SCHUR FUNCTION IDENTITIES, THEIR t-ANALOGS, AND k-SCHUR IRREDUCIBILITY

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Abstract. We obtain general identities for the product of two Schur functions in the case where one of the functions is indexed by a rectangular partition, and give their t-analogs using vertex operators. We study subspaces forming a filtration for the symmetric function space that lends itself to generalizing the theory of Schur functions and also provides a convenient environment for studying the Macdonald polynomials. We use our identities to prove that the vertex operators leave such subspaces invariant. We finish by showing that these operators act simply on the k-Schur functions, thus leading to a concept of irreducibility for these functions.

1. Introduction

Let Λ be the ring of symmetric functions in the variables $x_1, x_2, \dots$, with coefficients in $\mathbb{Q}(q,t)$. The Schur functions, $s_\lambda[X]$, form a fundamental basis of Λ, with central roles in fields such as representation theory and algebraic geometry. For example, the Schur functions can be identified with the characters of irreducible representations of the symmetric group, and their products are equivalent to the Pieri formulas for multiplying Schubert varieties in the intersection ring of a Grassmannian. Furthermore, the connection coefficients of the Schur function basis with various bases such as the homogeneous symmetric functions, the Hall-Littlewood polynomials, and the Macdonald polynomials, are positive and have representation theoretic interpretations. In the case of the Macdonald polynomials, $H_\lambda[X;q,t]$, this expansion takes the form

$$H_\lambda[X;q,t] = \sum_\mu K_{\mu\lambda}(q,t)\, s_\mu[X]\,, \qquad K_{\mu\lambda}(q,t) \in \mathbb{N}[q,t]\,, \tag{1.1}$$

where $K_{\mu\lambda}(q,t)$ are known as the q,t-Kostka polynomials. The representation theoretic interpretation for these polynomials is given in [2, 3].

Recent developments [5, 6] have suggested that a certain filtration of Λ provides a convenient environment for the generalization of natural properties held by the Schur functions, and for the study of the Macdonald polynomials. This filtration, $\Lambda_t^{(1)} \subseteq \Lambda_t^{(2)} \subseteq \cdots \subseteq \Lambda_t^{(\infty)} = \Lambda$, is given by

$$\Lambda_t^{(k)} = \mathcal{L}\left\{H_\lambda[X;t]\right\}_{\lambda;\lambda_1\le k} = \mathcal{L}\left\{H_\lambda[X;q,t]\right\}_{\lambda;\lambda_1\le k}\,, \tag{1.2}$$

where $H_\lambda[X;t]$ denote the Hall-Littlewood polynomials. Two bases for these spaces are introduced in [6]; the k-split polynomial basis, which is related to a t-generalization of Schur function products [9, 10, 11], and the k-Schur function basis, $s_\lambda^{(k)}[X;t]$ [5, 6]. The latter basis plays a role in $\Lambda_t^{(k)}$ analogous to the one played in Λ by the Schur functions. For example, work related to the k-Schur functions prompted a k-analog of partition conjugation, a refinement of formula (1.1),

$$H_\lambda[X;q,t] = \sum_\mu K_{\mu\lambda}^{(k)}(q,t)\, s_\mu^{(k)}[X;t]\,, \tag{1.3}$$

where $K_{\mu\lambda}^{(k)}(q,t)$ are conjectured to be in $\mathbb{N}[q,t]$, and a generalization of the Pieri and Littlewood-Richardson rules.

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Here, we study classical Schur function properties (and their t-analogs) in the context of the spaces $\Lambda_t^{(k)}$. In particular, the expansion of a product of two Schur functions in terms of Schur functions is explicitly known. In our case, we examine the expansion of such a product, where one Schur function is indexed by a rectangular partition, in terms of certain products of Schur functions. We find, for nonnegative integers a, r and m and partition ν with $\ell(\nu) \leq r$,

$$s_{a^{r+m}}\, s_\nu = \sum_{\mu:\, \ell(\mu)\leq r,\, \mu_1\leq m} (-1)^{|\mu|} s_{a^r+\mu}\, s_{a-\mu'_m,\dots,a-\mu'_1,\nu}\,, \tag{1.4}$$

where the summand vanishes if $(a-\mu'_m,\dots,a-\mu'_1)$ is not a partition. We also generalize this to

$$B_{a^{r+m}}\, B_\nu = \sum_{\mu:\, \ell(\mu)\leq r,\, \mu_1\leq m} (-t)^{|\mu|} B_{a^r+\mu}\, B_{a-\mu'_m,\dots,a-\mu'_1,\nu}\,, \tag{1.5}$$

where B_λ is a vertex operator that reduces to s_λ when $t = 1$ [12].

This result enables us to provide a thorough analysis of the operators B_λ in the context of our filtration. We derive commutation relations on B_λ as well as a number of other identities for these operators and the Schur functions. We find that operators (and Schur functions) indexed by rectangular partitions, i.e. partitions of the form $(\ell^{k+1-\ell})$, play a particularly important role in our study. For example, these operators leave fundamental subspaces of $\Lambda_t^{(k)}$ invariant.

Results concerning the k-Schur function basis arise as a consequence of our work with operators indexed by rectangular partitions. In the last section, we prove that the action of such an operator on a k-Schur function produces only one k-Schur function. Namely, for $\ell = 1, 2, \dots, k$,

$$B_{\ell^{k+1-\ell}}\, s_\lambda^{(k)}[X;t] = t^d s_\mu^{(k)}[X;t]\,, \tag{1.6}$$

where μ is the partition rearrangement of the entries in $(\ell^{k+1-\ell})$ and λ, and t^d is a positive power of t given explicitly in Theorem 26. This result has the important consequence of simplifying the construction of the k-Schur functions. In effect, for each k, there is a subset of $k!$ k-Schur functions called the irreducible k-Schur functions, from which all other $s_\lambda^{(k)}[X;t]$ may be constructed by successive application of operators indexed by rectangular partitions. That is,

$$s_\lambda^{(k)}[X;t] = t^c B_{R_1}\cdots B_{R_j}\, s_\mu^{(k)}[X;t]\,, \qquad c\in\mathbb{N}\,, \tag{1.7}$$

where $s_\mu^{(k)}[X;t]$ is an irreducible k-Schur function and $R_1,\dots,R_j$ are rectangular partitions.

Since the Hall-Littlewood polynomials at $t = 1$ are the homogeneous symmetric functions, $h_\lambda[X]$, $\Lambda_t^{(k)}$ reduces to the polynomial ring $\Lambda^{(k)} = \mathbb{Q}[h_1,\dots,h_k]$. Since B_R is simply multiplication by the Schur function s_R when $t = 1$, relation (1.7) reduces to

$$s_\lambda^{(k)}[X] = s_{R_1}[X] s_{R_2}[X]\dots s_{R_\ell}[X] s_\mu^{(k)}[X]\,. \tag{1.8}$$

It follows that the irreducible k-Schur functions thus constitute a natural basis for the quotient ring $\Lambda_t^{(k)}/\mathcal{I}_k$, where $\mathcal{I}_k$ is the ideal generated by Schur functions indexed by rectangular shapes of the type $(\ell^{k+1-\ell})$.

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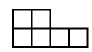
2. Definitions

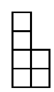
2.1. **Partitions.** Symmetric polynomials are indexed by partitions, sequences of non-negative integers $\lambda = (\lambda_1, \lambda_2, \dots)$ with $\lambda_1 \geq \lambda_2 \geq \dots$. The number of non-zero parts in λ is denoted $\ell(\lambda)$ and the degree of λ is $|\lambda| = \lambda_1 + \cdots + \lambda_{\ell(\lambda)}$. We use λ_L to denote $\lambda_{\ell(\lambda)}$. $\mathcal{P}^r_{\leq m}$ denotes the set of all partitions of length at most r, and whose first part is not larger than m. In this fashion, $\mathcal{P}^r_m$ is the set of partitions of length at most r, and whose first part is equal to m. The case $\mathcal{P}^r_{\leq\infty}$ will be denoted $\mathcal{P}^r$. Finally, for a partition $\mu = (\mu_1, \dots, \mu_m)$ of m entries (possibly zero), the reverse reading is denoted $\mu^R = (\mu_m, \dots, \mu_1)$.

We use the dominance order on partitions with $|\lambda| = |\mu|$, where $\lambda \leq \mu$ when $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$ for all i. Given two partitions λ and μ, $\lambda \cup \mu$ stands for the partition rearrangement of the parts of λ and μ, $\lambda \pm \mu$ stands for $(\lambda_1 \pm \mu_1, \lambda_2 \pm \mu_2, \dots)$, and (λ, μ) stands for the concatenation of λ and μ. Note that if $\lambda \leq \mu$ and $\nu \leq \omega$, then $\lambda \cup \nu \leq \mu \cup \omega$. We shall denote by δ_n (or simply δ when the value of n is clear) the partition $(n-1, n-2, \dots, 0)$.

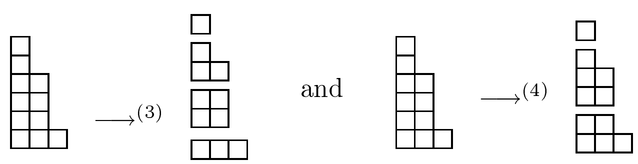
Any partition λ has an associated Ferrers diagram with λ_i lattice squares in the i^{th} row, from the bottom to top. For example,

$$\lambda = (4,2) =$$  . (2.1)

For each cell $s = (i,j)$ in the diagram of λ, let $\ell'(s), \ell(s), a(s)$ and $a'(s)$ be respectively the number of cells in the diagram of λ to the south, north, east and west of the cell s. The hook-length of any cell in λ, is $h_s(\lambda) = \ell(s) + a(s) + 1$. In the example, $h_{(1,2)}(4,2) = 2 + 1 + 1$. The *main hook-length* of λ, $h_M(\lambda)$, is the hook-length of the cell $s = (1,1)$ in the diagram of λ. Therefore, $h_M\big((4,2)\big) = 5$. The conjugate λ' of a partition λ is defined by the reflection of the Ferrers diagram about the main diagonal. For example, the conjugate of (4,2) is

$$\lambda' =$$  $$= (2,2,1,1)\,.$$ (2.2)

A partition λ is said to be *k-bounded* if its first part is not larger than k, i.e, if $\lambda_1 \leq k$. We associate to any k-bounded partition λ a sequence of partitions, $\lambda^{\to k}$, called the *k-split* of λ. $\lambda^{\to k} = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(r)})$ is obtained by partitioning λ (without rearranging the entries) into partitions $\lambda^{(i)}$ where $h_M(\lambda^{(i)}) = k$, for all $i < r$. For example, $(3,2,2,2,1,1)^{\to 3} = \big((3),(2,2),(2,1),(1)\big)$ and $(3,2,2,2,1,1)^{\to 4} = \big((3,2),(2,2,1),(1)\big)$. Equivalently, the diagram of λ is cut horizontally into partitions with main hook-length k.

