Affine Schubert Calculus

Jennifer Morse

Collaborators (in order of appearance):
Lapointe, Lascoux, Wachs, Shimozono, Lam

Other Contributors: Garsia, Stanton, Sottile

- A combinatorial analog for Schur functions
  Macdonald polynomials

- A geometric analog for Schur functions
  Gromov-Witten invariants

- Affine Schubert calculus

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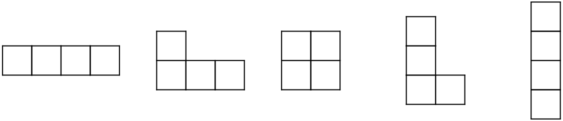
- Affine Schubert calculus

Schur functions

Partitions: $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$

Degree 4: (4) (3,1) (2,2) (2,1,1) (1,1,1,1)

Ferrers shapes:



Tableaux: Columns strictly increasing, rows non-decreasing

$$T = \begin{array}{|c|c|c|c|} \hline 5 & \\ \cline{1-2} 3 & 4 \\ \cline{1-2} 2 & 3 \\ \hline 1 & 2 & 2 & 5 \\ \hline \end{array}$$

Schur functions

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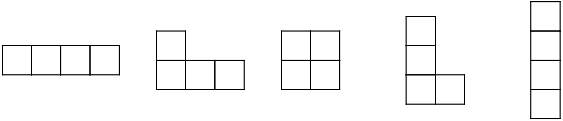


Tableaux: Columns strictly increasing, rows non-decreasing

Weight $= \ (\#ones, \#twos, \#threes, \ldots)$

$$T = \begin{array}{|c|c|c|c|} \hline 5 \\ \cline{1-1} 3 & 4 \\ \cline{1-2} 2 & 3 \\ \hline 1 & 2 & 2 & 5 \\ \hline \end{array}$$

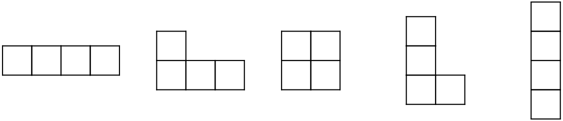
$$shape(T) = (4, 2, 2, 1)$$
$$weight(T) = (1, 3, 2, 1, 2)$$

Schur functions

Partitions: $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$

Degree 4: (4) (3,1) (2,2) (2,1,1) (1,1,1,1)

Ferrers shapes:



Tableaux: Columns strictly increasing, rows non-decreasing

Weight = $(\#ones, \#twos, \#threes, \ldots)$

$$T = \begin{array}{|c|c|c|c|} \hline 8 & & & \\ \hline 7 & 9 & & \\ \hline 3 & 4 & & \\ \hline 1 & 2 & 5 & 6 \\ \hline \end{array}$$

Standard tableaux have weight (1,1,...,1)

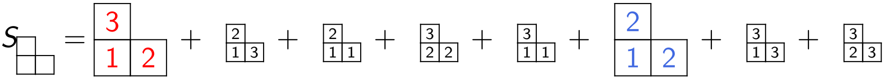
Schur functions

$$S_{\begin{array}{|c|c}\hline & \multicolumn{1}{c}{} \\ \hline & \multicolumn{1}{c|}{} \\ \hline\end{array}} = \begin{array}{|c|c|}\hline \color{red}{3} & \multicolumn{1}{c}{} \\ \hline \color{red}{1} & \color{red}{2} \\ \hline\end{array} + \begin{array}{|c|c|}\hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline\end{array} + \begin{array}{|c|c|}\hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline\end{array} + \begin{array}{|c|c|}\hline 3 & \multicolumn{1}{c}{} \\ \hline 2 & 2 \\ \hline\end{array} + \begin{array}{|c|c|}\hline 3 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline\end{array} + \begin{array}{|c|c|}\hline \color{blue}{2} & \multicolumn{1}{c}{} \\ \hline \color{blue}{1} & \color{blue}{2} \\ \hline\end{array} + \begin{array}{|c|c|}\hline 3 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline\end{array} + \begin{array}{|c|c|}\hline 3 & \multicolumn{1}{c}{} \\ \hline 2 & 3 \\ \hline\end{array}$$

$$= x_{\color{red}{1}}x_{\color{red}{2}}x_{\color{red}{3}} + x_1x_2x_3 + x_1x_1x_2 + x_2x_2x_3 + x_1x_1x_3 + x_{\color{blue}{1}}x_{\color{blue}{2}}x_{\color{blue}{2}} + x_1x_3x_3 + x_2x_3x_3$$

Basis for symmetric function space

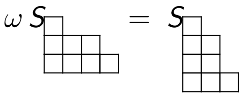
Schur functions



$$= x_1x_2x_3+x_1x_2x_3+x_1x_1x_2+x_2x_2x_3+x_1x_1x_3+x_1x_2x_2+x_1x_3x_3+x_2x_3x_3$$

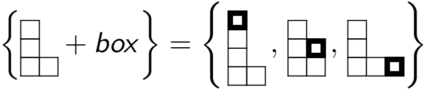
Basis for symmetric function space

- Involution *(shape conjugation)*



- Pieri rule:

$$S_1\, S_\lambda = \sum_{\mu=\lambda+box} S_\mu \quad \text{where}$$

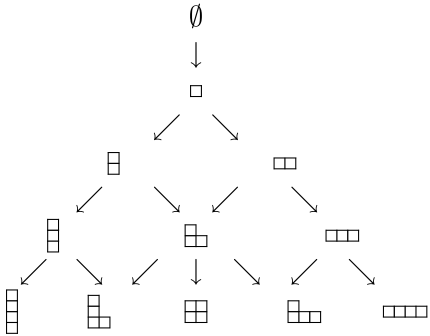


- Young partition lattice induced by Pieri rule

Young Partition Lattice: $\lambda \prec \mu$ if $\lambda \subset \mu$

$\lambda \prec\cdot\, \mu$ if $\mu = \lambda + box$

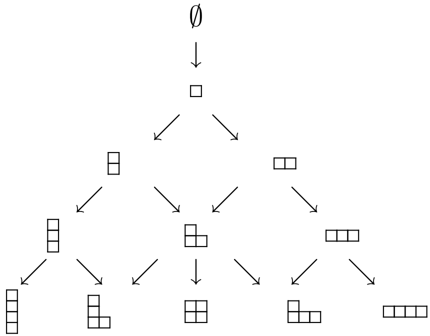
$\lambda \prec\cdot\, \mu$ if S_μ occurs in $S_1 S_\lambda$



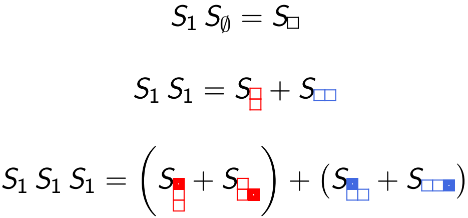
Young Partition Lattice: $\lambda \prec \mu$ if $\lambda \subset \mu$

$\lambda \prec\!\cdot\, \mu$ if $\mu = \lambda + \mathit{box}$

$\lambda \prec\!\cdot\, \mu$ if S_μ occurs in $S_1 S_\lambda$



$$S_1 \cdots S_1 = \sum_\mu (\#\ \mathit{of\ chains\ to}\ \mu)\, S_\mu$$



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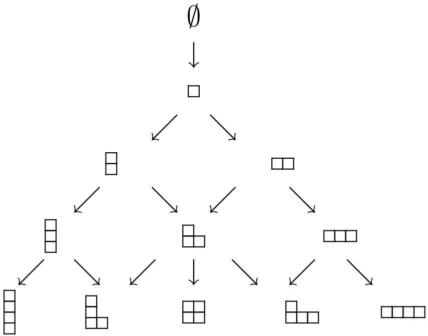


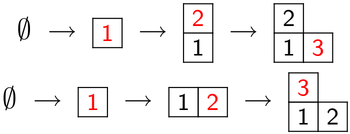
$$S_1 \cdots S_1 = \sum_\mu (\#\ of\ chains\ to\ \mu)\, S_\mu$$

$$S_1\, S_\emptyset = S_{\square}$$

$$S_1\, S_1 = S_{\square \atop \square} + S_{\square\square}$$

$$S_1\, S_1\, S_1 = \left(S_{\blacksquare \atop {\square \atop \square}} + S_{\square \atop \square\blacksquare}\right) + \left(S_{\blacksquare \atop \square\square} + S_{\square\square\blacksquare}\right)$$

# of chains = # of std tableaux:



Classical combinatorics of Schur functions

- Definition: $S_{\ydiagram{1,2}} = \ytableaushort{3,12} + \ytableaushort{2,13} + \ytableaushort{2,11} + \ytableaushort{3,22} + \ytableaushort{3,11} + \ytableaushort{2,12} + \cdots$

- Involution: $\omega\, S_{\ydiagram{1,3}} = S_{\ydiagram{1,1,2}}$

- Pieri rule:

$$S_1\, S_\lambda = \sum_{\mu=\lambda+box} S_\mu \quad \text{where} \quad \left\{ \ydiagram{1,1,2} + box \right\} = \left\{ \ytableaushort{\bullet,{},{},{}{}},\ \ytableaushort{{},{}\bullet,{}{}},\ \ytableaushort{{},{},{}{}\bullet} \right\}$$

- Young's Lattice

$$\begin{aligned} S_1 \cdots S_1 &= \sum_\mu (\#\ chains\ to\ shape\ \mu)\, S_\mu \\ &= \sum_\mu (\#\ std\ tab\ of\ shape\ \mu)\, S_\mu \end{aligned}$$

- Kostka Numbers:

$$S_{\lambda_1} \cdots S_{\lambda_\ell} = \sum_\mu (\#\ tab\ of\ shape\ \mu\ and\ weight\ \lambda)\, S_\mu$$

Macdonald Polynomials: $H_\lambda[X; q, t]$

- Representation theory
- Algebraic geometry
- Special functions
- Integrable systems
- Combinatorics

$$H_{2,2} = 2q(1 - t^5)\, x_1 x_2 x_3^2 + 3tq(q - t^3)\, x_2 x_4^3 - \frac{7t^3}{3tq - q^3}\, x_1^2 x_2^2 + \cdots$$

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$$\downarrow$$

Schur function expansion

$$\downarrow$$

$$H_{2,2} = t^2 S_{(4)} + (qt^2+qt+t)\,S_{(3,1)} + (q^2t^2+1)\,S_{(2,2)} + (q^2t+qt+q)\,S_{(2,1,1)} + q^2 S_{(1,1,1,1)}$$

Macdonald Polynomials: $H_\lambda[X; q, t]$

- Representation theory
- Algebraic geometry
- Special functions
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$$H_{2,2} = 2q(1 - t^5)\, x_1 x_2 x_3^2 + 3tq(q - t^3)\, x_2 x_4^3 - \frac{7t^3}{3tq - q^3}\, x_1^2 x_2^2 + \cdots$$

$$\downarrow$$

Schur function expansion

$$\downarrow$$

$$H_{2,2} = t^2 S_{(4)} + (qt^2 + qt + t)\, S_{(3,1)} + (q^2 t^2 + 1)\, S_{(2,2)} + (q^2 t + qt + q)\, S_{(2,1,1)} + q^2 S_{(1,1,1,1)}$$

