k)})\right]-\alpha_k DH\left(\overline{v}_{\alpha_k}\right)(\widehat{v}).$$

Then we have

$$F\begin{pmatrix}0\\0\end{pmatrix}=0\text{ and }DF\begin{pmatrix}0\\0\end{pmatrix}=0.$$

Now we can reformulate system (3.2) as the following system

$$\frac{dw(t)}{dt}=\mathcal{A}w(t)+F\left(w(t)\right),\quad w(0)=w_0\in\overline{D(\mathcal{A})}.\tag{3.3}$$

The following three lemmas are obtained in Magal and Ruan [25].

Lemma 3.1 *Let Assumption 1.1 be satisfied and assume that $\tau>0$. Then*

$$\sigma\left(B_{\alpha_k}\mid_{\widehat{\Pi}_c(X)}\right)=\{i\omega_k,-i\omega_k\},\quad\sigma\left(B_{\alpha_k}\mid_{(I-\widehat{\Pi}_c)(X)}\right)=\sigma\left(B_{\alpha_k}\right)\setminus\{i\omega_k,-i\omega_k\}$$

with

$$\widehat{\Pi}_{\pm i\omega_k}\begin{pmatrix}\delta\\\psi\end{pmatrix}=\begin{pmatrix}0\\\frac{d\Delta(\alpha,\pm i\omega_k)}{d\lambda}^{-1}\left[\delta+\int_0^{+\infty}\int_s^{+\infty}\gamma\left(l\right)e^{-(\pm i\omega_k+\mu)(l-s)}dl\psi(s)ds\right]e^{-(\pm i\omega_k+\mu)\cdot}\end{pmatrix}$$

and

$$\widehat{\Pi}_c \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}^{-1} e^{-(i\omega_k+\mu).} + \frac{d\Delta(\alpha_k, -i\omega_k)}{d\lambda}^{-1} e^{-(-i\omega_k+\mu).} \end{pmatrix}.$$

Set

$$\begin{cases} b_1 := e^{-(i\omega_k+\mu).} + e^{-(-i\omega_k+\mu).} \\ b_2 := \frac{e^{-(i\omega_k+\mu).} - e^{-(-i\omega_k+\mu).}}{i} \end{cases} \Leftrightarrow \begin{cases} 2e^{-(i\omega_k+\mu).} = b_1 + ib_2 \\ 2e^{-(-i\omega_k+\mu).} = b_1 - ib_2. \end{cases}$$

We also have

$$\Delta(\alpha_k, i\omega_k) = 0 \Leftrightarrow \eta(\alpha_k) \int_0^{+\infty} \gamma(a) e^{-(i\omega_k+\mu)a} da = 1$$

and

$$\begin{aligned} \frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda} &= \eta(\alpha_k) \int_0^{+\infty} a\gamma(a) e^{-(i\omega_k+\mu)a} da \\ &= \eta(\alpha_k) \int_0^{+\infty} (a-\tau)\gamma(a) e^{-(i\omega_k+\mu)a} da \\ &\quad + \tau\eta(\alpha_k) \int_0^{+\infty} \gamma(a) e^{-(i\omega_k+\mu)a} da \\ &= \eta(\alpha_k) \int_\tau^{+\infty} (a-\tau)^{n+1} e^{-\varsigma(a-\tau)} e^{-(i\omega_k+\mu)a} da + \tau \\ &= (n+1)!\eta(\alpha) \frac{e^{-(i\omega_k+\mu)\tau}}{(\varsigma + i\omega_k + \mu)^{n+2}} + \tau \\ &= \frac{(n+1)}{(\varsigma + i\omega_k + \mu)} + \tau. \end{aligned}$$

So

$$\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda} = \frac{(n+1)}{(\varsigma + i\omega_k + \mu)} + \tau = \frac{(n+1)}{\sqrt{(\varsigma+\mu)^2 + \omega_k^2}} [(\varsigma+\mu) - i\omega_k] + \tau.$$

Therefore, we have

$$\begin{aligned} \mathrm{Re}\left(\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}\right) &= \frac{(n+1)(\varsigma+\mu)}{\sqrt{(\varsigma+\mu)^2 + \omega_k^2}} + \tau, \\ \mathrm{Im}\left(\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}\right) &= -\frac{(n+1)\omega_k}{\sqrt{(\varsigma+\mu)^2 + \omega_k^2}}, \end{aligned}$$

and

$$\frac{d\Delta(\alpha_k, -i\omega_k)}{d\lambda} = \overline{\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}}.$$

Moreover

$$\widehat{\Pi}_c \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left| \frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda} \right|^{-2} \begin{pmatrix} 0 \\ \mathrm{Re}\left(\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}\right) b_1 + \mathrm{Im}\left(\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}\right) b_2 \end{pmatrix}.$$

The set $\left\{ \binom{0}{b_1}, \binom{0}{b_2} \right\}$ is a basis of $\mathcal{X}_c := \widehat{\Pi}_c(X)$. Observe that by construction we have

$$B_{\alpha_k} \begin{pmatrix} 0 \\ e^{-(\pm i\omega_k+\mu).} \end{pmatrix} = \begin{pmatrix} 0 \\ -\left(\frac{d}{da} + \mu I\right) e^{-(\pm i\omega_k+\mu).} \end{pmatrix} = \pm i\omega_k \begin{pmatrix} 0 \\ e^{-(\pm i\omega_k+\mu).} \end{pmatrix},$$

so

$$B_{\alpha_k}\begin{pmatrix}0\\ b_1\end{pmatrix}=-\omega_k\begin{pmatrix}0\\ b_2\end{pmatrix},\ B_{\alpha_k}\begin{pmatrix}0\\ b_2\end{pmatrix}=\omega_k\begin{pmatrix}0\\ b_1\end{pmatrix},$$

and we obtain that the matrix of $B_{\alpha_k}\mid_{\widehat{\Pi}_c(X)}$ with respect to basis $\left\{\binom{0}{b_1},\binom{0}{b_2}\right\}$ is

$$B_{\alpha_k}\mid_{\widehat{\Pi}_c(X)}=\begin{pmatrix}0 & \omega_k\\ -\omega_k & 0\end{pmatrix}.$$

Set

$$\widehat{\Pi}_h:=\left(I-\widehat{\Pi}_c\right).$$

Lemma 3.2 *Let Assumption 1.1 be satisfied and assume that $\tau>0$. Then we have*

$$\begin{aligned}\widehat{\Pi}_h\begin{pmatrix}1\\ 0\end{pmatrix}&=\begin{pmatrix}1\\ -\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}^{-1}e^{-(i\omega_k+\mu)\cdot}-\frac{d\Delta(\alpha_k,-i\omega_k)}{d\lambda}^{-1}e^{-(-i\omega_k+\mu)\cdot}\end{pmatrix}\\ &=\begin{pmatrix}1\\ -\left|\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}\right|^{-2}\left[\mathrm{Re}\left(\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}\right)b_1+\mathrm{Im}\left(\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}\right)b_2\right]\end{pmatrix}.\end{aligned}$$

For each $\lambda\in i\mathbb{R}\setminus\{-i\omega_k,i\omega_k\}$,

$$\begin{aligned}&\left(\lambda I-B^{\mathbb{C}}_{\alpha_k}\mid_{\widehat{\Pi}_h(X)}\right)^{-1}\widehat{\Pi}_h\begin{pmatrix}1\\ 0\end{pmatrix}\\ &=\begin{pmatrix}0\\ -\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}^{-1}\frac{e^{-(i\omega_k+\mu)\cdot}}{(\lambda-i\omega_k)}-\frac{d\Delta(\alpha_k,-i\omega_k)}{d\lambda}^{-1}\frac{e^{-(-i\omega_k+\mu)\cdot}}{(\lambda+i\omega_k)}+\Delta\left(\alpha_k,\lambda\right)^{-1}e^{-(\lambda+\mu)\cdot}\end{pmatrix}.\end{aligned}$$

Moreover, if $\lambda=i\omega_k$, we have

$$\begin{aligned}&\left(i\omega_k I-B^{\mathbb{C}}_{\alpha}\mid_{\widehat{\Pi}_h(X)}\right)^{-1}\widehat{\Pi}_h\begin{pmatrix}1\\ 0\end{pmatrix}\\ &=\begin{pmatrix}0\\ -\frac{d\Delta(\alpha_k,-i\omega_k)}{d\lambda}^{-1}\frac{e^{-(-i\omega_k+\mu)\cdot}}{2i\omega_k}+\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}^{-2}\left[-\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}\cdot-\frac{1}{2}\frac{d^2\Delta(\alpha_k,i\omega_k)}{d\lambda^2}\right]e^{-(i\omega_k+\mu)\cdot}\end{pmatrix},\end{aligned}$$

and if $\lambda=-i\omega_k$, we have

$$\begin{aligned}&\left(-i\omega_k I-B^{\mathbb{C}}_{\alpha}\mid_{\widehat{\Pi}_h(X)}\right)^{-1}\widehat{\Pi}_h\begin{pmatrix}1\\ 0\end{pmatrix}\\ &=\begin{pmatrix}0\\ \frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}^{-1}\frac{e^{-(i\omega_k+\mu)\cdot}}{2i\omega_k}+\frac{d\Delta(\alpha_k,-i\omega_k)}{d\lambda}^{-2}\left[-\frac{d\Delta(\alpha_k,-i\omega_k)}{d\lambda}\cdot-\frac{1}{2}\frac{d^2\Delta(\alpha_k,-i\omega_k)}{d\lambda^2}\right]e^{-(-i\omega_k+\mu)\cdot}\end{pmatrix}.\end{aligned}$$

Lemma 3.3 *Let Assumption 1.1 be satisfied and assume that $\tau>0$. Then*

$$\sigma\left(\mathcal{A}\right)=\sigma\left(B_{\alpha_k}\right)\cup\{0\}.$$

Moreover, we have for $\lambda\in\rho\left(\mathcal{A}\right)\cap\Omega=\Omega\setminus\left(\sigma\left(B_{\alpha_k}\right)\cup\{0\}\right)$ that

$$\left(\lambda I-\mathcal{A}\right)^{-1}\begin{pmatrix}r\\ \begin{pmatrix}\delta\\ \psi\end{pmatrix}\end{pmatrix}=\begin{pmatrix}\frac{r}{\lambda}\\ \left(\lambda I-B_{\alpha_k}\right)^{-1}\begin{pmatrix}\delta\\ \psi\end{pmatrix}\end{pmatrix}$$

and the eigenvalues 0 and $\pm i\omega_k$ of $\mathcal{A}$ are simple. The corresponding projectors $\Pi_0,\Pi_{\pm i\omega_k}:\mathcal{X}+i\mathcal{X}\to\mathcal{X}+i\mathcal{X}$ are defined by

$$\Pi_0 \begin{pmatrix} r \\ v \end{pmatrix} = \begin{pmatrix} r \\ 0 \end{pmatrix}, \ \Pi_{\pm i\omega_k} \begin{pmatrix} r \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ \widehat{\Pi}_{\pm i\omega_k} v \end{pmatrix}, \forall \begin{pmatrix} r \\ v \end{pmatrix} \in \mathcal{X} + i\mathcal{X}.$$