 (2.3)

The last partition in the sequence $\lambda^{\to k}$ may have main hook- length less than k. It is important to note that $\lambda^{\to k} = (\lambda)$ when $h_M(\lambda) \leq k$.

2.2. **Symmetric functions.** The power sum $p_i(x_1, x_2, \dots)$ is

$$p_i(x_1, x_2, \dots) = x_1^i + x_2^i + \cdots\,, \tag{2.4}$$

and for a partition $\lambda = (\lambda_1, \lambda_2, \dots)$,

$$p_\lambda(x_1, x_2, \dots) = p_{\lambda_1}(x_1, x_2, \dots)\, p_{\lambda_2}(x_1, x_2, \dots) \cdots\,. \tag{2.5}$$

We employ the notation of λ-rings, needing only the formal ring of symmetric functions Λ to act on the ring of rational functions in $x_1, \dots, x_N, q, t$, with coefficients in $\mathbb{R}$. The action of a power sum p_i on a rational function is, by definition,

$$p_i\left[\frac{\sum_\alpha c_\alpha u_\alpha}{\sum_\beta d_\beta v_\beta}\right] = \frac{\sum_\alpha c_\alpha u_\alpha^i}{\sum_\beta d_\beta v_\beta^i}, \tag{2.6}$$

with $c_\alpha, d_\beta \in \mathbb{R}$ and u_α, v_β monomials in $x_1, \dots, x_N, q, t$. Since the power sums form a basis of the ring Λ, any symmetric function has a unique expression in terms of power sums, and (2.6) extends to an action of Λ on rational functions. In particular $f[X]$, the action of a symmetric function f on the monomial $X = x_1 + \cdots + x_N$, is simply $f(x_1, \ldots, x_N)$. In the remainder of the article, we will always consider the number of variables N to be infinite, unless otherwise specified.

The monomial symmetric function $m_\lambda[X_n]$ is

$$m_\lambda[X_n] = \sum_{\sigma \in S_n; \sigma(\lambda) \text{ distinct}} x^{\sigma(\lambda)} . \tag{2.7}$$

The complete symmetric function $h_r[X]$ is

$$h_r[X] = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_r} x_{i_1} x_{i_2} \cdots x_{i_r} , \tag{2.8}$$

and $h_\lambda[X]$ stands for the homogeneous symmetric function

$$h_\lambda[X] = h_{\lambda_1}[X]\, h_{\lambda_2}[X] \cdots . \tag{2.9}$$

And, the elementary symmetric function $e_\lambda[X]$ is defined

$$e_\lambda[X] = e_{\lambda_1}[X]\, e_{\lambda_2}[X] \cdots , \quad \text{where} \quad e_r[X] = \sum_{1 \le i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} . \tag{2.10}$$

Although the Schur functions may be characterized in many ways, here it will be convenient to use the Jacobi-Trudi determinantal expression:

$$s_\lambda[X] = \det \Big| h_{\lambda_i + j - 1}[X] \Big|_{1 \le i,j \le \ell(\lambda)} \tag{2.11}$$

where $h_r[X] = 0$ if $r < 0$. Note, in particular, $s_r[X] = h_r[X]$.

We recall that the Macdonald scalar product, $\langle\ ,\ \rangle_{q,t}$, on $\Lambda \otimes \mathbb{Q}(q,t)$ is defined by setting

$$\langle p_\lambda[X], p_\mu[X] \rangle_{q,t} = \delta_{\lambda\mu}\, z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}} , \tag{2.12}$$

where for a partition λ with $m_i(\lambda)$ parts equal to i, we associate the number

$$z_\lambda = 1^{m_1} m_1!\, 2^{m_2} m_2! \cdots \tag{2.13}$$

If $q = t$, this expression no longer depends on a parameter and is then denoted $\langle , \rangle$, satisfying

$$\langle s_\lambda[X], s_\mu[X] \rangle = \delta_{\lambda\mu} . \tag{2.14}$$

The Macdonald integral forms $J_\lambda[X; q, t]$ are uniquely characterized [8] by

$$\text{(i)}\ \langle J_\lambda, J_\mu \rangle_{q,t} = 0, \qquad \text{if } \lambda \ne \mu, \tag{2.15}$$

$$\text{(ii)}\ J_\lambda[X; q, t] = \sum_{\mu \le \lambda} v_{\lambda\mu}(q,t) s_\mu[X] \quad \text{with} \quad v_{\lambda\mu}(q,t) \in \mathbb{Q}(q,t) , \tag{2.16}$$

$$\text{(iii)}\ v_{\lambda\lambda}(q,t) = \prod_{s \in \lambda} (1 - q^{a(s)} t^{\ell(s)+1}), \tag{2.17}$$

Here, we use a modification of the Macdonald integral forms that is obtained by setting

$$H_\lambda[X; q, t] = J_\lambda[X/(1-t); q, t] = \sum_\mu K_{\mu\lambda}(q,t)\, s_\mu[X] , \tag{2.18}$$

with the coefficients $K_{\mu\lambda}(q,t) \in \mathbb{N}[q,t]$ known as the q,t-Kostka polynomials. When $q = 0$, $J_\lambda[X; q, t]$ reduces to the Hall-Littlewood polynomial, $J_\lambda[X; 0, t] = Q_\lambda[X; t]$. Again, we use a modification;

$$H_\lambda[X; t] = H_\lambda[X; 0, t] = Q_\lambda[X/(1-t); t] = s_\lambda[X] + \sum_{\mu > \lambda} K_{\mu\lambda}(t)\, s_\mu[X] , \tag{2.19}$$

with the coefficients $K_{\mu\lambda}(t) \in \mathbb{N}[t]$ known as the Kostka-Foulkes polynomials. The Kostka numbers $K_{\mu\lambda} \in \mathbb{N}$ arise in the limit $t = 1$, as coefficients in the expansion

$$h_\lambda[X] = H_\lambda[X;1] = s_\lambda[X] + \sum_{\mu > \lambda} K_{\mu\lambda}\, s_\mu[X]\,. \tag{2.20}$$

The Kostka numbers also appear in the expansion

$$s_\lambda[X] = m_\lambda[X] + \sum_{\mu < \lambda} K_{\lambda\mu}\, m_\mu[X]\,. \tag{2.21}$$

3. Vertex operators and Schur functions

The ring of symmetric polynomials over rational functions in the parameter t has shown to be of interest in many fields of mathematics and physics. A natural basis for this space is given by the Hall-Littlewood polynomials, $H_\lambda[X;t]$; t-analogs of the homogeneous symmetric functions, $h_\lambda[X]$. Our approach to the study of this space employs vertex operators that arise in the recursive construction for the Hall-Littlewood polynomials [4]. These operators can be defined [12] for $\ell \in \mathbb{Z}$, by

$$B_\ell = \sum_{i=0}^{\infty} s_{i+\ell}[X]\, s_i[X(t-1)]^\perp\,, \tag{3.1}$$

where for f, g and h arbitrary symmetric functions, $f^\perp$ is such that on the scalar product (2.14),

$$\langle f^\perp g, h\rangle = \langle g, fh\rangle\,. \tag{3.2}$$

Note, $B_\ell \cdot 1 = 0$ if $\ell < 0$. The operators build the Hall-Littlewood polynomials by

$$H_\lambda[X;t] = B_{\lambda_1} H_{\lambda_2,\ldots,\lambda_\ell}[X;t]\,, \qquad \text{for} \qquad \lambda_1 \geq \lambda_2\,, \tag{3.3}$$

and, since they satisfy the relation

$$B_m B_n = tB_n B_m + tB_{m+1}B_{n-1} - B_{n-1}B_{m+1}\,, \qquad m, n \in \mathbb{Z}\,, \tag{3.4}$$

their action on Λ can be computed algebraically.

The definition for these operators was extended in [12] to any partition λ of length L, by

$$B_\lambda \equiv \prod_{1 \leq i < j \leq L} (1 - te_{ij}) B_{\lambda_1} \cdots B_{\lambda_L}\,, \tag{3.5}$$

where e_{ij} acts by

$$e_{ij}\left(B_{\lambda_1} \cdots B_{\lambda_L}\right) = B_{\lambda_1} \cdots B_{\lambda_i+1} \cdots B_{\lambda_j-1} \cdots B_{\lambda_L}\,. \tag{3.6}$$

It is important to note that (3.1) gives $B_\ell = s_\ell$ when $t = 1$, and thus (3.5) reduces to the Jacobi-Trudi formula (2.11) for $s_\lambda[X]$. Therefore, we have

$$B_\lambda = s_\lambda[X]\,, \qquad \text{when } t = 1\,. \tag{3.7}$$

3.1. **Identities on Schur functions and vertex operators.** Here we derive properties for the vertex operators, and consequently for the Schur functions. Our properties reveal information about the behavior of B_λ and as a by-product, allow us to prove conjectures relating to a filtration of the symmetric function space. The main result of this section is an explicit formula for special products of two vertex operators.

Theorem 1. *Let a, r and m be nonnegative integers and ν be a partition with $\ell(\nu) \leq r$. Then*

$$B_{a^{r+m}}\, B_\nu = \sum_{\mu \in \mathcal{P}^r_{\leq m}} (-t)^{|\mu|} B_{a^r + \mu}\, B_{a^m - (\mu')^R, \nu}\,, \tag{3.8}$$

where the summand vanishes if $(a^m - (\mu')^R)$ is not a partition.

Since $B_\lambda \to s_\lambda$ when $t=1$ by (3.7), an immediate consequence of this result is an explicit formula for a product of two Schur functions.

Corollary 2. *Let a, r and m be nonnegative integers and ν be a partition with $\ell(\nu) \le r$. Then*

$$s_{a^{r+m}}\, s_\nu = \sum_{\mu \in \mathcal{P}^r_{\le m}} (-1)^{|\mu|} s_{a^r+\mu}\, s_{a^m-(\mu')^R,\nu}\,, \tag{3.9}$$

where the summand vanishes if $(a^m - (\mu')^R)$ is not a partition.

