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Some Recent Developments of the Bartnik Mass

Pengzi Miao*

Abstract

We report some recent developments of the Bartnik mass. In particular, we describe an evolution formula of the Bartnik mass of a family of closed surfaces evolving in a given manifold with nonnegative scalar curvature. We also discuss an upper bound of the Bartnik mass when the surface is isometric to a round sphere but is allowed to have arbitrary positive mean curvature.

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1 Scalar curvature and the ADM Mass

Let (M, g) be a Riemannian manifold. The scalar curvature R of (M, g), in any coordinate chart, can be written as

$$R = \partial_i(X^i) + Q(\partial g, \partial g), \tag{1.1}$$

where

$$X^i = g^{il} g^{jk} (\partial_j g_{lk} - \partial_l g_{jk}), \tag{1.2}$$

and $Q(\partial g, \partial g)$ denotes some quantity that is quadratic in the coordinate derivatives of the metric coefficients. Though $\{X^i\}$ is not a geometric quantity, the presence of the divergence term in (1.1) still leads to many interesting geometric consequences. For instance, it explains why the Euler-Lagrange equation of the Einstein-Hilbert functional

$$\int_M R \ dV$$

is an equation of second order instead of fourth order.

*School of Mathematical Sciences, Monash University, Victoria 3800, Australia. E-mail: Pengzi.Miao@sci.monash.edu.au

A Riemannian 3-manifold (M, g) is said to be *asymptotically flat* (with one end) if there exists a compact set $K \subset M$ such that $M \setminus K$ is diffeomorphic to $\mathbb{R}^3 \setminus B_1(0)$ and, in the standard coordinates in $\mathbb{R}^3$, the metric g satisfies

$$|g_{ij} - \delta_{ij}| = O(|x|^{-1}), \quad |\partial g_{ij}| = O(|x|^{-2}), \quad |\partial\partial g_{ij}| = O(|x|^{-3}), \tag{1.3}$$

where ∂ denotes the usual partial derivative operator on $\mathbb{R}^3$.

Given an asymptotically flat manifold (M, g), one can consider the formula (1.1) in a coordinate chart that defines the asymptotically flatness of (M, g). One is naturally led to the limit of the flux integral

$$\lim_{r\to\infty} \oint_{|x|=r} X \cdot \nu \; d\sigma, \tag{1.4}$$

where ν is the Euclidean outward pointing unit normal to the coordinate sphere $\{|x| = r\}$ and $d\sigma$ is the Euclidean surface measure. By (1.3), one has

$$\lim_{r\to\infty} \oint_{|x|=r} X \cdot \nu \; d\sigma = \lim_{r\to\infty} \oint_{|x|=r} \sum_{i,j} (\partial_j g_{ij} - \partial_i g_{jj})\nu^i \; d\sigma. \tag{1.5}$$

The total mass of (M, g) ([1]) is defined by

$$m_{ADM}(g) = \lim_{r\to\infty} \frac{1}{16\pi} \oint_{|x|=r} \sum_{i,j} (\partial_j g_{ij} - \partial_i g_{jj})\nu^i \; d\sigma. \tag{1.6}$$

One sees immediately that $m_{ADM}(g)$ is well defined if $R \in L^1(M)$. The fact that $m_{ADM}(g)$ is independent of the choices of the rectangular coordinates and of exhaustion of M used to define the limit was shown in [2], [11].

The Positive Mass Theorem ([21], [23]) in its simplest form is

Theorem 1.1. *Let (M, g) be a complete, asymptotically flat 3-manifold with nonnegative scalar curvature. The total mass of (M, g) satisfies*

$$m_{ADM}(g) \geqslant 0,$$

and $m_{ADM}(g) = 0$ if and only if (M, g) is isometric to the Euclidean space $(\mathbb{R}^3, \delta_{ij})$.

2 The Bartnik mass and static metrics

There have been many approaches ([20], [13], [9], [10], [3], [14], [8], [15], etc) towards defining a *quasi-local* mass of a bounded region Ω in an asymptotically flat 3-manifold (M, g). In [3], a variational definition was proposed by Bartnik.