Note that we have

$$\overline{\Pi_{i\omega_k}(x)} = \Pi_{-i\omega_k}(\overline{x}), \quad \forall x \in \mathcal{X} + i\mathcal{X}.$$

In this context, the projectors $\Pi_c : \mathcal{X} \to \mathcal{X}$ and $\Pi_h : \mathcal{X} \to \mathcal{X}$ are defined by

$$\begin{aligned} \Pi_c(x) &:= \left(\Pi_0 + \Pi_{i\omega_k} + \Pi_{-i\omega_k}\right)(x), \quad \forall x \in \mathcal{X}, \\ \Pi_h(x) &:= (I - \Pi_c)(x), \quad \forall x \in \mathcal{X}, \end{aligned}$$

and we denote by

$$\mathcal{X}_c := \Pi_c(\mathcal{X}), \mathcal{X}_h := \Pi_h(\mathcal{X}), \mathcal{A}_c := \mathcal{A}|_{\mathcal{X}_c}, \mathcal{A}_h := \mathcal{A}|_{\mathcal{X}_h}.$$

Then by Lemmas 3.1 and 3.3, we have

$$\Pi_c \begin{pmatrix} 0_{\mathbb{R}} \\ \begin{pmatrix} 1 \\ 0_{L^1} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}} \\ \widehat{\Pi}_{i\omega_k} \begin{pmatrix} 1 \\ 0_{L^1} \end{pmatrix} + \widehat{\Pi}_{-i\omega_k} \begin{pmatrix} 1 \\ 0_{L^1} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}} \\ \widehat{\Pi}_c \begin{pmatrix} 1 \\ 0_{L^1} \end{pmatrix} \end{pmatrix}.$$

Define the basis of $\mathcal{X}_c = \mathcal{R}(\Pi_c)$ by

$$e_1 = \begin{pmatrix} 1 \\ 0_{\mathbb{R}} \\ 0_{L^1} \end{pmatrix}, e_2 = \begin{pmatrix} 0_{\mathbb{R}} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ b_1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ e^{-(\mu+i\omega_k)\cdot} + e^{-(\mu-i\omega_k)\cdot} \end{pmatrix} \end{pmatrix},$$

$$e_3 = \begin{pmatrix} 0_{\mathbb{R}} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ b_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0_{\mathbb{R}} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ \frac{e^{-(\mu+i\omega_k)\cdot} - e^{-(\mu-i\omega_k)\cdot}}{i} \end{pmatrix} \end{pmatrix}.$$

We can readily check the following lemma.

Lemma 3.4 *For $\lambda \in i\mathbb{R}$ we have*

$$(\lambda I - \mathcal{A}_h)^{-1} \Pi_h \begin{pmatrix} 0 \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ \left(\lambda I - B_\alpha^{\mathbb{C}}|_{\widehat{\Pi}_h(X)}\right)^{-1} \widehat{\Pi}_h \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}.$$

3.2 Computation of the Normal Form

We apply the method described in Liu et al. [21, Theorem 4.2] for $k = 2$. The main point is to compute $L_2 \in \mathcal{L}_s\left(\mathcal{X}_c^2, \mathcal{X}_h \cap D(\mathcal{A})\right)$ by solving the following equation for each $(w_1, w_2) \in \mathcal{X}_c^2$:

$$\frac{d}{dt}\left[L_2(e^{\mathcal{A}_c t} w_1, e^{\mathcal{A}_c t} w_2)\right](0) = \mathcal{A}_h L_2(w_1, w_2) + \frac{1}{2!}\Pi_h D^2 F(0)(w_1, w_2). \tag{3.4}$$

Note that

$$\frac{d}{dt}\left[L_2(e^{\mathcal{A}_c t} w_1, e^{\mathcal{A}_c t} w_2)\right](0) = L_2(\mathcal{A}_c w_1, w_2) + L_2(w_1, \mathcal{A}_c w_2).$$

So system (3.4) can be rewritten as

$$L_2(\mathcal{A}_c w_1, w_2) + L_2(w_1, \mathcal{A}_c w_2) = \mathcal{A}_h L_2(w_1, w_2) + \frac{1}{2!}\Pi_h D^2 F(0)(w_1, w_2). \tag{3.5}$$

We first observe that

$$D^2F(0)\,(w_1,w_2)=\begin{pmatrix}0_{\mathbb{R}}\\ D^2W(0)\,(w_1,w_2)\end{pmatrix}$$

and

$$D^3F(0)\,(w_1,w_2,w_3)=\begin{pmatrix}0_{\mathbb{R}}\\ D^3W(0)\,(w_1,w_2,w_3)\end{pmatrix}$$

for each

$$w_1:=\begin{pmatrix}\widehat{\alpha}_1\\ v_1\end{pmatrix},\ w_2:=\begin{pmatrix}\widehat{\alpha}_2\\ v_2\end{pmatrix},\ w_3:=\begin{pmatrix}\widehat{\alpha}_3\\ v_3\end{pmatrix}\in\overline{D(\mathcal{A})},$$

with $v_i=\begin{pmatrix}0_{\mathbb{R}}\\ \varphi_i\end{pmatrix}$, $i=1,2,3$, where

$$\begin{aligned}
&D^2W(0)\,(w_1,w_2)\\
&\quad=D^2W(0)\left(\begin{pmatrix}\widehat{\alpha}_1\\ v_1\end{pmatrix},\begin{pmatrix}\widehat{\alpha}_2\\ v_2\end{pmatrix}\right)\\
&\quad=\alpha_kD^2H\left(\overline{v}_{\alpha_k}\right)(v_1,v_2)+\widehat{\alpha}_2DH\left(\overline{v}_{\alpha_k}\right)(v_1)+\widehat{\alpha}_1DH\left(\overline{v}_{\alpha_k}\right)(v_2)\\
&\qquad+\widehat{\alpha}_2\alpha_kD^2H\left(\overline{v}_{\alpha_k}\right)\left(v_1,\ \left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}\right)\\
&\qquad+\widehat{\alpha}_1\alpha_kD^2H\left(\overline{v}_{\alpha_k}\right)\left(v_2,\ \left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}\right),
\end{aligned}\tag{3.6}$$

and

$$\begin{aligned}
&D^3W(0)\,(w_1,w_2,w_3)\\
&\quad=D^3W(0)\left(\begin{pmatrix}\widehat{\alpha}_1\\ v_1\end{pmatrix},\begin{pmatrix}\widehat{\alpha}_2\\ v_2\end{pmatrix},\begin{pmatrix}\widehat{\alpha}_3\\ v_3\end{pmatrix}\right)\\
&\quad=\widehat{\alpha}_1D^2H\left(\overline{v}_{\alpha_k}\right)(v_2,v_3)+\widehat{\alpha}_2D^2H\left(\overline{v}_{\alpha_k}\right)(v_1,v_3)+\widehat{\alpha}_3D^2H\left(\overline{v}_{\alpha_k}\right)(v_1,v_2)\\
&\qquad+2\widehat{\alpha}_2\widehat{\alpha}_3D^2H\left(\overline{v}_{\alpha_k}\right)\left(v_1,\ \left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}\right)+2\widehat{\alpha}_1\widehat{\alpha}_3D^2H\left(\overline{v}_{\alpha_k}\right)\left(v_2,\ \left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}\right)\\
&\qquad+2\widehat{\alpha}_1\widehat{\alpha}_2D^2H\left(\overline{v}_{\alpha_k}\right)\left(v_3,\ \left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}\right)+\widehat{\alpha}_2\widehat{\alpha}_3\alpha_kD^2H\left(\overline{v}_{\alpha_k}\right)\left(v_1,\ \left.\frac{d^2\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\left(\widehat{\alpha}\right)^2}\right|_{\widehat{\alpha}=0}\right)\\
&\qquad+\widehat{\alpha}_1\widehat{\alpha}_3\alpha_kD^2H\left(\overline{v}_{\alpha_k}\right)\left(v_2,\ \left.\frac{d^2\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\left(\widehat{\alpha}\right)^2}\right|_{\widehat{\alpha}=0}\right)+\widehat{\alpha}_1\widehat{\alpha}_2\alpha_kD^2H\left(\overline{v}_{\alpha_k}\right)\left(v_3,\ \left.\frac{d^2\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\left(\widehat{\alpha}\right)^2}\right|_{\widehat{\alpha}=0}\right)\\
&\qquad+\alpha_kD^3H\left(\overline{v}_{\alpha_k}\right)\left(v_1,v_2,v_3\right)+\widehat{\alpha}_3\alpha_kD^3H\left(\overline{v}_{\alpha_k}\right)\left(v_1,v_2,\ \left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}\right)\\
&\qquad+\widehat{\alpha}_2\alpha_kD^3H\left(\overline{v}_{\alpha_k}\right)\left(v_1,v_3,\ \left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}\right)+\widehat{\alpha}_1\alpha_kD^3H\left(\overline{v}_{\alpha_k}\right)\left(v_2,v_3,\ \left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}\right)\\
&\qquad+\widehat{\alpha}_2\widehat{\alpha}_3\alpha_kD^3H\left(\overline{v}_{\alpha_k}\right)\left(v_1,\ \left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0},\ \left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}\right)\\
&\qquad+\widehat{\alpha}_1\widehat{\alpha}_3\alpha_kD^3H\left(\overline{v}_{\alpha_k}\right)\left(v_2,\ \left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0},\ \left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}\right)\\
&\qquad+\widehat{\alpha}_1\widehat{\alpha}_2\alpha_kD^3H\left(\overline{v}_{\alpha_k}\right)\left(v_3,\ \left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0},\ \left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}\right),
\end{aligned}\tag{3.7}$$

with ($k = 1, 2, 3$)

$$D^k H\left(\overline{v}_{\alpha_k}\right)\left(\begin{pmatrix}0\\ \varphi_1\end{pmatrix},\ldots,\begin{pmatrix}0\\ \varphi_k\end{pmatrix}\right)$$
$$=\begin{pmatrix} h^{(k)}\left(\int_0^{+\infty}\gamma\left(a\right)\overline{u}_{\alpha_k}\left(a\right)da\right)\prod_{i=1}^{k}\int_0^{+\infty}\gamma\left(a\right)\varphi_i\left(a\right)da \\ 0\end{pmatrix},$$

