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TRAVELING WAVE SOLUTIONS FOR TIME PERIODIC REACTION-DIFFUSION SYSTEMS

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Abstract. This paper deals with traveling wave solutions for time periodic reaction-diffusion systems. The existence of traveling wave solutions is established by combining the fixed point theorem with super- and sub-solutions, which reduces the existence of traveling wave solutions to the existence of super- and sub-solutions. The asymptotic behavior is determined by the stability of periodic solutions of the corresponding initial value problems. To illustrate the abstract results, we investigate a time periodic Lotka-Volterra system with two species by presenting the existence and nonexistence of traveling wave solutions, which connect the trivial steady state to the unique positive periodic solution of the corresponding kinetic system.

1. **Introduction.** Traveling wave solutions of reaction-diffusion systems have been widely studied since 1937 [16, 22]. For autonomous reaction-diffusion systems, a typical example is

$$\frac{\partial w(x,t)}{\partial t} = D\frac{\partial^2 w(x,t)}{\partial x^2} + h(w),\ x \in \mathbb{R},\ t > 0, \tag{1}$$

where $w = (w_1, w_2, \cdots, w_n) \in \mathbb{R}^n$, $D = \mathrm{diag}\{d_1, d_2, \cdots, d_n\}$ is a matrix with positive constants $d_1, d_2, \cdots, d_n$, $h = (h_1, h_2, \cdots, h_n) : \mathbb{R}^n \to \mathbb{R}^n$ is a given function. Here, a traveling wave solution of (1) is a special entire solution (defined for all $t \in \mathbb{R}$) taking the form

$$w(x,t) = \rho(\xi),\ \xi = x + ct,$$

in which $c \in \mathbb{R}$ is the parameter of wave speed and ρ formulates the wave profile. Clearly, letting $w(x,t) = \rho(\xi)$, we obtain the following ODE system

$$D\rho''(\xi) - c\rho'(\xi) + h(\rho(\xi)) = 0,\ \xi \in \mathbb{R},$$

so the existence of traveling wave solutions can be studied by the theory of phase analysis, see Volpert et al. [42], Ye et al. [47, Chapter 1]. Besides phase analysis, there are also some other methods that can be used to study the existence of traveling wave solutions of (1), we refer to Ducrot et al. [9], Fife [14], Gilding and

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Kersner [17], Smoller [39], Volpert et al. [42], Wu [45] for many important results on this topic.

When the reaction-diffusion systems are not autonomous, one important case is the following time periodic system

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2} + h(t,u),\ x \in \mathbb{R},\ t > 0, \tag{2}$$

where $u = (u_1, u_2, \cdots, u_n) \in \mathbb{R}^n$, $h = (h_1, h_2, \cdots, h_n) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a given function and is T-periodic in t. For the nonautonomous periodic system (2), a time periodic traveling wave solution (for simplicity, a traveling wave solution) is a special entire solution taking the following form

$$u(x,t) = \psi(z,t),\ \psi(z,t) = \psi(z,t+T),\ z = x + ct,$$

in which c also denotes the wave speed while $\psi = (\psi_1, \psi_2, \cdots, \psi_n) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$ formulates the wave profile. Thus, the corresponding wave form system is

$$\frac{\partial \psi(z,t)}{\partial t} = D\frac{\partial^2 \psi(z,t)}{\partial z^2} - c\frac{\partial \psi(z,t)}{\partial z} + h(t,\psi), z \in \mathbb{R}, t \in \mathbb{R}. \tag{3}$$

Clearly, (3) is also a partial differential system. Comparing the nonautonomous cases with the autonomous ones, there are some significant differences. For example, (3) cannot be investigated by phase analysis of ODE systems.

Of course, there are many results concerning the existence of solutions of (3) as well as some other properties (e.g., uniqueness and stability). Alikakos et al. [1], Bates and Chen [5], Shen [36] and Xin [46] established the existence and global stability of periodic traveling waves with bistable nonlinearities. Fang and Zhao [12] proved the existence of bistable traveling wave solutions of monotone semiflows. Guo and Hamel [18] obtained the minimal wave speed of a lattice differential system with periodic parameters. Liang et al. [24] established propagation theory of monotone periodic evolution systems with the abstract conclusions in [25]. Nadin [31] and Nolen et al. [32] investigated the traveling wave solutions of a reaction-diffusion equation with time and space periodic parameters. Using monotone iterations, Zhao and Ruan [51, 52] investigated the traveling wave solutions of reaction-diffusion systems with time periodic parameters. Bao et al. [3] studied a nonlocal dispersal model, Bao and Wang [4] obtained the existence and stability of bistable traveling wave solutions of a Lotka-Volterra system. Wang et al. [44] considered the traveling wave solutions of a periodic and diffusive SIR epidemic model. The traveling wave solutions of almost periodic models were studied by Shen [34, 35].

Besides the traveling wave solutions, there are also some other results concerning the propagation theory of time periodic systems. In Liang et al. [24], the authors discussed the asymptotic speed of spread. Shen [37], Huang and Shen [21] considered the spreading speed in time almost periodic and space periodic KPP models. Nolen et al. [32] studied the asymptotic spreading as well as the traveling wave solutions in periodic cases. Moreover, some important propagation results on general reaction-diffusion systems were established, of which the results remain true for periodic systems. For example, Berestycki et al. [6, 7] investigated the asymptotic spreading of reaction-diffusion equations in heterogeneous excitable media. Sheng and Cao [38] obtained the entire solutions for periodic equations.

It should be noted that in most of the above results, the comparison principle appealing to cooperative systems plays a very important role. Moreover, due to the mutual property, traveling wave solutions are monotone such that their limit behavior is easy to confirm. It is well known that if the system is not cooperative, then the limit behavior of the solutions may be very complicated even if it is an ODE system. One typical example is the predator-prey system. On the traveling wave solutions, Berestycki et al. [8], Fang and Zhao [11], Faria and Trofimchuk [13] confirmed the existence of nonmonotone traveling wave solutions in scalar equations with nonlocality or time delay. When autonomous Lotka-Volterra competitive systems (see Tang and Fife [40]) are concerned, Lin and Ruan [26] proved the existence of nonmonotone traveling wave solutions connecting $(0,0)$ with the positive steady state under proper conditions.

The purpose of this paper is to investigate the traveling wave solutions of time periodic reaction-diffusion systems with more general monotone conditions, e.g., Lotka-Volterra competitive systems with two species. We first investigate the existence of traveling wave solutions. Motivated by [23, 44], we obtain the existence of traveling wave solutions by the fixed point theorem if the system satisfies proper monotone conditions. To confirm the asymptotic behavior of traveling wave solutions, we apply the stability results of periodic solutions of the corresponding initial value problems that usually can be investigated by the corresponding kinetic systems. Note that the stability of periodic solutions has been widely studied, then at least for some classical systems including cooperative systems and competitive systems, the verification of limit behavior for traveling wave solutions becomes an easy job.

In particular, for some autonomous nonmonotone systems (e.g., delayed equations [29], integral equations [10], abstract recursions [48]), there are also some methods which can be used to determine the limit behavior of nontrivial traveling wave solutions, e.g., perturbation method [20], contracting rectangles [26]. However, because the limit behavior of nonautonomous systems may depend on time (is not a constant), it is difficult to construct proper auxiliary functions similar to these results. Fortunately, our methods and ideas could apply to many nonmonotone systems although our proof seems to be very simple.

The rest of this paper is organized as follows. In Section 2, we investigate the existence of traveling wave solutions of some monotone systems by the fixed point theorem. The asymptotic behavior of traveling wave solutions is studied in Section 3. Finally, we consider a time periodic Lotka-Volterra competitive system by presenting the existence and nonexistence of traveling wave solutions.

2. **Existence of nonconstant wave solutions.** We use the standard partial ordering in $\mathbb{R}^n$. That is, if $u=(u_1,u_2,\cdots,u_n), v=(v_1,v_2,\cdots,v_n)\in\mathbb{R}^n$, then $u\geq v$ iff $u_i\geq v_i, i=1,2,\cdots,n$; $u>v$ iff $u\geq v$ but $u_i>v_i$ for some $i\in\{1,2,\cdots,n\}$; $u\gg v$ iff $u_i>v_i, i=1,2,\cdots,n$. Moreover, $|\cdot|$ is the supremum norm in $\mathbb{R}^n$.

In this section, we shall investigate the existence of traveling wave solutions, which is motivated by Wang et al. [44]. It should be noted that a periodic SIR epidemic model with diffusion and standard incidence was considered in [44], and the existence of traveling wave solutions was established by combining the fixed point theorem with super- and sub-solutions. To focus on our main idea, we first consider the following competitive system

$$\begin{cases} u_t(x,t)=d_1u_{xx}(x,t)+f(t,u(x,t),v(x,t)),\ x\in\mathbb{R},t>0,\\ v_t(x,t)=d_2v_{xx}(x,t)+g(t,u(x,t),v(x,t)),\ x\in\mathbb{R},t>0\end{cases}\tag{4}$$

under the following conditions

(A1) $d_1 > 0, d_2 > 0$;
(A2) f, g are T-periodic functions in t with some $T > 0$, that is,
$$f(t+T,\cdot,\cdot) = f(t,\cdot,\cdot),\ g(t+T,\cdot,\cdot) = g(t,\cdot,\cdot), t \in \mathbb{R};$$
(A3) there are constants $M_1 > 0, M_2 > 0$ such that
$$f(t,0,0) = 0,\ f(t,M_1,0) < 0,\ g(t,0,0) = 0,\ g(t,0,M_2) < 0,\ t \in [0,T];$$
(A4) $f(t,u,v)$ is nonincreasing in v while $g(t,u,v)$ is nonincreasing in u, where $u \in [0,M_1], v \in [0,M_2]$;
(A5) for some $\alpha \in (0,1)$, $f(t,\cdot,\cdot)$ and $g(t,\cdot,\cdot)$ are C^α in t, $f(\cdot,u,v)$ and $g(\cdot,u,v)$ are Lipschitz continuous in $u \in [0,M_1], v \in [0,M_2]$.