Definition 2.1. *Let $\mathcal{PM}$ denote the set of all asymptotically flat 3-manifolds (M, g) with nonnegative scalar curvature such that (M, g) has no closed minimal surfaces. Let $(M, g) \in \mathcal{PM}$ and $\Omega \subset (M, g)$ be a bounded, connected region with connected boundary $\partial\Omega$. Let $\mathcal{PM}(\Omega)$ denote the set of $(\tilde{M}, \tilde{g}) \in \mathcal{PM}$ such that Ω embeds isometrically into $(\tilde{M}, \tilde{g})$. The Bartnik mass is defined by*

$$m_B(\Omega) = \inf\{m_{ADM}(\tilde{g}) \mid (\tilde{M}, \tilde{g}) \in \mathcal{PM}(\Omega)\}. \tag{2.1}$$

The condition that $(M, g) \in \mathcal{PM}$ has no closed minimal surfaces is imposed to exclude examples which hide Ω inside an arbitrarily small neck, which would make $m_B(\Omega)$ trivially zero. A modification of $m_B(\Omega)$ was given in [14] to allow $\mathcal{PM}$ to contain manifolds with outermost minimal surface boundary .

The first immediate consequence of (2.1) is the monotonicity of the Bartnik mass: if $\Omega_1 \subset \Omega_2$, then $m_B(\Omega_1) \leqslant m_B(\Omega_2)$. The non-negativity of $m_B(\Omega)$ follows directly from the Positive Mass Theorem. The strict positivity of $m_B(\Omega)$ was shown in [14] with a slightly weaker rigidity conclusion that if $m_B(\Omega) = 0$, then Ω is locally flat. It was also shown in [14] that, if $\{\Omega_i\}_{i=1}^{\infty}$ is an exhaustion sequence of $(M, g) \in \mathcal{PM}$, then $\lim_{i\to\infty} m_B(\Omega_i) = m_{ADM}(g)$.

Although in many respects the definition of $m_B(\Omega)$ is quite satisfactory, it is not constructive. Hence it is necessary to determine computational methods. The following *conjecture* ([3], [14]) is the key to the computability of $m_B(\Omega)$.

Static Extension Conjecture

The infimum $m_B(\Omega)$ is realized by a unique, asymptotically flat 3-manifold (M^S, g^S) with boundary ∂M such that ∂M is isometric to $\partial\Omega$, g^S is a static metric in the interior of M^S, and the mean curvature of ∂M agrees with the mean curvature of $\partial\Omega$ under the boundary isometry.

A Riemannian metric g is called **static** if there is a (positive) function N such that the warped Lorentzian metric

$$\bar{g} = -N^2 dt^2 + g \tag{2.2}$$

is a solution to the Vacuum Einstein Equation, i.e. $Ric(\bar{g}) = 0$. Equivalently, g is static if the pair (g, N) satisfies the coupled system

$$\begin{cases} NRic(g) = \nabla^2 N \\ \quad \triangle N \quad = \quad 0, \end{cases} \tag{2.3}$$

where $Ric(g)$ is the Ricci curvature of g, $\nabla^2 N$ and $\triangle N$ denote the Hessian and Laplacian of N with respect to g. The function N is called the static potential of g. For example, as the 4-dimensional Schwarzschild spacetime metric

$$\bar{g}_m^S = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{1}{1 - \frac{2m}{r}} dr^2 + r^2 d\omega^2 \tag{2.4}$$

(where $d\omega^2$ is the round metric on the unit sphere $S^2 \subset \mathbb{R}^3$) satisfies $Ric(\bar{g}_m^S) = 0$, the 3-dimensional spatial Schwarzschild metric

$$g_m^S = \frac{1}{1 - \frac{2m}{r}} dr^2 + r^2 d\omega^2 \tag{2.5}$$

is static with the static potential given by $N = \sqrt{1 - \frac{2m}{r}}$.

The Riemannian Penrose inequality ([8], [14]) provides examples where the static extension conjecture holds for the modified Bartnik mass [14]. Other evidence supporting the conjecture comes from the result in [12] on scalar curvature deformation and the critical point analysis for the total mass functional in [5], [7]. Partial result on the existence of a static metric extension for small perturbation of Euclidean balls were given in [16].