$$h^{(1)}(x) = (1-\beta x)\exp\left(-\beta x\right),$$
$$h^{(2)}(x) = \left(\beta^2 x - 2\beta\right)\exp\left(-\beta x\right),$$
$$h^{(3)}(x) = \left(-\beta^3 x + 3\beta^2\right)\exp\left(-\beta x\right),$$
$$\overline{v}_{\alpha_k} = \begin{pmatrix}0\\ \overline{u}_{\alpha_k}\end{pmatrix} = \begin{pmatrix}0\\ \frac{\ln\left(\alpha_k\int_0^{+\infty}\gamma(a)e^{-\mu a}da\right)}{\beta\int_0^{+\infty}\gamma(a)e^{-\mu a}da}\exp(-\mu\cdot)\end{pmatrix},$$
$$\int_0^{+\infty}\gamma\left(a\right)\overline{u}_{\alpha_k}\left(a\right)da = \frac{\ln\left(\alpha_k\int_0^{+\infty}\gamma\left(a\right)e^{-\mu a}da\right)}{\beta},$$
$$\frac{d\overline{u}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}} = \frac{1}{\widehat{\alpha}+\alpha_k}\times\frac{\exp\left(-\mu\cdot\right)}{\beta\int_0^{+\infty}\gamma\left(a\right)e^{-\mu a}da},$$
$$\int_0^{+\infty}\gamma\left(a\right)\left.\frac{d\overline{u}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}(a)da = \frac{1}{\beta\alpha_k},$$
$$\frac{d^2\overline{u}_{\widehat{\alpha}+\alpha_k}}{d\left(\widehat{\alpha}\right)^2} = -\frac{1}{\left(\widehat{\alpha}+\alpha_k\right)^2}\times\frac{\exp\left(-\mu\cdot\right)}{\beta\int_0^{+\infty}\gamma\left(a\right)e^{-\mu a}da},$$
$$\int_0^{+\infty}\gamma\left(a\right)\left.\frac{d^2\overline{u}_{\widehat{\alpha}+\alpha_k}}{d\left(\widehat{\alpha}\right)^2}\right|_{\widehat{\alpha}=0}(a)da = -\frac{1}{\beta\left(\alpha_k\right)^2}.$$

To simplify the computation, we use the eigenfunctions of $\mathcal{A}$ in $\mathcal{X}_c$ and consider

$$\widehat{e}_1 := \begin{pmatrix}1\\ \begin{pmatrix}0_{\mathbb{R}}\\ 0_C\end{pmatrix}\end{pmatrix},\ \widehat{e}_2 := \begin{pmatrix}0_{\mathbb{R}}\\ \begin{pmatrix}0_{\mathbb{R}}\\ e^{-(\mu+i\omega_k)\cdot}\end{pmatrix}\end{pmatrix},\ \widehat{e}_3 =: \begin{pmatrix}0_{\mathbb{R}}\\ \begin{pmatrix}0_{\mathbb{R}}\\ e^{-(\mu-i\omega_k)\cdot}\end{pmatrix}\end{pmatrix}.$$

We have

$$\mathcal{A}\widehat{e}_1 = 0,\ \mathcal{A}\widehat{e}_2 = i\omega_k\widehat{e}_2,\ \text{and}\ \mathcal{A}\widehat{e}_3 = -i\omega_k\widehat{e}_3.$$

In order to simplify the notation, from now on we set

$$\chi := \int_0^{+\infty}\gamma\left(a\right)e^{-\mu a}da = \frac{n!\exp(-\mu\tau)}{(\mu+\varsigma)^{n+1}}. \tag{3.8}$$

(i) **Computation of** $L_2(\widehat{e}_1,\widehat{e}_1)$: We have

$$\Pi_h D^2F\left(0\right)\left(\widehat{e}_1,\widehat{e}_1\right) = 0$$

and

$$\mathcal{A}_c\widehat{e}_1 = 0.$$

By (3.5) we have

$$L_2\left(\mathcal{A}_c\widehat{e}_1,\widehat{e}_1\right) + L_2\left(\widehat{e}_1,\mathcal{A}_c\widehat{e}_1\right) = \mathcal{A}_h L_2(\widehat{e}_1,\widehat{e}_1) + \frac{1}{2!}\Pi_h D^2F\left(0\right)\left(\widehat{e}_1,\widehat{e}_1\right).$$

So

$$0 = \mathcal{A}_h L_2(\widehat{e}_1, \widehat{e}_1).$$

Since 0 belongs to the resolvent set of $\mathcal{A}_h$, we obtain

$$L_2(\widehat{e}_1, \widehat{e}_1) = 0. \tag{3.9}$$

(ii) **Computation of** $L_2(\widehat{e}_1, \widehat{e}_2)$: Since $\mathcal{A}_c\widehat{e}_1 = 0$ and $\mathcal{A}_c\widehat{e}_2 = i\omega_k\widehat{e}_2$, the equation

$$L_2\left(\mathcal{A}_c\widehat{e}_1, \widehat{e}_2\right) + L_2\left(\widehat{e}_1, \mathcal{A}_c\widehat{e}_2\right) = \mathcal{A}_h L_2(\widehat{e}_1, \widehat{e}_2) + \frac{1}{2!}\Pi_h D^2F\left(0\right)\left(\widehat{e}_1, \widehat{e}_2\right)$$

is equivalent to

$$\left(i\omega_k - \mathcal{A}_h\right) L_2\left(\widehat{e}_1, \widehat{e}_2\right) = \frac{1}{2!}\Pi_h D^2F\left(0\right)\left(\widehat{e}_1, \widehat{e}_2\right),$$

where

$$\begin{aligned}
&D^2F\left(0\right)\left(\widehat{e}_1, \widehat{e}_2\right)\\
&= \begin{pmatrix} 0_{\mathbb{R}} \\ D^2W\left(0\right)\left(\begin{pmatrix} 1 \\ 0_X \end{pmatrix}, \begin{pmatrix} 0_{\mathbb{R}} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ e^{-(\mu+i\omega_k)\cdot} \end{pmatrix} \end{pmatrix}\right) \end{pmatrix}\\
&= \begin{pmatrix} 0_{\mathbb{R}} \\ DH\left(\overline{v}_{\alpha_k}\right)\begin{pmatrix} 0_{\mathbb{R}} \\ e^{-(\mu+i\omega_k)\cdot} \end{pmatrix} \end{pmatrix}\\
&\quad + \begin{pmatrix} 0_{\mathbb{R}} \\ \alpha_k D^2H\left(\overline{v}_{\alpha_k}\right)\left(\begin{pmatrix} 0_{\mathbb{R}} \\ e^{-(\mu+i\omega_k)\cdot} \end{pmatrix}, \left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}\right) \end{pmatrix}\\
&= \begin{pmatrix} 0_{\mathbb{R}} \\ h^{(1)}\left(\int_0^{+\infty}\gamma\left(a\right)\overline{u}_{\alpha_k}\left(a\right)da\right)\int_0^{+\infty}\gamma\left(a\right)e^{-(\mu+i\omega_k)a}da \\ +\alpha_k h^{(2)}\left(\int_0^{+\infty}\gamma\left(a\right)\overline{u}_{\alpha_k}\left(a\right)da\right)\int_0^{+\infty}\gamma\left(a\right)e^{-(\mu+i\omega_k)a}da\int_0^{+\infty}\gamma\left(a\right)\left.\frac{d\overline{u}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}\left(a\right)da \\ 0_{L^1} \end{pmatrix}.
\end{aligned}$$

Thus, we have

$$D^2F\left(0\right)\left(\widehat{e}_1, \widehat{e}_2\right) = c_{12}\begin{pmatrix} 0_{\mathbb{R}} \\ \begin{pmatrix} 1 \\ 0_{L^1} \end{pmatrix} \end{pmatrix}$$

with

$$\begin{aligned}
c_{12} &= h^{(1)}\left(\int_0^{+\infty}\gamma\left(a\right)\overline{u}_{\alpha_k}\left(a\right)da\right)\int_0^{+\infty}\gamma\left(a\right)e^{-(\mu+i\omega_k)a}da\\
&\quad + \alpha_k h^{(2)}\left(\int_0^{+\infty}\gamma\left(a\right)\overline{u}_{\alpha_k}\left(a\right)da\right)\int_0^{+\infty}\gamma\left(a\right)e^{-(\mu+i\omega_k)a}da\\
&\quad \times \int_0^{+\infty}\gamma\left(a\right)\left.\frac{d\overline{u}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}\left(a\right)da\\
&= \left(1 - \beta\int_0^{+\infty}\gamma\left(a\right)\overline{u}_{\alpha_k}\left(a\right)da\right)\exp\left(-\beta\int_0^{+\infty}\gamma\left(a\right)\overline{u}_{\alpha_k}\left(a\right)da\right)\frac{\chi}{1-\ln\left(\alpha\chi\right)}\\
&\quad + \alpha_k\left(\beta^2\int_0^{+\infty}\gamma\left(a\right)\overline{u}_{\alpha_k}\left(a\right)da - 2\beta\right)\exp\left(-\beta\int_0^{+\infty}\gamma\left(a\right)\overline{u}_{\alpha_k}\left(a\right)da\right)
\end{aligned}$$

$$
\begin{aligned}
&\times \frac{\chi}{1-\ln(\alpha\chi)} \times \frac{1}{\alpha_k}\frac{1}{\beta} \\
&= \frac{\chi}{1-\ln(\alpha\chi)}\left[\left(1-\beta\int_0^{+\infty}\gamma(a)\,\overline{u}_{\alpha_k}(a)\,da\right)\exp\left(-\beta\int_0^{+\infty}\gamma(a)\,\overline{u}_{\alpha_k}(a)\,da\right)\right. \\
&\quad \left.+\left(\beta\int_0^{+\infty}\gamma(a)\,\overline{u}_{\alpha_k}(a)\,da-2\right)\exp\left(-\beta\int_0^{+\infty}\gamma(a)\,\overline{u}_{\alpha_k}(a)\,da\right)\right] \\
&= -\frac{\chi}{1-\ln(\alpha\chi)}\exp\left(-\beta\int_0^{+\infty}\gamma(a)\,\overline{u}_{\alpha_k}(a)\,da\right) \\
&= -\frac{\chi}{1-\ln(\alpha\chi)}\left(\alpha_k\int_0^{+\infty}\gamma(a)\,e^{-\mu a}da\right)^{-1} \\
&= -\frac{1}{\alpha_k(1-\ln(\alpha\chi))}.
\end{aligned}
$$