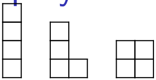
$$\downarrow$$

$\#$ of monomial terms $= \#$ of std tableaux

$$\downarrow$$

$$(S_1)^4 = S_{(4)} + (1 + 1 + 1)\, S_{(3,1)} + \left(\begin{array}{|c|c|} \hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array} + \begin{array}{|c|c|} \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array} \right) S_{(2,2)} + (1 + 1 + 1)\, S_{(2,1,1)} + S_{(1,1,1,1)}$$

k-bounded Macdonald polynomials

$2-bounded:$  $bad =$ 

$$H_{1,1,1,1}=t^6S_{(4)}+\left(t^3+t^4+t^5\right)S_{(3,1)}+\left(t^2+t^4\right)S_{(2,2)}+\left(t+t^2+t^3\right)S_{(2,1,1)}+S_{(1,1,1,1)}$$

$$H_{2,1,1}=t^3S_{(4)}+\left(t+t^2+qt^3\right)S_{(3,1)}+\left(t+qt^2\right)S_{(2,2)}+\left(1+qt+qt^2\right)S_{(2,1,1)}+qS_{(1,1,1,1)}$$

$$H_{2,2}=t^2S_{(4)}+\left(qt^2+qt+t\right)S_{(3,1)}+\left(q^2t^2+1\right)S_{(2,2)}+\left(q^2t+qt+q\right)S_{(2,1,1)}+q^2S_{(1,1,1,1)}$$

k-bounded Macdonald polynomials

$$2-\textit{bounded}: \quad (1,1,1,1) \quad (2,1,1) \quad (2,2) \qquad \textit{bad} = (4) \quad (3,1)$$

$$H_{1^4} = t^4\left(S_{(2,2)} + tS_{(3,1)} + t^2 S_{(4)}\right) + (t^2+t^3)\left(S_{(2,1,1)} + tS_{(3,1)}\right) + \left(S_{(1,1,1,1)} + tS_{(2,1,1)} + t^2 S_{(2,2)}\right)$$

$$H_{2,1,1} = t\left(S_{(2,2)} + tS_{(3,1)} + t^2 S_{(4)}\right) + (1+qt^2)\left(S_{(2,1,1)} + tS_{(3,1)}\right) + q\left(S_{(1,1,1,1)} + tS_{(2,1,1)} + t^2 S_{(2,2)}\right)$$

$$H_{2,2} = \left(S_{(2,2)} + tS_{(3,1)} + t^2 S_{(4)}\right) + (tq+q)\left(S_{(2,1,1)} + tS_{(3,1)}\right) + q^2\left(S_{(1,1,1,1)} + tS_{(2,1,1)} + t^2 S_{(2,2)}\right)$$

Coefficients = positive sum of monomials

Factors = t-positive sum of Schur functions

k-bounded Macdonald polynomials

$$2 - bounded: \quad (1,1,1,1) \quad (2,1,1) \quad (2,2) \qquad bad = (4) \quad (3,1)$$

$$H_{1^4} = t^4\left(S_{(2,2)} + tS_{(3,1)} + t^2 S_{(4)}\right) + \left(t^2 + t^3\right)\left(S_{(2,1,1)} + tS_{(3,1)}\right) + \left(S_{(1,1,1,1)} + tS_{(2,1,1)} + t^2 S_{(2,2)}\right)$$

$$H_{2,1,1} = t\left(S_{(2,2)} + tS_{(3,1)} + t^2 S_{(4)}\right) + \left(1 + qt^2\right)\left(S_{(2,1,1)} + tS_{(3,1)}\right) + q\left(S_{(1,1,1,1)} + tS_{(2,1,1)} + t^2 S_{(2,2)}\right)$$

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$$2-\textit{bounded}: \quad (1,1,1,1) \quad (2,1,1) \quad (2,2) \qquad \textit{bad} = (4) \quad (3,1)$$

$$H_{1^4} = t^4\left(S_{2,2} + tS_{3,1} + t^2 S_{4}\right) + \left(t^2+t^3\right)\left(S_{2,1,1} + tS_{3,1}\right) + \left(S_{1,1,1,1} + tS_{2,1,1} + t^2 S_{2,2}\right)$$

$$H_{2,1,1} = t\left(S_{2,2} + tS_{3,1} + t^2 S_{4}\right) + \left(1+qt^2\right)\left(S_{2,1,1} + tS_{3,1}\right) + q\left(S_{1,1,1,1} + tS_{2,1,1} + t^2 S_{2,2}\right)$$

$$H_{2,2} = \underbrace{\left(S_{2,2} + tS_{3,1} + t^2 S_{4}\right)}_{S^{(2)}_{2,2}} + (tq+q)\underbrace{\left(S_{2,1,1} + tS_{3,1}\right)}_{S^{(2)}_{2,1,1}} + q^2\underbrace{\left(S_{1,1,1,1} + tS_{2,1,1} + t^2 S_{2,2}\right)}_{S^{(2)}_{1,1,1,1}}$$

Coefficients = positive sum of monomials

Factors = t-positive sum of Schur functions

New basis? $S^{(k)}_\lambda$ *for* $\lambda_1 \leq k$

k-Schur functions for $\lambda_1 \leq k$??

- Involution: $\omega S_\lambda^{(k)} = S_{\lambda^{\omega_k}}^{(k)}$
- k-Pieri rule:

$$S_1 \, S_\lambda^{(k)} = \sum_{certain \;\; partitions \;\; \mu} S_\mu^{(k)}$$

- k-Kostka numbers:

$$S_{\lambda_1} \cdots S_{\lambda_\ell} = \sum_\mu (positive \;\; integers) \; S_\mu^{(k)}$$

- Macdonald expansion:

$$H_\lambda[X; q, t] = \sum_{\mu_1 \leq k} (positive \;\; sums \;\; of \;\; monomials \;\; in \;\; q, t) \; S_\mu^{(k)}$$

k-Schur functions for $\lambda_1 \leq k$??

- Involution: $\omega S_\lambda^{(k)} = S_{\lambda^{\omega_k}}^{(k)}$
- k-Pieri rule:

$$S_1\, S_\lambda^{(k)} = \sum_{\textit{certain partitions } \mu} S_\mu^{(k)}$$

$\downarrow$ iterate $\downarrow$

- k-Kostka numbers:

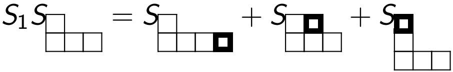
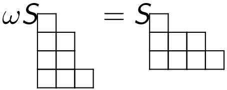
$$S_1 \cdots S_1 = \sum_\mu (\textit{positive integers})\; S_\mu^{(k)}$$

$\downarrow$ t, q-generalize $\downarrow$

- Macdonald expansion:

$$H_\lambda[X; q, t] = \sum_{\mu_1 \leq k} (\textit{positive sums of monomials in } q, t)\; S_\mu^{(k)}$$

k-bounded partitions and cores

- Pieri rule: 
- Involution: 

k-bounded partitions and cores

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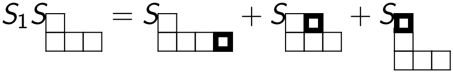
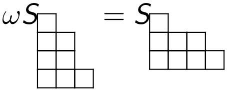
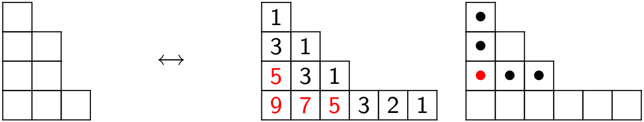


- Involution:



Bijection: k-bounded partitions $\leftrightarrow$ $k+1$-core shapes

(no $k+1$-hooks)



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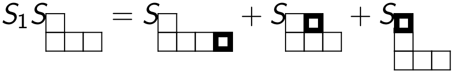
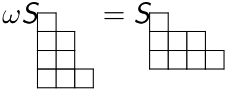


- Involution:



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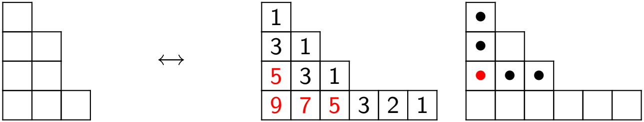
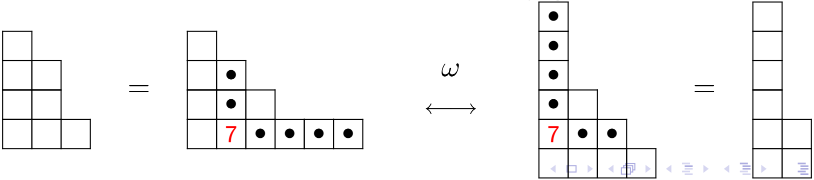


Involution on k-bounded partitions: conjugation of cores



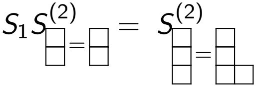
k-Pieri Rule

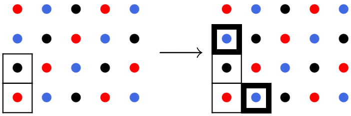
$$S_1 S_\lambda^{(k)} = \sum_{\text{some } \mu} S_\mu^{(k)}$$

$$S_1 S^{(2)}_{\begin{array}{|c|}\hline \, \\ \hline \, \\ \hline \end{array}} = S^{(2)}_{\begin{array}{|c|}\hline \, \\ \hline \, \\ \hline \, \\ \hline \end{array}}$$

k-Pieri Rule

$$S_1 S_\lambda^{(k)} = \sum_{core(\mu)=core(\lambda)+box/colored} S_\mu^{(k)}$$

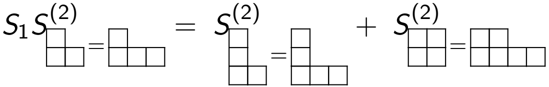


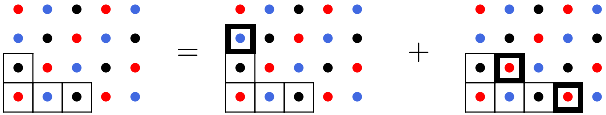


$core(\lambda) \rightarrow core(\mu)$ + box (and all others of the same color)

k-Pieri Rule

$$S_1 S_\lambda^{(k)} = \sum_{core(\mu)=core(\lambda)+box/colored} S_\mu^{(k)}$$





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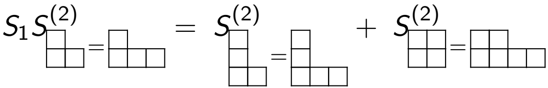
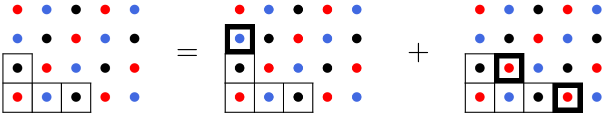


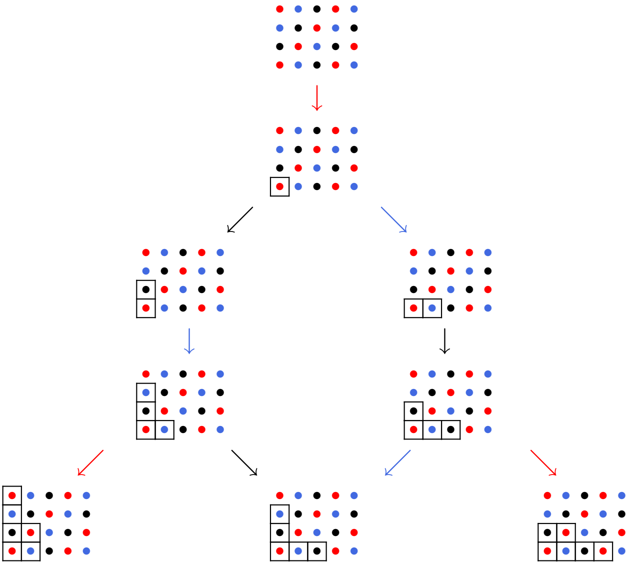


$core(\lambda) \to core(\mu)$ + box (and all others of the same color)