3 Evolution of the Bartnik mass

Let $(M, g) \in \mathcal{PM}$ and let $\{\Sigma_t\}$ be a family of closed 2-surfaces evolving in (M^3, g) according to the equation

$$\frac{\partial X}{\partial t} = \eta\nu, \tag{3.1}$$

where ν is the outward pointing unit normal to Σ_t and η is the speed. For each t, let Ω_t be the region enclosed by Σ_t. *Assume* the static extension conjecture holds, the Bartnik mass $m_(\Omega_t)$ is then determined only by the induced metric on Σ_t and the mean curvature of Σ_t in Ω_t. For this reason, one writes $m_B(\Omega_t)$ as $m_B(\Sigma_t)$.

Under the assumption that the static extension conjecture holds, the following evolution formula of $m_B(\Sigma_t)$ is derived in [6].

Theorem 3.1. *Assume the static extension conjecture holds, the Bartnik mass $m_B(\Sigma_t)$ satisfies*

$$\frac{d}{dt} m_B(\Sigma_t) = \frac{1}{16\pi} \oint_{\Sigma_t} N_t^S (R + |\Pi_t^S - \Pi_t|^2)\eta \; d\mu, \tag{3.2}$$

where $d\mu$ is the surface measure of the induced metric on Σ_t, R is the scalar curvature of (M, g), N_t^S is the static potential of the unique static extension *of Σ_t, Π_t and Π_t^S denote the second fundamental form of Σ_t in (M, g) and in the static extension.*

Integrate (3.2) and apply the co-area formula, one has the following corollary.

Corollary 3.1. *Assume the static extension conjecture holds. Suppose $\{\Sigma_t\}$ evolves with a positive speed. For any $t_2 > t_1$, the Bartnik mass of Σ_{t_2} and Σ_{t_1} are related by*

$$m_B(\Sigma_{t_2}) - m_B(\Sigma_{t_1}) = \frac{1}{16\pi} \int_{\Omega_{[t_1,t_2]}} N_t^S (R + |\Pi_t^S - \Pi_t|^2) \; dV_g, \tag{3.3}$$

where $\Omega_{[t_1,t_2]}$ is the region between Σ_{t_1} and Σ_{t_2}.

The interesting feature about (3.3) is that, although the integrand

$$G = N_t^S (R + |\Pi_t^S - \Pi_t|^2) \tag{3.4}$$

defines a function on $\Omega_{[t_1,t_2]}$ through the foliation $\{\Sigma_t\}_{t_1 \leqslant t \leqslant t_2}$, the integral

$$\int_{\Omega_{[t_1,t_2]}} G \; dV_g \tag{3.5}$$

is foliation independent.

To derive (3.2), the following lemma derived in [7] plays a key role.

Lemma 3.1. *Let* $(M^3, \hat{g})$ *be an asymptotically flat manifold with boundary* Σ*. Consider the functional*

$$\mathcal{H}(g, N) = 16\pi m_{ADM}(g) - \int_M NR\, dV_g \tag{3.6}$$

defined for $(g, N) \in \mathcal{G} \times \mathcal{N}$*, where* $\mathcal{G}$ *is the space of asymptotically flat metrics on* M *and* $\mathcal{N}$ *is the set of functions on* M *that approaches* 1 *at the infinity of* M*. Then, at* $(g, N) \in \mathcal{G} \times \mathcal{N}$*, the linearization of* $\mathcal{H}(\cdot, \cdot)$ *with respect to its first variable is given by*

$$\begin{aligned} D_g\mathcal{H}(g, N)(h) = &- \int_M \langle DR(g)^*(N) + \frac{1}{2}NRg, h\rangle\, dV_g \\ &+ \oint_\Sigma \{(\nabla_\nu N)(tr_\Sigma h) - N[\langle h|_\Sigma, \Pi\rangle + 2DH(h)]\}\, d\mu, \end{aligned} \tag{3.7}$$