Let
$$u(x,t) = U(z,t),\ v(x,t) = V(z,t),\ z = x + ct$$
be a traveling wave solution of (4). Then the corresponding wave system is
$$\begin{cases} U_t(z,t) = d_1 U_{zz}(z,t) - cU_z(z,t) + f(t,U,V),\ z \in \mathbb{R},\ t \in \mathbb{R}, \\ V_t(z,t) = d_2 V_{zz}(z,t) - cV_z(z,t) + g(t,U,V),\ z \in \mathbb{R},\ t \in \mathbb{R}, \end{cases} \tag{5}$$
and (U,V) satisfies the following periodic conditions
$$U(z,t) = U(z,t+T),\ V(z,t) = V(z,t+T),\ z \in \mathbb{R},\ t \in \mathbb{R}. \tag{6}$$

Let X, Y be
$$\begin{aligned} X &= \{u\colon \mathbb{R} \to \mathbb{R} \text{ is bounded and uniformly continuous}\}, \\ Y &= \{v\colon \mathbb{R} \times [0,T] \to \mathbb{R} \text{ is bounded and uniformly continuous}\}, \end{aligned}$$
which are Banach spaces equipped with the standard super norms $|\cdot|_X, |\cdot|_Y$, so for X^2, Y^2. Let μ be a positive constant. Define
$$B_\mu\left(\mathbb{R} \times [0,T], \mathbb{R}^2\right) := \left\{ u = (u_1,u_2) : \begin{array}{l} u_i \in Y, \sup_{t\in[0,T],x\in\mathbb{R}}[e^{-\mu|x|}|u_i(x,t)|] < \infty, \\ u_i(x,0) = u_i(x,T), x \in \mathbb{R},\ i = 1,2 \end{array} \right\}$$
equipped with the norm
$$|u|_\mu := \max\left\{ \sup_{t\in[0,T],x\in\mathbb{R}} [e^{-\mu|x|}|u_1(x,t)|], \sup_{t\in[0,T],x\in\mathbb{R}} [e^{-\mu|x|}|u_2(x,t)|] \right\},$$
which is a Banach space. Furthermore, it has the following nice property.

Proposition 1. *Assume that $D \subset Y^2$. For any given bounded interval I, if D restricted on $I \times [0,T]$ is precompact in the sense of the supremum norm, then D is precompact in the sense of $|\cdot|_\mu$.*

Let $K > 0$ be a constant such that
$$f(t,u,v) + Ku$$
is monotone increasing in $u \in [0,M_1]$ for any $t \in [0,T]$ and $v \in [0,M_2]$, and
$$g(t,u,v) + Kv$$
is monotone increasing in $v \in [0,M_2]$ for any $t \in [0,T]$ and $u \in [0,M_1]$. Define
$$\left(T_i(t)w\right)(x) = e^{-Kt}\frac{1}{\sqrt{4\pi d_i t}}\int_{\mathbb{R}} e^{-\frac{(x-ct-y)^2}{4d_i t}} w(y)dy,\ i = 1,2 \tag{7}$$

for $t > 0, x \in \mathbb{R}, w \in X$. Then $T_i : X \to X, i = 1, 2$, are analytic semigroups generated by

$$A_i u = d_i u_{zz} - c u_z - K u,\ i = 1, 2,$$

respectively. In particular, we have

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |\left(T_i(t)w\right)(x)| = 0,\ i = 1, 2$$

for any given $w \in X$. For details, see the Appendix.

By the smooth condition (A5), periodic condition (A2) and the theory of analytic semigroups (see the Appendix), the existence of (5)-(6) can be obtained by the existence of mild solutions of the following integral system

$$\begin{cases} U(z,t) = \left(T_1(t)U(\cdot,0)\right)(z) + \int_0^t T_1(t-s)F(U,V,\cdot,s)ds(z), \\ V(z,t) = \left(T_2(t)V(\cdot,0)\right)(z) + \int_0^t T_2(t-s)G(U,V,\cdot,s)ds(z) \end{cases} \tag{8}$$

with (6) for $z \in \mathbb{R}, t \in [0, T]$, where

$$F(U,V,z,t) = KU(z,t) + f(t, U(z,t), V(z,t)),$$
$$G(U,V,z,t) = KV(z,t) + g(t, U(z,t), V(z,t)).$$

For the purpose, we shall use the fixed point theorem and try to define a nonlinear operator on a proper convex set. According to the monotone condition (A4), we now give the following definition of super- and sub-solutions.

Definition 2.1. Assume that $(\overline{U}(z,t), \overline{V}(z,t))$ and $(\underline{U}(z,t), \underline{V}(z,t))$ are continuous functions for $z \in \mathbb{R}, t \in [0, T^*]$ with some $T^* > 0$. Then they are *a pair of super- and sub-solutions* of (8) if

(1) $(0,0) \le (\underline{U}(z,0), \underline{V}(z,0)) \le (\overline{U}(z,0), \overline{V}(z,0)) \le (M_1, M_2)$ for $z \in \mathbb{R}$;
(2) $(0,0) \le (\underline{U}(z,t), \underline{V}(z,t)), (\overline{U}(z,t), \overline{V}(z,t)) \le (M_1, M_2)$ for $z \in \mathbb{R}, t \in [0, T^*]$;
(3) they satisfy

$$\begin{cases} \overline{U}(z,t) \ge \left(T_1(t-r)\overline{U}(\cdot,r)\right)(z) + \int_r^t T_1(t-s)F(\overline{U},\underline{V},\cdot,s)ds(z), \\ \overline{V}(z,t) \ge \left(T_2(t-r)\underline{V}(\cdot,r)\right)(z) + \int_r^t T_2(t-s)G(\underline{U},\overline{V},\cdot,s)ds(z), \\ \underline{U}(z,t) \le \left(T_1(t-r)\underline{U}(\cdot,r)\right)(z) + \int_r^t T_1(t-s)F(\underline{U},\overline{V},\cdot,s)ds(z), \\ \underline{V}(z,t) \le \left(T_2(t-r)\underline{V}(\cdot,r)\right)(z) + \int_r^t T_2(t-s)G(\overline{U},\underline{V},\cdot,s)ds(z) \end{cases} \tag{9}$$

for all $0 \le r < t \le T^*, z \in \mathbb{R}$.

By the monotone condition, we have the following conclusion.

Proposition 2. *Assume that* $(\overline{U}(z,t), \overline{V}(z,t))$ *and* $(\underline{U}(z,t), \underline{V}(z,t))$ *are a pair of super- and sub-solutions of* (8). *Then*

$$(0,0) \le (\underline{U}(z,t), \underline{V}(z,t)) \le (\overline{U}(z,t), \overline{V}(z,t)) \le (M_1, M_2), z \in \mathbb{R}, t \in [0, T^*].$$

Theorem 2.2. *Assume that (A1)-(A5) hold. If*

$$(\overline{U}(z,t), \overline{V}(z,t)), (\underline{U}(z,t), \underline{V}(z,t))$$

are a pair of super- and sub-solutions of (8) *for* $z \in \mathbb{R}, t \in [0, 2T]$ *such that*

$$\begin{cases} (\overline{U}(z,0), \overline{V}(z,0)) = (\overline{U}(z,T), \overline{V}(z,T)) = (\overline{U}(z,2T), \overline{V}(z,2T)), \\ (\underline{U}(z,0), \underline{V}(z,0)) = (\underline{U}(z,T), \underline{V}(z,T)) = (\underline{U}(z,2T), \underline{V}(z,2T)) \end{cases} \tag{10}$$

for $z \in \mathbb{R}$, *then* (5) *has a positive solution* $(U(z,t), V(z,t))$ *satisfying* (6) *and*

$$(\underline{U}(z,t), \underline{V}(z,t)) \le (U(z,t), V(z,t)) \le (\overline{U}(z,t), \overline{V}(z,t)),\ z \in \mathbb{R}, t \in [0, T].$$

Proof. Define

$$D_1 = \left\{(U,V) \in B_\mu : (\underline{U},\underline{V}) \le (U,V) \le (\overline{U},\overline{V})\right\}$$

and

$$D_2 = \left\{(\phi,\psi) \in X^2 : (\underline{U}(z,0),\underline{V}(z,0)) \le (\phi,\psi) \le (\overline{U}(z,0),\overline{V}(z,0))\right\}.$$

Then D_1 and D_2 are nonempty and convex by (10). Moreover, they are bounded and closed in the sense of the corresponding norms. Firstly, for any given $(U_1,V_1) \in D_1$, we consider the following integral system

$$\begin{cases} U_2(z,t) = \left(T_1(t)U_2(\cdot,0)\right)(z) + \int_0^t T_1(t-s)F(U_1,V_1,\cdot,s)ds(z), \\ V_2(z,t) = \left(T_2(t)V_2(\cdot,0)\right)(z) + \int_0^t T_2(t-s)G(U_1,V_1,\cdot,s)ds(z). \end{cases} \tag{11}$$

Our question is the existence and uniqueness of $(U_2(z,t),V_2(z,t)) \in D_1$. By the smooth condition (see the Appendix, Theorem A) as well as the comparison principle, for any given $(U_2(z,0),V_2(z,0)) \in D_2$, there exists $(U_2(z,t),V_2(z,t))$ satisfying

$$(\underline{U}(z,t),\underline{V}(z,t)) \le (U_2(z,t),V_2(z,t)) \le (\overline{U}(z,t),\overline{V}(z,t)).$$

Furthermore, let $(U_3(z,0),V_3(z,0)),(U_4(z,0),V_4(z,0)) \in D_2$, and

$$\begin{cases} U_i(z,t) = \left(T_1(t)U_i(\cdot,0)\right)(z) + \int_0^t T_1(t-s)F(U_1,V_1,\cdot,s)ds(z), \\ V_i(z,t) = \left(T_2(t)V_i(\cdot,0)\right)(z) + \int_0^t T_2(t-s)G(U_1,V_1,\cdot,s)ds(z) \end{cases}$$

for $i = 3,4$. Then

$$\begin{aligned} |U_3(z,T) - U_4(z,T)| &\le e^{-KT}|U_3(\cdot,0) - U_4(\cdot,0)|_X, \\ |V_3(z,T) - V_4(z,T)| &\le e^{-KT}|V_3(\cdot,0) - V_4(\cdot,0)|_X, \end{aligned}$$

which further imply the existence and uniqueness of $(U_2(z,t),V_2(z,t)) \in D_1$ with

$$(U_2(z,0),V_2(z,0)) = (U_2(z,T),V_2(z,T)), z \in \mathbb{R} \tag{12}$$

by contracting mapping principle on D_2.

Due to the above discussion, for each $(U_1,V_1) \in D_1$, (11) defines a unique $(U_2,V_2) \in D_1$ satisfying (12). Denote it by $F : D_1 \to D_1$. Thus, we only need to prove the existence of a fixed point of F in D_1.