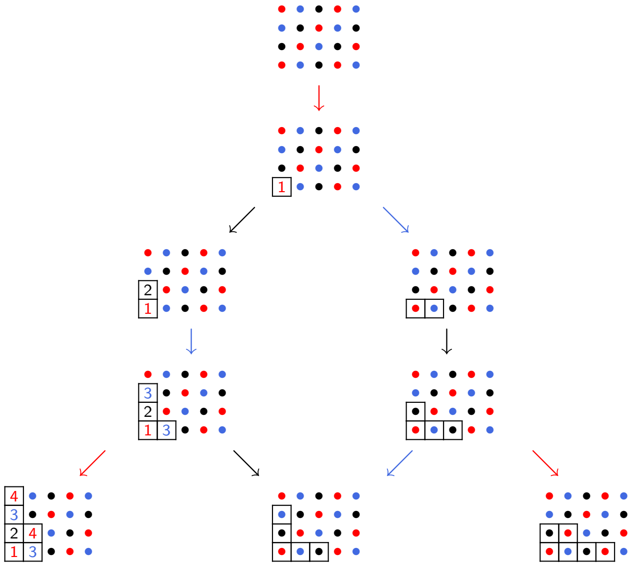
Iterate to determine k-Kostka numbers:

$$S_1 \cdots S_1 = \sum_\mu (positive\ integers)\, S_\mu^{(k)}$$

$\lambda \prec\!\cdot\, \mu$ when $S_\mu^{(k)}$ in $S_1 S_\lambda^{(k)}$

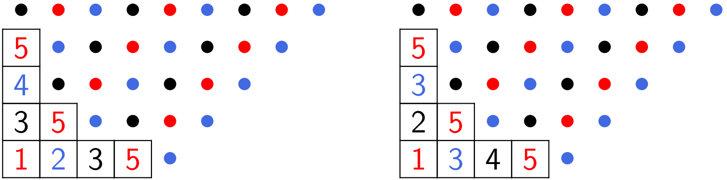


$\lambda \prec\!\cdot\, \mu$ when $S_\mu^{(k)}$ in $S_1 S_\lambda^{(k)}$



k-Tableaux

- Shape is a $k+1$-core
- Columns and rows strictly increasing
- Multiplicities have same color ($k+1$-residue)



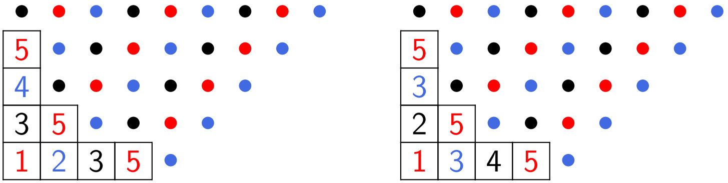
$k = 2$

k-Tableaux

Shape is a $k+1$-core

Columns and rows strictly increasing

Multiplicities have same color ($k+1$-residue)



$k = 2$

Iterated k-Pieri rule:

$$S_1 \cdots S_1 = \sum_{\mu} (\#k\text{-tableaux of shape core}(\mu))\ S_\mu^{(k)}$$

$\downarrow$ t, q-generalize $\downarrow$

Macdonald expansion:

$$H_\lambda[X; q, t] = \sum_{\mu_1 \le k} (\textit{positive sums of monomials in } q, t)\ S_\mu^{(k)}$$

Approach to Macdonald coefficients

$$S_1S_1S_1S_1 = \begin{array}{|c|c|c|c|}\hline 3&4\\\cline{1-2} 1&2&3&4\\\hline\end{array}\; S^{(2)}_{(2,2)} + \left(\begin{array}{|c|c|c|}\hline 4\\\cline{1-1} 3\\\cline{1-1} 1&2&4\\\hline\end{array} + \begin{array}{|c|c|c|}\hline 4\\\cline{1-1} 2\\\cline{1-1} 1&3&4\\\hline\end{array} \right) S^{(2)}_{(2,1,1)} + \begin{array}{|c|c|}\hline 4\\\cline{1-1} 3\\\cline{1-1} 2&4\\\hline 1&3\\\hline\end{array}\; S^{(2)}_{(1,1,1,1)}$$

$$\downarrow$$

$$Map \;\; T \;\; to \;\; q^{??}\, t^{??}$$

$$\downarrow$$

$$H_{2,1,1} = t\, S^{(2)}_{(2,2)} + \left(1 + qt^2\right) S^{(2)}_{(2,1,1)} + q\, S^{(2)}_{(1,1,1,1)}$$

Compare:

$$S_1 \cdots S_1 = \sum_{\mu} (\#k\text{-tableaux of shape core}(\mu))\; S^{(k)}_{\mu}$$

$$S_1 \cdots S_1 = \sum_{\mu} (\#\text{std tableaux of shape } \mu)\; S_{\mu}$$

$$\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline\end{array} \quad \begin{array}{|c|c|c|}\hline 4\\\cline{1-1} 1&2&3\\\hline\end{array} \quad \begin{array}{|c|c|c|}\hline 2\\\cline{1-1} 1&3&4\\\hline\end{array} \quad \begin{array}{|c|c|c|}\hline 3\\\cline{1-1} 1&2&4\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 3&4\\\hline 1&2\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 2&4\\\hline 1&3\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 4\\\cline{1-1} 3\\\cline{1-1} 1&2\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 4\\\cline{1-1} 2\\\cline{1-1} 1&3\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 3\\\cline{1-1} 2\\\cline{1-1} 1&4\\\hline\end{array} \quad \begin{array}{|c|}\hline 4\\\hline 3\\\hline 2\\\hline\end{array}$$

k-bounded Macdonald polynomials

$$H_{1^4} = t^4\left(S_{(2,2)} + tS_{(3,1)} + t^2 S_{(4)}\right) + (t^2+t^3)\left(S_{(2,1,1)} + tS_{(3,1)}\right) + \left(S_{(1,1,1,1)} + tS_{(2,1,1)} + t^2 S_{(2,2)}\right)$$

$$H_{2,1,1} = t\left(S_{(2,2)} + tS_{(3,1)} + t^2 S_{(4)}\right) + (1+qt^2)\left(S_{(2,1,1)} + tS_{(3,1)}\right) + q\left(S_{(1,1,1,1)} + tS_{(2,1,1)} + t^2 S_{(2,2)}\right)$$

$$H_{2,2} = \underbrace{\left(S_{(2,2)} + tS_{(3,1)} + t^2 S_{(4)}\right)}_{S^{(2)}_{(2,2)}} + (tq+q)\underbrace{\left(S_{(2,1,1)} + tS_{(3,1)}\right)}_{S^{(2)}_{(2,1,1)}} + q^2\underbrace{\left(S_{(1,1,1,1)} + tS_{(2,1,1)} + t^2 S_{(2,2)}\right)}_{S^{(2)}_{(1,1,1,1)}}$$

Coefficients q, t-count k-tableaux

Factors? Characterize $S^{(k)}_\lambda$ for $\lambda_1 \le k$?

k-Schur function conjectures: $\lambda_1 \leq k$

- Involution: $\omega S_\lambda^{(k)} = S_{\lambda^{\omega_k}}^{(k)}$
- k-Pieri rule:

$$S_1 \, S_\lambda^{(k)} = \sum_{core(\mu) = core(\lambda) \,+\, box/colored} S_\mu^{(k)}$$

- k-Kostka numbers:

$$S_1 \cdots S_1 = \sum_\mu (\# \ k\text{-tableaux shape core}(\mu)) \, S_\mu^{(k)}$$

- Macdonald expansion:

$$H_\lambda[X; q, t] = \sum_{\mu_1 \leq k} (q, t \text{ count } k\text{-tableaux shape core}(\mu)) \, S_\mu^{(k)}$$

k-Schur function conjectures: $\lambda_1 \leq k$

- Basis for *Span* $\{S_{\lambda_1} S_{\lambda_2} \cdots S_{\lambda_\ell}\}_{\lambda=(\lambda_1,\ldots,\lambda_\ell)}$
- Involution: $\omega S_\lambda^{(k)} = S_{\lambda^{\omega_k}}^{(k)}$
- k-Pieri rule:

$$S_1\, S_\lambda^{(k)} = \sum_{core(\mu)=core(\lambda)\;+\;box/colored} S_\mu^{(k)}$$

- k-Kostka numbers:

$$S_1 \cdots S_1 = \sum_\mu (\#\ k\text{-tableaux shape core}(\mu))\, S_\mu^{(k)}$$

More generally: $S_{\lambda_1} \cdots S_{\lambda_n} = S_\lambda^{(k)} + \sum_{\mu > \lambda} K_{\mu\lambda}^{(k)}\, S_\mu^{(k)}$

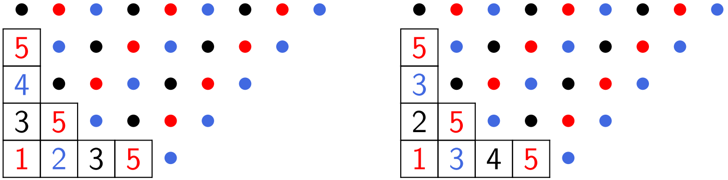
$$\begin{bmatrix} S_1 S_1 S_1 \\ \\ S_2 S_1 \end{bmatrix} = \begin{bmatrix} 1 & ?? \\ \\ 0 & 1 \end{bmatrix} \begin{bmatrix} S_{1,1,1}^{(2)} \\ \\ S_{2,1}^{(2)} \end{bmatrix}$$

k-Tableaux

Shape is a $k+1$-core

Columns strictly increasing, rows non-decreasing

Multiplicities have same color ($k+1$-residue)



$k = 2$

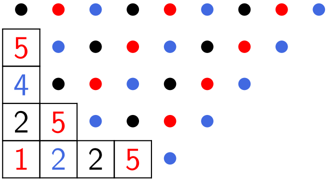
each letter occurs with one color (standard)

k-Tableaux

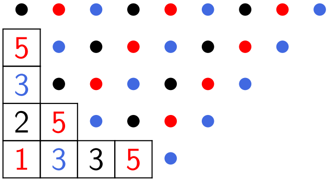
Shape is a $k+1$-core

Columns strictly increasing, rows non-decreasing

Multiplicities have same color ($k+1$-residue)



(1,2,0,1,1)



(1,1,2,0,1)

$k=2$

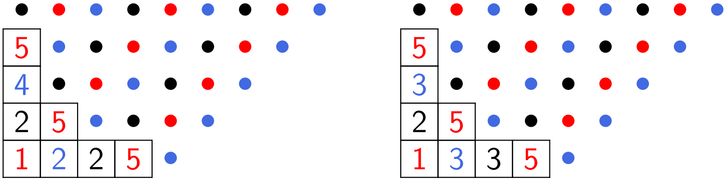
weight $= (\#\ of\ \ colors\ \ of\ \ 1's, \#\ \ of\ \ colors\ \ of\ \ 2's, \ldots)$

k-Tableaux

Shape is a $k+1$-core

Columns strictly increasing, rows non-decreasing

Multiplicities have same color ($k+1$-residue)