where $DR(g)^*$ *is the formal* L^2 dV_g*-adjoint of the linearization of the scalar curvature map* R *at* g*,* ν *is the* ∞*-pointing unit normal to* Σ *in* (M, g)*,* $\nabla_\nu N$ *is the directional derivative of* N *along* ν*,* $tr_\Sigma h$ *is the trace of* $h|_\Sigma$*, which is the restriction of* h *to* Σ*,* Π *denotes the second fundamental form of* Σ *in* (M, g)*, defined by* $\Pi_{\alpha\beta} = \langle\nabla_{\partial_\alpha}\nu, \partial_\beta\rangle$ *and* $DH(h)$ *is the linearization of the mean curvature* H *of* Σ*.*

We explain how (3.2) is derived from (3.7). *Assume* the static extension conjecture holds. For each $\Sigma_t \in (M, g)$, there is a unique static extension (M_t^S, g_t^S) with boundary $\partial M_t^S = \Sigma_t$ such that

$$m_{ADM}(g_t^S) = m_B(\Sigma_t) \tag{3.8}$$

and

$$g_t^S|_{\Sigma_t} = g|_{\Sigma_t},\ H_t^S = H_t, \tag{3.9}$$

where $g_t^S|_{\Sigma_t}$, $g|_{\Sigma_t}$ denote the induced metric on Σ_t in (M_t^S, g_t^S), (M, g), and H_t^S, H_t denote the mean curvature of Σ_t in (M_t^S, g_t^S), (M, g).

Let N_t^S be the static potential of (M_t^S, g_t^S). As g_t^S has zero scalar curvature, one has

$$\mathcal{H}(g_t^S, N_t^S) = 16\pi m_{ADM}(g_t^S) = 16\pi m_B(\Sigma_t). \tag{3.10}$$

Therefore

$$\begin{aligned} 16\pi \frac{d}{dt} m_B(\Sigma_t) &= \frac{d}{dt}\mathcal{H}(g_t^S, N_t^S) \\ &= D_g\mathcal{H}(g_t^S, N_t^S)(h_t^S), \end{aligned} \tag{3.11}$$

where the last equality holds because $R(g_t^S) = 0$, and

$$h_t^S = \frac{d}{dt} g_t^S \tag{3.12}$$

denotes the variation of the family of the static metrics $\{g_t^S\}$. It follows from (3.11), (3.7) and the fact $DR(g_t^S)^*(N_t^S) = 0$ ([12]) that

$$16\pi \frac{d}{dt} m_B(\Sigma_t) = \oint_{\Sigma_t} (\nabla_\nu N_t^S)(tr_{\Sigma_t} h_t^S) \, d\mu_t - \oint_{\Sigma_t} N_t^S [\langle h_t^S|_{\Sigma_t}, \Pi_t^S \rangle + 2DH(h_t^S)] \, d\mu_t, \tag{3.13}$$

where ν is the ∞-pointing unit normal to Σ_t in (M_t^S, g_t^S), Π_t^S is the second fundamental form of Σ_t in (M_t^S, g_t^S), and $d\mu_t$ is the surface measure on Σ_t.

Applying the geometric boundary condition (3.9), one has

$$h_t^S|_{\Sigma_t} = \frac{d}{dt} (g|_{\Sigma_t}), \; DH(h_t^S) = \frac{d}{dt} H_t. \tag{3.14}$$

On the other hand, the following formulas governing the evolution of $g|_{\Sigma_t}$ and H_t are well known

$$\frac{d}{dt} (g|_{\Sigma_t}) = 2\eta \Pi_t, \tag{3.15}$$

and

$$\frac{d}{dt} H_t = -\triangle_{\Sigma_t} \eta - (|\Pi_t|^2 + Ric(n,n))\eta, \tag{3.16}$$

where Π_t is the second fundamental form of Σ_t in (M, g), n is the ∞-pointing unit normal to Σ_t in (M, g) and $Ric(n,n)$ is the Ricci curvature of (M, g) along n.