Clearly, $F : D_1 \to D_1$ is continuous in the sense of B_μ. Moreover, for each fixed $(U_1,V_1) \in D_1$, although the smoothness of $(U_2(z,t),V_2(z,t)), z \in \mathbb{R}, t \in [0,T]$ can not be well formulated, the regularity for $z \in \mathbb{R}, t \in [T,2T]$ will be improved by the periodicity. Letting

$$(U_1(z,t),V_1(z,t)) = (U_1(z,t+T),V_1(z,t+T)), z \in \mathbb{R}, t \in [0,T],$$

then

$$(U_2(z,t),V_2(z,t)) = (U_2(z,t+T),V_2(z,t+T)), z \in \mathbb{R}, t \in [0,T].$$

Moreover, they are uniformly bounded in $C^{2\alpha,\alpha}$ (see the Appendix). Therefore, by Ascoli-Arzela lemma, for any bounded interval $I \subset \mathbb{R}$, $\{(U_2(z,t),V_2(z,t)), z \in I, t \in [0,T]\}$ is precompact in the sense of supremum norm, and so $F : D_1 \to D_1$ is compact in the sense of B_μ by Proposition 1. The existence of a fixed point follows by Schauder's fixed point theorem. The proof is complete. □

However, it is difficult to verify (9) due to the nonlocality. Fortunately, the process can be finished by some differential inequalities. We now introduce the following definition of the differential systems (see [15, 30, 33, 40, 47]).

Definition 2.3. Assume that

$$\overline{U}(z,t)=\min\{\overline{U}_1(z,t),\cdots,\overline{U}_l(z,t)\},\ \overline{V}(z,t)=\min\{\overline{V}_1(z,t),\cdots,\overline{V}_l(z,t)\}$$

and

$$\underline{U}(z,t)=\max\{\underline{U}_1(z,t),\cdots,\underline{U}_l(z,t)\},\ \underline{V}(z,t)=\max\{\underline{V}_1(z,t),\cdots,\underline{V}_l(z,t)\}$$

for some integer l, $z\in\mathbb{R}, t\in[0,T^*]$ with $T^*>0$, are continuous functions. Then they are *a pair of super- and sub-solutions* of (5) if

(1) $(0,0)\le(\underline{U}(z,0),\underline{V}(z,0))\le(\overline{U}(z,0),\overline{V}(z,0))\le(M_1,M_2)$, and
$(0,0)\le(\underline{U}(z,t),\underline{V}(z,t)),(\overline{U}(z,t),\overline{V}(z,t))\le(M_1,M_2), z\in\mathbb{R}, t\in[0,T^*]$;

(2) for any given $(z_0,t_0)\in\mathbb{R}\times(0,T^*)$, if $\underline{U}(z_0,t_0)=\underline{U}_j(z_0,t_0)$ for some $j\in\{1,\cdots,l\}$, then there exists a neighborhood $B(z_0,t_0)\in\mathbb{R}\times(0,T^*)$ such that
$$\underline{U}_{j,t}(z,t)\le d_1\underline{U}_{j,zz}(z,t)-c\underline{U}_{j,z}(z,t)+f(t,\underline{U}_j,\overline{V})$$
when $(z,t)\in B(z_0,t_0)$;

(3) for any given $(z_0,t_0)\in\mathbb{R}\times(0,T^*)$, if $\underline{V}(z_0,t_0)=\underline{V}_j(z_0,t_0)$ for some $j\in\{1,\cdots,l\}$, then there exists a neighborhood $B(z_0,t_0)\in\mathbb{R}\times(0,T^*)$ such that
$$\underline{V}_{j,t}(z,t)\le d_2\underline{V}_{j,zz}(z,t)-c\underline{V}_{j,z}(z,t)+g(t,\overline{U},\underline{V}_j)$$
when $(z,t)\in B(z_0,t_0)$;

(4) for any given $(z_0,t_0)\in\mathbb{R}\times(0,T^*)$, if $\overline{U}(z_0,t_0)=\overline{U}_j(z_0,t_0)$ for some $j\in\{1,\cdots,l\}$, then there exists some neighborhood $B(z_0,t_0)\in\mathbb{R}\times(0,T^*)$ such that
$$\overline{U}_{j,t}(z,t)\ge d_1\overline{U}_{j,zz}(z,t)-c\overline{U}_{j,z}(z,t)+f(t,\overline{U}_j,\underline{V})$$
when $(z,t)\in B(z_0,t_0)$;

(5) for any given $(z_0,t_0)\in\mathbb{R}\times(0,T^*)$, if $\overline{V}(z_0,t_0)=\overline{V}_j(z_0,t_0)$ for some $j\in\{1,\cdots,l\}$, then there exists some neighborhood $B(z_0,t_0)\in\mathbb{R}\times(0,T^*)$ such that
$$\overline{V}_{j,t}(z,t)\ge d_2\overline{V}_{j,zz}(z,t)-c\overline{V}_{j,z}(z,t)+g(t,\underline{U},\overline{V}_j)$$
when $(z,t)\in B(z_0,t_0)$.

In light of the positivity of (T_1,T_2), we have the following conclusion.

Theorem 2.4. *Theorem 2.2 remains true if we replace the super- and sub-solutions in Definition 2.1 by those in Definition 2.3.*

3. Asymptotic behavior of nontrivial wave solutions.

In this section, we study the asymptotic behavior of nontrivial solutions of (3). Firstly, we consider the corresponding reaction system as follows

$$\frac{du(t)}{dt}=h(t,u),\ t>0,\tag{13}$$

where h is same as that in (2) and $h(t,\cdot)$ is C^α in t for some $\alpha\in(0,1)$, $h(\cdot,u)$ is Lipschitz continuous in u. In particular, we assume that

$$u^*(t)=(u_1^*(t),u_2^*(t),\cdots,u_n^*(t))\tag{14}$$

is a T-periodic solution of (13).

Further consider the initial value problem

$$\begin{cases}\frac{\partial u(x,t)}{\partial t}=D\frac{\partial^2 u(x,t)}{\partial x^2}+h(t,u),\ x\in\mathbb{R},\ t>0,\\ u(x,0)=u_0(x),\ x\in\mathbb{R},\end{cases}\tag{15}$$

where the initial value $u_0 = (u_{1,0}, u_{2,0}, \cdots, u_{n,0}) : \mathbb{R} \to \mathbb{R}^n$ is bounded and uniformly continuous. Moreover, we assume that there exist $A, B \in \mathbb{R}^n$ such that (15) has a unique global solution

$$A \le u(x,t) \le B,\ x \in \mathbb{R},\ t > 0$$

if

$$A \le u_0(x) \le B,\ x \in \mathbb{R}.$$

For this system, there are many important results about the long time behavior, see some results in [14, 39, 47, 49, 53]. In particular, the stability of $u^*(t)$ in (15) can be obtained by that in (13) under proper conditions, at least for some monotone systems including different Lotka-Volterra type systems, see Teng and Chen [41] for some results.

Our main results are presented as follows.

Theorem 3.1. *Assume that there exist $a, b \in \mathbb{R}^n$ with*

$$a = (a_1, a_2, \cdots, a_n),\ b = (b_1, b_2, \cdots, b_n)$$

and

$$A \le a \le b \le B.$$

Let $\psi = (\psi_1, \psi_2, \cdots, \psi_n)$ be a solution of (3) with

$$a_i < \inf_{t\in[0,T]} \liminf_{z\to\infty} \psi_i(z,t) \le \sup_{t\in[0,T]} \limsup_{z\to\infty} \psi_i(z,t) < b_i,\ i = 1, 2, \cdots, n. \tag{16}$$

If $u^*(t)$ satisfies

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u(x,t) - u^*(t)| = 0$$

when

$$a \le u_0(x) \le b,\ x \in \mathbb{R},$$

then

$$\lim_{z\to\infty} \psi(z,t) = u^*(t),$$

in which the convergence is uniform in $t \in \mathbb{R} (t \in [0,T])$.

Before proving our conclusion, we give the following result.

Lemma 3.2. *Assume that $u^*(t)$ satisfies*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |w(x,t) - u^*(t)| = 0,$$

where $w(x,t)$ is the solution of

$$\begin{cases} \frac{\partial w(x,t)}{\partial t} = D\frac{\partial^2 w(x,t)}{\partial x^2} + h(t,w),\ x \in \mathbb{R}, t > 0, \\ w(x,0) = w(x), x \in \mathbb{R} \end{cases}$$

with uniformly continuous $w(x) = (w_1(x), w_2(x), \cdots, w_n(x))$ and

$$a \le w(x) \le b, x \in \mathbb{R}.$$

Then for any $\epsilon > 0$, there exist $T_1 > 0$ and $N = N(T_1, \epsilon) > 0$ such that

$$|u(x,t) - u^*(t)| < \epsilon,\ x \ge 0,\ t \in [T_1, T_1 + 2T]$$

provided that

$$a \le u_0(x) \le b,\ x > -N,\ \text{and } A \le u_0(x) \le B,\ x \in \mathbb{R}. \tag{17}$$