From Theorem 1, we derive a number of identities for the product of vertex operators. We now state and prove these identities, postponing the proof of our theorem to the end of this section. We start by giving several properties of the vertex operators B_v that arise from the fact that they satisfy the same reordering relations as the Schur functions. Namely (see Proposition 3 of [12]), for an integral vector v,

$$B_v = -B_{v_1,\dots,v_{i-1},v_{i+1}-1,v_i+1,v_{i+2},\dots,v_r}\,. \tag{3.10}$$

This reordering relation allows us to prove the following properties:

Property 3. *If $v=(v_1,\dots,v_r)\in\mathbb{Z}^r$ is such that $v_j - v_i = j-i$ for $i \neq j$, then $B_v = 0$.*

Proof. Without loss of generality, assume $j > i$. Given B_v, we successively apply (3.10) to move v_j from position j to position $i+1$;

$$\begin{aligned} B_v &= (-1)^{j-i-1} B_{v_1,\dots,v_i,v_j-(j-i-1),v_{i+1}+1,\dots,v_{j-1}+1,v_{j+1},\dots,v_r} && (3.11)\\ &= (-1)^{j-i-1} B_{v_1,\dots,v_i,v_i+1,v_{i+1}+1,\dots,v_{j-1}+1,v_{j+1},\dots,v_r}\,, && (3.12)\end{aligned}$$

since $v_j - v_i = j - i$. Further, switching the entries in position i and $i+1$, we obtain

$$B_{v_1,\dots,v_i,v_i+1,v_{i+1}+1,\dots,v_{j-1}+1,v_{j+1},\dots,v_r} = -B_{v_1,\dots,v_i,v_i+1,v_{i+1}+1,\dots,v_{j-1}+1,v_{j+1},\dots,v_r}\,, \tag{3.13}$$

which implies that $B_{v_1,\dots,v_i,v_i+1,v_{i+1}+1,\dots,v_{j-1}+1,v_{j+1},\dots,v_r} = 0$. Consequently, B_v is also null. $\square$

Property 4. *Let μ and ν be partitions of lengths m and r respectively. If there is a non-zero partition reordering, $\pm B_\lambda$, of $B_{\mu,\nu}$, then $\lambda_1 = \max\{\mu_1, \nu_1 - m\}$ and $\lambda_{m+r} = \min\{\mu_m + r, \nu_r\}$.*

Proof. An element of v in B_v moved i steps to the left (right) with (3.10), is decreased (increased) by i. Therefore, the only possible entries in the first position of any reordering of $B_{\mu,\nu}$ are

$$\mu_i - (i-1)\,, \quad i = 1,\dots,m \quad \text{or} \quad \nu_j - (j+m-1)\,, \quad j=1,\dots,r\,. \tag{3.14}$$

Assume the first entry in a reordering B_λ of $B_{\mu,\nu}$ is not the largest of these, i.e. $\lambda_1 < \max\{\mu_1,\nu_1 - m\}$. Since the entries $\mu_1 + i$ for some $i \ge 0$ and $\nu_1 - j$ for some $j \le m$ must occur in B_λ, and $\lambda_1 < \max\{\mu_1, \nu_1 - m\} \le \max\{\mu_1 + i, \nu_1 - j\}$, λ_1 cannot be the largest entry. Therefore λ is not a partition unless $\lambda_1 = \max\{\mu_1, \nu_1 - m\}$. Similar reasoning applies for the smallest entry λ_{m+r}. $\square$

The contrapositive of Property 4 then gives the following result:

Property 5. *Let μ and ν be partitions of lengths m and r, respectively. If $\max\{\mu_1, \nu_1 - m\} < \min\{\mu_m + r, \nu_r\}$ then $B_{\mu,\nu} = 0$.*

These properties of B_v allow us to give several identities concerning the product of an operator B_ν with an operator indexed by partitions of the form $(\ell^{k+1-\ell})$, $\ell = 1,\dots,k$, hereafter referred to as k-rectangles. Our identities are derived from particular cases of Theorem 1. The first is a t-commutation relation.

Identity 6. *Let i, k and ℓ be nonnegative integers. For $\ell \le i \le k$, we have*

$$B_{\ell^{k+1-\ell}}\, B_i = t^{i-\ell}\, B_i\, B_{\ell^{k+1-\ell}}\,. \tag{3.15}$$

Proof. Theorem 1, with $a = \ell$, $r = 1$, $m = k - \ell$, and $\nu = (i)$, gives

$$B_{\ell^{k+1-\ell}}\, B_i = \sum_{\mu \in \mathcal{P}^1_{\le k-\ell}} (-t)^{|\mu|} B_{\ell+\mu}\, B_{\ell^{k-\ell}-(\mu')^R, i} = \sum_{j=0}^{k-\ell} (-t)^j B_{\ell+j}\, B_{\ell^{k-\ell}-(1^j)^R, i}\,. \tag{3.16}$$

Since $(\ell^{k-\ell} - (1^j)^R, i) = (\ell^{k-\ell-j}, (\ell-1)^j, i)$, it suffices to show $B_{\ell^{k-\ell-j},(\ell-1)^j,i} = 0$ unless $j = i - \ell$, in which case $B_{\ell^{k-i},(\ell-1)^{i-\ell},i} = (-1)^{i-\ell} B_{\ell^{k+1-\ell}}$ by relation (3.10). When $k \ge i > j + \ell$, positions $k-\ell+1$ and $k-i+1$ of $(\ell^{k-\ell-j}, (\ell-1)^j, i)$ contain the entries i and ℓ, resp. Since these entries differ by $i - \ell$, Property 3 gives $B_{\ell^{k-\ell-j},(\ell-1)^j,i} = 0$. Similarly, when $\ell \le i < \ell + j$, the entries in positions $k - \ell + 1$ and $k - i$, resp i and $\ell - 1$, differ by $i - \ell + 1$ and again we have $B_{\ell^{k-\ell-j},(\ell-1)^j,i} = 0$. □

Identity 7. *Let ν be a partition of length r where $h_M(\nu) \le k$ and $\nu_1 \ge \ell$ for $\ell \in \mathbb{N}$. Then*

$$B_{\ell^{k+1-\ell}}\, B_\nu = \sum_i (-t)^{c(i)} B_{\rho^{(i)}}\, B_{\gamma^{(i)}}\,, \tag{3.17}$$

where $c(i) \in \mathbb{N}$, and $\rho^{(i)}$ and $\gamma^{(i)}$ are partitions such that $\rho^{(i)}_1 = \nu_1$, $\rho^{(i)}_L \ge \ell$, $h_M(\rho^{(i)}) = k, \gamma^{(i)}_1 = \ell$, and $h_M(\gamma^{(i)}) \le k$ (equality holds only when $h_M(\nu) = k$).

Proof. Let $m = \nu_1 - \ell$, $r = k + 1 - \nu_1$, and $a = \ell$ in Theorem 1. We then have

$$B_{\ell^{k+1-\ell}}\, B_\nu = \sum_{\substack{\ell(\mu) \le k+1-\nu_1 \\ \mu_1 \le \nu_1 - \ell}} (-t)^{|\mu|} B_{\ell^{k+1-\nu_1}+\mu}\, B_{\ell^{\nu_1-\ell}-(\mu')^R, \nu}\,. \tag{3.18}$$

If $\mu_1 < \nu_1 - \ell$ (equivalently $\mu'_{\nu_1-\ell} = 0$), then the first and $\nu_1 - \ell + 1^{th}$ entries of $(\ell^{\nu_1-\ell} - (\mu')^R, \nu)$ are $\ell - \mu'_{\nu_1-\ell} = \ell$ and ν_1 respectively. Since these entries differ by $\nu_1 - \ell$, Property 3 implies that $B_{\ell^{\nu_1-\ell}-(\mu')^R,\nu} = 0$ in this case. We thus have,

$$B_{\ell^{k+1-\ell}}\, B_\nu = \sum_{\substack{\ell(\mu) \le k+1-\nu_1 \\ \mu_1 = \nu_1 - \ell}} (-t)^{|\mu|} B_{\ell^{k+1-\nu_1}+\mu}\, B_{\ell^{\nu_1-\ell}-(\mu')^R, \nu}\,. \tag{3.19}$$

If we let $(\ell^{k+1-\nu_1} + \mu) = \rho^{(i)}$ then $\rho^{(i)}_1 = \mu_1 + \ell = \nu_1$. Moreover, $\ell(\mu) \le k+1-\nu_1$ gives that $\rho^{(i)}_L \ge \ell$ and $h_M(\rho^{(i)}) = \nu_1 + (k+1-\nu_1) - 1 = k$. Further, since $(\ell^{\nu_1-\ell} - (\mu')^R, \nu)$ is the concatenation of two weakly decreasing sequences, Property 4 implies any non-zero partition reordering, $\pm B_{\gamma^{(i)}}$, of $B_{\ell^{\nu_1-\ell}-(\mu')^R,\nu}$ has $\gamma^{(i)}_1 = \max\{\ell - \mu'_{\nu_1-\ell}, \nu_1 - (\nu_1 - \ell)\} = \ell$. Then, $h_M(\gamma^{(i)}) = \ell + (\nu_1 - \ell + r) - 1 = \nu_1 + r - 1 \le k$ since $h_M(\nu) = \nu_1 + r - 1 \le k$ (equality holds only when $h_M(\nu) = k$). □

We now give two identities regarding the product of an operator indexed by a k-rectangle, $(\ell^{k+1-\ell})$, with an operator indexed by any partition with main hook-length exactly k.

Identity 8. *Let ν be a partition of length r where $h_M(\nu) = k$, $\nu_1 \ge \ell$, and $\nu_r < \ell$. Then*

$$B_{\ell^{k+1-\ell}}\, B_\nu = \sum_i (-t)^{c(i)} B_{\rho^{(i)}}\, B_{\gamma^{(i)}}\,, \tag{3.20}$$

where $c(i) \in \mathbb{N}$, and $\rho^{(i)}$ and $\gamma^{(i)}$ are partitions where $\rho^{(i)}_1 = \nu_1$, $\rho^{(i)}_L \ge \ell$, $h_M(\rho^{(i)}) = h_M(\gamma^{(i)}) = k$, $\gamma^{(i)}_1 = \ell$, and $\gamma^{(i)}_L = \nu_r$.