Plug (3.14), (3.15) and (3.16) into formula (3.13), one has

$$16\pi \frac{d}{dt} m_B(\Sigma_t) = \oint_\Sigma 2\eta H_t (\nabla_\nu N_t^S) - N_t^S 2\eta \langle \Pi_t, \Pi_t^S \rangle \, d\mu_t + \oint_\Sigma 2N_t^S [\triangle_{\Sigma_t} \eta + (|\Pi_t|^2 + Ric(n,n))\eta] d\mu_t. \tag{3.17}$$

Integrating by parts,

$$16\pi \frac{d}{dt} m_B(\Sigma_t) = \oint_\Sigma 2\eta [\triangle_{\Sigma_t} N_t^S + H_t(\nabla_\nu N_t^S) - N_t^S \langle \Pi_t^S, \Pi_t \rangle] \, d\mu_t + \oint_{\Sigma_t} 2\eta (|\Pi_t|^2 + Ric(n,n)) N_t^S \, d\mu_t. \tag{3.18}$$

To proceed, one makes use of the following identity ([17])

$$\triangle_{\Sigma_t} N_t^S + H_t^S (\nabla_\nu N_t^S) + Ric_t^S(\nu,\nu) N_t^S = 0, \tag{3.19}$$

where $Ric_t^S(\nu,\nu)$ is the Ricci curvature of (M_t^S, g_t^S) along ν. Applying the mean curvature matching condition $H_t^S = H_t$, one has

$$\triangle_{\Sigma_t} N_t^S + H_t(\nabla_\nu N_t^S) = -Ric_t^S(\nu,\nu) N_t^S. \tag{3.20}$$

Therefore, the right side of (3.18) is reduced to

$$\oint_\Sigma 2\eta N_t^S \left\{ -Ric_t^S(\nu,\nu) - \langle \Pi_t^S, \Pi_t \rangle + |\Pi_t|^2 + Ric(n,n) \right\} \, d\mu. \tag{3.21}$$

Finally, one applies the Gauss equation to Σ_t in (M,g) and in (M_t^S, g_t^S) to get

$$2K_t = R - 2Ric(n,n) + H_t^2 - |\Pi_t|^2 \tag{3.22}$$

$$2K_t = 0 - 2Ric_t^S(\nu,\nu) + (H_t^S)^2 - |\Pi_t^S|^2, \tag{3.23}$$

where K_t is the Gaussian curvature of Σ_t. After applying the mean curvature matching condition $H_t^S = H_t$ again, one has

$$Ric(n,n) - Ric_t^S(\nu,\nu) = \frac{1}{2}(R + |\Pi_t^S|^2 - |\Pi_t|^2). \tag{3.24}$$

One concludes

$$\begin{aligned} &16\pi \frac{d}{dt} m_B(\Sigma_t) \\ = &\oint_\Sigma \eta N_t^S \left\{ -2\langle \Pi_t^S, \Pi_t\rangle + 2|\Pi_t|^2 + R + |\Pi_t^S|^2 - |\Pi_t|^2 \right\} \ d\mu_t \\ = &\oint_\Sigma N_t^S (R + |\Pi_t^S - \Pi_t|^2)\eta \ d\mu_t. \end{aligned} \tag{3.25}$$

4 An upper bound of the Bartnik mass

In this section we discuss an upper bound of $m_B(\Omega)$ under the assumption that $\partial\Omega$ is isometric to a round sphere.

In general, if $\partial\Omega$ has positive Gaussian curvature, one can isometrically embed $\partial\Omega$ into $\mathbb{R}^3$ as a convex surface by the Weyl embedding theorem ([19]). The Brown-York mass ([9]) of $\partial\Omega$ is then defined by

$$m_{BY}(\partial\Omega) = \frac{1}{8\pi} \oint_{\partial\Omega} (H_0 - H) \ d\mu, \tag{4.1}$$

where H, H_0 is the mean curvature of $\partial\Omega$ in Ω, $\mathbb{R}^3$ respectively. If Ω has nonnegative scalar curvature, it was shown in [22] that $m_{BY}(\partial\Omega) \geqslant 0$ and equality holds if and only if Ω is isometric to a Euclidean domain. In fact, the method and result in [22] directly implies that

$$m_B(\Omega) \leqslant m_{BY}(\partial\Omega). \tag{4.2}$$

In the special case when $\partial\Omega$ is isometric to a round sphere, one has a refined estimate of $m_B(\Omega)$ ([18]).