So

$$
L_2(\widehat{e}_1,\widehat{e}_2) = -\frac{1}{2\alpha_k(1-\ln(\alpha\chi))}(i\omega_k-\mathcal{A}_h)^{-1}\,\Pi_h\begin{pmatrix}0_{\mathbb{R}}\\ 1\\ 0_{L^1}\end{pmatrix}.
$$

By using a similar method together with Lemmas 3.2 and 3.3, we obtain the following results:

$$
L_2(\widehat{e}_1,\widehat{e}_2) = L_2(\widehat{e}_2,\widehat{e}_1) = \begin{pmatrix}0\\ \begin{pmatrix}0\\ \psi_{1,2}\end{pmatrix}\end{pmatrix}, \tag{3.10}
$$

$$
L_2(\widehat{e}_1,\widehat{e}_3) = L_2(\widehat{e}_3,\widehat{e}_1) = \begin{pmatrix}0\\ \begin{pmatrix}0\\ \psi_{1,3}\end{pmatrix}\end{pmatrix}, \tag{3.11}
$$

$$
L_2(\widehat{e}_2,\widehat{e}_3) = L_2(\widehat{e}_3,\widehat{e}_2) = \begin{pmatrix}0\\ \begin{pmatrix}0\\ \psi_{2,3}\end{pmatrix}\end{pmatrix}, \tag{3.12}
$$

$$
L_2(\widehat{e}_2,\widehat{e}_2) = \begin{pmatrix}0\\ \begin{pmatrix}0\\ \psi_{2,2}\end{pmatrix}\end{pmatrix}, \tag{3.13}
$$

$$
L_2(\widehat{e}_3,\widehat{e}_3) = \begin{pmatrix}0\\ \begin{pmatrix}0\\ \psi_{3,3}\end{pmatrix}\end{pmatrix}, \tag{3.14}
$$

where

$$
\begin{aligned}
\psi_{1,2}(a) = \overline{\psi_{1,3}(a)} &:= -\frac{1}{2\alpha_k(1-\ln(\alpha_k\chi))} \\
&\quad\times\begin{pmatrix}-\frac{d\Delta(\alpha_k,-i\omega_k)}{d\lambda}^{-1}\dfrac{e^{-(-i\omega_k+\mu)a}}{2i\omega_k} \\ +\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}^{-2}\left[-\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}a-\frac{1}{2}\frac{d^2\Delta(\alpha_k,i\omega_k)}{d\lambda^2}\right]e^{-(i\omega_k+\mu)a}\end{pmatrix}, \\
\psi_{2,2}(a) = \overline{\psi_{3,3}(a)} &:= \frac{\beta\chi(\ln(\alpha_k\chi)-2)}{2(1-\ln(\alpha_k\chi))^2} \\
&\quad\times\begin{pmatrix}-\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}^{-1}\frac{e^{-(i\omega_k+\mu)a}}{i\omega_k} \\ -\frac{d\Delta(\alpha_k,-i\omega_k)}{d\lambda}^{-1}\frac{e^{-(-i\omega_k+\mu)a}}{3i\omega_k}+\Delta(\alpha_k,2i\omega_k)^{-1}e^{-(2i\omega_k+\mu)a}\end{pmatrix},
\end{aligned}
$$

and

$$\psi_{2,3}(a) = \frac{\beta\chi\left(\ln(\alpha_k\chi) - 2\right)}{2\left(1 - \ln(\alpha_k\chi)\right)^2} \times \begin{pmatrix} \frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}^{-1} \frac{e^{-(i\omega_k+\mu)a}}{i\omega_k} - \\ \frac{d\Delta(\alpha_k, -i\omega_k)}{d\lambda}^{-1} \frac{e^{-(-i\omega_k+\mu)a}}{i\omega_k} + \Delta(\alpha_k, 0)^{-1} e^{-\mu a} \end{pmatrix}.$$

By using (3.9)–(3.14), and the fact that

$$e_1 = \widehat{e}_1,\ e_2 = \widehat{e}_2 + \widehat{e}_3,\ \text{and } e_3 = \frac{\widehat{e}_2 - \widehat{e}_3}{i},$$

we obtain the following lemma.

Lemma 3.5 *The symmetric and bilinear map $L_2 : \mathcal{X}_c^2 \to \mathcal{X}_h \cap D(\mathcal{A})$ is defined by*

(a) $L_2(e_1, e_1) = 0$;

(b) $L_2(e_1, e_2)$ *and* $L_2(e_2, e_1)$ *are defined by*

$$L_2(e_1, e_2) = L_2(e_2, e_1) = \begin{pmatrix} 0_{\mathbb{R}} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ 2\mathrm{Re}\psi_{1,2} \end{pmatrix} \end{pmatrix};$$

(c) $L_2(e_1, e_3)$ *and* $L_2(e_3, e_1)$ *are defined by*

$$L_2(e_1, e_3) = L_2(e_3, e_1) = \begin{pmatrix} 0_{\mathbb{R}} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ 2\mathrm{Im}\psi_{1,2} \end{pmatrix} \end{pmatrix};$$

(d) $L_2(e_2, e_2)$ *is defined by*

$$L_2(e_2, e_2) = \begin{pmatrix} 0_{\mathbb{R}} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ 2\mathrm{Re}\psi_{2,2} + 2\psi_{2,3} \end{pmatrix} \end{pmatrix};$$

(e) $L_2(e_2, e_3)$ *and* $L_2(e_3, e_2)$ *are defined by*

$$L_2(e_2, e_3) = L_2(e_3, e_2) = \begin{pmatrix} 0_{\mathbb{R}} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ 2\mathrm{Im}\psi_{2,2} \end{pmatrix} \end{pmatrix};$$

(f) $L_2(e_3, e_3) = \begin{pmatrix} 0_{\mathbb{R}} \\ \begin{pmatrix} 0_{\mathbb{R}} \\ -2\mathrm{Re}\psi_{2,2} + 2\psi_{2,3} \end{pmatrix} \end{pmatrix}$.

We define $G_2 : \mathcal{X} \to \mathcal{X}_h \cap D(\mathcal{A})$ by

$$G_2(\Pi_c w) := L_2(\Pi_c w, \Pi_c w), \quad \forall w \in \mathcal{X},$$

and the change of variable $\xi_2 : \mathcal{X} \to \mathcal{X}$ and $\xi_2^{-1} : \mathcal{X} \to \mathcal{X}$ by

$$\xi_2(w) := w - G_2(\Pi_c w) \text{ and } \xi_2^{-1}(w) := w + G_2(\Pi_c w), \quad \forall w \in \mathcal{X},$$

and $F_2 : \overline{D(\mathcal{A})} \to \mathcal{X}$ by

$$F_2(w) := F\left(\xi_2^{-1}(w)\right) + \mathcal{A}G_2(\Pi_c w) - DG_2(\Pi_c w)\mathcal{A}_c\Pi_c w - DG_2(\Pi_c w)\Pi_c F\left(\xi_2^{-1}(w)\right).$$

By applying in Liu et al. [21, Theorem 4.2] to (3.3) for $k = 2$, we obtain the following theorem.

Theorem 3.6 *By using the change of variables*

$$w_2(t) = w(t) - G_2\left(\Pi_c w(t)\right) \Leftrightarrow w(t) = w_2(t) + G_2\left(\Pi_c w_2(t)\right),$$

the map $t \to w(t)$ is an integrated solution of the Cauchy problem (3.3) if and only if $t \to w_2(t)$ is an integrated solution of the Cauchy problem

$$\begin{cases} \dfrac{dw_2(t)}{dt} = \mathcal{A}w_2(t) + F_2(w_2(t)), \quad t \geq 0, \\ w_2(0) = w_2 \in \overline{D(\mathcal{A})}. \end{cases} \tag{3.15}$$

Moreover, the reduced equation of the Cauchy problem (3.15) is given by the ordinary differential equation on $\mathbb{R} \times X_c$:

$$\begin{cases} \dfrac{d\widehat{\alpha}(t)}{dt} = 0, \\ \dfrac{dy_c(t)}{dt} = B_{\alpha_k}|_{\widehat{\Pi}_c(X)} y_c(t) + \widehat{\Pi}_c W\left(I + G_2\right)\begin{pmatrix} \widehat{\alpha}(t) \\ y_c(t) \end{pmatrix} + \widehat{R}_c\begin{pmatrix} \widehat{\alpha}(t) \\ y_c(t) \end{pmatrix}, \end{cases} \tag{3.16}$$

where $\widehat{R}_c \in C^4\left(\mathbb{R} \times X_c, X_c\right)$, and $\widehat{R}_c\begin{pmatrix} \widehat{\alpha}(t) \\ y_c(t) \end{pmatrix}$ is a remainder term of order 4, that is,

$$\widehat{R}_c\begin{pmatrix} \widehat{\alpha} \\ y_c \end{pmatrix} = \|(\widehat{\alpha}, y_c)\|^4\, O\left(\widehat{\alpha}, y_c\right),$$

where $O\left(\widehat{\alpha}, y_c\right)$ is a function of $(\widehat{\alpha}, y_c)$ which remains bounded when $(\widehat{\alpha}, y_c)$ goes to 0, or equivalently,

$$D^j \widehat{R}_c\,(0) = 0 \ \textit{ for each } \ j = 1, 2, 3.$$

Furthermore,

$$\frac{\partial^j \widehat{R}_c\,(0)}{\partial^j \widehat{\alpha}} = 0, \quad \forall j = 1, 2, 3, 4,$$

which implies that

$$\widehat{R}_c\begin{pmatrix} \widehat{\alpha} \\ y_c \end{pmatrix} = O\left(\widehat{\alpha}^3 \|y_c\| + \widehat{\alpha}^2 \|y_c\|^2 + \widehat{\alpha}\, \|y_c\|^3 + \|y_c\|^4\right).$$

In the following theorem we compute the Taylor's expansion of the reduced system (3.16) by using the formula obtained for L_2 in Lemma 3.5.