Proof. This result follows from formula (3.19) in Identity 7 with $h_M(\nu) = \nu_1 + r - 1 = k$ and $\nu_r < \ell$. Since $\ell(\mu) \le k + 1 - \nu_1 = r$ implies $\mu'_1 \le r$, Property 4 proves $\gamma^{(i)}_L = \min\{\ell - \mu'_1 + r, \nu_r\} = \nu_r$ since $\ell - \mu'_1 + r \ge \ell - r + r = \ell > \nu_r$. □

Lemma 9. *Let μ, ν, and λ be partitions, and let $\lambda = w\big((\mu,\nu)+\delta)\big) - \delta$ for some permutation w. Let γ be a partition such that $\gamma \neq \mu$, and such that $\ell(\mu) = \ell(\gamma) = n$, where both partitions may contain zeroes. Then $\lambda \neq \sigma\big((\gamma,\nu)+\delta)\big) - \delta$ for all permutations σ.*

Proof. Assume there exist permutations w and σ where $\lambda = w\big((\mu,\nu)+\delta)\big) - \delta = \sigma\big((\gamma,\nu)+\delta)\big) - \delta$ for $\mu \neq \gamma$. This implies that $(\mu,\nu)+\delta = \sigma'\big((\gamma,\nu)+\delta)\big)$ for some permutation σ'. For all $i > n$, $\big((\mu,\nu)+\delta\big)_i = \big((\gamma,\nu)+\delta\big)_i$, since $\ell(\gamma) = \ell(\mu) = n$ implies $\gamma_i = \mu_i = 0$. Thus, $\mu + \delta_n = \sigma'(\gamma + \delta_n)$, or equivalently $\mu = \sigma'(\gamma+\delta_n) - \delta_n$, for some $\sigma' \in S_n$. However, since γ is the only element that is a partition in the set $\{\tau(\gamma+\delta_n) - \delta_n \,:\, \tau \in S_n\}$, we arrive at the contradiction $\mu = \gamma$. $\square$

Identity 10. *Let ν be a partition of length r where $h_M(\nu) = k$ and $\nu_r \geq \ell$. Then*

$$B_{\ell^{k+1-\ell}}\, B_\nu = t^{|\nu|-r\ell} B_\nu\, B_{\ell^{k+1-\ell}}\,. \tag{3.21}$$

Proof. For γ indexing the partition reordering of $B_{\ell^{\nu_1-\ell}-(\mu')^R,\nu}$, Identity 7 implies that (3.19) holds with $\gamma_1 = \ell$. Since $h_M(\nu) = \nu_1 + r - 1 = k$, we have $\ell(\gamma^{(i)}) = \nu_1 - \ell + r = k+1-\ell$. Also, $\ell(\mu) \leq k+1-\nu_1 = r$ leads to $\mu'_1 \leq r$. Therefore, $\nu_r \geq \ell$ implies by Property 4 that $\gamma_{k+1-\ell} = \min\{\ell - \mu'_1 + r, \nu_r\} \geq \ell$ since $\ell - \mu'_1 + r \geq \ell - r + r = \ell$. Thus, from Property 5, $\gamma_1 = \gamma_{k+1-\ell} = \ell$ implies that all non-zero $B_{\ell^{\nu_1-\ell}-(\mu')^R,\nu}$ equal $\pm B_\gamma$ where $\gamma = (\ell^{k+1-\ell})$.

It now suffices to show that $B_{\ell^{\nu_1-\ell}-(\mu')^R,\nu} = \pm B_{(\ell^{k+1-\ell})}$ only when $\mu = (\nu_1-\ell,\ldots,\nu_r-\ell)$. Our claim will follow since (3.19) then simplifies to

$$B_{\ell^{k+1-\ell}}\, B_\nu = \pm\, t^{|\nu|-r\ell} B_\nu\, B_{\ell^{k+1-\ell}}\,, \tag{3.22}$$

where the sign must be positive since when $t=1$, this relation becomes $s_{\ell^{k+1-\ell}} s_\nu = s_\nu s_{\ell^{k+1-\ell}}$. In fact, we only need to show that when $\mu = (\nu_1-\ell,\ldots,\nu_r-\ell)$, there exists some permutation w where

$$w\big((\ell^{\nu_1-\ell} - (\mu')^R, \nu) + \delta\big) - \delta = (\ell^{k+1-\ell})\,. \tag{3.23}$$

Then by Lemma 9, since there exists no other partition $(\ell^{\nu_1-\ell} - (\mu')^R)$ such that $w\big((\ell^{\nu_1-\ell} - (\mu')^R, \nu) + \delta\big) - \delta = (\ell^{k+1-\ell})$, non-zero terms occur only when $\mu = (\nu_1-\ell,\ldots,\nu_r-\ell)$.

To prove (3.23), it is equivalent to show that there exists some permutation w' where

$$w'\delta + (\ell^{k+1-\ell}) = (\ell^{\nu_1-\ell} - (\mu')^R, \nu) + \delta\big) \tag{3.24}$$

Since $\mu = (\nu_1-\ell,\ldots,\nu_r-\ell) \implies \nu = (\mu_1+\ell,\ldots,\mu_r+\ell)$, and $\delta = (k-\ell,\ldots,0) = (r+\mu_1-1,\ldots,0)$ given $r = k-\nu_1+1$ (from $h_M(\nu)=k$), we must show there is some permutation w' where

$$\begin{aligned} w'\delta + (\ell^{k+1-\ell}) &= (\ell - \mu'_{\mu_1} + r + \mu_1 - 1, \ldots, \ell - \mu'_1 + r, \mu_1 + \ell + r - 1, \ldots, \mu_r + \ell) \\ &= (\ell^{k+1-\ell}) + (-\mu'_{\mu_1} + r + \mu_1 - 1, \ldots, -\mu'_1 + r, \mu_1 + r - 1, \ldots, \mu_r)\,. \end{aligned} \tag{3.25}$$

The last r entries of $u = (-\mu'_{\mu_1} + r + \mu_1 - 1, \ldots, -\mu'_1 + r, \mu_1 + r - 1, \ldots, \mu_r)$ are $\mu_i + r - i$, $i = 1,\ldots,r$, and the first μ_1 entries are $r - 1 + j - \mu'_j$, $j = \mu_1,\ldots,1$. Since $r \geq \mu'_1$, a vector of this type is known [8] to be a permutation of $\delta_{r+\mu_1}$. Thus, there is some w' such that $u = w'\delta$, and (3.25) follows. $\square$

3.2. **Proof of Theorem 1.** We use several lemmas that rely on properties for the Kostka matrix. These properties are derived in Appendix 6 for lack of reference. Here, we use E_m^d to denote the set of vectors of length m, with d ones and $m-d$ zeroes. Then E_m^λ is the set of vectors $v = v_1 + v_2 + \ldots$, where $v_1 \in E_m^{\lambda_1}$, $v_2 \in E_m^{\lambda_2}, \ldots$.

Lemma 11. *For a partition $\lambda \in \mathcal{P}^r$ and any partition ν, we have*

$$\sum_{\sigma \in S_r;\sigma(\lambda)\ \text{distinct}} B_{\sigma(\lambda)+b^r,\nu} = \sum_{\mu\in\mathcal{P}^r} B_{\mu+b^r,\nu} K^{-1}_{\lambda\mu}\,. \tag{3.26}$$

Proof. Using Formula 33, we have

$$m_{\lambda+b^r} = \sum_{\sigma\in S_r;\sigma(\lambda+b^r)\ \text{distinct}} s_{\sigma(\lambda+b^r)} = \sum_{\mu\in\mathcal{P}^r} K^{-1}_{\lambda+b^r\ \mu+b^r}\, s_{\mu+b^r}\,. \tag{3.27}$$

With $K^{-1}_{\lambda\mu} = K^{-1}_{\lambda+b^r\mu+b^r}$, from Formula 30, this gives

$$\sum_{\sigma\in S_r;\sigma(\lambda)\ \text{distinct}} s_{\sigma(\lambda)+b^r} = \sum_{\mu\in\mathcal{P}^r} K^{-1}_{\lambda\mu}\, s_{\mu+b^r}\,.$$

We formally replace s_* by $B_{*,\nu}$ since $s_{\sigma(\lambda)}$ and $B_{\sigma(\lambda),\nu}$ both obey the reordering relation (3.10). $\square$

Lemma 12. *For $\lambda\in\mathcal{P}^m_{\leq a}$, we have*

$$\sum_{E\in E^\lambda_m} B_{a^m-E,\nu} = \sum_\omega \sum_{\gamma\in\mathcal{P}^m} \sum_{\rho\in P^m_{\leq a}} K^{-1}_{\gamma\rho} K_{\omega\lambda} K_{\omega'\gamma}\, B_{a^m-\rho^R,\nu}\,. \tag{3.28}$$

Proof. It is known [8] that for $\lambda\in\mathcal{P}^m$,

$$e_\lambda = \sum_{\gamma,\omega} K_{\omega\lambda}K_{\omega'\gamma} m_\gamma \quad\text{and}\quad \sum_{E\in E^\lambda_m} x^E = e_\lambda[x_1+\cdots+x_m]\,. \tag{3.29}$$

Equivalently, we thus have

$$\sum_{E\in E^\lambda_m} x^E = \sum_\omega \sum_{\gamma\in\mathcal{P}^m} K_{\omega\lambda}K_{\omega'\gamma} \sum_{\sigma\in S_m;\sigma(\gamma)\ \text{distinct}} x^{\sigma(\gamma)}\,. \tag{3.30}$$

Formally replacing x^* by $B_{a^m-*,\nu}$, this implies

$$\sum_{E\in E^\lambda_m} B_{a^m-E,\nu} = \sum_\omega \sum_{\gamma\in\mathcal{P}^m} K_{\omega\lambda}K_{\omega'\gamma} \sum_{\sigma\in S_m;\sigma(\gamma)\ \text{distinct}} B_{a^m-\sigma(\gamma),\nu}\,. \tag{3.31}$$

Since $a^m-\gamma^R$ is a partition, we can use Lemma 11 with $\lambda = a^m-\gamma^R$ and $b=0$ to obtain

$$\sum_{E\in E^\lambda_m} B_{a^m-E,\nu} = \sum_\omega \sum_{\gamma\in\mathcal{P}^m} K_{\omega\lambda}K_{\omega'\gamma} \sum_{\mu\in\mathcal{P}^m} B_{\mu,\nu} K^{-1}_{a^m-\gamma^R,\mu}. \tag{3.32}$$

Since $K^{-1}_{a^m-\gamma^R,\mu}=0$ if $a^m-\gamma^R\not\geq\mu$, we have $a\geq\mu_1$. Thus, $\mu = a^m-\rho^R$ for some $\rho\in\mathcal{P}^m_{\leq a}$ and

$$\sum_{E\in E^\lambda_m} B_{a^m-E,\nu} = \sum_\omega \sum_{\gamma\in\mathcal{P}^m} K_{\omega\lambda}K_{\omega'\gamma} \sum_{\rho\in\mathcal{P}^m_{\leq a}} B_{a^m-\rho^R,\nu} K^{-1}_{a^m-\gamma^R,a^m-\rho^R}. \tag{3.33}$$

The property then follows from Formula 32, which gives $K^{-1}_{a^m-\gamma^R,a^m-\rho^R} = K^{-1}_{\gamma,\rho}$. $\square$

The last lemma needed to derive our expression for the product of Schur functions and operators uses methods presented in [1] and [12]. Here, $v\in[n]$ is a vector with entries from $0,1,\ldots,n$.