Proof. By the property of $w(x,t)$, there exists $T_1 > 0$ such that

$$\sup_{x\in\mathbb{R}} |w(x,t) - u^*(t)| < \frac{\epsilon}{2},\ t \geq T_1.$$

Consider (15) for $t \in [0, T_1 + 2T]$, then the result is true once there exists $N > 0$ such that

$$\sup_{x\geq 0, t\in[T_1, T_1+2T]} |u(x,t) - w(x,t)| < \frac{\epsilon}{2} \tag{18}$$

with $u_0(x)$ satisfying (17). In fact, let us consider $\varphi = u - w := (\varphi_1, \varphi_2, \cdots, \varphi_n)$, then

$$\begin{cases} \frac{\partial \varphi(x,t)}{\partial t} = D\Delta\varphi(x,t) + H(x,t), x \in \mathbb{R}, t > 0, \\ \varphi(x,0) = \varphi(x) = u_0(x) - w(x), x \in \mathbb{R}, \end{cases} \tag{19}$$

where $H(x,t) = h(t,u) - h(t,w) := (\widetilde{h}_1(x,t), \cdots, \widetilde{h}_n(x,t))$ and

$$\varphi(x,0) = 0, x \geq -N,$$

in which N will be clarified later. By the Lipschitz continuity of h, there exists a constant $L > 0$ such that

$$-L(|\varphi_1| + |\varphi_2| + \cdots + |\varphi_n|) \leq \widetilde{h}_i(x,t) \leq L(|\varphi_1| + |\varphi_2| + \cdots + |\varphi_n|)$$

for all $i = 1, 2, \cdots, n, t > 0, x \in \mathbb{R}$. In what follows, we denote

$$\mathcal{H}(\varphi) = L(|\varphi_1| + |\varphi_2| + \cdots + |\varphi_n|).$$

According to the selection of A, B, we see that $\varphi(x,t)$ is bounded and well defined for all $t > 0, x \in \mathbb{R}$. To estimate the property of $\varphi(x,t)$, we shall construct a pair of generalized upper and lower solutions for (19). In particular, $\overline{\varphi} := (\overline{\varphi}_1, \overline{\varphi}_2, \cdots, \overline{\varphi}_n)$ and $\underline{\varphi} := (\underline{\varphi}_1, \underline{\varphi}_2, \cdots, \underline{\varphi}_n)$ are called a pair of *generalized upper and lower solutions* of (19) if

$$\begin{cases} \frac{\partial \overline{\varphi}_i(x,t)}{\partial t} \geq d_i \frac{\partial^2 \overline{\varphi}_i(x,t)}{\partial x^2} + \widetilde{h}_i(x,t), x \in \mathbb{R}, t > 0, \\ \overline{\varphi}_i(x,0) \geq \varphi_i(x), x \in \mathbb{R} \end{cases} \tag{20}$$

and

$$\begin{cases} \frac{\partial \underline{\varphi}_i(x,t)}{\partial t} \leq d_i \frac{\partial^2 \underline{\varphi}_i(x,t)}{\partial x^2} + \widetilde{h}_i(x,t), x \in \mathbb{R}, t > 0, \\ \underline{\varphi}_i(x,0) \leq \varphi_i(x), x \in \mathbb{R} \end{cases} \tag{21}$$

for all $i \in \{1, 2, \cdots, n\}$.

By (19), we have

$$-\mathcal{H}(\varphi) \leq \widetilde{h}_i(x,t) \leq \mathcal{H}(\varphi)$$

for all $i = 1, 2, \cdots, n, t > 0, x \in \mathbb{R}$. Then the comparison principle (or the positivity of semigroup generated by Δ operator) implies

$$\begin{cases} \frac{\partial \overline{\varphi}_i(x,t)}{\partial t} = d_i \Delta\overline{\varphi}_i(x,t) + \mathcal{H}(\overline{\varphi}), x \in \mathbb{R}, t > 0, \\ \overline{\varphi}_i(x,0) = \sum_{i=1}^n |u_{i,0}(x) - w_i(x)|, x \in \mathbb{R}, \end{cases} \tag{22}$$

and

$$\begin{cases} \frac{\partial \underline{\varphi}_i(x,t)}{\partial t} = d_i \Delta\underline{\varphi}_i(x,t) - \mathcal{H}(\underline{\varphi}), x \in \mathbb{R}, t > 0, \\ \underline{\varphi}_i(x,0) = -\sum_{i=1}^n |u_{i,0}(x) - w_i(x)|, x \in \mathbb{R}, \end{cases} \tag{23}$$

formulate a pair of generalized upper and lower solutions of (19). Furthermore, we have

$$\overline{\varphi}_i(x,t) = -\underline{\varphi}_i(x,t) \geq 0, i = 1, 2, \cdots, n, t > 0, x \in \mathbb{R},$$

and

$$\overline{\varphi}_i(x,t) \geq \varphi_i(x,t) \geq -\underline{\varphi}_i(x,t) \geq 0, i = 1,2,\cdots,n, t>0, x\in\mathbb{R}.$$

Now, we shall further estimate $\overline{\varphi}_i(x,t), i=1,2,\cdots,n, t>0, x\in\mathbb{R}$. Let $c'>0$ and $\lambda>0$ be positive constants such that

$$c'\lambda - d_i\lambda^2 \geq Ln \text{ for all } i\in\{1,2,\cdots,n\}.$$

Further define continuous functions

$$\begin{aligned} \overline{\phi}_i(x,t) \quad &: \quad = \min\left\{Ke^{Lnt}, Ke^{\lambda(-x-N+c't)}\right\}, i\in\{1,2,\cdots,n\}, \\ \underline{\phi}_i(x,t) \quad &: \quad = \max\left\{-Ke^{Lnt}, -Ke^{\lambda(-x-N+c't)}\right\}, i\in\{1,2,\cdots,n\} \end{aligned}$$

for $t\geq 0, x\in\mathbb{R}$, where $K>0$ (that is independent of N) such that

$$\sum_{i=1}^{n} |u_{i,0}(x) - w_i(x)| \leq \min\left\{K, Ke^{\lambda(-x-N)}\right\}, x\in\mathbb{R}.$$

By direct calculations, if they are differentiable, we see that

$$\frac{\partial\overline{\phi}_i(x,t)}{\partial t} \geq d_i\frac{\partial^2\overline{\phi}_i(x,t)}{\partial x^2} + Ln|\overline{\phi}_i(x,t)| \geq d_i\frac{\partial^2\overline{\phi}_i(x,t)}{\partial x^2} + \mathcal{H}(\rho),$$

and

$$\frac{\partial\underline{\phi}_i(x,t)}{\partial t} \leq d_i\frac{\partial^2\underline{\phi}_i(x,t)}{\partial x^2} - Ln|\underline{\phi}_i(x,t)| \leq d_i\frac{\partial^2\underline{\phi}_i(x,t)}{\partial x^2} - \mathcal{H}(\rho)$$

for $i=1,2,\cdots,n$ and any continuous vector function ρ satisfying

$$(\underline{\phi}_1(x,t),\cdots,\underline{\phi}_n(x,t)) \leq \rho(x,t) \leq (\overline{\phi}_1(x,t),\cdots,\overline{\phi}_n(x,t)), t>0, x\in\mathbb{R}.$$

Then we obtain a generalized upper and lower solutions of (22) and (23). According to the classical theory of reaction-diffusion systems (see [47]), $\varphi(x,t)$ satisfies

$$\underline{\phi}_i(x,t) \leq \underline{\varphi}_i(x,t) \leq \varphi_i(x,t) \leq \overline{\varphi}_i(x,t) \leq \overline{\phi}_i(x,t), t>0, x\in\mathbb{R}.$$

Let $N>0$ such that

$$Ke^{\lambda(-N+c'(T_1+2T))} < \frac{\epsilon}{2},$$

then (18) holds. The proof is complete. □

Remark 1. The result fails if $T\to\infty$ but N is finite. One typical example is the propagation theory of bistable reaction-diffusion equations with negative sign of wave speed [2].

We now give the proof of Theorem 3.1.

Proof. From the definition,

$$u(x,t) = \psi(x+ct,t), x\in\mathbb{R}, t>0$$

is the solution of the following initial value problem

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2} + h(t,u),\ x\in\mathbb{R},\ t>0, \\ u(x,0) = \psi(x,0),\ x\in\mathbb{R}. \end{cases} \tag{24}$$

According to (16), there exists a positive constant $Z>0$ such that

$$a_i \leq \psi_i(x,0) \leq b_i,\ x>Z,\ i=1,2,\cdots,n.$$

By Lemma 3.2, for any $\epsilon>0$, there exist $Z_1>Z$ and $T_1>0$ such that

$$|u(x,t) - u^*(t)| < \epsilon,\ x>Z_1,\ t\in[T_1, T_1+T].$$

Again by the invariant form of traveling wave solutions, we have

$$|\psi(z,t) - u^*(t)| < \epsilon,\ z > Z_1 + |c|(T_1 + T),\ t \in [0, T].$$

The proof is complete. □

4. **Applications to time periodic Lotka-Volterra competitive models.** In this section, we shall investigate the existence and nonexistence of traveling wave solutions of the following competitive system

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d_1 u_{xx}(x,t) + u(x,t)\left[r_1(t) - a_1(t)u(x,t) - b_1(t)v(x,t)\right], \\ \frac{\partial v(x,t)}{\partial t} = d_2 v_{xx}(x,t) + v(x,t)\left[r_2(t) - a_2(t)u(x,t) - b_2(t)v(x,t)\right], \end{cases} \tag{25}$$

where $x \in \mathbb{R}, t > 0$. The parameters satisfy the following assumptions:

(C1) For $i = 1, 2$ and some $\theta \in (0,1)$, $r_i(t), a_i(t), b_i(t) \in C^{\frac{\theta}{2}}(\mathbb{R}, \mathbb{R})$ are T-periodic functions in $t \in \mathbb{R}$ with some $T > 0$;
(C2) $d_1 > 0, d_2 > 0$;
(C3) $a_1(t) > 0, b_2(t) > 0, t \in [0, T]$;
(C4) $b_1(t) \geq 0, a_2(t) \geq 0, t \in [0, T]$;

The corresponding kinetic system

$$\begin{cases} \frac{du(t)}{dt} = u(t)\left[r_1(t) - a_1(t)u(t) - b_1(t)v(t)\right], \\ \frac{dv(t)}{dt} = v(t)\left[r_2(t) - a_2(t)u(t) - b_2(t)v(t)\right] \end{cases} \tag{26}$$

admits a trivial solution $(0,0)$ and two nonnegative semi-trivial periodic solutions $(p(t), 0)$ and $(0, q(t))$, where

$$\begin{cases} p(t) = \frac{p_0 e^{\int_0^t r_1(s)ds}}{1 + p_0 \int_0^t e^{\int_0^s r_1(\tau)d\tau} a_1(s)ds}, & p_0 = \frac{e^{\int_0^T r_1(s)ds} - 1}{\int_0^T e^{\int_0^s r_1(\tau)d\tau} a_1(s)ds} > 0, \\ q(t) = \frac{q_0 e^{\int_0^t r_2(s)ds}}{1 + q_0 \int_0^t e^{\int_0^s r_2(\tau)d\tau} b_2(s)ds}, & q_0 = \frac{e^{\int_0^T r_2(s)ds} - 1}{\int_0^T e^{\int_0^s r_2(\tau)d\tau} b_2(s)ds} > 0. \end{cases}$$

In addition, under proper conditions, it also admits a positive periodic solution $(u^*(t), v^*(t))$ that is stable or unstable if the initial value of (26) satisfies

$$(u(0), v(0)) \gg (0, 0).$$

In what follows, we say that $(u^*(t), v^*(t))$ is *asymptotically stable* if $(u(t), v(t))$ of (26) satisfies

$$\lim_{t \to +\infty} [|u(t) - u^*(t)| + |v(t) - v^*(t)|] = 0$$

with the initial value $(u(0), v(0)) \gg (0, 0)$. There are many sufficient conditions (e.g. $\overline{r}_1 > \max_{t \in [0,T]} \left(\frac{b_1(t)}{b_2(t)}\right) \overline{r}_2, \overline{r}_2 > \max_{t \in [0,T]} \left(\frac{a_2(t)}{a_1(t)}\right) \overline{r}_1$) on the asymptotic stability of $(u^*(t), v^*(t))$ in (26). In Hess [19] and Lisena [27], we can find some stable results of these periodic solutions and we do not list them here.