$k=2$

weight $= (\#\ of\ colors\ of\ 1's, \#\ of\ colors\ of\ 2's, \ldots)$

$$K^{(k)}_{\mu\lambda} = \#\text{ k-tableaux of shape core}(\mu)\text{ and weight }\lambda = \begin{cases} 0 & \text{for } \lambda > \mu \\ 1 & \text{for } \lambda = \mu \end{cases}$$

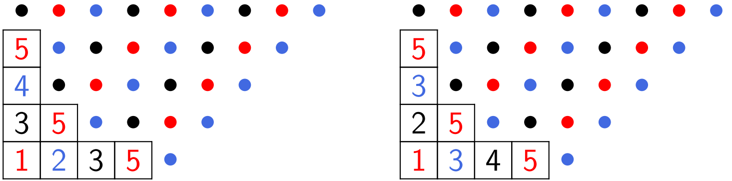
Example: weight = (4,1) requires 4 colors for the 1's

k-Tableaux

Shape is a $k+1$-core

Columns strictly increasing, rows non-decreasing

Multiplicities have same color ($k+1$-residue)



$k = 2$

weight $= (\#\ of\ colors\ of\ 1's, \#\ of\ colors\ of\ 2's, \ldots)$

$$K_{\mu\lambda}^{(k)} = \# \text{ k-tableaux of shape core}(\mu) \text{ and weight } \lambda = \begin{cases} 0 & \text{for } \lambda > \mu \\ 1 & \text{for } \lambda = \mu \end{cases}$$

Define k-Schurs by inverting expression

$$S_{\lambda_1} \cdots S_{\lambda_\ell} = \sum_{\mu \geq \lambda} K_{\mu\lambda}^{(k)}\, s_{\mu}^{(k)}$$

A combinatorial analog: $\lambda_1 \leq k$

- Basis for $Span\{S_{\lambda_1} S_{\lambda_2} \cdots S_{\lambda_\ell}\}_{\lambda=(\lambda_1,\ldots,\lambda_\ell)}$
- Involution: $\omega S_\lambda^{(k)} = S_{\lambda^{\omega_k}}^{(k)}$
- k-Pieri rule:
$$S_\ell \, S_\lambda^{(k)} = \sum_{\mu=\lambda \; + \; boxes \;\; of \;\; \ell \;\; colors} S_\mu^{(k)}$$
- k-Kostka numbers
$$S_{\lambda_1} \cdots S_{\lambda_\ell} = \sum_{\mu \geq \lambda} (\# \, k\text{-tableaux shape core}(\mu) \text{ and weight } \lambda) \; S_\mu^{(k)}$$
- Conjectured Macdonald expansion:
$$H_\lambda[X; q, t] = \sum_\mu (q, t \;\; count \;\; std \;\; k\text{-}tab \;\; of \;\; shape \;\; core(\mu)) \; S_\mu^{(k)}$$

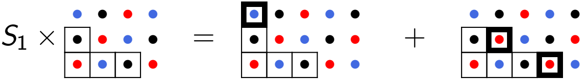
Outline

- A combinatorial analog for Schur functions
  Macdonald polynomials
- A geometric analog for Schur functions
  Gromov-Witten invariants
- Affine Schubert calculus

k-Schurs are a combinatorial analog for Schurs

- Basis for $Span\{S_{\lambda_1} S_{\lambda_2} \cdots S_{\lambda_\ell}\}_{\lambda=(\lambda_1,\ldots,\lambda_\ell)}$
- Involution: $\omega S_\lambda^{(k)} = S_{\lambda^{\omega_k}}^{(k)}$
- k-Pieri rule:

$$S_\ell \, S_\lambda^{(k)} = \sum_{\mu=\lambda \,+\, boxes \;\, of \;\, \ell \;\, colors} S_\mu^{(k)}$$



- k-Kostka numbers

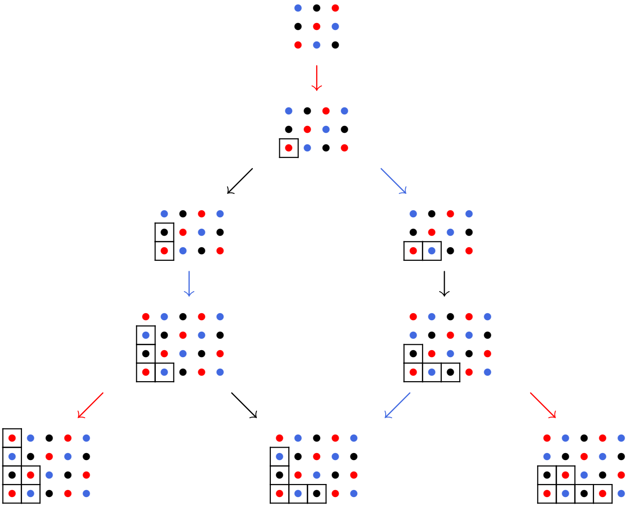
$$S_{\lambda_1} \cdots S_{\lambda_\ell} = \sum_{\mu \geq \lambda} (\#\, k\text{-tableaux shape core}(\mu) \text{ and weight } \lambda)\; S_\mu^{(k)}$$

- Conjectured Macdonald expansion:

$$H_\lambda[X; q, t] = \sum_\mu (q, t \;\, count \;\, std \;\, k\text{-}tab \;\, of \;\, shape \;\, core(\mu))\; S_\mu^{(k)}$$

k-Young Lattice

$\lambda \prec\cdot \mu$ when $S_\mu^{(k)}$ occurs in $S_1\, S_\lambda^{(k)}$



Outline

- A combinatorial analog for Schur functions
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  Gromov-Witten invariants
- Affine Schubert calculus

Geometry of the Grassmannian

How many lines in $\mathbb{P}^{n-1}$ intersect certain linear subspaces?

Cohomology of the Grassmannian:

Grassmannian: $Gr_{a,n}$ = set of a-dimensional subspaces of $\mathbb{C}^n$

Schubert varieties indexed by $\lambda \subseteq a \times (n-a)$

- Addition: $H^*(Gr_{a,n}) = \bigoplus_{\lambda \subseteq a \times (n-a)} \mathbb{Z}\,\sigma_\lambda$

- Multiplication: $\sigma_\lambda \cup \sigma_\mu = \sum_\nu c_{\lambda\mu}^{\nu} \sigma_\nu$

$c_{\lambda\mu}^{\nu}$ = number of points in intersection of Schubert varieties

What are the structure constants $c_{\lambda\mu}^{\nu}$?

Cohomology connected to symmetric functions

$$H^*(Gr_{a,n}) \cong \Lambda/\mathcal{I} \quad \text{where } \mathcal{I} = \langle e_{n-a+1}, \ldots, e_n \cup \{h_i : i > n\}\rangle$$

$$\sigma_\lambda \leftrightarrow S_\lambda \quad \text{when} \quad \lambda \subseteq a \times (n-a) \ \textit{rectangle}$$

Schubert basis realized by Schur functions

Cohomology connected to symmetric functions

$$H^*(Gr_{a,n}) \cong \Lambda/\mathcal{I} \quad \text{where } \mathcal{I} = \langle e_{n-a+1}, \ldots, e_n \cup \{h_i : i > n\}\rangle$$

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Schubert basis realized by Schur functions

Cohomology structure: $\sigma_\lambda \cup \sigma_\mu \leftrightarrow S_\lambda S_\mu \mod \mathcal{I}$

$$S_\lambda \, S_\mu = \sum_\nu c_{\lambda\mu}^\nu \, S_\nu$$

$$\downarrow$$

$$S_\lambda \, S_\mu \mod \mathcal{I} = \sum_{\nu \subseteq rect} (???) \, S_\nu$$

$$\updownarrow$$

$$\sigma_\lambda \cup \sigma_\mu = \sum_{\nu \subseteq rect} (???) \, \sigma_\nu$$

Cohomology connected to symmetric functions

$$H^*(Gr_{a,n}) \cong \Lambda/\mathcal{I} \quad \text{where } \mathcal{I} = \langle e_{n-a+1}, \ldots, e_n \cup \{h_i : i > n\}\rangle$$

$$\sigma_\lambda \leftrightarrow S_\lambda \quad \text{when} \quad \lambda \subseteq a \times (n-a) \textit{ rectangle}$$

Schubert basis realized by Schur functions

Cohomology structure: $\sigma_\lambda \cup \sigma_\mu \leftrightarrow S_\lambda S_\mu \mod \mathcal{I}$

$$S_\lambda S_\mu = \sum_{\nu \subseteq rect} c_{\lambda\mu}^{\nu} S_\nu + \sum_{\nu \not\subseteq rect} c_{\lambda\mu}^{\nu} S_\nu$$

$$\downarrow$$

$KEY : S_\nu \equiv 0$ for $\nu \not\subseteq rect$

$$S_\lambda S_\mu \mod \mathcal{I} = \sum_{\nu \subseteq rect} c_{\lambda\mu}^{\nu} S_\nu$$

$$\updownarrow$$

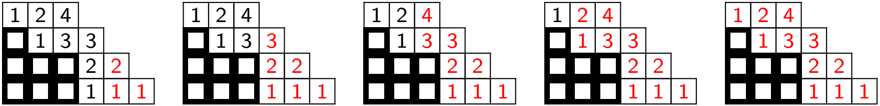
$$\sigma_\lambda \cup \sigma_\mu = \sum_{\nu \subseteq rect} c_{\lambda\mu}^{\nu} \sigma_\nu$$

Littlewood-Richardson coefficients

$$S_\lambda \, S_\mu \;=\; \sum_\nu c^\nu_{\lambda\mu} \, S_\nu$$

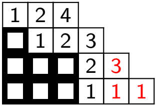
yamanouchi skew-tableaux

at any cut: $(\#\ \textit{ones} \geq \#\ \textit{twos} \geq \#\ \textit{threes} \geq \#\ \textit{fours}\ \cdots)$



(2,1) (3,2,1) (3,2,2,1) (4,3,2,1) (5,3,2,1)

Bad Guy:



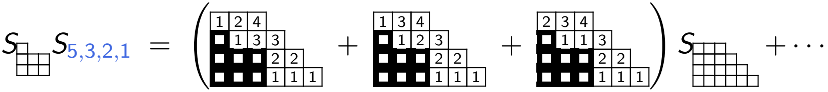
(2,0,1)

Littlewood-Richardson Rule

Product of Schur functions:

$$S_\lambda \, S_\mu \;=\; \sum_{\nu} c^{\nu}_{\lambda\mu} \, S_\nu$$

$c^{\nu}_{\lambda\mu}$ counts yamanouchi skews of shape ν/λ and weight μ



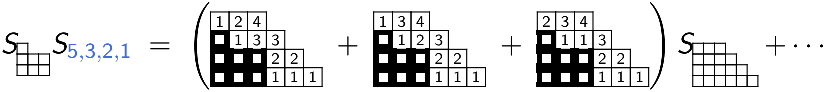
count yamanouchi skews with 5 ones, 3 twos, 2 threes, 1 four

Littlewood-Richardson Rule

Product of Schur functions:

$$S_\lambda \, S_\mu \; = \; \sum_{\nu} c^{\nu}_{\lambda\mu} \, S_\nu$$

$c^{\nu}_{\lambda\mu}$ counts yamanouchi skews of shape ν/λ and weight μ



count yamanouchi skews with 5 ones, 3 twos, 2 threes, 1 four

Multiplicative structure of cohomology:

$$\sigma_\lambda \cup \sigma_\mu = \sum_{\nu \subseteq a \times (n-a)} c^{\nu}_{\lambda\mu} \sigma_\nu$$

$c^{\nu}_{\lambda\mu}$ counts points in intersection of Schubert varieties $X_\lambda \cap X_\mu \cap X_{\hat{\nu}}$

Quantum cohomology: $QH^*(Gr_{a,n})$

How may lines in $\mathbb{P}^2$ pass through 2 fixed points?