Lemma 13. *Let ν,μ,γ be any partitions with $\ell(\nu)=n$, $\ell(\mu)=r$ and $\ell(\gamma)=m$. Then*

$$\sum_{I=(i_1,\ldots,i_m)\in[n]} \sum_{E\in E^I_n} (-t)^{|I|} B_{\mu,\gamma+I}\, B_{\nu-E} = \sum_{I=(i_1,\ldots,i_r)\in[m]} \sum_{E\in E^I_m} (-t)^{|I|} B_{\mu+I}\, B_{\gamma-E,\nu}\,. \tag{3.34}$$

Proof. Identity (20) in [12] implies that

$$H(U^r,V^m)H(Z^n)\prod_{i=1}^m\prod_{j=1}^n\left(1-\frac{t}{v_i}z_j\right) = H(U^r)H(V^m,Z^n)\prod_{i=1}^r\prod_{j=1}^m\left(1-\frac{t}{u_i}v_j\right), \tag{3.35}$$

where $H(X^\ell)$ is a formal Laurent series in an ordered set of variables $X^\ell=(x_1,\ldots,x_\ell)$ with coefficients given by operators which act on $P\in\Lambda$ by

$$H(X^\ell)P[Y] = P\left[Y-(1-q)\sum_{i=1}^\ell x_i^{-1}\right]\prod_{i,j}\frac{1}{(1-y_ix_j)}\prod_{1\leq i<j\leq\ell}(1-x_j/x_i)\,. \tag{3.36}$$

Binomial expansion of (3.35) then gives

$$H(U^r,V^m)H(Z^n)\prod_{i=1}^{m}\sum_{0\le l_i\le n}\left(\frac{-t}{v_i}\right)^{l_i}\sum_{E\in E_n^{l_i}} z^E = H(U^r)H(V^m,Z^n)\prod_{i=1}^{r}\sum_{0\le l_i\le m}\left(\frac{-t}{u_i}\right)^{l_i}\sum_{E\in E_m^{l_i}} v^E\,.$$

Equivalently, by expanding the products, we have

$$\begin{aligned}H(U^r,V^m)H(Z^n)&\sum_{0\le l_1\le n}\cdots\sum_{0\le l_m\le n}\frac{(-t)^{l_1+\cdots+l_m}}{v_1^{l_1}\cdots v_m^{l_m}}\sum_{E\in E_n^{(l_1,\cdots,l_m)}} z^E\\ &= H(U^r)H(V^m,Z^n)\sum_{0\le l_1\le m}\cdots\sum_{0\le l_r\le m}\frac{(-t)^{l_1+\cdots+l_r}}{u_1^{l_1}\cdots u_r^{l_r}}\sum_{E\in E_m^{(l_1,\cdots,l_r)}} v^E\end{aligned}$$

It is known [12] that $H(X^\ell)P[Y]\big|_{x^\lambda} = B_\lambda P[Y]$. Therefore, for partitions μ,γ,ν with $\ell(\mu)=r,\ell(\gamma)=m,\ell(\nu)=n$, we take the coefficient of u^μ, v^γ and z^ν in both sides to obtain

$$\begin{aligned}&\sum_{0\le l_1\le n}\cdots\sum_{0\le l_m\le n}(-t)^{l_1+\cdots+l_m}B_{\mu,\gamma_1+l_1,\ldots,\gamma_m+l_m}\sum_{E\in E_n^{(l_1,\cdots,l_m)}} B_{\nu-E}\\ &\qquad= \sum_{0\le l_1\le m}\cdots\sum_{0\le l_r\le m}(-t)^{l_1+\cdots+l_r}B_{\mu_1+l_1,\cdots,\mu_r+l_r}\sum_{E\in E_m^{(l_1,\cdots,l_r)}} B_{\gamma-E,\nu}\,.\end{aligned}$$

This completes the proof. □

We can now prove Theorem 1 using these lemmas and our properties for the vertex operators.

Proof of Theorem 1. With $\ell(\nu)=n\le r$, letting $\mu=(a^r)$ and $\gamma=(a^m)$ in Lemma 13, we have

$$\sum_{L=(l_1,\ldots,l_m)\in[n]}\sum_{E\in E_n^L}(-t)^{|L|}B_{a^r,a+l_1,\ldots,a+l_m}B_{\nu-E} = \sum_{L=(l_1,\ldots,l_r)\in[m]}\sum_{E\in E_m^L}(-t)^{|L|}B_{a+l_1,\cdots,a+l_r}B_{a^m-E,\nu}\,. \tag{3.37}$$

Property 3 implies $B_v = B_{a^r,a+l_1,\ldots,a+l_m}=0$ if $1\le l_1\le r$, since $v_{r+1}-v_{r+1-l_1}=l_1$. Thus $B_v=0$ unless l_1 is zero since $0\le l_1\le n\le r$. Given $l_1=0$, Property 3 implies $B_v=B_{a^{r+1},a+l_2,\ldots,a+l_m}=0$ if $1\le l_2\le r$, since $v_{r+2}-v_{r+2-l_2}=l_2$. Thus $B_v=0$ unless l_2 is zero since $0\le l_2\le n\le r$. Repeating this argument, $B_{a^r,a+l_1,\ldots,a+l_m}=0$ unless $l_1=l_2=\cdots=l_m=0$, and we have

$$\sum_{E\in E_n^{0,\ldots,0}} B_{a^r,a^m}\,B_{\nu-E} = B_{a^{r+m}}\,B_\nu = \sum_{L=(l_1,\ldots,l_r)\in[m]}\sum_{E\in E_m^L}(-t)^{|L|}B_{a+l_1,\cdots,a+l_r}\,B_{a^m-E,\nu}\,. \tag{3.38}$$

Moreover, since $E_m^L = E_m^{\sigma L}$ for any permutation σ, (3.38) can be written as

$$B_{a^{r+m}}\,B_\nu = \sum_{\lambda\in\mathcal{P}^r_{\le m}}(-t)^{|\lambda|}\left(\sum_{\sigma\in S_r;\sigma(\lambda)\ \text{distinct}} B_{\sigma(\lambda)+a^r}\right)\left(\sum_{E\in E_m^\lambda} B_{a^m-E,\nu}\right)\,. \tag{3.39}$$

We can now use Lemmas 11 and 12 to obtain

$$\begin{aligned}B_{a^{r+m}}\,B_\nu &= \sum_{\lambda\in\mathcal{P}^r_{\le m}}(-t)^{|\lambda|}\left(\sum_{\mu\in\mathcal{P}^r} B_{\mu+a^r}K^{-1}_{\lambda\mu}\right)\left(\sum_{\omega}\sum_{\gamma\in\mathcal{P}^m}\sum_{\rho\in\mathcal{P}^m_{\le a}} K^{-1}_{\gamma\rho}K_{\omega\lambda}K_{\omega'\gamma}\,B_{a^m-\rho^R,\nu}\right)\\ &= \sum_{\rho\in\mathcal{P}^m_{\le a}}\sum_{\gamma\in\mathcal{P}^m}\sum_{\mu\in\mathcal{P}^r}(-t)^{|\mu|}B_{\mu+a^r}\sum_{\omega}\left(\sum_{\lambda\in\mathcal{P}^r_{\le m}}K_{\omega\lambda}K^{-1}_{\lambda\mu}\right)K^{-1}_{\gamma\rho}K_{\omega'\gamma}\,B_{a^m-\rho^R,\nu}\end{aligned} \tag{3.40}$$

It happens that we can sum over all λ since any term $\lambda \notin \mathcal{P}^r_{\leq m}$ vanishes by the known property $K_{\lambda\mu} = K^{-1}_{\lambda\mu} = 0$ if $\lambda \ngeq \mu$. That is, if $\lambda \notin \mathcal{P}^r$, $K^{-1}_{\lambda\mu} = 0$ since $\mu \in \mathcal{P}^r$. Moreover, if $\lambda_1 > m$, $K_{\omega\lambda} = 0$ if $\omega_1 \ngtr m$. But if $\omega_1 > m$, we have $\omega' \notin \mathcal{P}^m$, and thus $K_{\omega'\gamma} = 0$ since $\gamma \in \mathcal{P}^m$. Therefore,

$$
\begin{aligned}
B_{a^{r+m}}\, B_\nu &= \sum_{\rho \in \mathcal{P}^m_{\leq a}} \sum_{\gamma \in \mathcal{P}^m} \sum_{\mu \in \mathcal{P}^r} (-t)^{|\mu|} B_{\mu + a^r} \sum_{\omega} \left(\sum_{\lambda} K_{\omega\lambda} K^{-1}_{\lambda\mu} \right) K^{-1}_{\gamma\rho} K_{\omega'\gamma}\, B_{a^m - \rho^R, \nu} \\
&= \sum_{\rho \in \mathcal{P}^m_{\leq a}} \sum_{\mu \in \mathcal{P}^r} (-t)^{|\mu|} B_{\mu + a^r} \left(\sum_{\gamma \in \mathcal{P}^m} K_{\mu'\gamma} K^{-1}_{\gamma\rho} \right) B_{a^m - \rho^R, \nu}\,, \qquad (3.41)
\end{aligned}
$$

since $\sum_\omega \delta_{\omega\mu} K_{\omega'\gamma} = K_{\mu'\gamma}$. Again, if $\gamma \notin \mathcal{P}^m$, we have $K^{-1}_{\gamma\rho} = 0$ since $\rho \in \mathcal{P}^m_{\leq a}$. Thus, we have

$$
\begin{aligned}
B_{a^{r+m}}\, B_\nu &= \sum_{\rho \in \mathcal{P}^m_{\leq a}} \sum_{\mu \in \mathcal{P}^r} (-t)^{|\mu|} B_{\mu + a^r} \left(\sum_{\gamma} K_{\mu'\gamma} K^{-1}_{\gamma\rho} \right) B_{a^m - \rho^R, \nu} \\
&= \sum_{\rho \in \mathcal{P}^m_{\leq a}} \sum_{\mu \in \mathcal{P}^r} (-t)^{|\rho|}\, \delta_{\mu'\rho}\, B_{\mu + a^r}\, B_{a^m - \rho^R, \nu} \\
&= \sum_{\mu \in \mathcal{P}^r_{\leq m}} (-t)^{|\mu|}\, B_{\mu + a^r}\, B_{a^m - (\mu')^R, \nu}\,, \qquad (3.42)
\end{aligned}
$$

with the restriction that the summand vanishes if $(a^m - (\mu')^R)$ is not a partition. □