Let

$$u(x,t) = U(z,t),\ v(x,t) = V(z,t),\ z = x + ct$$

be a traveling wave solution of (25). Then the corresponding wave system is

$$\begin{cases} U_t(z,t) = d_1 U_{zz}(z,t) - cU_z(z,t) + U(z,t)\left[r_1(t) - a_1(t)U(z,t) - b_1(t)V(z,t)\right], \\ V_t(z,t) = d_2 V_{zz}(z,t) - cV_z(z,t) + V(z,t)\left[r_2(t) - a_2(t)U(z,t) - b_2(t)V(z,t)\right], \\ U(z,t) = U(z, t+T),\ V(z,t) = V(z, t+T) \end{cases} \tag{27}$$

for $z \in \mathbb{R}, t \in \mathbb{R}$.

The existence of (27) has been well investigated under different conditions. If $(u^*(t), v^*(t))$ vanishes in (26), then Zhao and Ruan [51, 52] studied the existence, uniqueness and stability of positive solutions connecting $(p(t), 0)$ with $(0, q(t))$. With conditions in [51, 52], one of $(p(t), 0)$ and $(0, q(t))$ is stable while the other is unstable. When $(u^*(t), v^*(t))$ exists and is unstable, Bao and Wang [4] considered the existence and stability of positive solutions connecting $(p(t), 0)$ with $(0, q(t))$, in which both $(p(t), 0)$ and $(0, q(t))$ are locally stable. Moreover, there are some important results on the propagation theory of competitive systems when the habitat and environments depend on t, x, we refer to some very recent conclusions by Wang and Zhang [43], Yu and Zhao [50].

We now consider the positive solutions of (27) connecting $(0, 0)$ with $(u^*(t), v^*(t))$ when $(u^*(t), v^*(t))$ is *an asymptotically stable periodic solution* of (26). Let

$$M_1 = \frac{\max_{t\in[0,T]} r_1(t)}{\min_{t\in[0,T]} a_1(t)},\ M_2 = \frac{\max_{t\in[0,T]} r_2(t)}{\min_{t\in[0,T]} b_2(t)}.$$

Then (A1)-(A5) hold. For $i = 1, 2$, we define

$$\overline{r}_i = \frac{1}{T}\int_0^T r_i(t)dt,$$

then they are positive constants. If $c > c^* := \max\{2\sqrt{d_1\overline{r}_1}, 2\sqrt{d_2\overline{r}_2}\}$, then we define constants

$$\gamma_i = \frac{c-\sqrt{c^2-4d_i\overline{r}_i}}{2d_i},\ \gamma_{i+2} = \frac{c+\sqrt{c^2-4d_i\overline{r}_i}}{2d_i},\ i = 1, 2,$$

and continuous functions

$$\phi_i(t) = e^{\int_0^t [d_i\gamma_i^2 - c\gamma_i + r_i(s)]ds},\ i = 1, 2,$$

then $\phi_1(t), \phi_2(t)$ are T-periodic functions in t. It should be noted that $d_i\gamma_i^2 - c\gamma_i + \overline{r}_i = 0$, then

$$\phi_i(t) = e^{\int_0^t [r_i(s)-\overline{r}_i]ds} = e^{\int_0^t [d_i(\gamma_i+\epsilon)^2 - c(\gamma_i+\epsilon)+r_i(s)-(d_i(\gamma_i+\epsilon)^2-c(\gamma_i+\epsilon)+\overline{r}_i)]ds},\ i = 1, 2.$$

Lemma 4.1. *Assume that $\overline{r}_1 > 0, \overline{r}_2 > 0$. If $c > c^*$, then (27) has a nontrivial solution.*

Proof. Let $\epsilon > 0$ be a constant such that

$$\gamma_i + \epsilon < \min\{2\gamma_i, \gamma_1 + \gamma_2, \gamma_{i+2}\}.$$

Clearly, it is admissible.

Define continuous functions

$$\begin{aligned}
\overline{U}(z,t) &= \min\{\phi_1(t)e^{\gamma_1 z}, p(t)\},\\
\overline{V}(z,t) &= \min\{\phi_2(t)e^{\gamma_2 z}, q(t)\},\\
\underline{U}(z,t) &= \max\{\phi_1(t)e^{\gamma_1 z} - L\phi_1(t)e^{(\gamma_1+\epsilon)z}, 0\},\\
\underline{V}(z,t) &= \max\{\phi_2(t)e^{\gamma_2 z} - L\phi_2(t)e^{(\gamma_2+\epsilon)z}, 0\}
\end{aligned}$$

for $L > 0$ clarified later. Firstly, let $L_1 > 1$ be a constant such that

$$(\overline{U}, \overline{V}) \geq (\underline{U}, \underline{V}), L > L_1,$$

and

$$\phi_1(t)e^{\gamma_1 z} - L_1\phi_1(t)e^{(\gamma_1+\epsilon)z} < 0, \phi_2(t)e^{\gamma_2 z} - L_1\phi_2(t)e^{(\gamma_2+\epsilon)z} < 0$$

if $t \in [0, T], z \geq 0$.

We now verify that $(\overline{U}, \overline{V})$ and $(\underline{U}, \underline{V})$ are a pair of super- and sub-solutions of (27).

(1) If $\overline{U} = \phi_1(t)e^{\gamma_1 z}$, then it is differentiable in any neighborhood of (z, t) and

$$\overline{U}_t(z,t) = \phi_1(t)e^{\gamma_1 z}(d_1\gamma_1^2 - c\gamma_1 + r_1(t)),$$

while

$$\begin{aligned}
&d_1\overline{U}_{zz}(z,t) - c\overline{U}_z(z,t) + \overline{U}(z,t)\left[r_1(t) - a_1(t)\overline{U}(z,t) - b_1(t)\underline{V}(z,t)\right] \\
&\quad \le d_1\overline{U}_{zz}(z,t) - c\overline{U}_z(z,t) + \overline{U}(z,t)r_1(t) \\
&\quad = [d_1\gamma_1^2 - c\gamma_1 + r_1(t)]\phi_1(t)e^{\gamma_1 z} \\
&\quad = \overline{U}_t(z,t).
\end{aligned}$$

If $\overline{U} = p(t)$, then

$$\begin{aligned}
&d_1\overline{U}_{zz}(z,t) - c\overline{U}_z(z,t) + \overline{U}(z,t)\left[r_1(t) - a_1(t)\overline{U}(z,t) - b_1(t)\underline{V}(z,t)\right] \\
&\quad = p(t)\left[r_1(t) - a_1(t)p(t) - b_1(t)\underline{V}(z,t)\right] \\
&\quad \le p(t)\left[r_1(t) - a_1(t)p(t)\right] \\
&\quad = \overline{U}_t(z,t).
\end{aligned}$$

(2) Similar to that in (1), we have

$$\overline{V}_t(z,t) \ge d_2\overline{V}_{zz}(z,t) - c\overline{V}_z(z,t) + \overline{V}(z,t)\left[r_2(t) - a_2(t)\underline{U}(z,t) - b_2(t)\overline{V}(z,t)\right]$$

with $\overline{V} = \phi_2(t)e^{\gamma_2 z}$ or $\overline{V} = q(t)$.

(3) If $\underline{U} = 0$, then

$$\underline{U}_t(z,t) = d_1\underline{U}_{zz}(z,t) - c\underline{U}_z(z,t) + \underline{U}(z,t)\left[r_1(t) - a_1(t)\underline{U}(z,t) - b_1(t)\overline{V}(z,t)\right] = 0.$$

Otherwise,

$$\begin{aligned}
\underline{U}_t(z,t) \quad = \quad & (d_1\gamma_1^2 - c\gamma_1 + r_1(t))\phi_1(t)e^{\gamma_1 z} \\
& - L(d_1(\gamma_1+\epsilon)^2 - c(\gamma_1+\epsilon) + r_1(t))\phi_1(t)e^{(\gamma_1+\epsilon)z} \\
& + L[d_1(\gamma_1+\epsilon)^2 - c(\gamma_1+\epsilon) + \overline{r}_1]\phi_1(t)e^{(\gamma_1+\epsilon)z}
\end{aligned}$$

and

$$\begin{aligned}
&d_1\underline{U}_{zz}(z,t) - c\underline{U}_z(z,t) + \underline{U}(z,t)\left[r_1(t) - a_1(t)\underline{U}(z,t) - b_1(t)\overline{V}(z,t)\right] \\
&\quad \ge d_1\underline{U}_{zz}(z,t) - c\underline{U}_z(z,t) + \underline{U}(z,t)r_1(t) \\
&\qquad - a_1(t)\phi_1^2(t)e^{2\gamma_1 z} - b_1(t)\phi_1(t)\phi_2(t)e^{(\gamma_1+\gamma_2)z} \\
&\quad = \phi_1(t)e^{\gamma_1 z}(d_1\gamma_1^2 - c\gamma_1 + r_1(t)) \\
&\qquad - L\phi_1(t)e^{(\gamma_1+\epsilon)z}(d_1(\gamma_1+\epsilon)^2 - c(\gamma_1+\epsilon)\gamma_1 + r_1(t)) \\
&\qquad - a_1(t)\phi_1^2(t)e^{2\gamma_1 z} - b_1(t)\phi_1(t)\phi_2(t)e^{(\gamma_1+\gamma_2)z}.
\end{aligned}$$