How may conics in $\mathbb{P}^2$ pass through 5 fixed points?

Gromov-Witten invariants

# of rational curves of degree d in X passing through fixed points

Quantum cohomology

- Addition: $QH^*(Gr_{a,n}) = H^*(Gr_{a,n}) \otimes \mathbb{Z}[q]$
- Multiplication:

$$\sigma_\lambda * \sigma_\mu = \sum_{\substack{\nu \subseteq a \times (n-a) \\ |\nu| = |\lambda| + |\mu| - dn}} q^d \, C_{\lambda\mu}^{\nu,d} \, \sigma_\nu$$

$C_{\lambda\mu}^{\nu,d} = \#$ rational curves of degree d in $Gr_{a,n}$ that meet fixed generic translates of the Schubert varieties X_λ, X_μ and $X_{\hat{\nu}}$

Connection to symmetric functions

$$QH^*(Gr_{a,n}) \cong \Lambda \otimes \mathbb{Z}[q]/\mathcal{I}_q$$

$$\text{where} \quad \mathcal{I}_q = \langle e_{n-a+1}, \ldots, e_{n-1}, e_n + (-1)^a q \cup \{h_i : i > n\} \rangle$$

$$\sigma_\lambda \leftrightarrow S_\lambda \quad \text{when} \quad \lambda \subseteq a \times (n-a) \ \textit{rectangle}$$

Cohomology structure: $\sigma_\lambda * \sigma_\mu \leftrightarrow S_\lambda S_\mu \mod \mathcal{I}_q$

$$S_\lambda S_\mu = \sum_\nu c_{\lambda\mu}^{\nu} S_\nu$$

$$\downarrow$$

$$S_\lambda S_\mu \mod \mathcal{I}_q = \sum_{\nu \subseteq rect} ??? \, S_\nu$$

$$\updownarrow$$

$$\sigma_\lambda * \sigma_\mu = \sum_{\nu \subseteq rect} ??? \, \sigma_\nu$$

Connection to symmetric functions

$$QH^*(Gr_{a,n}) \quad \cong \quad \Lambda \otimes \mathbb{Z}[q]/\mathcal{I}_q$$

$$\textit{where} \quad \mathcal{I}_q = \langle e_{n-a+1}, \ldots, e_{n-1}, e_n + (-1)^a q \cup \{h_i : i > n\} \rangle$$

$$\sigma_\lambda \leftrightarrow S_\lambda \quad \text{when} \quad \lambda \subseteq a \times (n-a) \ \textit{rectangle}$$

Cohomology structure: $\sigma_\lambda * \sigma_\mu \quad \leftrightarrow \quad S_\lambda S_\mu \mod \mathcal{I}_q$

$$S_\lambda S_\mu = \sum_{\nu \subseteq rect} c_{\lambda\mu}^{\nu} S_\nu + \sum_{\nu \not\subseteq rect} c_{\lambda\mu}^{\nu} S_\nu$$

$$\downarrow$$

Problem: $S_\nu \equiv \pm q^* S_{\hat{\nu}}$ for $\nu \not\subseteq rect$

$$S_\lambda S_\mu \mod \mathcal{I}_q = \sum_{\nu \subseteq rect} c_{\lambda\mu}^{\nu} S_\nu + \sum_{\hat{\nu} \subseteq rect} c_{\lambda\mu}^{\nu} (\pm q^*) S_{\hat{\nu}}$$

$$\updownarrow$$

$$\sigma_\lambda * \sigma_\mu = \sum_{\nu \subseteq rect} ??? \ \sigma_\nu$$

Gromov-Witten invariants

- Quantum cohomology of the Grassmannian
- Wess-Zumino-Witten model of conformal field theory
- Hecke algebras at roots of unity
- Knot invariants for 3-manifolds
- $S_\lambda \, S_\mu \mod \mathcal{I}_q = \sum_{\nu \subseteq rect} ???? \, S_\nu$

Gromov-Witten invariants

- Quantum cohomology of the Grassmannian
- Wess-Zumino-Witten model of conformal field theory
- Hecke algebras at roots of unity
- Knot invariants for 3-manifolds
- $S_\lambda S_\mu \mod \mathcal{I}_q = \sum_{\nu \subseteq rect} ???? \, S_\nu$

What the hecke do *k*-Schur functions have to do with this?

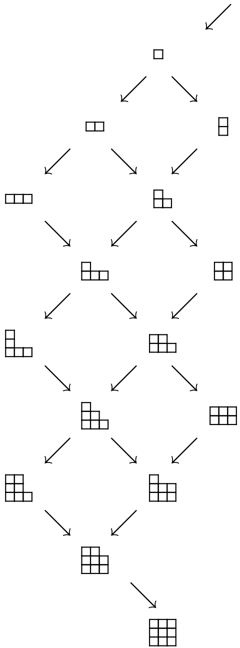
Order ideals in the Young lattice

Young Lattice
order: $\lambda <\cdot \mu$ when $\lambda \subset \mu$

Principal Order ideal
Generator: rectangle
vertices $=$ *all partitions that fit inside rectangle*
order $=$ *shape containment*

- Graded lattice
- Self dual and distributive
- Total number of vertices $=$ binomial number
- Rank generating function $=$ q-binomial
- Unimodal

Order ideals in k-Young lattice



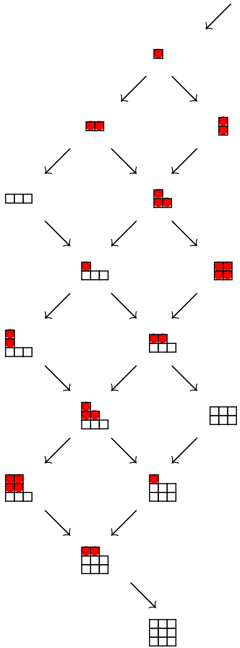
Principal Order ideal

order: $\lambda \prec\!\cdot\, \mu$ *when* $S_\mu^{(k)}$ *in* $S_1 S_\lambda^{(k)}$

generator: $a \times m$ rectangle

- *Graded lattice of rank am*
- *Self dual and distributive*
- *Total number of vertices* $\binom{k+1}{m-1} + (a+m-k)\binom{k}{m-1}$
- *Rank generating function* $$\begin{bmatrix} k \\ m \end{bmatrix}_q + q^{k+1}\frac{1-q^{m(a+m-k-1)}}{1-q^m}\begin{bmatrix} k \\ m-1 \end{bmatrix}_q$$
- *Conjecture* : *Unimodal for* $k \neq -1 \bmod p$
- *sieved sums of q-binomials*

Order ideals in k-Young lattice



Principal Order ideal

order: $\lambda \prec\!\cdot\, \mu$ *when* $S_\mu^{(k)}$ *in* $S_1 S_\lambda^{(k)}$

generator: $a \times m$ rectangle

order: $\lambda \prec\!\cdot\, \mu$ *when* $\mu = \lambda + box$

vertices: *shapes in extended rectangles*

$(a^*, \nu \subseteq (a-1) \times (k+1-a))$

Order ideals and Gromov-Wittens

Gromov-Witten invariants:

$$S_\lambda\, S_\mu = \sum_{\nu \subseteq rect} c_{\lambda\mu}^{\nu}\, S_\nu + \sum_{\nu \not\subseteq rect} c_{\lambda\mu}^{\nu}\, S_\nu$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \text{mod } \mathcal{I}_q$$

$$S_\lambda\, S_\mu = \sum_{\nu \subseteq rect} c_{\lambda\mu}^{\nu}\, S_\nu + \sum_{\hat{\nu} \subseteq rect} c_{\lambda\mu}^{\nu}\, (\pm q^*\, S_{\hat{\nu}}) \; = \sum_{\nu \subseteq rect} q^*\, c_{\lambda\mu}^{\nu}\, S_\nu$$

Order ideals and Gromov-Wittens

Gromov-Witten invariants:

$$S_\lambda\, S_\mu = \sum_{\nu=(a^*,\hat{\nu}\subseteq rect)} c^{\nu}_{\lambda\mu}\, S_\nu + \sum_{\nu\neq(a^*,\hat{\nu}\subseteq rect)} c^{\nu}_{\lambda\mu}\, S_\nu$$

$\downarrow \qquad \downarrow$ mod a subideal $\mathcal{I}$ of $\mathcal{I}_q$

$$S_\lambda\, S_\mu = \sum_{\nu=(a^*,\hat{\nu}\subseteq rect)} c^{\nu}_{\lambda\mu}\, S_\nu + \sum_{\eta=(a^*,\hat{\eta}\subseteq rect)} c^{\nu}_{\lambda\mu}\, (\pm S_\eta) = \sum_{\nu=(a^*,\hat{\nu}\subseteq rect)} c^{\nu}_{\lambda\mu}\, S_\nu$$

$\downarrow \qquad \downarrow$

$\downarrow$ S mod $\mathcal{I}_q = q^2$ S $\downarrow$

$$S_{\hat{\lambda}}\, S_{\hat{\mu}} \qquad = \sum_{\hat{\nu}\subseteq rect} c^{\nu}_{\lambda\mu}\, q^*\, S_{\hat{\nu}}$$

Order ideals and Gromov-Wittens

Known case: $$S_1\, S_\lambda \mod \mathcal{I} = \sum_{\substack{\lambda\subseteq\mu \\ \mu=(a^*,\nu\subseteq rect)}} S_\mu$$

k-Pieri Rule : $$S_1\, S_\lambda^{(k)} = \sum_{\substack{\lambda\subseteq\mu \\ \mu=(a^*,\nu\subseteq rect)}} S_\mu^{(k)} + \sum_{\substack{\lambda\prec\mu \\ \mu\neq(a^*,\nu\subseteq rect)}} S_\mu^{(k)}$$

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Theorem:

$$S_\lambda^{(k)} \mod \mathcal{I} = \begin{cases} S_\lambda & \text{if } \lambda = (a^*, \nu \subseteq rect) \\ 0 & \text{otherwise} \end{cases}$$

A geometric analog of Schur functions

$$S^{(k)}_\lambda\, S^{(k)}_\mu \;=\; \sum_{\nu=(a^*,\hat\nu\subseteq rect)} c^{\nu,k}_{\lambda\mu}\, S^{(k)}_\nu + \sum_{\nu\neq(a^*,\hat\nu\subseteq rect)} c^{\nu,k}_{\lambda\mu}\, S^{(k)}_\nu$$

$$\downarrow$$

$$S_\lambda\, S_\mu \mod \mathcal{I} \;=\; \sum_{\nu=(a^*,\hat\nu\subseteq rect)} c^{\nu,k}_{\lambda\mu}\, S_\nu$$

$$\downarrow$$

$$S_{\hat\lambda}\, S_{\hat\mu} \mod \mathcal{I}_q \;=\; \sum_{\hat\nu\subseteq rect} q^d\, c^{\nu,k}_{\lambda\mu}\, S_{\hat\nu}$$

$$\updownarrow$$

$$\sigma_{\hat\lambda} * \sigma_{\hat\mu} \;=\; \sum_{\hat\nu\subseteq rect} q^d\, c^{\nu,k}_{\lambda\mu}\, \sigma_{\hat\nu}$$