Theorem 3.7 *The reduced system (3.16) expressed in terms of the basis $\{e_1, e_2, e_3\}$ has the following form*

$$\begin{cases} \dfrac{d\widehat{\alpha}(t)}{dt} = 0, \\ \dfrac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = M_c\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \left(\widetilde{H}_2 + \widetilde{H}_3 + \widehat{R}_c\right)\begin{pmatrix} \widehat{\alpha}(t) \\ \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \end{pmatrix}, \end{cases} \tag{3.17}$$

where

$$M_c = \begin{bmatrix} 0 & \omega_k \\ -\omega_k & 0 \end{bmatrix};$$

he map $\widetilde{H}_2 : \mathbb{R}^3 \to \mathbb{R}^2$ *is defined by*

$$\widetilde{H}_2 \begin{pmatrix} \widehat{\alpha} \\ x \\ y \end{pmatrix} = \chi_2 (\widehat{\alpha}, x, y) \left| \frac{d\Delta (\alpha_k, i\omega_k)}{d\lambda} \right|^{-2} \begin{bmatrix} \mathrm{Re} \left(\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda} \right) \\ \mathrm{Im} \left(\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda} \right) \end{bmatrix},$$

in which

$$\chi_2 (\widehat{\alpha}, x, y) = -\frac{2}{\alpha_k [1 - \ln (\alpha_k \chi)]} \widehat{\alpha} x + \frac{2\chi\beta (\ln (\alpha_k \chi) - 2)}{[1 - \ln (\alpha_k \chi)]^2} x^2;$$

the map $\widetilde{H}_3 : \mathbb{R}^3 \to \mathbb{R}^2$ *is defined by*

$$\widetilde{H}_3 \begin{pmatrix} \widehat{\alpha} \\ x \\ y \end{pmatrix} = \chi_3 (\widehat{\alpha}, x, y) \left| \frac{d\Delta (\alpha_k, i\omega_k)}{d\lambda} \right|^{-2} \begin{bmatrix} \mathrm{Re} \left(\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda} \right) \\ \mathrm{Im} \left(\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda} \right) \end{bmatrix},$$

in which

$$\begin{aligned} &\chi_3 (\widehat{\alpha}, x, y) \\ &= \left(-\frac{2\widehat{\alpha}}{\alpha_k \chi} + \frac{4\beta (\ln (\alpha_k \chi) - 2) x}{1 - \ln (\alpha_k \chi)} \right) \times \left[\left(x^2 - y^2 \right) \int_0^{+\infty} \gamma (a) \mathrm{Re}\psi_{2,2} (a) \, da \right. \\ &\quad + \left(x^2 + y^2 \right) \int_0^{+\infty} \gamma (a) \psi_{2,3} (a) \, da + 2xy \int_0^{+\infty} \gamma (a) \mathrm{Im}\psi_{2,2} (a) \, da \\ &\quad \left. + 2\widehat{\alpha} x \int_0^{+\infty} \gamma (a) \mathrm{Re}\psi_{1,2} (a) \, da + 2\widehat{\alpha} y \int_0^{+\infty} \gamma (a) \mathrm{Im}\psi_{1,2} (a) \, da \right] \\ &\quad + \frac{1}{(\alpha_k)^2 (1 - \ln (\alpha_k \chi))} \widehat{\alpha}^2 x + \frac{2\beta\chi}{\alpha_k (1 - \ln (\alpha_k \chi))^2} \widehat{\alpha} x^2 \\ &\quad + \frac{4\beta^2 (-\ln (\alpha_k \chi) + 3) \chi^2}{3 (1 - \ln (\alpha_k \chi))^3} x^3; \end{aligned}$$

and the remainder term $\widehat{R}_c \in C^4 \left(\mathbb{R}^3, \mathbb{R}^2 \right)$ *satisfies*

$$\widehat{R}_c \begin{pmatrix} \widehat{\alpha} \\ x \\ y \end{pmatrix} = O \left(\widehat{\alpha}^3 \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| + \widehat{\alpha}^2 \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2 + \widehat{\alpha} \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|^3 + \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|^4 \right). \tag{3.18}$$

Proof We firstly prove that the reduced system (3.16) expressed in terms of the basis $\{e_1, e_2, e_3\}$ has the following form

$$\begin{cases} \dfrac{d\widehat{\alpha}(t)}{dt} = 0, \\ \dfrac{d}{dt} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = M_c \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \left(\widehat{H}_2 + \widehat{H}_3 + \widehat{R}_c \right) \begin{pmatrix} \widehat{\alpha}(t) \\ \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \end{pmatrix}, \end{cases} \tag{3.19}$$

where the map $\widehat{H}_2 : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by

$$\widehat{H}_2 \begin{pmatrix} \widehat{\alpha} \\ x \\ y \end{pmatrix} = \widetilde{\psi} \left| \frac{d\Delta (\alpha_k, i\omega_k)}{d\lambda} \right|^{-2} \begin{bmatrix} \mathrm{Re} \left(\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda} \right) \\ \mathrm{Im} \left(\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda} \right) \end{bmatrix} \tag{3.20}$$

with

$$\widetilde{\psi} = -\frac{\widehat{\alpha}}{\alpha_k \chi} \int_0^{+\infty} \gamma (a) \psi (a) \, da + \frac{\beta (\ln (\alpha_k \chi) - 2)}{2\chi} \left(\int_0^{+\infty} \gamma (a) \psi (a) \, da \right)^2$$

and

$$\begin{aligned}
&\int_0^{+\infty} \gamma(a)\,\psi(a)\,da \\
&= x\frac{2\chi}{1-\ln(\alpha_k\chi)} + 2\left(x^2-y^2\right)\int_0^{+\infty}\gamma(a)\,\mathrm{Re}\psi_{2,2}(a)\,da \\
&\quad + 2\left(x^2+y^2\right)\int_0^{+\infty}\gamma(a)\,\psi_{2,3}(a)\,da + 4xy\int_0^{+\infty}\gamma(a)\,\mathrm{Im}\psi_{2,2}(a)\,da \\
&\quad + 4\widehat{\alpha}x\int_0^{+\infty}\gamma(a)\,\mathrm{Re}\psi_{1,2}(a)\,da + 4\widehat{\alpha}y\int_0^{+\infty}\gamma(a)\,\mathrm{Im}\psi_{1,2}(a)\,da.
\end{aligned}$$

The map $\widehat{H}_3 : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by

$$\widehat{H}_3\begin{pmatrix}\widehat{\alpha}\\ x\\ y\end{pmatrix} = \widehat{\psi}\left|\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}\right|^{-2}\begin{bmatrix}\mathrm{Re}\left(\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}\right)\\ \mathrm{Im}\left(\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}\right)\end{bmatrix}, \tag{3.21}$$

where

$$\widehat{\psi} = \frac{1}{(\alpha_k)^2(1-\ln(\alpha_k\chi))}\widehat{\alpha}^2 x + \frac{2\beta\chi}{\alpha_k(1-\ln(\alpha_k\chi))^2}\widehat{\alpha}x^2 + \frac{4\beta^2(-\ln(\alpha_k\chi)+3)\chi^2}{3(1-\ln(\alpha_k\chi))^3}x^3.$$

By using the Taylor's expansion of W around 0, the reduced system (3.16) can be rewritten as follows:

$$\begin{aligned}
\frac{d\widehat{\alpha}(t)}{dt} &= 0,\\
\frac{dy_c(t)}{dt} &= B_{\alpha_k}|_{\widehat{\Pi}_c(X)}y_c(t) + \frac{1}{2!}\widehat{\Pi}_c D^2W(0)\left((I+G_2)\begin{pmatrix}\widehat{\alpha}(t)\\ y_c(t)\end{pmatrix}\right)^2\\
&\quad + \frac{1}{3!}\widehat{\Pi}_c D^3W(0)\left((I+G_2)\begin{pmatrix}\widehat{\alpha}(t)\\ y_c(t)\end{pmatrix}\right)^3 + \widetilde{R}_c\begin{pmatrix}\widehat{\alpha}(t)\\ y_c(t)\end{pmatrix}.
\end{aligned}$$

Set

$$y_c = \begin{pmatrix}0\\ xb_1+yb_2\end{pmatrix} = \begin{pmatrix}0\\ x\left(e^{-(\mu+i\omega_k)\cdot}+e^{-(\mu-i\omega_k)\cdot}\right)+y\left(\frac{e^{-(\mu+i\omega_k)\cdot}-e^{-(\mu-i\omega_k)\cdot}}{i}\right)\end{pmatrix}.$$

Since we consider $\{e_1, e_2, e_3\}$ as the basis for $\mathcal{X}_c = \mathcal{R}(\Pi_c)$, i.e., $\left\{\binom{0}{b_1}, \binom{0}{b_2}\right\}$ is a basis of $\mathcal{X}_c := \widehat{\Pi}_c(\mathcal{X})$, we obtain that

$$M_c = \begin{bmatrix}0 & \omega_k\\ -\omega_k & 0\end{bmatrix}.$$

Now we compute $\widehat{H}_2(\mathcal{X})$. We have

$$\begin{aligned}
(I+G_2)\begin{pmatrix}\widehat{\alpha}\\ y_c\end{pmatrix} &= \widehat{\alpha}e_1 + xe_2 + ye_3 + L_2(\widehat{\alpha}e_1 + xe_2 + ye_3, \widehat{\alpha}e_1 + xe_2 + ye_3)\\
&= \widehat{\alpha}e_1 + xe_2 + ye_3 + \widehat{\alpha}^2L_2(e_1,e_1) + x^2L_2(e_2,e_2) + y^2L_2(e_3,e_3)\\
&\quad + 2\widehat{\alpha}xL_2(e_1,e_2) + 2\widehat{\alpha}yL_2(e_1,e_3) + 2xyL_2(e_2,e_3).
\end{aligned}$$

By Lemma 3.5, it follows that

$$
\begin{aligned}
&(I+G_2)\begin{pmatrix}\widehat{\alpha}\\ y_c\end{pmatrix}=\begin{pmatrix}\widehat{\alpha}\\ \begin{pmatrix}0\\ \psi\end{pmatrix}\end{pmatrix}\\
&\Leftrightarrow \psi(a)=x\left(e^{-(\mu+i\omega_k)a}+e^{-(\mu-i\omega_k)a}\right)+y\left(\frac{e^{-(\mu+i\omega_k)a}-e^{-(\mu-i\omega_k)a}}{i}\right)\\
&\qquad +2\left(x^2-y^2\right)\mathrm{Re}\left(\psi_{2,2}(a)\right)+2\left(x^2+y^2\right)\psi_{2,3}(a)\\
&\qquad +4\widehat{\alpha}x\mathrm{Re}\left(\psi_{1,2}(a)\right)+4\widehat{\alpha}y\mathrm{Im}\left(\psi_{1,2}(a)\right)+4xy\mathrm{Im}\left(\psi_{2,2}(a)\right). \quad (3.22)
\end{aligned}
$$