Theorem 4.1. *Suppose $\partial\Omega$ is isometric to a round sphere and has positive mean curvature H, then*

$$m_B(\Omega) \leqslant \sqrt{\frac{|\partial\Omega|}{16\pi}} \left[1 - \frac{1}{16\pi|\partial\Omega|} \left(\oint_\Sigma H d\mu \right)^2 \right], \tag{4.3}$$

where $|\partial\Omega|$ is the area of $\partial\Omega$.

This bound is sharp when $\partial\Omega$ has constant mean curvature. A similar but weaker estimate was given in [4] (Theorem 8 in Section 5) where $\frac{1}{|\partial\Omega|}\left(\oint_\Sigma H d\mu\right)^2$ is replaced by $|\partial\Omega| \min_{\partial\Omega} H^2$.

The proof of (4.3) is a slight modification of the proof in [22]. The main idea is as follows. Suppose $\partial\Omega$ is isometric to a round sphere of radius r_0 in $\mathbb{R}^3$. Consider a 3-dimensional spatial Schwarzschild manifold

$$(M^S_m, g^S_m) = ([2m, \infty) \times S^2, \frac{1}{1-\frac{2m}{r}} dr^2 + r^2 d\omega^2), \tag{4.4}$$

where m is chosen in $(-\infty, \frac{1}{2}r_0)$. One identifies $\partial\Omega$ with the spherically symmetric coordinate sphere $\{r = r_0\}$ in (M^S_m, g^S_m). Write the metric g^S_m outside Σ as

$$g^S_m = d\rho^2 + g_\rho, \tag{4.5}$$

where ρ is the distance to Σ. Following [22], one considers a function u defined on M^S_m outside Σ such that the warped metric

$$g^u = u^2 d\rho^2 + g_\rho \tag{4.6}$$

has zero scalar curvature and the mean curvature of Σ with respect to g^u agrees with H, the mean curvature of $\Sigma = \partial\Omega$ in Ω. As (M^S_m, g^S_m) is static, one considers its static potential function N, given by

$$N = \sqrt{1 - \frac{2m}{r}}. \tag{4.7}$$

A key observation is that

$$\oint_{\Sigma_\rho} N(H^S - H^u)\ d\mu \tag{4.8}$$

is monotone decreasing as $\rho \to \infty$ and

$$\lim_{\rho\to\infty} \oint_{\Sigma_\rho} N(H^S - H^u)\ d\mu = 8\pi(m_{ADM}(g^u) - m), \tag{4.9}$$

where H^S, H^u denote the mean curvature of the distance level set Σ_ρ with respect to g^S_m, g^u respectively. Note that when $m = 0$, the above is reduced to the original monotonicity in [22]. Thus, at Σ one has

$$\oint_\Sigma N(H^S - H^u)\ d\mu \geqslant 8\pi(m_{ADM}(g^u) - m). \tag{4.10}$$

In particular, this implies

$$m + \frac{1}{8\pi}\oint_\Sigma N(H^S - H^u)\ d\mu \geqslant m_{ADM}(g^u) \geqslant m_B(\Omega). \tag{4.11}$$

Minimizing the left side of (4.11) over $m \in (-\infty, \frac{1}{2}r_0)$ gives the estimate (4.3).

The following conjecture is motivated by (4.11). If true, it would provide a natural generalization of (4.2).

Conjecture 4.1. *Suppose $\partial\Omega$ can be isometrically embedded into (M, g), where (M, g) is an asymptotically flat 3-manifold and g is a static metric. Then the Bartnik mass $m_B(\Omega)$ satisfies*

$$m_B(\Omega) \leqslant m_{ADM}(g) + \frac{1}{8\pi} \oint_{\partial\Omega} N(H^S - H)\ d\mu, \tag{4.12}$$

where N is the static potential of (M, g), H^S and H are the mean curvature of $\partial\Omega$ in (M, g) and Ω.

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