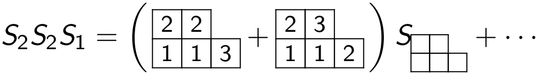
Gromov-Witten invariants = k-Littlewood Richardson coefficients

- k-Schur functions
- Quantum cohomology of the Grassmannian
- Wess-Zumino-Witten model of conformal field theory
- Hecke algebras at roots of unity

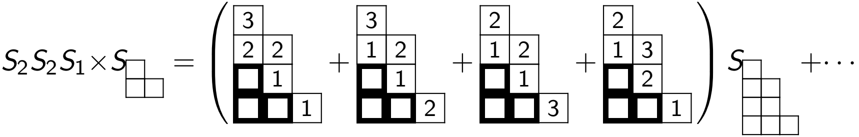
Approach to Gromov-Witten invariants

Approach to Usual Littlewood-Richardson Coefficients

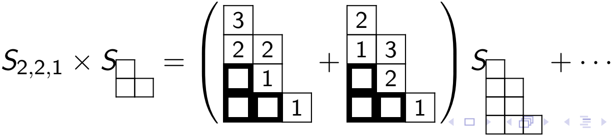
Count tableaux: $S_{\lambda_1} S_{\lambda_2} \cdots S_{\lambda_\ell}$



Count skew tableaux: $S_{\lambda_1} S_{\lambda_2} \cdots S_{\lambda_\ell} \, S_\mu$



Count yamanouchi skew tableaux: $S_\lambda \, S_\mu$



Approach to Gromov-Witten invariants

Find k-Littlewood-Richardson Rule

Count k-tableaux

$$S_{\lambda_1} S_{\lambda_2} \cdots S_{\lambda_\ell} = \sum_{\mu} (\# \text{ of } k\text{-tab of weight } \lambda \text{ and shape core}(\mu))\, S_\nu^{(k)}$$

Count skew k-tableaux

$$S_{\lambda_1} \cdots S_{\lambda_\ell}\, S_\mu^{(k)} = \sum_{\nu} (\# \text{ of skew } k\text{-tab of weight } \lambda \text{ and shape core}(\nu/\mu))\, S_\nu^{(k)}$$

OPEN: Find a notion of *yamanouchi* skew k-tableaux

$$S_\lambda^{(k)}\, S_\mu^{(k)} = \sum_{\nu} c_{\lambda\mu}^{\nu,k}\, S_\nu^{(k)} \quad \text{where } c_{\lambda\mu}^{\nu,k} \text{ count a subset of skew } k\text{-tableaux}$$

Outline

- Macdonald polynomials and a combinatorial analog for Schur functions
- Gromov-Witten invariants and a geometric analog for Schur functions
- Affine Schubert calculus

Schubert Calculus

Basic framework

Enumerative geometry questions

- Identify with cohomology ring
- Create cohomology/polynomial dictionary

Combinatorics describes cohomology/polynomial computation

Grown to include representation theory and physics

Example - Grassmannian

1879: Schubert curious about enumerative geometry

How many lines in $\mathbb{P}^{n-1}$ intersect certain linear subspaces?

Cohomology of the Grassmannian:

Grassmannian: $Gr_{a,n}$ = set of a-dimensional subspaces of $\mathbb{C}^n$

Schubert varieties indexed by $\lambda \subseteq a \times (n-a)$

- Addition: $H^*(Gr_{a,n}) = \bigoplus_{\lambda \subseteq a \times (n-a)} \mathbb{Z}\sigma_\lambda$
- Multiplication: $\sigma_\lambda \cup \sigma_\mu = \sum_\nu c^\nu_{\lambda\mu}\sigma_\nu$

$c^\nu_{\lambda\mu}$= number of points in intersection of Schubert varieties

What are the structure constants $c^\nu_{\lambda\mu}$?

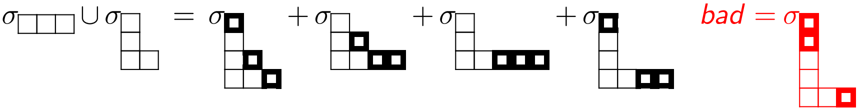
Multiplication on the ring

Giambelli's Formula: $\sigma_\mu = det\left(\sigma_{\mu_i+j-1}\right)_{i,j}$

Schubert class in terms of *special classes*

$$\sigma_{5,3,3} = det\begin{pmatrix} \sigma_5 & \sigma_6 & \sigma_7 \\ \sigma_2 & \sigma_3 & \sigma_4 \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix}$$

Pieri Rule : $\sigma_r \cup \sigma_\lambda = \sum_{\mu=\lambda\ +\ r-strip} \sigma_\mu$



Pieri + Giambelli determines multiplication $\sigma_\mu \cup \sigma_\lambda$

Connect to symmetric functions

Schubert basis identified with Schur functions (1947)

$$\sigma_\lambda \leftrightarrow S_\lambda$$

$$S_{\begin{smallmatrix}\square & \\ \square & \square\end{smallmatrix}} = \begin{smallmatrix}3 & \\ 1 & 2\end{smallmatrix} + \begin{smallmatrix}2 & \\ 1 & 3\end{smallmatrix} + \begin{smallmatrix}2 & \\ 1 & 1\end{smallmatrix} + \begin{smallmatrix}3 & \\ 2 & 2\end{smallmatrix} + \begin{smallmatrix}3 & \\ 1 & 1\end{smallmatrix} + \begin{smallmatrix}2 & \\ 1 & 2\end{smallmatrix} + \begin{smallmatrix}3 & \\ 1 & 3\end{smallmatrix} + \begin{smallmatrix}3 & \\ 2 & 3\end{smallmatrix}$$

- Cohomology structure reduced to polynomial computation
- Combinatorial developments settle geometric questions

  Robinson-Schensted-Knuth insertion algorithm on tableaux
  Monoid on tableaux $\longrightarrow$ *Littlewood-Richardon rule*

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Schubert calculus (Grassmannian)

cohomology		symmetric function ring
Schubert basis	$\longleftarrow$ partitions $\longrightarrow$	Schur functions
Pieri rule	$\longleftarrow$ multiplication $\longrightarrow$	Pieri rule
geometric quantity	$\longleftarrow$ structure constants $\longrightarrow$	L-R rule

Example - Flag

Set of complete flags (chains of vector spaces) in $\mathbb{C}^n$

$GL(n, \mathbb{C})/B$ where $B =$ subgroup of upper triangular matrices

- **Schubert basis** for cohomology is indexed by permutations

  Generators: $S_n = \langle s_1, \ldots, s_{n-1} \rangle$ where $s_i = (1, 2, .., i+1, i, .., n)$

  Coxeter relations:
  $$\begin{aligned} s_i\, s_{i+1}\, s_i &= s_{i+1}\, s_i\, s_{i+1} \\ s_i\, s_j &= s_j\, s_i \quad \text{(for non-adjacent } i, j) \\ s_i^2 &= id \end{aligned}$$

- **Schubert polynomials** $\mathfrak{S}_w$ give explicit realization

$$\mathfrak{S}_w = \partial_{w^{-1}w_0}\left(x_1^{n-1} x_2^{n-2} \cdots x_n^0\right) \quad \text{where}$$

$$\partial_j(f) = \frac{f(x_1, \ldots, x_n) - f(x_1, \ldots, x_{j+1}, x_j, \ldots)}{x_j - x_{j+1}}$$

Multiplicative structure

Strong (Bruhat) order: $w \lessdot u$

$$u = w\,\tau_{a,b} \quad \text{and} \quad \ell(u) = \ell(w) + 1$$

$$(1\,3\,2)\tau_{1,3} = (2\,3\,1) \quad \text{or} \quad (s_2)\,\tau_{1,3} = s_1\,s_2$$

- **Monk's formula** determines multiplication combinatorially

$$\mathfrak{S}_{s_i}\mathfrak{S}_w = \sum_{\substack{w \lessdot\, u = w\,\tau_{a,b} \\ a \leq i < b}} \mathfrak{S}_u$$

$$\mathfrak{S}_{s_1}\mathfrak{S}_{(1\,3\,2)} = \mathfrak{S}_{(1\,3\,2)\tau_{1,2}} + \mathfrak{S}_{(1\,3\,2)\tau_{1,3}} = \mathfrak{S}_{(3\,1\,2)} + \mathfrak{S}_{(2\,3\,1)}$$

- **Littlewood-Richardson** combinatorics being developed

Schubert Calculus

cohomology		symmetric function ring
cohomology		polynomial ring
Schubert basis	partitions	Schur functions
Schubert basis	permutations	Schubert polynomials
Pieri rule	multiplication	Pieri rule
Monk's formula		Monk's formula
geometric quantity	structure constants	L-R rule
geometric quantity		???

Affine Grassmannian

- $\tilde{G}r = SL(n, \mathbb{C}((t)))/SL(n, \mathbb{C}[[t]])$
- (co)homology identified with symmetric functions *[Bott'58]*

$$H^*(\tilde{G}r) \cong \Lambda/\langle m_\lambda : \lambda_1 \geq n \rangle \qquad H_*(\tilde{G}r) \cong \mathbb{Z}[S_1, \ldots, S_{n-1}]$$

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- Schubert bases *[Peterson, Kostant and Kumar, and Graham]*

Affine permutations

$\sigma : \mathbb{Z} \to \mathbb{Z}$

Example $\sigma \in \tilde{S}_3$:

$$\{\,-\infty, \dots, -2, -1,\ 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ \dots, \infty\}$$
$$\{\dots\dots\dots, -4,\ 0,\ 1,\ -1,\ 3,\ 4,\ 2,\ 6,\ \dots\dots\}$$

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Affine Grassmannian permutations: $\sigma(1) < \sigma(2) < \cdots < \sigma(n)$

Enumerative geometry

Do geometric quantities arise in (co)homology $H(\tilde{G}r)$?

Recall:
quantum cohomology $QH^*(Gr_{a,n}) \leftrightarrow$ Gromov-Witten invariants

$$\sigma_\lambda * \sigma_\mu = \sum_{\substack{\nu \subseteq a \times (n-a) \\ |\nu| = |\lambda| + |\mu| - dn}} q^d \, C_{\lambda\mu}^{\nu,d} \, \sigma_\nu$$

[Peterson'97]
Surjective homomorphism: $H_*(\tilde{G}r) \twoheadrightarrow QH^*(Gr_{a,n})$

Some homology structure constants = Gromov-Wittens

Affine Schubert Calculus (Grassmannian)

cohomology		$\Lambda / \langle m_\lambda : \lambda_1 \geq n \rangle$
homology		$\mathbb{Z}[S_1, \ldots, S_{n-1}]$
Schubert basis	affine permutations	symmetric functions
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???	multiplication	???
???	multiplication	???
geometric quantities	structure constants	???
Gromov-Wittens and ???		???