By (3.6) we deduce that

$$
\begin{aligned}
&\frac{1}{2!}D^2W(0)\begin{pmatrix}\widehat{\alpha}\\ \begin{pmatrix}0\\ \psi\end{pmatrix}\end{pmatrix}^2\\
&=\widehat{\alpha}DH\left(\overline{v}_{\alpha_k}\right)\begin{pmatrix}0\\ \psi\end{pmatrix}+\frac{1}{2}\alpha_k D^2H\left(\overline{v}_{\alpha_k}\right)\left(\begin{pmatrix}0\\ \psi\end{pmatrix},\begin{pmatrix}0\\ \psi\end{pmatrix}\right)\\
&\quad+\widehat{\alpha}\alpha_k D^2H\left(\overline{v}_{\alpha_k}\right)\left(\begin{pmatrix}0\\ \psi\end{pmatrix},\begin{pmatrix}0\\ \frac{1}{\alpha_k}\times\frac{\exp(-\mu\cdot)}{\beta\int_0^{+\infty}\gamma(a)e^{-\mu a}da}\end{pmatrix}\right)=\begin{pmatrix}\widetilde{\psi}\\ 0\end{pmatrix},
\end{aligned}
$$

where

$$
\widetilde{\psi}=-\frac{\widehat{\alpha}}{\alpha_k\chi}\int_0^{+\infty}\gamma(a)\psi(a)\,da+\frac{\beta\left(\ln(\alpha_k\chi)-2\right)}{2\chi}\left(\int_0^{+\infty}\gamma(a)\psi(a)\,da\right)^2
$$

with

$$
\begin{aligned}
&\int_0^{+\infty}\gamma(a)\psi(a)\,da\\
&=x\frac{2\chi}{1-\ln(\alpha_k\chi)}+2\left(x^2-y^2\right)\int_0^{+\infty}\gamma(a)\mathrm{Re}\psi_{2,2}(a)\,da\\
&\quad+2\left(x^2+y^2\right)\int_0^{+\infty}\gamma(a)\psi_{2,3}(a)\,da+4xy\int_0^{+\infty}\gamma(a)\mathrm{Im}\psi_{2,2}(a)\,da\\
&\quad+4\widehat{\alpha}x\int_0^{+\infty}\gamma(a)\mathrm{Re}\psi_{1,2}(a)\,da+4\widehat{\alpha}y\int_0^{+\infty}\gamma(a)\mathrm{Im}\psi_{1,2}(a)\,da.
\end{aligned}
$$

By projecting on X_c and using Lemma 3.1 and the same identification as above, we obtain

$$
\begin{aligned}
&\frac{1}{2!}\widehat{\Pi}_c D^2W(0)\left((I+G_2)\begin{pmatrix}\widehat{\alpha}\\ y_c\end{pmatrix}\right)^2=\widetilde{\psi}\widehat{\Pi}_c\begin{pmatrix}1\\ 0\end{pmatrix}\\
&=\widetilde{\psi}\left|\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}\right|^{-2}\begin{bmatrix}0\\ \mathrm{Re}\left(\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}\right)b_1+\mathrm{Im}\left(\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}\right)b_2\end{bmatrix},
\end{aligned}
$$

and (3.20) follows. Set

$$
\begin{aligned}
\widehat{R}_c\begin{pmatrix}\widehat{\alpha}\\ y_c\end{pmatrix}&=\widetilde{R}_c\begin{pmatrix}\widehat{\alpha}\\ y_c\end{pmatrix}\\
&\quad+\frac{1}{3!}\widehat{\Pi}_c\left\{D^3W(0)\left((I+G_2)\begin{pmatrix}\widehat{\alpha}\\ y_c\end{pmatrix}\right)^3-D^3W(0)\begin{pmatrix}\widehat{\alpha}\\ y_c\end{pmatrix}^3\right\}.
\end{aligned}
$$

Then by (3.22) and (3.7), we deduce that the remainder term satisfies the order condition (3.18). Thus, it only remains to compute $\frac{1}{3!}D^3W(0)\begin{pmatrix}\widehat{\alpha}\\ y_c\end{pmatrix}^3$. In order to compute $\widehat{H}_3(\mathcal{X})$, we consider

$$\begin{aligned}
&D^3W(0)\begin{pmatrix}\widehat{\alpha}\\ y_c\end{pmatrix}^3\\
&\quad= 3\widehat{\alpha}D^2H\left(\overline{v}_{\alpha_k}\right)(y_c,y_c)+6\left(\widehat{\alpha}\right)^2D^2H\left(\overline{v}_{\alpha_k}\right)\left(y_c,\left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}\right)\\
&\qquad+3\left(\widehat{\alpha}\right)^2\alpha_kD^2H\left(\overline{v}_{\alpha_k}\right)\left(y_c,\left.\frac{d^2\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\left(\widehat{\alpha}\right)^2}\right|_{\widehat{\alpha}=0}\right)+\alpha_kD^3H\left(\overline{v}_{\alpha_k}\right)(y_c,y_c,y_c)\\
&\qquad+3\widehat{\alpha}\alpha_kD^3H\left(\overline{v}_{\alpha_k}\right)\left(y_c,y_c,\left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}\right)\\
&\qquad+3\left(\widehat{\alpha}\right)^2\alpha_kD^3H\left(\overline{v}_{\alpha_k}\right)\left(y_c,\left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0},\left.\frac{d\overline{v}_{\widehat{\alpha}+\alpha_k}}{d\widehat{\alpha}}\right|_{\widehat{\alpha}=0}\right).
\end{aligned}$$

Using the same notation as above for y_c and after some computation, we deduce that

$$\frac{1}{3!}D^3W(0)\begin{pmatrix}\widehat{\alpha}\\ y_c\end{pmatrix}^3=\begin{pmatrix}\widehat{\psi}\\ 0\end{pmatrix}$$

with

$$\begin{aligned}
\widehat{\psi}&=\frac{1}{6}h^{(2)}\left(\int_0^{+\infty}\gamma(a)\,\overline{u}_{\alpha_k}(a)\,da\right)\left(3\widehat{a}\left(\frac{2x\chi}{1-\ln(\alpha_k\chi)}\right)^2+\frac{3\left(\widehat{\alpha}\right)^2}{\alpha_k\beta}\frac{2x\chi}{1-\ln(\alpha_k\chi)}\right)\\
&\quad+\frac{1}{6}h^{(3)}\left(\int_0^{+\infty}\gamma(a)\,\overline{u}_{\alpha_k}(a)\,da\right)\times\left[\alpha_k\left(\frac{2x\chi}{1-\ln(\alpha_k\chi)}\right)^3\right.\\
&\quad\left.+\frac{3\widehat{\alpha}}{\beta}\left(\frac{2x\chi}{1-\ln(\alpha_k\chi)}\right)^2+\frac{3\left(\widehat{\alpha}\right)^2}{\alpha_k\beta^2}\frac{2x\chi}{1-\ln(\alpha_k\chi)}\right]\\
&=\frac{1}{(\alpha_k)^2\left(1-\ln(\alpha_k\chi)\right)}\widehat{\alpha}^2x+\frac{2\beta\chi}{\alpha_k\left(1-\ln(\alpha_k\chi)\right)^2}\widehat{\alpha}x^2\\
&\quad+\frac{4\beta^2\left(-\ln(\alpha_k\chi)+3\right)\chi^2}{3\left(1-\ln(\alpha_k\chi)\right)^3}x^3.
\end{aligned}$$

By Lemma 3.1, we obtain

$$\begin{aligned}
\frac{1}{3!}\widehat{\Pi}_cD^3W(0)\begin{pmatrix}\widehat{\alpha}\\ y_c\end{pmatrix}^3&=\widehat{\psi}\widehat{\Pi}_c\begin{pmatrix}1\\ 0\end{pmatrix}\\
&=\widehat{\psi}\left|\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}\right|^{-2}\begin{bmatrix}0\\ Re\left(\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}\right)b_1+\mathrm{Im}\left(\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}\right)b_2\end{bmatrix}
\end{aligned}$$

and (3.21) follows. Moreover, (3.19) can be rewritten as (3.17). □

From Theorem 3.7, dropping the auxiliary equation for the parameter, we obtain the following equations

$$\begin{aligned}\frac{d}{dt}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} &= M_c\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}\\ &+\chi_2\left(\widehat{\alpha},x,y\right)\left|\frac{d\Delta\left(\alpha_k,i\omega_k\right)}{d\lambda}\right|^{-2}\begin{bmatrix}\mathrm{Re}\left(\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}\right)\\ \mathrm{Im}\left(\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}\right)\end{bmatrix}\\ &+\chi_3\left(\widehat{\alpha},x,y\right)\left|\frac{d\Delta\left(\alpha_k,i\omega_k\right)}{d\lambda}\right|^{-2}\begin{bmatrix}\mathrm{Re}\left(\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}\right)\\ \mathrm{Im}\left(\frac{d\Delta(\alpha_k,i\omega_k)}{d\lambda}\right)\end{bmatrix}\\ &+\widehat{R}_c\begin{pmatrix}\widehat{\alpha}\\ x\\ y\end{pmatrix},\end{aligned}\tag{3.23}$$

where M_c, $\chi_2\left(\widehat{\alpha},x,y\right)$, $\chi_3\left(\widehat{\alpha},x,y\right)$ and $\widehat{R}_c$ are defined in Theorem 3.7 and $\widehat{\alpha}$ is the parameter here.