Then

$$\underline{U}_t(z,t) \le d_1\underline{U}_{zz}(z,t) - c\underline{U}_z(z,t) + \underline{U}(z,t)\left[r_1(t) - a_1(t)\underline{U}(z,t) - b_1(t)\overline{V}(z,t)\right]$$

if

$$\begin{aligned}
&(d_1\gamma_1^2 - c\gamma_1 + r_1(t))\phi_1(t)e^{\gamma_1 z} - L(d_1(\gamma_1+\epsilon)^2 - c(\gamma_1+\epsilon) + r_1(t))\phi_1(t)e^{(\gamma_1+\epsilon)z} \\
&+ L[d_1(\gamma_1+\epsilon)^2 - c(\gamma_1+\epsilon) + \overline{r}_1]\phi_1(t)e^{(\gamma_1+\epsilon)z} \\
&\quad \le \phi_1(t)e^{\gamma_1 z}(d_1\gamma_1^2 - c\gamma_1 + r_1(t)) \\
&\qquad - L\phi_1(t)e^{(\gamma_1+\epsilon)z}(d_1(\gamma_1+\epsilon)^2 - c(\gamma_1+\epsilon)\gamma_1 + r_1(t)) \\
&\qquad - a_1(t)\phi_1^2(t)e^{2\gamma_1 z} - b_1(t)\phi_1(t)\phi_2(t)e^{(\gamma_1+\gamma_2)z}
\end{aligned}$$

or

$$\begin{aligned} &L[d_1(\gamma_1+\epsilon)^2 - c(\gamma_1+\epsilon) + \overline{r}_1]\phi_1(t)e^{(\gamma_1+\epsilon)z} \\ &\quad \le -a_1(t)\phi_1^2(t)e^{2\gamma_1 z} - b_1(t)\phi_1(t)\phi_2(t)e^{(\gamma_1+\gamma_2)z}, \end{aligned}$$

which is true when

$$L > -\frac{\max_{t\in[0,T]}\{a_1(t)\phi_1(t) + b_1(t)\phi_2(t)\}}{d_1(\gamma_1+\epsilon)^2 - c(\gamma_1+\epsilon) + \overline{r}_1} + L_1 := L_2.$$

(4) In a similar way, we can prove that

$$\underline{V}_t(z,t) \le d_2\underline{V}_{zz}(z,t) - c\underline{V}_z(z,t) + \underline{V}(z,t)\left[r_2(t) - a_2(t)\overline{U}(z,t) - b_2(t)\underline{V}(z,t)\right]$$

with $\underline{V} = \phi_2(t)e^{\gamma_2 z} - L\phi_2(t)e^{(\gamma_2+\epsilon)z}$ or $\underline{V} = 0$ if

$$L > -\frac{\max_{t\in[0,T]}\{a_2(t)\phi_1(t) + b_2(t)\phi_2(t)\}}{d_2(\gamma_2+\epsilon)^2 - c(\gamma_2+\epsilon) + \overline{r}_2} + L_1 := L_3.$$

By what we have done, if $L = L_2 + L_3$, then we obtain a pair of super- and sub-solutions of (27). Moreover, they also satisfy the other conditions in Theorem 2.2, and the existence of traveling wave solution is proved. The proof is complete. □

Furthermore, we can establish the existence of traveling wave solutions of (27) with $c = c^*$. Without loss of generality, we first assume that $d_1\overline{r}_1 > d_2\overline{r}_2$, which implies that $c^* = 2\sqrt{d_1\overline{r}_1}$. Define positive constants

$$\lambda_1 = \frac{c^*}{2d_1} = \sqrt{\frac{\overline{r}_1}{d_1}}, \gamma_2 = \frac{c^* - \sqrt{(c^*)^2 - 4d_2\overline{r}_2}}{2d_2}, \gamma_4 = \frac{c^* + \sqrt{(c^*)^2 - 4d_2\overline{r}_2}}{2d_2},$$

further define continuous functions

$$\phi_1(t) = e^{\int_0^t [d_1\lambda_1^2 - c^*\lambda_1 + r_1(s)]ds},\ \phi_2(t) = e^{\int_0^t [d_2\gamma_2^2 - c^*\gamma_2 + r_2(s)]ds},$$

which are T-periodic in t.

Lemma 4.2. *Assume that $\overline{r}_1 > 0, \overline{r}_2 > 0$ and $d_1\overline{r}_1 > d_2\overline{r}_2$. If $c = c^*$, then (27) has a nontrivial solution.*

Proof. Let $\epsilon > 0$ be a constant such that

$$\gamma_2 + \epsilon < \min\left\{\gamma_2 + \frac{\lambda_1}{2}, 2\gamma_2, \gamma_4\right\}.$$

Clearly, it is admissible.

Denote by $\rho > 0$ a positive constant such that

$$\inf_{t\in[0,T]} \phi_1(t) \sup_{z\in\mathbb{R}} \left\{-\rho z e^{\lambda_1 z}\right\} > \sup_{t\in[0,T]} p(t),$$

and $z_1(t)$ the smaller root of $-\rho\phi_1(t)ze^{\lambda_1 z} = p(t)$. Construct continuous functions

$$\begin{aligned} \overline{U}(z,t) &= \begin{cases} -\rho\phi_1(t)ze^{\lambda_1 z}, z \le z_1(t), \\ p(t), z > z_1(t), \end{cases} \\ \overline{V}(z,t) &= \min\{\phi_2(t)e^{\gamma_2 z}, q(t)\}, \\ \underline{U}(z,t) &= \begin{cases} (-\rho z - L\sqrt{-z})\phi_1(t)e^{\lambda_1 z}, z \le -\left(\frac{L}{\rho}\right)^2, \\ 0, z > -\left(\frac{L}{\rho}\right)^2, \end{cases} \\ \underline{V}(z,t) &= \max\{\phi_2(t)e^{\gamma_2 z} - L\phi_2(t)e^{(\gamma_2+\epsilon)z}, 0\}, \end{aligned}$$

where $L > 0$ will be clarified latter. Let $L_1 > 1$ be a constant such that

$$(\overline{U}, \overline{V}) \geq (\underline{U}, \underline{V}) \text{ if } L > L_1$$

and

$$\phi_2(t)e^{\gamma_2 z} - L_1\phi_2(t)e^{(\gamma_2+\epsilon)z} < 0$$

if $t \in [0, T], z \geq 0$.

We now verify that $(\overline{U}, \overline{V})$ and $(\underline{U}, \underline{V})$ are a pair of super- and sub-solutions of (27).

(1) Similar to that in (2) of Lemma 4.1, we have

$$\overline{V}_t(z,t) \geq d_2\overline{V}_{zz}(z,t) - c^*\overline{V}_z(z,t) + \overline{V}(z,t)\left[r_2(t) - a_2(t)\underline{U}(z,t) - b_2(t)\overline{V}(z,t)\right]$$

with $\overline{V} = \phi_2(t)e^{\gamma_2 z}$ or $\overline{V} = q(t)$.

(2) If $\overline{U} = -\rho\phi_1(t)ze^{\lambda_1 z}$, then it is differentiable in any neighborhood of (z, t) and

$$\overline{U}_t(z,t) = \overline{U}(z,t)(d_1\lambda_1^2 - c^*\lambda_1 + r_1(t)).$$

In addition,

$$\begin{aligned}\overline{U}_z(z,t) &= -\rho\phi_1(t)e^{\lambda_1 z} + \lambda_1\overline{U}(z,t),\\ \overline{U}_{zz}(z,t) &= -2\rho\lambda_1\phi_1(t)e^{\lambda_1 z} + \lambda_1^2\overline{U}(z,t),\end{aligned}$$

and

$$\begin{aligned}&d_1\overline{U}_{zz}(z,t) - c^*\overline{U}_z(z,t) + \overline{U}(z,t)\left[r_1(t) - a_1(t)\overline{U}(z,t) - b_1(t)\underline{V}(z,t)\right]\\ &\quad\leq d_1\overline{U}_{zz}(z,t) - c^*\overline{U}_z(z,t) + \overline{U}(z,t)r_1(t)\\ &\quad= d_1\left[-2\rho\lambda_1\phi_1(t)e^{\lambda_1 z} + \lambda_1^2\overline{U}(z,t)\right]\\ &\qquad -c^*\left[-\rho\phi_1(t)e^{\lambda_1 z} + \lambda_1\overline{U}(z,t)\right] + \overline{U}(z,t)r_1(t)\\ &\quad= \overline{U}_t(z,t).\end{aligned}$$

If $\overline{U} = p(t)$, then

$$\begin{aligned}&d_1\overline{U}_{zz}(z,t) - c\overline{U}_z(z,t) + \overline{U}(z,t)\left[r_1(t) - a_1(t)\overline{U}(z,t) - b_1(t)\underline{V}(z,t)\right]\\ &\quad= p(t)\left[r_1(t) - a_1(t)p(t) - b_1(t)\underline{V}(z,t)\right]\\ &\quad\leq p(t)\left[r_1(t) - a_1(t)p(t)\right]\\ &\quad= \overline{U}_t(z,t).\end{aligned}$$

(3) If $\underline{V} = 0$, then

$$\underline{V}_t(z,t) = d_2\underline{V}_{zz}(z,t) - c^*\underline{V}_z(z,t) + \underline{V}(z,t)\left[r_2(t) - a_2(t)\overline{U}(z,t) - b_2(t)\underline{V}(z,t)\right].$$

Otherwise, we can find a positive number $L_2 > 0$ such that if $L > L_1 + L_2$, then $e^{\gamma_2 z} > Le^{(\gamma_2+\epsilon)z}$ implies that $-\rho\phi_1(t)ze^{\lambda_1 z} < \phi_1(t)e^{\lambda_1 z/2}$. A directly calculation yields

$$\begin{aligned}\underline{V}_t(z,t) \quad = \quad &(d_2\gamma_2^2 - c^*\gamma_2 + r_2(t))\phi_2(t)e^{\gamma_2 z}\\ &-L(d_2(\gamma_2+\epsilon)^2 - c^*(\gamma_2+\epsilon) + r_2(t))\phi_2(t)e^{(\gamma_2+\epsilon)z}\\ &+L[d_2(\gamma_2+\epsilon)^2 - c^*(\gamma_2+\epsilon) + \overline{r}_2]\phi_2(t)e^{(\gamma_2+\epsilon)z}\end{aligned}$$

and

$$\begin{aligned}
&d_2\underline{V}_{zz}(z,t) - c^*\underline{V}_z(z,t) + \underline{V}(z,t)\left[r_2(t) - a_2(t)\overline{U}(z,t) - b_2(t)\underline{V}(z,t)\right] \\
&\quad \geq d_2\underline{V}_{zz}(z,t) - c^*\underline{V}_z(z,t) + \underline{V}(z,t)r_2(t) \\
&\qquad -a_2(t)\phi_1(t)\phi_2(t)e^{\left(\gamma_2+\frac{\lambda_1}{2}\right)z} - b_2(t)\phi_2^2(t)e^{2\gamma_2 z} \\
&\quad \geq \phi_2(t)e^{\gamma_2 z}(d_2\gamma_2^2 - c^*\gamma_2 + r_2(t)) \\
&\qquad -L\phi_2(t)e^{(\gamma_2+\epsilon)z}(d_2(\gamma_2+\epsilon)^2 - c^*(\gamma_2+\epsilon)\gamma_2 + r_2(t)) \\
&\qquad -a_2(t)\phi_1(t)\phi_2(t)e^{\left(\gamma_2+\frac{\lambda_1}{2}\right)z} - b_2(t)\phi_2^2(t)e^{2\gamma_2 z}.
\end{aligned}$$