4. k-split polynomials

Recent developments in the study of the symmetric function space have centered around a filtration, $\Lambda^{(1)}_t \subseteq \Lambda^{(2)}_t \subseteq \cdots \subseteq \Lambda^{(\infty)}_t = \Lambda$, given by the subspaces

$$
\Lambda^{(k)}_t = \mathcal{L}\left\{ H_\lambda[X;t] \right\}_{\lambda; \lambda_1 \leq k} = \mathcal{L}\left\{ H_\lambda[X;q,t] \right\}_{\lambda; \lambda_1 \leq k}\,. \qquad (4.1)
$$

This filtration provides a convenient environment for the generalization of the theory of Schur functions, and for the study of Macdonald polynomials [5, 6]. Several new families of polynomials were introduced in [6] as an approach to studying the spaces $\Lambda^{(k)}_t$. In this section, we use results from Section 3 to prove properties related to one of these families; the k-split polynomials.

Definition 14. *For a k-bounded partition λ, let $\lambda^{\to k} = (\lambda^{(1)}, \lambda^{(2)}, \dots)$. The k-split polynomials are defined recursively by*

$$
G^{(k)}_\lambda[X;t] = B_{\lambda^{(1)}}\, G^{(k)}_{(\lambda^{(2)}, \lambda^{(3)}, \dots)}[X;t]\,, \quad \textit{with} \quad G^{(k)}_{()} = 1\,. \qquad (4.2)
$$

It was shown in [6] that the k-split polynomials form a basis for $\Lambda^{(k)}_t$, and that vertex operators indexed by partitions with hook-length not larger than k leave this space invariant. More precisely,

Property 15. [6] *If λ is a partition with $h_M(\lambda) \leq k$, then $B_\lambda\, f \in \Lambda^{(k)}_t$ for any $f \in \Lambda^{(k)}_t$.*

Further, it was shown that B_i acts invariantly on subspaces of $\Lambda^{(k)}_t$, defined for integers $a \leq k$ by

$$
\Lambda^{(a,k)}_t = \mathcal{L}\{ H_\lambda[X;t] \}_{a \leq \lambda_1 \leq k}\,. \qquad (4.3)
$$

Note that in the case $a \leq 0$, we simply have $\Lambda^{(a,k)}_t = \Lambda^{(k)}_t$, and that $\Lambda^{(a,k)}_t$ can also be defined as

$$
\Lambda^{(a,k)}_t = \mathcal{L}\{ G_\lambda[X;t] \}_{a \leq \lambda_1 \leq k}\,, \qquad (4.4)
$$

since the transition matrix between the two bases is upper triangular [6].

Property 16. [6] *If i is an integer such that $i \leq k$, then $B_i\, f \in \Lambda^{(i,k)}_t \subseteq \Lambda^{(k)}_t$ for all $f \in \Lambda^{(k)}_t$.*

We start by extending these properties and then discover, more specifically, that there exist subspaces of $\Lambda_t^{(a,k)}$ that are invariant under a special set of the B_λ operators. These subspaces are defined, for nonnegative integers $j \leq k$,

$$\Omega_j^{(k)} = \mathcal{L}\big\{G_\lambda^{(k)}[X;t]\big\}_{\lambda_1 = j}\,. \tag{4.5}$$

Our generalization of Property 15 is

Property 17. *If λ is a partition with $h_M(\lambda) \leq k$, then $B_\lambda\, f \in \Lambda_t^{(\lambda_1,k)}$ for all $f \in \Lambda_t^{(k)}$.*

Proof. Definition (3.5) gives

$$\begin{aligned} B_\lambda &= \prod_{2\leq j\leq \ell(\lambda)} (1-te_{1j}) B_{\lambda_1} \prod_{2\leq i<j\leq \ell(\lambda)} (1-te_{ij}) B_{\lambda_2} \cdots B_{\lambda_{\ell(\lambda)}} \\ &= \prod_{2\leq j\leq \ell(\lambda)} (1-te_{1j}) B_{\lambda_1} B_{\hat\lambda} \end{aligned} \tag{4.6}$$

Since e_{1j} increases the index of B_{λ_1}, we have

$$B_\lambda = \sum_{i=0}^{\ell(\lambda)-1} c_i(t)\, B_{\lambda_1+i}\, O_i\,, \qquad\qquad c_i(t) \in \mathbb{Z}[t]\,, \tag{4.7}$$

where O_i is a product of B_j's with $j \leq k$. Let $f \in \Lambda_t^{(k)}$ and note that Property 16 implies $B_j\, f \in \Lambda_t^{(k)}$ for $j \leq k$. Therefore, $O_i \cdot f \in \Lambda_t^{(k)}$ and again by Property 16, $B_{\lambda_1+i}\, O_i \cdot f \in \lambda_t^{(\lambda_1+i,k)} \subseteq \Lambda_t^{(\lambda_1,k)}$ since $\lambda_1 + i \leq \lambda_1 + \ell(\lambda) - 1 = h_M(\lambda) \leq k$. □

Property 18. *If i is an integer such that $1 \leq i < k$ then $B_i\, f \in \Lambda_t^{(i+1,k)}$ for all $f \in \Lambda_t^{(i+1,k)}$.*

Proof. Let $f \in \Lambda_t^{(i+1,k)}$ and assume without loss of generality that $f = H_\lambda[X;t] = B_{\lambda_1} H_{\hat\lambda}[X;t]$ for λ with $\lambda_1 > i$. When $\lambda_1 = i+1$, we have $B_i f = B_i B_{i+1} H_{\hat\lambda}[X;t] = B_{i+1} B_i H_{\hat\lambda}[X;t]$ by the commutation relation (3.4). Property 16 then implies that $B_i f = B_{i+1} B_i H_{\hat\lambda}[X;t] \in \Lambda_t^{(i+1,k)}$. In the case that $k \geq \lambda_1 > i+1$, again by relation (3.4), we have

$$B_i\, H_\lambda[X;t] = B_i\, B_{\lambda_1}\, H_{\hat\lambda}[X;t] = \big(tB_{\lambda_1} B_i + tB_{i+1} B_{\lambda_1-1} - B_{\lambda_1-1} B_{i+1}\big) \cdot H_{\hat\lambda}\,. \tag{4.8}$$

The three terms in the right hand side are all of the type $B_a B_b H_{\hat\lambda}$, where $a > i$ and $b \leq k$. Therefore, $B_a B_b H_{\hat\lambda} \in \Lambda_t^{(i+1,k)}$ again by Property 16. □

We now prove that the subspaces $\Omega_j^{(k)}$ are invariant under the action of operators indexed by k-rectangles, using the following lemma:

Lemma 19. *If λ is a partition with $h_M(\lambda) = k$ and $\lambda_L \geq j$ then $B_\lambda\, f \in \Omega_{\lambda_1}^{(k)}$ for all $f \in \Omega_j^{(k)}$.*

Proof. Letting $f = G_\mu^{(k)}[X;t]$ with $\mu_1 = j$, we have that (λ,μ) is a partition since $\mu_1 \leq \lambda_L$. If $\mu^{\rightarrow k} = (\mu^{(1)}, \mu^{(2)}, \dots)$, we thus have $\big(\lambda,\mu\big)^{\rightarrow k} = (\lambda, \mu^{(1)}, \mu^{(2)}, \dots)$ since $h_M(\lambda) = k$. Therefore,

$$B_\lambda\, f = B_\lambda \left(B_{\mu^{(1)}}\, B_{\mu^{(2)}} \cdots\right) \cdot 1 = G_{(\lambda,\mu)}^{(k)}[X;t] \tag{4.9}$$

by definition, and thus $B_\lambda\, f \in \Omega_{\lambda_1}^{(k)}$. □

We are now prepared to prove the final result of this section.

Theorem 20. *If $j \leq k$ is a nonnegative integer then $B_{\ell^{k+1-\ell}}\, f \in \Omega_{\max(j,\ell)}^{(k)}$ for all $f \in \Omega_j^{(k)}$.*

Proof. We have either (a) $\max(j,\ell)=\ell$ or (b) $\max(j,\ell)=j$. In case (a), letting $\lambda=(\ell^{k+1-\ell})$ in Lemma 19, the assertion holds. Thus assume $j>\ell$ and let $f=G_\nu^{(k)}[X;t]$ where $\nu_1=j$. If $\nu^{\to k}=(\nu^{(1)},\nu^{(2)},\dots)$ then there are three possibilities: (i) $h_M(\nu^{(1)})<k$, (ii) $h_M(\nu^{(1)})=k$ and $\nu_L^{(1)}<\ell$, or (iii) $h_M(\nu^{(1)})=k$ and $\nu_L^{(1)}\geq\ell$, where $L=\ell(\nu^{(1)})$.