3.3 Direction and Stability of Hopf Bifurcation

We now study the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions following the Hopf bifurcation theorem presented in Hassard et al. [13, p. 16]. We first make some preliminary remarks. Rewrite system (3.23) as follows

$$\frac{dX}{dt}=F(X,\widehat{\alpha}),\tag{3.24}$$

where the equilibrium point is $X=0\in\mathbb{R}^2$ and the critical value of the bifurcation parameter $\widehat{\alpha}$ is 0. Since the equilibrium solutions belong to the center mainfold, we have for each $|\widehat{\alpha}|$ small enough that

$$F(0,\widehat{\alpha})=0.$$

Notice that $\partial_x F\left(0,\widehat{\alpha}\right)$ is unknown whenever $\widehat{\alpha}\neq 0$. The system (3.23) only provides an approximation of order 2 for $\partial_x F\left(0,\widehat{\alpha}\right)$ with respect to $\widehat{\alpha}$. Nevertheless by using Magal and Ruan [25, Proposition 4.22], we know that the eigenvalues $\lambda(\widehat{\alpha})$ of $\partial_x F\left(0,\widehat{\alpha}\right)$ are the roots of the original characteristic equation

$$1=\eta(\alpha)\int_0^{+\infty}\gamma\left(a\right)e^{-(\mu+\lambda)a}da\Leftrightarrow 1=\eta\left(\widehat{\alpha}+\alpha_k\right)\int_0^{+\infty}\gamma\left(a\right)e^{-(\mu+\lambda)a}da\tag{3.25}$$

with

$$\eta(\alpha)=\frac{1-\ln\left(\alpha\int_0^{+\infty}\gamma\left(a\right)e^{-\mu a}da\right)}{\int_0^{+\infty}\gamma\left(a\right)e^{-\mu a}da}=\frac{1-\ln\left(\alpha\chi\right)}{\chi},$$

and

$$\lambda(0)=\pm\omega_k i.$$

The implicit function theorem implies that the characteristic equation has a unique pair of complex conjugate roots $\lambda(\widehat{\alpha})$, $\overline{\lambda(\widehat{\alpha})}$ close to $i\omega_k$, $-i\omega_k$ for $\widehat{\alpha}$ in a neighborhood of 0. Here $\lambda(\widehat{\alpha})=a(\widehat{\alpha})+ib(\widehat{\alpha})$, $a(0)=0$ and $ib(0)=i\omega_k$ (where $\omega_k>0$ are provided by Proposition 2.3 for $k\in\mathbb{N}$). From (3.25), we have

$$\eta'\left(\widehat{\alpha}+\alpha_k\right)\int_0^{+\infty}\gamma\left(a\right)e^{-(\mu+\lambda(\widehat{\alpha}))a}da-\eta\left(\widehat{\alpha}+\alpha_k\right)\int_0^{+\infty}a\gamma\left(a\right)e^{-(\mu+\lambda)a}da\frac{d\lambda(\widehat{\alpha})}{d\widehat{\alpha}}=0$$

and

$$\begin{aligned}\int_0^{+\infty} a\gamma(a)\, e^{-(\mu+\lambda)a} da &= \int_\tau^{+\infty} a\,(a-\tau)^n\, e^{-(\mu+\lambda)a} da \\ &= \int_\tau^{+\infty} (a-\tau)^{n+1}\, e^{-(\mu+\lambda)a} da + \tau \int_\tau^{+\infty} (a-\tau)^n\, e^{-(\mu+\lambda)a} da \\ &= (n+1)!\frac{e^{-(\lambda+\mu)\tau}}{(\varsigma+\lambda+\mu)^{n+2}} + \tau n!\frac{e^{-(\lambda+\mu)\tau}}{(\varsigma+\lambda+\mu)^{n+1}} \\ &= \left[\frac{(n+1)}{(\varsigma+\lambda+\mu)} + \tau\right] n!\frac{e^{-(\lambda+\mu)\tau}}{(\varsigma+\lambda+\mu)^{n+1}} \\ &= \left[\frac{(n+1)}{(\varsigma+\lambda+\mu)} + \tau\right] \int_0^{+\infty} \gamma(a)\, e^{-(\mu+\lambda)a} da.\end{aligned}$$

Thus

$$\eta'(\widehat{\alpha}+\alpha_k) - \eta(\widehat{\alpha}+\alpha_k)\left[\frac{(n+1)}{(\varsigma+\lambda+\mu)} + \tau\right]\frac{d\lambda(\widehat{\alpha})}{d\widehat{\alpha}} = 0$$

and

$$\begin{aligned}\frac{d\lambda(0)}{d\widehat{\alpha}} &= \frac{\eta'(\alpha_k)}{\eta(\alpha_k)}\left[\frac{(n+1)}{(\varsigma+i\omega_k+\mu)} + \tau\right]^{-1} = \frac{\eta'(\alpha_k)}{\eta(\alpha_k)}\frac{(\varsigma+i\omega_k+\mu)}{(n+1)+\tau(\varsigma+i\omega_k+\mu)} \\ &= \frac{\eta'(\alpha_k)}{\eta(\alpha_k)}\frac{(\varsigma+i\omega_k+\mu)\left[(n+1)+\tau(\varsigma+\mu)\right] - i\tau\omega_k]}{\left[(n+1)+\tau(\varsigma+\mu)\right]^2+\tau^2\omega_k^2} \\ a'(0) &= \mathrm{Re}\left[\frac{\eta'(\alpha_k)}{\eta(\alpha_k)}\left[\frac{(n+1)}{(\varsigma+i\omega_k+\mu)} + \tau\right]^{-1}\right] \\ &= \frac{\eta'(\alpha_k)}{\eta(\alpha_k)}\mathrm{Re}\left[\frac{(\varsigma+i\omega_k+\mu)}{(n+1)+\tau(\varsigma+i\omega_k+\mu)}\right]\end{aligned}$$

It follows that

$$a'(0) = \frac{\alpha_k\chi}{[\ln(\alpha_k\chi)-1]}\frac{(\varsigma+\mu)\left[(n+1)+\tau(\varsigma+\mu)\right]+\tau\omega_k^2}{\left[(n+1)+\tau(\varsigma+\mu)\right]^2+\tau^2\omega_k^2} > 0. \tag{3.26}$$

Finally, the spectrum of $\partial_x F(0,\widehat{\alpha})$ is

$$\sigma(\partial_x F(0,\widehat{\alpha})) = \left\{\lambda(\widehat{\alpha}), \overline{\lambda(\widehat{\alpha})}\right\}.$$

Using a procedure as in the proof of Kuznetsov [18, Lemma 3.3 on p. 92] and introducing a complex variable z, we rewrite system (3.24) for sufficiently small $|\widehat{\alpha}|$ as a single equation:

$$\dot{z} = \lambda(\widehat{\alpha})z + g(z,\bar{z};\widehat{\alpha}), \tag{3.27}$$

where

$$\lambda(\widehat{\alpha}) = a(\widehat{\alpha}) + ib(\widehat{\alpha}),\ g(z,\bar{z},\widehat{\alpha}) = \sum_{i+j=2}^{3}\frac{1}{i!j!}g_{ij}(\widehat{\alpha})z^i\bar{z}^j + O(|z|^3).$$

On can verify that system (3.24) satisfies

(1) $F(0,\widehat{\alpha}) = 0$ for $\widehat{\alpha}$ in an open interval containing 0, and $0 \in \mathbb{R}^2$ is an isolated stationary point of F;
(2) $F(X,\widehat{\alpha})$ is jointly $C^{L+2}(L \geq 2)$ in X and $\widehat{\alpha}$ in a neighborhood of $(0,0) \in \mathbb{R}^2 \times \mathbb{R}$;

(3) $A(\widehat{\alpha}) = D_X F(0, \widehat{\alpha})$ has a pair of complex conjugate eigenvalues λ and $\bar{\lambda}$ such that $\lambda(\widehat{\alpha}) = a(\widehat{\alpha}) + ib(\widehat{\alpha})$, where $b(0) = \omega_0 > 0,\ a(0) = 0,\ a'(0) \neq 0$,

then by Hassard et al. [13, Theorem II, p. 16], there exist an $\varepsilon_p > 0$ and a $C^{L+1}-$function

$$\widehat{\alpha}(\varepsilon) = \sum_{1}^{[\frac{L}{2}]} \widehat{\alpha}_{2i}\varepsilon^{2i} + O(\varepsilon^{L+1}), \qquad 0 < \varepsilon < \varepsilon_p, \tag{3.28}$$

such that for each $\varepsilon \in (0, \varepsilon_p)$ system (3.24) has a family of periodic solutions $P_\varepsilon(t)$ with period $T(\varepsilon)$ occurring for $\widehat{\alpha} = \widehat{\alpha}(\varepsilon)$. The period $T(\varepsilon)$ of $P_\varepsilon(t)$ is a $C^{L+1}-$function given by

$$T(\varepsilon) = \frac{2\pi}{\omega_0}\left[1 + \sum_{1}^{[\frac{L}{2}]} \tau_{2i}\varepsilon^{2i}\right] + O(\varepsilon^{L+1}), \qquad 0 < \varepsilon < \varepsilon_p. \tag{3.29}$$

Two of Floquet exponents of $P_\varepsilon(t)$ approach 0 as $\varepsilon \downarrow 0$. One is 0 for $\varepsilon \in (0, \varepsilon_p)$ and the other is a $C^{L+1}-$function

$$\kappa(\varepsilon) = \sum_{1}^{[\frac{L}{2}]} \kappa_{2i}\varepsilon^{2i} + O(\varepsilon^{L+1}), \qquad 0 < \varepsilon < \varepsilon_p. \tag{3.30}$$

Moreover, $P_\varepsilon(t)$ is orbitally asymptotically stable with asymptotic phase if $\kappa(\varepsilon) < 0$ and unstable if $\kappa(\varepsilon) > 0$.

Next we need to compute the coefficients $\widehat{\alpha}_{2i}$ and κ_{2i} in (3.28) and (3.30). If the Poincaré normal form of (3.27) is

$$\dot{\xi} = \lambda(\widehat{\alpha})\xi + \sum_{j=1}^{[L/2]} c_j(\widehat{\alpha})\xi|\xi|^{2j} + O(|\xi||(\xi, \widehat{\alpha})|^{L+1}) \equiv C(\xi, \bar{\xi}, \widehat{\alpha}), \tag{3.31}$$

where $C(\xi, \bar{\xi}, \widehat{\alpha})$ is C^{L+2} jointly in $(\xi, \bar{\xi}, \widehat{\alpha})$ in a neighborhood of $0 \in \mathbb{C} \times \mathbb{C} \times \mathbb{R}$, then the results in Hassard et al. [13, p. 32 and p. 44] imply that the periodic solution of period $T(\varepsilon)$ such that $\xi(0, \widehat{\alpha}) = \varepsilon$ of (3.31) has the form

$$\xi = \varepsilon \exp[2\pi i t/T(\varepsilon)] + O(\varepsilon^{L+2}),$$

where

$$T(\varepsilon) = \frac{2\pi}{\omega_0}\left[1 + \sum_{1}^{L} \tau_i\varepsilon^i\right] + O(\varepsilon^{L+1}) \tag{3.32}$$

and

$$\widehat{\alpha}(\varepsilon) = \sum_{1}^{L} \widehat{\alpha}_i\varepsilon^i + O(\varepsilon^{L+1}). \tag{3.33}$$