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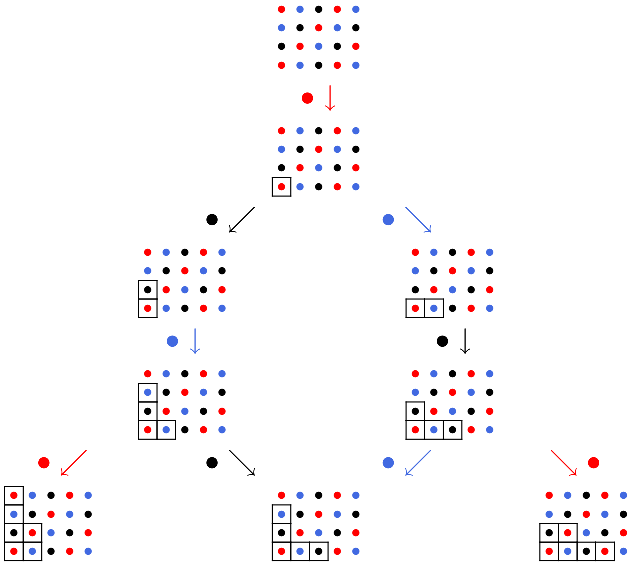
k-Schur functions for k-bounded partitions

- Basis for $\mathbb{Z}[S_1, S_2, \ldots, S_k]$
- k-Pieri rule:

$$S_\ell \, S_\lambda^{(k)} = \sum_{\mu=\lambda+\ell \;\; colors} S_\mu^{(k)}$$

- combinatorial analog of Schur functions

Connection with affine world



chains = standard k-tableaux = sequence of colors

Words for affine permutations

Generators: $\tilde{S}_n = \langle s_0, s_1, \ldots, s_{n-1} \rangle$ where

$$\forall\, i > 0 \quad s_i = [1, 2, \ldots, i+1, i, \ldots, n]$$

$$\textit{and} \quad s_0 = [0, 2, 3, \ldots, n-1, n+1]$$

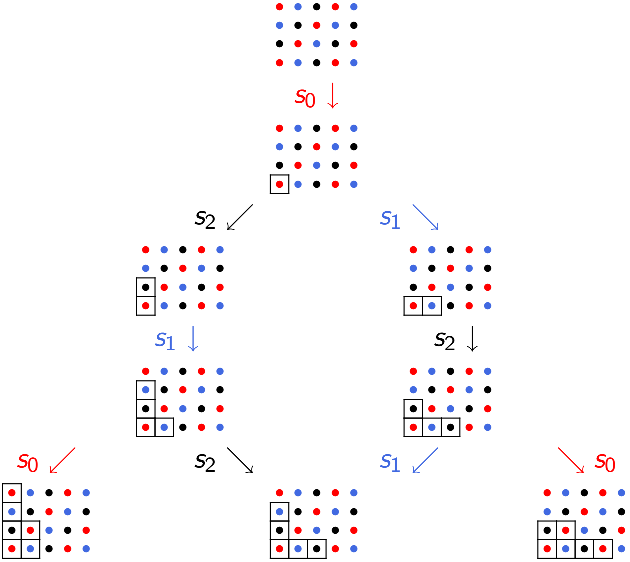
Satisfy Coxeter relations:

$$\begin{aligned} s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \quad \text{(0 and n-1 adjacent)} \\ s_i s_j &= s_j s_i \quad \text{(for non-adjacent } i, j) \\ s_i^2 &= id \end{aligned}$$

Grassmannian case:

Permutations whose reduced words all end with s_0

Connection with affine world



chains = standard k-tableaux = words for affine perms

Std k-tableaux $\leftrightarrow$ words for affine Grassmannian perms

$T \to s_{i_n} s_{i_{n-1}} \cdots s_{i_1}$ *where* i_a *is the* $k+1$*-residue of letter* a *in* T

7					
4	8				
2	6	7			
1	3	4	5	6	7

$\to$ 8 7 6 5 4 3 2 1 $\to$ ● ● ● ● ● ● ● ● $= s_3\, s_1\, s_0\, s_3\, s_2\, s_1\, s_3\, s_0$

Std k-tableaux $\leftrightarrow$ words for affine Grassmannian perms

$T \longrightarrow s_{i_n} s_{i_{n-1}} \cdots s_{i_1}$ *where* i_a *is the* $k+1$*-residue of letter* a *in* T

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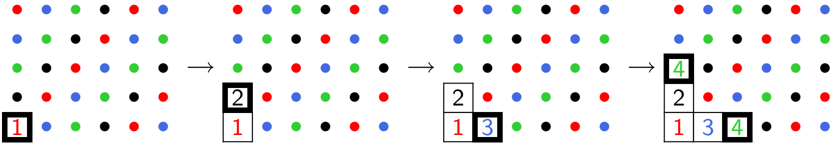
$\longrightarrow$ 8 7 6 5 4 3 2 1 $\longrightarrow$ ● ● ● ● ● ● ● ● $= s_3\, s_1\, s_0\, s_3\, s_2\, s_1\, s_3\, s_0$

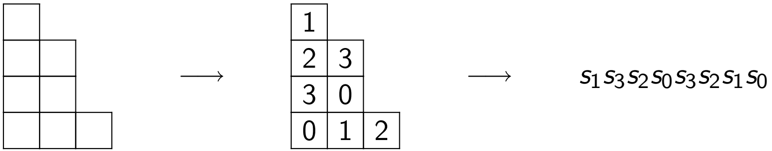
$s_3\, s_1\, s_0\, s_3\, s_2\, s_1\, s_3\, s_0 \longrightarrow$ 8 7 6 5 4 3 2 1 $\longrightarrow$



Consequences

Bijection:

k-**bounded partitions** $\leftrightarrow$ **affine Grassmannian perms**

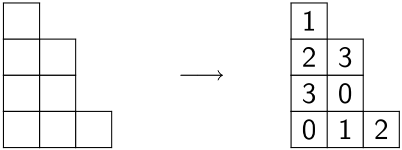


equivalent but algorithmically distinct to [Björner and Brenti]

Consequences

Bijection:

***k*-bounded partitions ↔ affine Grassmannian perms**



$\longrightarrow \quad s_1 s_3 s_2 s_0 s_3 s_2 s_1 s_0$

equivalent but algorithmically distinct to [Björner and Brenti]

Weak order on affine permutations

$\sigma \lessdot \tau$ when $\tau = \sigma\, s_i$ and $\ell(\tau) = \ell(\sigma) + 1$

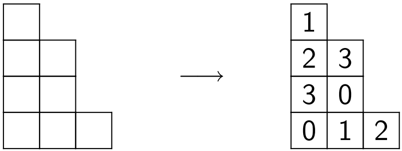
chains in weak order = reduced words for affine permutations

$\emptyset \lessdot s_0 \lessdot s_0 s_1 \lessdot s_0 s_1 s_2 \lessdot s_0 s_1 s_2 s_0$

Consequences

Bijection:

k-**bounded partitions** $\leftrightarrow$ **affine Grassmannian perms**



$\longrightarrow$ $s_1 s_3 s_2 s_0 s_3 s_2 s_1 s_0$

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Weak order on affine permutations

$\sigma \lessdot \tau$ when $\tau = \sigma\, s_i$ and $\ell(\tau) = \ell(\sigma) + 1$

chains in weak order $=$ reduced words for affine permutations

$\emptyset \lessdot s_0 \lessdot s_0 s_1 \lessdot s_0 s_1 s_2 \lessdot s_0 s_1 s_2 s_0 \quad \updownarrow$

chains in k-Young lattice $=$ k-tableaux

k-**Young lattice** $\cong$ **weak order on affine Grassmannian perms**

Corollary: rephrase k-Pieri rule

k-Pieri rule

$$S_1\, S_\lambda^{(k)} = \sum_{core(\mu)=core(\lambda)+box/colored} S_\mu^{(k)}$$

- k-Young lattice $\cong$ weak order on affine Grass permutations

Weak order:

- affine permutations
  $w \longrightarrow w\, s_i$ if $\ell(w) < \ell(w\, s_i)$
- cores
  $core(\lambda) \rightarrow core(\lambda)$ + box (and others of the same color)

- k-bounded partition $\leftrightarrow$ $k+1$-core $\leftrightarrow$ affine Grass permutation

k-Pieri rule

For any affine grassmannian permuation w,

$$S_{s_0}\, S_w^{(k)} = \sum_{\substack{w \prec u \\ u=grassmannian}} S_u^{(k)}$$

Discoveries about k-Schur functions

- Basis for $\mathbb{Z}[S_1, S_2, \ldots, S_k]$
- Indexed by affine Grassmannian permutations

  k-bounded partitions ↔ affine Grassmannian perms
- k-Pieri rule

  weak order on affine Grassmannian permutations

$$S_{s_0}\, S_w^{(k)} = \sum_{\substack{w \prec u \\ u = grassmannian}} S_u^{(k)}$$

- Gromov-Witten invariants = k-Littlewood Richardson coeffs

  a subset of k-Schurs functions give Schubert class structure constants of $QH^*(Gr_{a,k+1})$

Discoveries about k-Schur functions

- Basis for $\mathbb{Z}[S_1, S_2, \ldots, S_k]$ $\cong H_*(\tilde{G}r)$
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$$H_*(\tilde{G}r) \twoheadrightarrow QH^*(Gr_{a,k+1})$$

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  a subset of k-Schurs functions give Schubert class structure constants of $QH^*(Gr_{a,k+1})$
- *[Lam],[Shimozono (conjecture)]*

  k-Schurs give Schubert classes for $H_*(\tilde{G}r)$

Schubert calculus in affine Grassmannian

homology of the affine Grassmannian

homology		$\mathbb{Z}[S_1, \ldots, S_{n-1}]$
Schubert basis	aff Grass permutations	k-Schur functions
k-Pieri rule	multiplication	k-Pieri rule
Gromov-Wittens/???	structure constants	???

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Gromov-Wittens/???	structure constants	???

cohomology of the affine Grassmannian?

cohomology ring		$\Lambda/\langle m_\lambda : \lambda_1 > n\rangle$
Schubert basis	aff Grass permutations	symmetric functions
k-Pieri rule	multiplication	k-Pieri rule
geometric quantity	structure constants	LR-rule?