In case (i), $h_M(\nu^{(1)})<k$ implies $\nu^{\to k}=(\nu^{(1)})$ and thus $f=B_{\nu^{(1)}}\cdot 1$. Identity 7 then gives

$$B_{\ell^{k+1-\ell}}\,B_{\nu^{(1)}}\cdot 1=\sum_i(-t)^{c(i)}B_{\rho^{(i)}}\,B_{\gamma^{(i)}}\cdot 1\,, \tag{4.10}$$

where the partitions $\rho^{(i)}$ and $\gamma^{(i)}$ are such that $(\rho^{(i)},\gamma^{(i)})^{\to k}=(\rho^{(i)},\gamma^{(i)})$ and $\rho_1^{(i)}=j$. Therefore,

$$B_{\ell^{k+1-\ell}}\,B_{\nu^{(1)}}\cdot 1=\sum_i(-t)^{c(i)}G^{(k)}_{(\rho^{(i)},\gamma^{(i)})}[X;t]\in\Omega_j^{(k)}\,. \tag{4.11}$$

For case (ii), we have $f=B_{\nu^{(1)}}B_{\nu^{(2)}}\cdots 1$. Identity 8 states that

$$B_{\ell^{k+1-\ell}}\,B_{\nu^{(1)}}=\sum_i(-t)^{c(i)}B_{\rho^{(i)}}\,B_{\gamma^{(i)}}\,, \tag{4.12}$$

where $\rho^{(i)}$ and $\gamma^{(i)}$ are such that $(\rho^{(i)},\gamma^{(i)},\nu^{(2)},\dots)^{\to k}=(\rho^{(i)},\gamma^{(i)},\nu^{(2)},\dots)$ and $\rho_1^{(i)}=j$. Thus

$$B_{\ell^{k+1-\ell}}\,f=\sum_i(-t)^{c(i)}G_{(\rho^{(i)},\gamma^{(i)},\nu^{(2)},\dots)}[X;t]\in\Omega_j^{(k)}\,. \tag{4.13}$$

Finally in case (iii), first t-commute $B_{\ell^{k+1-\ell}}$ with the operators $B_{\nu^{(i)}}$, using Identity 10, until

$$B_{\ell^{k+1-\ell}}\,B_{\nu^{(1)}}\cdots B_{\nu^{(m)}}\,B_{\nu^{(m+1)}}\cdots 1=t^*B_{\nu^{(1)}}\cdots B_{\nu^{(m)}}\,B_{\ell^{k+1-\ell}}\,B_{\nu^{(m+1)}}\cdots 1\,, \tag{4.14}$$

where $*$ is a power of t, and where $\nu^{(m)}$ is such that $\nu_L^{(m)}\geq\ell$, while $\nu_1^{(m+1)}<\ell$, $h_M(\nu^{(m+1)})<k$ or $\nu_L^{(m+1)}<\ell$. In any of these scenarios, if $\mu^{\to k}=(\nu^{(m+1)},\dots)$ then μ satisfies the conditions of cases (a), (i) or (ii), resp. We thus have

$$f=B_{\ell^{k+1-\ell}}B_{\nu^{(m+1)}}\cdots 1=B_{\ell^{k+1-\ell}}\,G_\mu^{(k)}[X;t]\ \in\ \Omega^{(k)}_{\max\{\ell,\mu_1=\nu_1^{(m+1)}\}}\,. \tag{4.15}$$

Since $\max\{\ell,\nu_1^{(m+1)}\}\leq\nu_L^{(m)}$, applying $B_{\nu^{(m)}}$ to f then produces an element of $\Omega^{(k)}_{\nu_1^{(m)}}$ by Lemma 19. Applying $B_{\nu^{(1)}},\dots,B_{\nu^{(m-1)}}$ by the same argument gives $B_{\nu^{(1)}}\cdots B_{\nu^{(m)}}\,B_{\ell^{k+1-\ell}}\,B_{\nu^{(m+1)}}\cdots 1\in\Omega^{(k)}_{\nu_1^{(1)}}=\Omega_\nu^{(k)}$ proves the theorem from (4.14). □

5. k-Schur functions

The k-split polynomials play a crucial role in the generalization of the theory of symmetric functions since they are essential for the construction of another family of polynomials called the k-Schur functions. The characterization of this family relies on a projection operator that acts linearly on $\Lambda_t^{(k)}$, for nonnegative integers $j\leq k$, by

$$T_j^{(k)}\,G_\lambda^{(k)}[X;t]=\begin{cases}G_\lambda^{(k)}[X;t] & \text{if } \lambda_1=j\\ 0 & \text{otherwise}\end{cases}\,. \tag{5.1}$$

Definition 21. *Let λ be a k-bounded partition. The k-Schur functions are defined recursively by,*

$$s_\lambda^{(k)}[X;t]=T^{(k)}_{\lambda_1}B_{\lambda_1}s^{(k)}_{(\lambda_1,\lambda_2,\dots)}[X;t]\,,\quad \textit{where}\quad s^{(k)}_{()}[X;t]=1\,. \tag{5.2}$$

The k-Schur functions are believed to play a role in $\Lambda_t^{(k)}$ that is analogous to the role the Schur functions have in Λ. It was shown that these functions form a basis for $\Lambda_t^{(k)}$ and that they reduce to the Schur functions themselves when $k \to \infty$. That is, $s_\lambda^{(k)}[X;t] = s_\lambda[X]$ when $h_M(\lambda) \le k$. Several other properties supporting the claim that the k-Schur functions generalize the theory of Schur functions are given in [5, 6].

A continuation of the study in Section 4 regarding the action of operators indexed by k-rectangles led to an observation that not all of the k-Schur functions need to be constructed using Definition 21. We found that for each k, there is a subset of $s_\lambda^{(k)}[X;t]$ called the irreducible k-Schur functions, from which all other k-Schur functions may be constructed by simply applying a succession of operators indexed by k-rectangles. This subset consists of the special set of k-Schur functions indexed by irreducible partitions; k-bounded partitions with no more than i parts equal to $k-i$, for $i = 0, \ldots, k-1$.

Definition 22. *A k-Schur function indexed by an irreducible partition is said to be a irreducible. Otherwise, the k-Schur function is called reducible.*

For example, the irreducible k-Schur functions for $k = 1, 2, 3$ are

$$\begin{aligned} k=1: &\quad s_0^{(1)}\,, \\ k=2: &\quad s_0^{(2)}\,,\quad s_1^{(2)}\,, \\ k=3: &\quad s_0^{(3)}\,,\quad s_1^{(3)}\,,\quad s_2^{(3)}\,,\quad s_{1,1}^{(3)}\,,\quad s_{2,1}^{(3)}\,,\quad s_{2,1,1}^{(3)}\,. \end{aligned} \tag{5.3}$$

These examples support the following property;

Property 23. [5] *There are $k!$ distinct k-irreducible partitions.*

The main result of this section is to prove that an operator indexed by a k-rectangle R sends, up to a constant, $s_\lambda^{(k)}[X;t]$ to $s_{R\cup\lambda}^{(k)}[X;t]$. Consequently, given the $k!$ irreducible k-Schur functions, any reducible k-Schur function can be obtained by applying a sequence of operators indexed by k-rectangles to the appropriate irreducible k-Schur function. We first give several properties of the operators B_λ and $T_j^{(k)}$ and then, using our results from Section 4, we will prove our main result.

Property 24. [6] *If λ is a partition with $h_M(\lambda) \le k$ then $T_{\lambda_1}^{(k)} B_\lambda f = T_{\lambda_1}^{(k)} B_{\lambda_1} B_{\hat\lambda} f$ for all $f \in \Lambda_t^{(k)}$.*

Property 25. *If $k \ge j > \ell$ are nonnegative integers, $T_j^{(k)} B_{\ell^{k+1-\ell}} f = B_{\ell^{k+1-\ell}} T_j^{(k)} f$ for all $f \in \Lambda_t^{(k)}$.*

Proof. It suffices to consider $f = G_\lambda^{(k)}[X;t]$. By the definition of $T_j^{(k)}$, we have

$$B_{\ell^{k+1-\ell}}\, T_j^{(k)}\, f = \begin{cases} 0 & \text{if } j \neq \lambda_1 \\ B_{\ell^{k+1-\ell}}\, f & \text{if } j = \lambda_1 \end{cases}. \tag{5.4}$$

On the other hand, consider $T_j^{(k)} B_{\ell^{k+1-\ell}} f$. By Theorem 20, $B_{\ell^{k+1-\ell}}\, f \in \Omega^{(k)}_{\max\{\ell,\lambda_1\}}$. If $j = \lambda_1 > \ell$, then $B_{\ell^{k+1-\ell}}\, f \in \Omega_j^{(k)} \implies T_j^{(k)} B_{\ell^{k+1-\ell}}\, f = B_{\ell^{k+1-\ell}}\, f$. If $j \neq \lambda_1$ then either $B_{\ell^{k+1-\ell}}\, f \in \Omega^{(k)}_{\ell<j}$ or $B_{\ell^{k+1-\ell}}\, f \in \Omega^{(k)}_{\lambda_1 \neq j}$. Both cases vanish under the action of $T_j^{(k)}$ and thus we prove our claim. □

We can now show that an operator indexed by a k-rectangle acts simply on a k-Schur function.

Theorem 26. *If μ, ν, λ are partitions where $\lambda = (\mu, \nu)$ and $\mu_L > \ell \ge \nu_1$, then*

$$B_{\ell^{k+1-\ell}}\, s_\lambda^{(k)}[X;t] = t^{|\mu| - \ell(\mu)\ell}\, s^{(k)}_{(\ell^{k+1-\ell})\cup\lambda}[X;t]\,. \tag{5.5}$$

Proof. Let $\ell(\mu) = M$. Since $\lambda_M = \mu_L > \ell$ and $\lambda_{M+1} = \nu_1 \le \ell$, we have

$$B_{\ell^{k+1-\ell}}\, s_\lambda^{(k)}[X;t] = t^{\lambda_1+\cdots+\lambda_M - M\ell} T_{\lambda_1}^{(k)} B_{\lambda_1} T_{\lambda_2}^{(k)} B_{\lambda_2} \ldots T_{\lambda_M}^{(k)} B_{\lambda_M} B_{\ell^{k+1-\ell}}\, s_\nu^{(k)}[X;t]\,, \tag{5.6}$$

by iterating the following argument M times: If $\hat\lambda = (\lambda_2, \ldots, \lambda_M)$, by definition of $s_\lambda^{(k)}$ we have

$$B_{\ell^{k+1-\ell}}\, s_\lambda^{(k)}[X;t] = B_{\ell^{k+1-\ell}}\, T_{\lambda_1}^{(k)}\, B_{\lambda_1}\, s_{\hat\lambda}^{(k)}[X;t]\,. \tag{5.7}$$

Since $s_{\hat\lambda}^{(k)} \in \Lambda_t^{(k)}$, Property 15 gives that $B_{\lambda_1} s_{\hat\lambda}^{(k)} \in \Lambda_t^{(k)}$. Property 25 then implies that $T_{\lambda_1}^{(k)}$ commutes with $B_{\ell^{k-\ell+1}}$ since $\lambda_1 > \ell$. We thus have

$$B_{\ell^{k+1-\ell}}\, s_\lambda^{(k)}[X;t] = T_{\lambda_1}^{(k)} B_{\ell^{k+1-\ell}} B_{\lambda_1}\, s_{\hat\lambda}^{(k)}[X;t]\,. \tag{5.8}$$