Furthermore, the coefficients are given by the following formulae:

$$\begin{aligned}
\widehat{\alpha}_1 &= 0,\\
\widehat{\alpha}_2 &= -\frac{\mathrm{Re}c_1(0)}{a'(0)},\\
\widehat{\alpha}_3 &= 0,\\
\widehat{\alpha}_4 &= -\frac{1}{a'(0)}\left[\mathrm{Re}c_2(0) + \widehat{\alpha}_2\mathrm{Re}c_1'(0) + \frac{a''(0)}{2}\widehat{\alpha}_2^2\right],
\end{aligned}$$

$$
\begin{aligned}
\tau_1 &= 0,\\
\tau_2 &= \frac{-1}{\omega_0}\left[\mathrm{Im}c_1(0) + \widehat{\alpha}_2 b'(0)\right],\\
\tau_3 &= 0,\\
\tau_4 &= -\frac{1}{\omega_0}\left[a'(0)\widehat{\alpha}_4 + \frac{a''(0)}{2}\widehat{\alpha}_2^2 + \mathrm{Im}c_1'(0)\widehat{\alpha}_2 + \mathrm{Im}c_2(0) - \omega_0\tau_2^2\right],\\
\kappa_1 &= 0\\
\kappa_2 &= 2\mathrm{Re}c_1(0),
\end{aligned}
$$

where

$$
c_1(0) = \frac{i}{2\omega_0}\left(g_{20}(0)g_{11}(0) - 2\left|g_{11}(0)\right|^2 - \frac{1}{3}\left|g_{02}(0)\right|^2\right) + \frac{g_{21}(0)}{2}. \tag{3.34}
$$

Applying the results in [13, pp. 45–51], we can change Eq. (3.27) into the Poincaré normal form (3.31) by using the following transformation:

$$
z = \xi + \chi(\xi, \bar{\xi}; \widehat{\beta}) = \xi + \sum_{i+j=2}^{L+1} \frac{1}{i!j!}\chi_{ij}(\widehat{\beta})\xi^i\bar{\xi}^j, \ \chi_{ij} \equiv 0 \text{ for } i = j+1.
$$

To use the bifurcation formulae for $\kappa(\varepsilon), \widehat{\alpha}(\varepsilon)$ and $T(\varepsilon)$, we need only to compute $c_1(0)$, $c_1'(0)$, and $c_2(0)$. For sufficiently small ε, if $\kappa_2 \neq 0$, $\widehat{\alpha}_2 \neq 0$, the stability of the bifurcating periodic solutions and the direction of the Hopf bifurcation are determined by the signs of κ_2 and $\widehat{\alpha}_2$.

By introducing a complex variable $z = x + iy$, when $\widehat{\alpha} = 0$ the system (3.23) reduces to

$$
\dot{z} = -i\omega_k z(t) + \left[\chi_2\left(0, \mathrm{Re}\left(z\right), \mathrm{Im}\left(z\right)\right) + \chi_3\left(0, \mathrm{Re}\left(z\right), \mathrm{Im}\left(z\right)\right)\right]\left(\overline{\frac{d\Delta\left(\alpha_k, i\omega_k\right)}{d\lambda}}\right)^{-1} + h.o.t.
$$

Set $\overline{z}(t) := \overline{z(t)}$, then we obtain

$$
\begin{aligned}
\frac{d\overline{z}(t)}{dt} = {} & i\omega_k\overline{z}(t) + \left[\chi_2\left(0, \mathrm{Re}\left(\overline{z}\right), -\mathrm{Im}\left(\overline{z}\right)\right) + \chi_3\left(0, \mathrm{Re}\left(\overline{z}\right), -\mathrm{Im}\left(\overline{z}\right)\right)\right]\left(\frac{d\Delta\left(\alpha_k, i\omega_k\right)}{d\lambda}\right)^{-1}\\
& + h.o.t.
\end{aligned} \tag{3.35}
$$

where

$$
\begin{aligned}
\chi_2\left(0, \mathrm{Re}\left(\overline{z}\right), -\mathrm{Im}\left(\overline{z}\right)\right) &= \frac{2\chi\beta\left(\ln\left(\alpha_k\chi\right) - 2\right)}{\left[1 - \ln\left(\alpha_k\chi\right)\right]^2}\left(\mathrm{Re}\left(\overline{z}\right)\right)^2,\\
\chi_3\left(0, \mathrm{Re}\left(\overline{z}\right), -\mathrm{Im}\left(\overline{z}\right)\right) &= \frac{4\beta\left(\ln\left(\alpha_k\chi\right) - 2\right)}{1 - \ln\left(\alpha_k\chi\right)}\mathrm{Re}\left(\overline{z}\right)\\
&\quad \times\left[\left(\left(\mathrm{Re}\left(\overline{z}\right)\right)^2 - \left(\mathrm{Im}\left(\overline{z}\right)\right)^2\right)\int_0^{+\infty}\gamma\left(a\right)\mathrm{Re}\psi_{2,2}\left(a\right)da\right.\\
&\quad + \left(\left(\mathrm{Re}\left(\overline{z}\right)\right)^2 + \left(\mathrm{Im}\left(\overline{z}\right)\right)^2\right)\int_0^{+\infty}\gamma\left(a\right)\psi_{2,3}\left(a\right)da\\
&\quad \left. - 2\mathrm{Re}\left(\overline{z}\right)\mathrm{Im}\left(\overline{z}\right)\int_0^{+\infty}\gamma\left(a\right)\mathrm{Im}\psi_{2,2}\left(a\right)da\right]\\
&\quad + \frac{4\beta^2\left(-\ln\left(\alpha_k\chi\right) + 3\right)\chi^2}{3\left(1 - \ln\left(\alpha_k\chi\right)\right)^3}\left(\mathrm{Re}\left(\overline{z}\right)\right)^3,
\end{aligned}
$$

with

$$\psi_{2,2}(a) = \frac{\beta\chi\left(\ln(\alpha_k\chi) - 2\right)}{2\left(1 - \ln(\alpha_k\chi)\right)^2} \times \left(\begin{array}{l} -\dfrac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}^{-1} \dfrac{e^{-(i\omega_k+\mu)a}}{i\omega_k} - \dfrac{d\Delta(\alpha_k, -i\omega_k)}{d\lambda}^{-1} \dfrac{e^{-(-i\omega_k+\mu)a}}{3i\omega_k} \\ + \Delta(\alpha_k, 2i\omega_k)^{-1} e^{-(2i\omega_k+\mu)a} \end{array} \right),$$

$$\psi_{2,3}(a) = \frac{\beta\chi\left(\ln(\alpha_k\chi) - 2\right)}{2\left(1 - \ln(\alpha_k\chi)\right)^2} \times \left(\begin{array}{l} \dfrac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}^{-1} \dfrac{e^{-(i\omega_k+\mu)a}}{i\omega_k} - \dfrac{d\Delta(\alpha_k, -i\omega_k)}{d\lambda}^{-1} \dfrac{e^{-(-i\omega_k+\mu)a}}{i\omega_k} \\ + \Delta(\alpha_k, 0)^{-1} e^{-\mu a} \end{array} \right).$$

Now by considering Eq. (3.35), we obtain that after some computations

$$g_{11} = \frac{\chi\beta\left(\ln(\alpha_k\chi) - 2\right)}{\left[1 - \ln(\alpha_k\chi)\right]^2} \left(\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}\right)^{-1}.$$

Moreover, we deduce that

$$\begin{aligned} g_{20} &= g_{11}, g_{02} = g_{11}, \\ g_{21} &= \frac{ice + cf + be - ibf + 3ae - i3af}{4} \end{aligned} \tag{3.36}$$

where

$$\begin{aligned} a &= \frac{4\chi^2\beta^2\left(3 - \ln(\alpha_k\chi)\right)}{3\left[1 - \ln(\alpha_k\chi)\right]^3} \\ &\quad + \frac{4\beta\left(\ln(\alpha_k\chi) - 2\right)}{1 - \ln(\alpha_k\chi)} \left[\int_0^{+\infty} \gamma(a)\,\mathrm{Re}\psi_{2,2}(a)\,da + \int_0^{+\infty} \gamma(a)\,\psi_{2,3}(a)\,da\right], \\ b &= \frac{4\beta\left(\ln(\alpha_k\chi) - 2\right)}{1 - \ln(\alpha_k\chi)} \left[-\int_0^{+\infty} \gamma(a)\,\mathrm{Re}\psi_{2,2}(a)\,da + \int_0^{+\infty} \gamma(a)\,\psi_{2,3}(a)\,da\right], \\ c &= \frac{8\beta\left(\ln(\alpha_k\chi) - 2\right)}{1 - \ln(\alpha_k\chi)} \int_0^{+\infty} \gamma(a)\,\mathrm{Im}\psi_{2,2}(a)\,da, \\ e &= \left|\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}\right|^{-2} \mathrm{Re}\left(\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}\right), \\ f &= \left|\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}\right|^{-2} \mathrm{Im}\left(\frac{d\Delta(\alpha_k, i\omega_k)}{d\lambda}\right). \end{aligned}$$

Hence, we obtain

$$c_1(0) = \frac{i}{2\omega_0}\left(g_{20}(0)g_{11}(0) - 2\left|g_{11}(0)\right|^2 - \frac{1}{3}\left|g_{02}(0)\right|^2\right) + \frac{g_{21}(0)}{2}$$

and

$$\widehat{\alpha}_2 = -\frac{\mathrm{Re}c_1(0)}{\alpha'(0)}, \kappa_2 = 2\mathrm{Re}c_1(0), \tau_2 = \frac{-1}{\omega_k}\left[\mathrm{Im}c_1(0) + \widehat{\alpha}_2 b'(0)\right].$$

We summarize the above discussions into a theorem on the direction and stability of Hopf bifurcation in the age structured model (1.1).

Theorem 3.8 *The direction of the Hopf bifurcation described in Theorem 2.4 is determined by the sign of* $\widehat{\alpha}_2$ *: if* $\widehat{\alpha}_2 > 0(<0)$, *then the bifurcating periodic solutions exist for* $\alpha > \alpha_k(\alpha < \alpha_k)$. *The bifurcating periodic solutions are stable (unstable) if* $\kappa_2 < 0(>0)$. *The period of the bifurcating periodic solutions of the age structured model (1.1) increases (decreases) if* $\tau_2 > 0(<0)$.

We would like to mention that though normal forms have been developed for some partial differential equations by Kokubu [16] and Eckmann et al. [9], but their results are for parabolic equations and do not apply to our age structured model (1.1) which is a hyperbolic equation. The normal form theory developed in Liu et al. [21] is for general abstract semilinear equations with non-dense domain which can be applied to several types of equations. In this paper, we have applied the theory to study the normal form of a class of hyperbolic partial differential equations. We believe that this normal form theory can be applied to some other types of equations including delay differential equations (Faria and Magalhães [10,11]), transport equations (Perthame [29]), reaction–diffusion equations (Kokubu [16], Eckmann et al. [9]), and partial differential equations with delay (Ducrot et al. [8]).

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