Dual k-Schur functions

$$\mathfrak{S}^{(2)}_{\begin{smallmatrix}\square & \\ \square & \square\end{smallmatrix}} = \begin{array}{|c|cc}\hline 3 & \multicolumn{2}{c}{} \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} \\ \hline\end{array} + \begin{array}{|c|cc}\hline 2 & \multicolumn{2}{c}{} \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} \\ \hline\end{array} + \begin{array}{|c|cc}\hline 3 & \multicolumn{2}{c}{} \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{3} \\ \hline\end{array} + \begin{array}{|c|cc}\hline 3 & \multicolumn{2}{c}{} \\ \hline 2 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} \\ \hline\end{array} + \begin{array}{|c|cc}\hline 3 & \multicolumn{2}{c}{} \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{3} \\ \hline\end{array} + \begin{array}{|c|cc}\hline 2 & \multicolumn{2}{c}{} \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} \\ \hline\end{array} + \begin{array}{|c|cc}\hline 3 & \multicolumn{2}{c}{} \\ \hline 1 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{3} \\ \hline\end{array} + \begin{array}{|c|cc}\hline 3 & \multicolumn{2}{c}{} \\ \hline 2 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} \\ \hline\end{array}$$

$$= x_1 x_2 x_3 + x_1 x_1 x_2 + x_1 x_1 x_3 + x_2 x_2 x_3 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2^2 x_3$$

Dual k-Schur functions

$$\mathfrak{S}^{(2)}_{\tiny\begin{array}{|c|c|}\hline \, & \multicolumn{1}{c}{} \\ \cline{1-1} \, & \, \\ \hline\end{array}} = \begin{array}{|c|c|c|}\hline 3 & \multicolumn{2}{c}{} \\ \hline {\color{red}1} & {\color{blue}2} & 3 \\ \hline\end{array} + \begin{array}{|c|c|c|}\hline 2 & \multicolumn{2}{c}{} \\ \hline {\color{red}1} & {\color{blue}1} & 2 \\ \hline\end{array} + \begin{array}{|c|c|c|}\hline 3 & \multicolumn{2}{c}{} \\ \hline {\color{red}1} & {\color{blue}1} & 3 \\ \hline\end{array} + \begin{array}{|c|c|c|}\hline 3 & \multicolumn{2}{c}{} \\ \hline {\color{red}2} & {\color{blue}2} & 3 \\ \hline\end{array} + \begin{array}{|c|c|c|}\hline 3 & \multicolumn{2}{c}{} \\ \hline {\color{red}1} & {\color{blue}1} & 3 \\ \hline\end{array} + \begin{array}{|c|c|c|}\hline 2 & \multicolumn{2}{c}{} \\ \hline {\color{red}1} & {\color{blue}2} & 2 \\ \hline\end{array} + \begin{array}{|c|c|c|}\hline 3 & \multicolumn{2}{c}{} \\ \hline {\color{red}1} & {\color{blue}3} & 3 \\ \hline\end{array} + \begin{array}{|c|c|c|}\hline 3 & \multicolumn{2}{c}{} \\ \hline {\color{red}2} & {\color{blue}2} & 3 \\ \hline\end{array}$$

$$= {\color{red}x_1}{\color{blue}x_2}x_3 + x_1x_1x_2 + {\color{red}x_1}{\color{blue}x_1}x_3 + x_2x_2x_3 + x_1^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2^2x_3$$

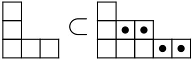
cohomology of the affine Grassmannian

cohomology ring		$\Lambda/\langle m_\lambda : \lambda_1 > n\rangle$
Schubert basis	aff Grass permutations	dual k-Schurs
k-Pieri rule	multiplication	k-Pieri rule
geometric quantity	structure constants	LR-rule?

Dual k-Pieri rule

Strong order: core containment $\gamma \subset \rho$

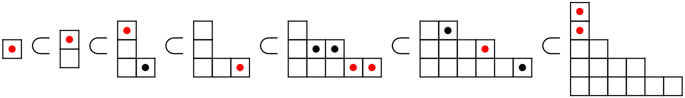
$k = 2 :$



covering: number of k-bounded cells increases by one

ρ/γ is a translated sequence of ribbons

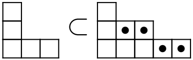
ρ is a "marked cover" of γ if one ribbon of γ/ρ is marked



Dual k-Pieri rule

Strong order: core containment $\gamma \subset \rho$

$k = 2$:



covering: number of k-bounded cells increases by one

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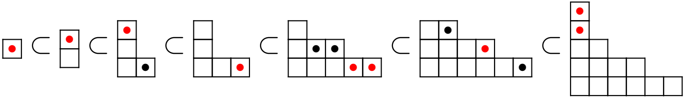
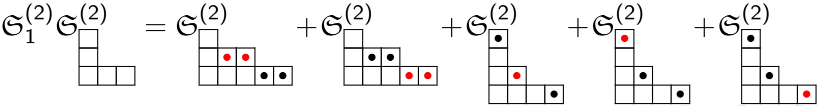


dual k-Pieri rule:

$$\mathfrak{S}_1^{(2)}\,\mathfrak{S}_\lambda^{(k)} = \sum_{core(\mu)\ \ marked\ \ cover\ \ of\ \ core(\lambda)} \mathfrak{S}_\mu^{(k)}$$



Affine Schubert calculus

cohomology of the affine Grassmannian

cohomology ring		$\Lambda / \langle m_\lambda : \lambda_1 > n \rangle$
Schubert basis	aff Grass permutations	dual k-Schurs
strong Pieri rule	multiplication	dual k-Pieri rule
geometric quantity	structure constants	LR-rule?

Affine Schubert calculus

cohomology of the affine Grassmannian

cohomology ring		$\Lambda/\langle m_\lambda : \lambda_1 > n\rangle$
Schubert basis	aff Grass permutations	dual k-Schurs
strong Pieri rule	multiplication	dual k-Pieri rule
geometric quantity	structure constants	LR-rule?

More general cases

co(homology)		polynomial ring
Schubert basis	affine perms	some polynomials
(dual) Monk's formula	multiplication	(dual) Monk's formula
geometric quantity	structure constants	L-R rule

Back to Macdonald polynomials

$$(S_1)^n = \sum_{\mu} K^{(k)}_{\mu 1^n} \, S^{(k)}_{\mu} \quad \text{where } K^{(k)}_{\mu 1^n} = \#\, \textit{standard } k - \textit{tableaux}$$
$$= \#\, \textit{reduced words}$$

$$S_1 S_1 S_1 S_1 = \begin{array}{|c|c|cc} \hline 3 & 4 & & \\ \hline 1 & 2 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} \\ \hline \end{array} \; S^{(2)}_{\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}} + \left(\begin{array}{|c|cc} \hline 4 & & \\ \hline 3 & & \\ \hline 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{4} \\ \hline \end{array} + \begin{array}{|c|cc} \hline 4 & & \\ \hline 2 & & \\ \hline 1 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} \\ \hline \end{array} \right) S^{(2)}_{\begin{smallmatrix}\square & \\ \square & \\ \square & \square\end{smallmatrix}} + \begin{array}{|c|c} \hline 4 & \\ \hline 3 & \\ \hline 2 & \multicolumn{1}{c|}{4} \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \; S^{(2)}_{\begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix}}$$

$$S_1 S_1 S_1 S_1 = s_0\, s_1\, s_2\, s_0 \; S^{(2)}_{\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}} + (s_0\, s_1\, s_2\, s_1 + s_0\, s_2\, s_1\, s_2)\; S^{(2)}_{\begin{smallmatrix}\square & \\ \square & \\ \square & \square\end{smallmatrix}} + s_0\, s_2\, s_1\, s_0 \; S^{(2)}_{\begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix}}$$

$$\downarrow$$

$$\textit{Map } \; T \; \textit{ to } \; q^{??}\, t^{??}$$

$$\downarrow$$

$$H_{2,1,1} = t\, S^{(2)}_{\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}} + \left(1 + qt^2\right) S^{(2)}_{\begin{smallmatrix}\square & \\ \square & \\ \square & \square\end{smallmatrix}} + q\, S^{(2)}_{\begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix}}$$

What about the t in k-Schur functions?

$$S_1 S_1 S_1 S_1 = s_0\, s_1\, s_2\, s_0\; S^{(2)}_{(2,2)} + (s_0\, s_1\, s_2\, s_1 + s_0\, s_2\, s_1\, s_2)\, S^{(2)}_{(2,1,1)} + s_0\, s_2\, s_1\, s_0\; S^{(2)}_{(1,1,1,1)}$$

$$\downarrow$$

$$Map \;\; T \;\; to \;\; q^{??}\, t^{??}$$

$$\downarrow$$

$$H_{2,1,1} \; = \; t \;\; S^{(2)}_{(2,2)} + \left(1 + q\, t^2\right) S^{(2)}_{(2,1,1)} + q \;\; S^{(2)}_{(1,1,1,1)}$$

What about the t in k-Schur functions?

$$S_1 S_1 S_1 S_1 = s_0\, s_1\, s_2\, s_0\; S^{(2)}_{(2,2)} + (s_0\, s_1\, s_2\, s_1 + s_0\, s_2\, s_1\, s_2)\, S^{(2)}_{(2,1,1)} + s_0\, s_2\, s_1\, s_0\; S^{(2)}_{(1,1,1,1)}$$

$$\overbrace{S_{(2,2)} + S_{(3,1)} + S_{(4)}} \qquad \overbrace{S_{(2,1,1)} + S_{(3,1)}} \qquad \overbrace{S_{(1,1,1,1)} + S_{(2,1,1)} + S_{(2,2)}}$$

$$\downarrow$$

$$Map\ \ T\ \ to\ \ q^{??}\, t^{??}$$

$$\downarrow$$

$$H_{2,1,1} = t\; S^{(2)}_{(2,2)} + \left(1 + q\, t^2\right) S^{(2)}_{(2,1,1)} + q\; S^{(2)}_{(1,1,1,1)}$$

$$\overbrace{S_{(2,2)} + tS_{(3,1)} + t^2 S_{(4)}} \qquad \overbrace{S_{(2,1,1)} + tS_{(3,1)}} \qquad \overbrace{S_{(1,1,1,1)} + tS_{(2,1,1)} + t^2 S_{(2,2)}}$$

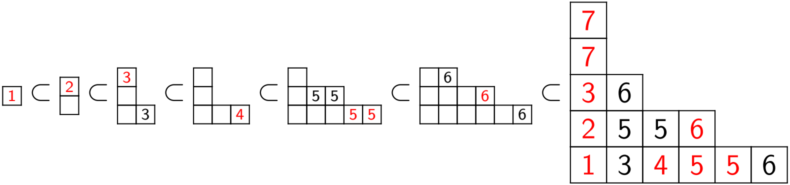
Dual k-tableaux

From a saturated chain of $k+1$-cores in Bruhat order

$$\emptyset \subset \gamma^{(1)} \subset \gamma^{(2)} \subset \ldots \subset \gamma^{(\ell)} = \gamma$$

Fill γ with letter i in cells of $\gamma^{(i+1)}/\gamma^{(i)}$

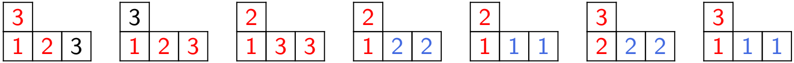
Mark one ribbon containing letter i



k-Schur functions:

$$s_\lambda^{(k)} = \sum_{\substack{T=dual\ \ k-tab \\ of\ \ shape\ \ core(\lambda)}} t^{\text{easy statistic}(T)} x^{weight(T)}$$

$s^{(2)}_{\text{(2,1)}} =$



Future Work

- **Combinatorial Problems**
  - Macdonald polynomials
    reduced words for affine permutations
    k-tableaux
  - Gromov-Witten invariants (more generally, constants for $H_*(\tilde{G}r)$)
    k-yamanouchi tableaux
  - Involution on k-tableaux
  - plactic monoid on k-tableaux
    word $\leftrightarrow (P, Q)$ for $P = k$-tableaux and $Q =$dual k-tab of same shape
  - Schur expansion for LLT polynomials

Future Work

- **Geometry**
  - Find geometric quantities encoded by structure constants of $H(\tilde{G}r)$
  - Affine Schubert polynomials that reduce to k-Schur functions in the grassmannian
  - the t in k-Schurs (Schubert classes) for the affine Grassmannian
- Representation theory
  - Schur functions = irreducible GL_n or S_n representation
  - Macdonald coefficients = bigraded characters of S_n-modules
  - LLT polynomials = Fock space representations
  - Crystal structure on dual k-tableaux