Then

$$\underline{V}_t(z,t) \leq d_2\underline{V}_{zz}(z,t) - c^*\underline{V}_z(z,t) + \underline{V}(z,t)\left[r_2(t) - a_2(t)\overline{U}(z,t) - b_2(t)\underline{V}(z,t)\right]$$

if

$$\begin{aligned}
&L[d_2(\gamma_2+\epsilon)^2 - c^*(\gamma_2+\epsilon) + \overline{r}_2]\phi_2(t)e^{(\gamma_2+\epsilon)z} \\
&\quad \leq -a_2(t)\phi_1(t)\phi_2(t)e^{\left(\gamma_2+\frac{\lambda_1}{2}\right)z} - b_2(t)\phi_2^2(t)e^{2\gamma_2 z}.
\end{aligned}$$

This is true when

$$L > -\frac{\max_{t\in[0,T]}\{a_2(t)\phi_1(t) + b_2(t)\phi_2(t)\}}{d_2(\gamma_2+\epsilon)^2 - c^*(\gamma_2+\epsilon) + \overline{r}_2} + L_1 := L_3.$$

(4) If $\underline{U} = 0$, then

$$\underline{U}_t(z,t) = d_1\underline{U}_{zz}(z,t) - c^*\underline{U}_z(z,t) + \underline{U}(z,t)\left[r_1(t) - a_1(t)\underline{U}(z,t) - b_1(t)\overline{V}(z,t)\right] = 0.$$

If $\underline{U} = (-\rho z - L\sqrt{-z})\phi_1(t)e^{\lambda_1 z}$, then

$$\underline{U}_t(z,t) = (d_1\lambda_1^2 - c^*\lambda_1 + r_1(t))\underline{U}(z,t),$$

and

$$\begin{aligned}
&\underline{U}_z(z,t) = \lambda_1\underline{U}(z,t) + \left[\frac{L}{2}(-z)^{-1/2} - \rho\right]\phi_1(t)e^{\lambda_1 z}, \\
&\underline{U}_{zz}(z,t) = \lambda_1^2\underline{U}(z,t) + \left[\lambda_1 L(-z)^{-1/2} - 2\lambda_1\rho + \frac{L}{4}(-z)^{-3/2}\right]\phi_1(t)e^{\lambda_1 z}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&d_1\underline{U}_{zz}(z,t) - c^*\underline{U}_z(z,t) + \underline{U}(z,t)\left[r_1(t) - a_1(t)\underline{U}(z,t) - b_1(t)\overline{V}(z,t)\right] \\
&\quad = d_1\left(\lambda_1^2\underline{U}(z,t) + \left[\lambda_1 L(-z)^{-1/2} - 2\lambda_1\rho + \frac{L}{4}(-z)^{-3/2}\right]\phi_1(t)e^{\lambda_1 z}\right) \\
&\qquad -c^*\left(\lambda_1\underline{U}(z,t) + \left[\frac{L}{2}(-z)^{-1/2} - \rho\right]\phi_1(t)e^{\lambda_1 z}\right) \\
&\qquad +\underline{U}(z,t)\left[r_1(t) - a_1(t)\underline{U}(z,t) - b_1(t)\overline{V}(z,t)\right] \\
&\quad = \underline{U}_t(z,t) + d_1\frac{L}{4}(-z)^{-3/2}\phi_1(t)e^{\lambda_1 z} - \underline{U}(z,t)\left[a_1(t)\underline{U}(z,t) + b_1(t)\overline{V}(z,t)\right],
\end{aligned}$$

which implies that

$$\underline{U}_t(z,t) \leq d_1\underline{U}_{zz}(z,t) - c^*\underline{U}_z(z,t) + \underline{U}(z,t)\left[r_1(t) - a_1(t)\underline{U}(z,t) - b_1(t)\overline{V}(z,t)\right]$$

if

$$\begin{aligned}
&d_1\frac{L}{4}(-z)^{-3/2}\phi_1(t)e^{\lambda_1 z} \\
&\quad \geq a_1(t)\rho^2 z^2\phi_1^2(t)e^{2\lambda_1 z} + b_1(t)(-\rho z)\phi_1(t)\phi_2(t)e^{(\lambda_1+\gamma_2)z}.
\end{aligned}$$

This is true if

$$L > \frac{4}{d_1} \sup_{z\le 0, t\in[0,T]} \left\{ (-z)^{3/2} \left[a_1(t)\rho^2 z^2 \phi_1(t) e^{\lambda_1 z} + b_1(t)(-\rho z)\phi_2(t) e^{\gamma_2 z} \right] \right\} := L_4.$$

Take $L = L_1 + L_2 + L_3 + L_4$, then we obtain a pair of super- and sub-solutions of (27). Similar to that in Lemma 4.1, the existence of traveling wave solution is proved. The proof is complete. $\square$

Lemma 4.3. *Assume that* $\overline{r}_1 > 0, \overline{r}_2 > 0$ *and* $d_2\overline{r}_2 > d_1\overline{r}_1$. *If* $c = c^*$, *then* (27) *has a nontrivial solution.*

Proof. By following exactly the same argument in Lemma 4.2, the proof is complete. $\square$

Lemma 4.4. *Assume that* $\overline{r}_1 > 0, \overline{r}_2 > 0$ *and* $d_1\overline{r}_1 = d_2\overline{r}_2$. *If* $c = c^*$, *then* (27) *has a nontrivial solution.*

Proof. Define positive constants

$$\lambda_1 = \frac{c^*}{2d_1} = \sqrt{\frac{\overline{r}_1}{d_1}}, \quad \lambda_2 = \frac{c^*}{2d_2} = \sqrt{\frac{\overline{r}_2}{d_2}},$$

further define continuous functions

$$\phi_i(t) = e^{\int_0^t [d_i\lambda_i^2 - c^*\lambda_i + r_i(s)]ds}, i = 1, 2,$$

which are T-periodic functions in t.

Let $\rho > 0$ be a positive constant such that

$$\inf_{t\in[0,T]} \phi_1(t) \sup_{z\in\mathbb{R}} \left\{ -\rho z e^{\lambda_1 z} \right\} > \sup_{t\in[0,T]} p(t),$$
$$\inf_{t\in[0,T]} \phi_2(t) \sup_{z\in\mathbb{R}} \left\{ -\rho z e^{\lambda_2 z} \right\} > \sup_{t\in[0,T]} q(t),$$

and $z_1(t)(z_2(t))$ be the smaller root of $-\rho\phi_1(t)ze^{\lambda_1 z} = p(t)(-\rho\phi_2(t)ze^{\lambda_2 z} = q(t))$. Construct continuous functions

$$\begin{aligned}
\overline{U}(z,t) &= \begin{cases} -\rho\phi_1(t)ze^{\lambda_1 z}, z \le z_1(t), \\ p(t), z > z_1(t), \end{cases} \\
\overline{V}(z,t) &= \begin{cases} -\rho\phi_2(t)ze^{\lambda_2 z}, z \le z_2(t), \\ q(t), z > z_2(t), \end{cases} \\
\underline{U}(z,t) &= \begin{cases} (-\rho z - L\sqrt{-z})\phi_1(t)e^{\lambda_1 z}, z \le -\left(\frac{L}{\rho}\right)^2, \\ 0, z > -\left(\frac{L}{\rho}\right)^2, \end{cases} \\
\underline{V}(z,t) &= \begin{cases} (-\rho z - L\sqrt{-z})\phi_2(t)e^{\lambda_2 z}, z \le -\left(\frac{L}{\rho}\right)^2, \\ 0, z > -\left(\frac{L}{\rho}\right)^2. \end{cases}
\end{aligned}$$

Similar to that of $d_1\overline{r}_1 \ne d_2\overline{r}_2$, there exists $L > 0$ such that $(\overline{U}, \overline{V}), (\underline{U}, \underline{V})$ are a pair of super- and sub- solutions of (27). Hence, (27) admits a nontrivial solution. $\square$

To continue our discussion, we now recall the following initial value problem involving Fisher nonlinearity with time periodic parameters

$$\begin{cases} w_t(x,t) = dw_{xx}(x,t) + w(x,t)[r(t) - a(t)w(x,t)], \\ w(x,0) = w_0(x), \end{cases} \tag{28}$$

in which $x \in \mathbb{R}, t > 0$, all the parameters are T-periodic and C^θ functions in t for some $T > 0$ and $\theta \in (0,1)$. Moreover, $d > 0, a(t) > 0$ and $\int_0^T r(t)dt > 0$. Under these conditions, the corresponding kinetic equation admits a positive periodic solution $w^*(t)$, which attracts all solutions with positive initial values. For this equation, we have the following results on asymptotic spreading [6, 24, 31].

Lemma 4.5. *Assume that $w_0(x) \in [0, w^*(0)]$ is a positive continuous function with nonempty compact support. If*

$$c_1^* = 2\sqrt{d\int_0^T r(t)dt/T},$$

then

$$\lim_{t\to\infty} \sup_{|x|<(c_1^*-\epsilon)t} |w(x,t) - w^*(t)| = 0, \quad \lim_{t\to\infty} \sup_{|x|>(c_1^*+\epsilon)t} w(x,t) = 0$$

for any $\epsilon \in (0, c_1^*)$.

Lemma 4.6. *Assume that $(U(z,t), V(z,t))$ is given by Lemmas 4.1-4.4. If*

$$\int_0^T [r_1(t) - b_1(t)q(t)]\, dt > 0, \ \int_0^T [r_2(t) - a_2(t)p(t)]\, dt > 0, \tag{29}$$

then

$$\liminf_{z\to\infty} \inf_{t\in[0,T]} (U(z,t), V(z,t)) \gg (0,0).$$

Moreover, if $(u^*(t), v^*(t))$ is asymptotically stable, then

$$\lim_{z\to\infty} (U(z,t), V(z,t)) = (u^*(t), v^*(t))$$

uniformly in $t \in \mathbb{R}$.

Proof. Let $m^*(t)$ be the unique positive periodic solution of

$$\frac{dm(t)}{dt} = m(t)\left[r_1(t) - b_1(t)q(t) - a_1(t)m(t)\right],$$

of which the existence and stability are obtained by (29).

By Lemma 4.1, $u(x,t) = U(x+ct,t)$ satisfies

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} \geq d_1 u_{xx}(x,t) + u(x,t)\left[r_1(t) - b_1(t)q(t) - a_1(t)u(x,t)\right], \\ u(x,0) = U(x,0) > 0 \end{cases}$$

for $x \in \mathbb{R}, t > 0$. By the comparison principle and Lemma 4.5, we have

$$\liminf_{t\to\infty} (u(0,t) - m^*(t)) \geq 0,$$

that is

$$\liminf_{t\to\infty} u(0,t) = \liminf_{z\to\infty} U(z,t) \geq m^*(t)$$

by the definition of traveling wave solution and $z = 0 + ct \to \infty$ when $t \to \infty$.

Similarly, we can obtain a small positive constant ϵ such that

$$\epsilon < \liminf_{z\to\infty} \inf_{t\in[0,T]} U(z,t) < \max_{t\in[0,T]} p(t) := \widehat{p}$$

and

$$\epsilon < \liminf_{z\to\infty} \inf_{t\in[0,T]} V(z,t) < \max_{t\in[0,T]} q(t) := \widehat{q}.$$

In the corresponding kinetic system of (25), $(u^*(t), v^*(t))$ is asymptotically stable if the initial value belongs to the set $[\epsilon, \widehat{p}] \times [\epsilon, \widehat{q}]$, then Theorem 3.1 implies what we wanted. The proof is complete. □

Lemma 4.7. *If $c < c^*$, then (27) does not have a positive solution $(U(z,t), V(z,t))$ such that*

$$\lim_{z\to-\infty}(U(z,t),V(z,t)) = (0,0), \;\; \liminf_{z\to\infty} \inf_{t\in[0,T]}(U(z,t),V(z,t)) \gg (0,0). \tag{30}$$

Proof. Without loss of generality, we assume that $c^* = 2\sqrt{d_1\overline{r}_1}$. Were the statement false, then there exists some $c' \in (0, c^*)$ such that (27) with $c = c'$ has a bounded positive T-periodic solution $(U(z,t), V(z,t))$. Let $\epsilon > 0$ be a small constant such that

$$c' < 2\sqrt{d_1(\overline{r}_1 - 2\epsilon)} < c^*.$$

For convenience, we denote $2\sqrt{d_1(\overline{r}_1 - 2\epsilon)} := c''$.

By (30), there exist $z_0 \in \mathbb{R}, M > 0$ such that

$$b_1(t)V(z,t) < \epsilon, \;\; z < z_0, \;\; t \in [0,T],$$

and

$$a_1(t)U(z,t) + b_1(t)V(z,t) < MU(z,t), \;\; z \geq z_0, \;\; t \in [0,T].$$

Thus, $u(x,t) = U(x + c't, t)$ satisfies

$$\begin{cases} u_t(x,t) \geq d_1 u_{xx}(x,t) + u(x,t)\left[r_1(t) - \epsilon - Mu(x,t)\right], \\ u(x,0) = U(x,0). \end{cases}$$

By Lemma 4.5, we see that

$$\liminf_{t\to\infty} u(-c''t, t) \geq \min_{t\in[0,T]} \underline{u}(t) > 0,$$

where $\underline{u}(t)$ is the unique positive solution of

$$\frac{du(t)}{dt} = u(t)\left[r_1(t) - \epsilon - Mu(t)\right].$$

On the other hand, $x = -c''t$ implies that

$$z = x + c't = (c' - c'')t \to -\infty \text{ as } t \to \infty,$$

which further indicates that

$$\lim_{t\to\infty} u(-c''t, t) = \lim_{z\to-\infty} U(z,t) = 0,$$

and a contradiction occurs. The proof is complete. □

Summarizing the above analysis, we have the following results.

Theorem 4.8. *Assume that (C1)-(C4) hold.*

(1) If (29) is true, then for any $c \geq c^*$, (27) has a positive solution satisfying

$$\lim_{z\to-\infty}(U(z,t),V(z,t)) = (0,0), \; \liminf_{z\to\infty} \inf_{t\in[0,T]}(U(z,t),V(z,t)) \gg (0,0).$$

Moreover, if $(u^*(t), v^*(t))$ is asymptotically stable, then

$$\lim_{z\to\infty}(U(z,t),V(z,t)) = (u^*(t), v^*(t))$$

uniformly in $t \in \mathbb{R}$.

(2) If $c < c^*$, then (27) does not have a positive solution satisfying (30).

Remark 2. From our results, we can see that the solution of (27) with $c = c^*$ does not decay exponentially as $z \to -\infty$, while it decays exponentially as $z \to -\infty$ if $c > c^*$.

Appendix. In this part, we collect some optimal regularity results in [28]. For convenience, we use the notations similar to those in [28]. Assume that the Banach spaces X, Y are given by Section 2, we consider the following initial problem

$$\begin{cases} u_t(x,t) = \mathcal{A}u(x,t) + f(x,t), \ 0 < t \le T, \ x \in \mathbb{R}, \\ u(x,0) = u_0(x), \ x \in \mathbb{R}, \end{cases} \tag{31}$$

where $f : \mathbb{R} \times [0,T] \mapsto \mathbb{R}$ is a continuous function, $\mathcal{A} = d\partial_{xx} + c\partial_x + a$ is the second order elliptic operator with constant coefficients $d, c, a \in \mathbb{R}$. We write problem (31) as an evolution equation in the space X by setting $u(t) = u(\cdot,t), f(t) = f(\cdot,t)$ and

$$\begin{cases} D(A) = \left\{ \varphi \in \bigcap_{p \ge 1} W^{2,p}_{loc}(\mathbb{R},\mathbb{R}) : \varphi, \mathcal{A}\varphi \in X \right\}, \\ A : D(A) \mapsto X, A\varphi = \mathcal{A}\varphi. \end{cases}$$

The realization A of $\mathcal{A}$ in X is a sectorial operator, and that $D(A) = C^2(\mathbb{R},\mathbb{R})$. Moreover, $\overline{D(A)} = X$. Let $f : \mathbb{R} \times [0,T] \mapsto \mathbb{R}$ be a continuous function such that $t \mapsto f(\cdot,t)$ belongs to $C([0,T],X)$, and let $u_0 \in X$. Then

$$u(x,t) = (e^{tA}u_0)(x) + \int_0^T e^{(t-s)A} f(\cdot,s)ds(x), 0 \le t \le T, x \in \mathbb{R} \tag{32}$$

defines a mild solution of (31) (see [28, Section 5.1.1]).

Theorem A. ([28, Theorem 5.1.2]) *Let $f : \mathbb{R} \times [0,T] \mapsto \mathbb{R}$ be a continuous function such that $t \mapsto f(\cdot,t)$ belongs to $C([0,T],X)$, and let $u_0 \in X$. Then the function u defined by (32) belongs to $Y \cap C^{2\theta,\theta}(\mathbb{R} \times [\varepsilon,T],\mathbb{R})$ for every $\varepsilon \in (0,T)$ and some $\theta \in (0,1)$, and there are $C > 0, C(\varepsilon,\theta) > 0$ such that*

$$\|u\|_\infty \le C\left(\|u_0\|_\infty + \|f\|_\infty\right),$$

$$\|u\|_{C^{2\theta,\theta}(\mathbb{R}\times[\varepsilon,T],\mathbb{R})} \le C(\varepsilon,\theta)\left(\varepsilon^{-\theta}\|u_0\|_\infty + \|f\|_\infty\right).$$

In addition, if $u_0 \in C^{2\theta}(\mathbb{R},\mathbb{R})$, with $0 < \theta < 1$, then u belongs to $C^{2\theta,\theta}(\mathbb{R}\times[0,T],\mathbb{R})$, and

$$\|u\|_{C^{2\theta,\theta}(\mathbb{R}\times[0,T],\mathbb{R})} \le C\left(\|u_0\|_{C^{2\theta}(\mathbb{R},\mathbb{R})} + \|f\|_\infty\right).$$

Theorem B. ([28, Theorem 5.1.4]) *Let $f \in Y \cap C^{\alpha,0}(\mathbb{R} \times [0,T],\mathbb{R})$, with $\alpha \in (0,2), \alpha \ne 1$, and let $u_0 \in X$. Then the mild solution u of problem (31) is differentiable with respect to t in $\mathbb{R}\times(0,T]$, $u(\cdot,t)$ belongs to $W^{2,p}_{loc}(\mathbb{R})$ for every $p \ge 1$. Moreover, u satisfies pointwise (31), and it is the unique solution of (31) belonging to Y and enjoying the above regularity properties. In addition, $u \in C^{2+\alpha,1}(\mathbb{R} \times [\varepsilon,T],\mathbb{R})$ for every $\varepsilon \in (0,T)$, and*

$$\|u\|_{C^{2+\alpha,1}(\mathbb{R}\times[\varepsilon,T],\mathbb{R})} \le \frac{C}{\varepsilon^{\alpha/2+1}}\left(\|u_0\|_\infty + \|f\|_{C^{\alpha,0}(\mathbb{R}\times[0,T],\mathbb{R})}\right).$$

If also $u_0 \in C^{2+\alpha}(\mathbb{R},\mathbb{R})$, then $u \in C^{2+\alpha,1}(\mathbb{R} \times [0,T],\mathbb{R})$, and

$$\|u\|_{C^{2+\alpha,1}(\mathbb{R}\times[0,T],\mathbb{R})} \le C\left(\|u_0\|_{C^{2+\alpha}(\mathbb{R},\mathbb{R})} + \|f\|_{C^{\alpha,0}(\mathbb{R}\times[0,T],\mathbb{R})}\right).$$

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