Furthermore, $B_{\ell^{k-\ell+1}}$ t-commutes with B_{λ_1} by Identity 6. Therefore,

$$B_{\ell^{k+1-\ell}}\, s_\lambda^{(k)}[X;t] = t^{\lambda_1-\ell} T_{\lambda_1}^{(k)} B_{\lambda_1} B_{\ell^{k+1-\ell}}\, s_{\hat\lambda}^{(k)}[X;t]\,. \tag{5.9}$$

Now if we can show, for a partition ν with $\nu_1 \le \ell$, that

$$B_{\ell^{k+1-\ell}}\, s_\nu^{(k)}[X;t] = \left(T_\ell^{(k)}\, B_\ell\right)^{k+1-\ell}\, s_\nu^{(k)}[X;t]\,, \tag{5.10}$$

then putting this into (5.6) implies our result by definition, since $\lambda_M > \ell \ge \lambda_{M+1}$. To prove (5.10), we have $s_\nu^{(k)} \in \Omega_{\nu_1}^{(k)}$ by definition, which implies $B_{\ell^{k+1-\ell}} s_\nu^{(k)} \in \Omega_\ell^{(k)}$ by Theorem 20 since $\nu_1 \le \ell$. Therefore, $B_{\ell^{k+1-\ell}} s_\nu^{(k)}$ is invariant under $T_\ell^{(k)}$ and (5.10) is equivalent to

$$T_\ell^{(k)} B_{\ell^{k+1-\ell}} s_\nu^{(k)}[X;t] = \left(T_\ell^{(k)}\, B_\ell\right)^{k+1-\ell}\, s_\nu^{(k)}[X;t]\,. \tag{5.11}$$

Using Property 24, the right hand side may be written

$$T_\ell^{(k)}\, B_{\ell^{k+1-\ell}}\, s_\nu^{(k)}[X;t] = T_\ell^{(k)}\, B_\ell\, B_{\ell^{k-\ell}}\, s_\nu^{(k)}[X;t]\,. \tag{5.12}$$

Now, $B_{\ell^{k-\ell}} s_\nu^{(k)}[X;t] \in \Lambda_t^{(\ell,k)}$ by Property 17. Further, since any element of $\Lambda_t^{(\ell,k)}$ can be decomposed into the sum of two functions, $f \in \Omega_\ell^{(k)}$ and $g \in \Lambda_t^{(\ell+1,k)}$, we have $B_{\ell^{k-\ell}}\, s_\nu^{(k)}[X;t] = f + g$. Therefore,

$$T_\ell^{(k)}\, B_\ell\, (f+g) = T_\ell^{(k)}\, B_\ell\, T_\ell^{(k)}\, f = T_\ell^{(k)}\, B_\ell\, T_\ell^{(k)}\, (f+g) \tag{5.13}$$

since $B_\ell \cdot g \in \Lambda_t^{(\ell+1,k)}$ by Property 18 implies $T_\ell^{(k)} B_\ell \cdot g = 0$. Thus, (5.12) gives

$$T_\ell^{(k)}\, B_{\ell^{k+1-\ell}}\, s_\nu^{(k)}[X;t] = T_\ell^{(k)} B_\ell\, T_\ell^{(k)} B_{\ell^{k-\ell}}\, s_\nu^{(k)}[X;t]\,. \tag{5.14}$$

Repeating this argument $k - \ell$ times proves (5.10). □

When $t = 1$, our theorem reduces to the simple expression:

Corollary 27. *If λ is a k-bounded partition, then*

$$s_{\ell^{k+1-\ell}}[X]\, s_\lambda^{(k)}[X] = s_{(\ell^{k+1-\ell})\cup\lambda}^{(k)}[X]\,. \tag{5.15}$$

The role of Schur functions indexed by k-rectangles in the subring $\Lambda^{(k)}$ leads naturally to the study of the quotient ring $\Lambda^{(k)}/\mathcal{I}_k$, where $\mathcal{I}_k$ denotes the ideal generated by $s_{(\ell^{k+1-\ell})}[X]$. It is known that the dimension of this quotient ring is $k!$ since

Proposition 28. [5] *The homogeneous functions indexed by k-irreducible partitions form a basis of the quotient ring $\Lambda^{(k)}/\mathcal{I}_k$.*

Corollary 27 then implies

Theorem 29. *The irreducible k-Schur functions form a basis of the quotient ring $\Lambda^{(k)}/\mathcal{I}_k$.*

The irreducible k-Schur function basis thus offers a simple method of performing operations in this quotient ring: first work in $\Lambda^{(k)}$ using k-Schur functions and then replace by zero all the k-Schur functions indexed by partitions which are not k-irreducible.

6. Appendix

Formula 30. *Let $\mu, \lambda \in \mathcal{P}^m$. The Kostka matrix is such that*

$$K^{-1}_{\lambda\mu} = K^{-1}_{\lambda+a^m,\mu+a^m} \,. \tag{6.1}$$

Proof. We have

$$\begin{aligned} m_{\lambda+a^m}[x_1+\cdots+x_m] &= (x_1\cdots x_m)^a\, m_\lambda\,[x_1+\cdots+x_m] \\ &= (x_1\cdots x_m)^a \sum_\mu K^{-1}_{\lambda\mu}\, s_\mu\,[x_1+\cdots+x_m] \\ &= \sum_\mu K^{-1}_{\lambda\mu}\, s_{\mu+a^m}\,[x_1+\cdots+x_m]\,, \end{aligned} \tag{6.2}$$

since $(x_1\cdots x_m)^a\, s_\mu\,[x_1+\cdots+x_m] = s_{\mu+a^m}\,[x_1+\cdots+x_m]$. By definition

$$m_{\lambda+a^m}[x_1+\cdots+x_m] = \sum_\nu K^{-1}_{\lambda+a^m,\nu}\, s_\nu\,[x_1+\cdots+x_m]\,. \tag{6.3}$$

Taking the coefficient of $s_{\mu+a^m}$ in (6.2) and (6.3), we get $K^{-1}_{\lambda\mu} = K^{-1}_{\lambda+a^m,\mu+a^m}$, as claimed. $\square$

Formula 31. *Let λ be a partition of at most m parts, and let $a \geq \lambda_1$. Then*

$$s_\lambda\left[\frac{1}{x_1}+\cdots+\frac{1}{x_m}\right](x_1\cdots x_m)^a = s_{a^m-\lambda^R}[x_1+\cdots+x_m]\,. \tag{6.4}$$

Proof. In [7] ($Sf5$), one finds

$$\begin{aligned} s_\lambda\left[\frac{1}{x_1}+\cdots+\frac{1}{x_m}\right] &= s_{(a^m/\lambda')'}[x_1+\cdots+x_m]/s_{1^m}[x_1+\cdots+x_m]^a \\ &= s_{(a^m/\lambda')'}[x_1+\cdots+x_m]/(x_1\cdots x_m)^a\,. \end{aligned} \tag{6.5}$$

Following from the Jacobi-Trudi determinantal expression for skew Schur functions, we have

$$\begin{aligned} s_{\mu/\lambda} = \det\left|s_{\mu_i-\lambda_j-i+j}\right|_{1\leq i,j\leq m} &= \det\left|s_{\mu_{n+1-j}-\lambda_{n+1-i}-(n+1-j)+(n+1-i)}\right|_{1\leq i,j\leq m} \\ &= \det\left|s_{\mu_{n+1-j}-\lambda_{n+1-i}-i+j}\right|_{1\leq i,j\leq m} \qquad . \end{aligned} \tag{6.6}$$

Therefore, if $\mu = (a^m)$, we obtain

$$s_{a^m/\lambda} = \det\left|s_{a-\lambda_{n+1-i}-i+j}\right|_{1\leq i,j\leq m} = s_{a^m-\lambda^R}\,, \tag{6.7}$$

completing the proof. $\square$

Formula 32. *Let $\mu, \lambda \in \mathcal{P}^m$, and let a be an integer such that $a \geq \lambda_1$ and $a \geq \mu_1$. Then, the Kostka matrix is such that*

$$K^{-1}_{\lambda\mu} = K^{-1}_{a^m-\lambda^R,a^m-\mu^R}\,. \tag{6.8}$$

Proof. Using Formula 31, we have

$$\begin{aligned} m_{a^m-\lambda^R}[x_1+\cdots+x_m] &= (x_1\cdots x_m)^a\, m_\lambda\left[\frac{1}{x_1}+\cdots+\frac{1}{x_m}\right] \\ &= (x_1\cdots x_m)^a \sum_\mu K^{-1}_{\lambda\mu}\, s_\mu\left[\frac{1}{x_1}+\cdots+\frac{1}{x_m}\right] \\ &= \sum_\mu K^{-1}_{\lambda\mu}\, s_{a^m-\mu^R}\,[x_1+\cdots+x_m] \end{aligned} \tag{6.9}$$

Since, by definition,

$$m_{a^m-\lambda^R}[x_1+\cdots+x_m]=\sum_{\nu} K^{-1}_{a^m-\lambda^R,\nu}\, s_\nu\, [x_1+\cdots+x_m]\ , \tag{6.10}$$

taking the coefficient of $s_{a^m-\mu^R}$ in (6.9) and (6.10), we get $K^{-1}_{\lambda\mu}=K^{-1}_{a^m-\lambda^R,a^m-\mu^R}$, as claimed. □

Formula 33. *If* $\lambda=(\lambda_1,\ldots,\lambda_n)$, *then*

$$m_\lambda=\sum_{\sigma\in S_n;\sigma(\lambda)\ \text{distinct}} x^{\sigma(\lambda)}=\sum_{\sigma\in S_n;\sigma(\lambda)\ \text{distinct}} s_{\sigma(\lambda)} \tag{6.11}$$

Proof. The formula holds if and only if

$$\sum_{\sigma\in S_n} x^{\sigma(\lambda)}=\sum_{\sigma\in S_n} s_{\sigma(\lambda)}\ , \tag{6.12}$$

since summing over all elements of S_n adds the same symmetry factor on each side of the equation. If we insert

$$s_{\sigma(\lambda)}=\frac{\sum_{w\in S_n}\epsilon(w)\,x^{w(\sigma(\lambda)+\delta)}}{\sum_{w\in S_n}\epsilon(w)\,x^{w(\delta)}}\ , \tag{6.13}$$

where $\epsilon(w)$ is the sign of the permutation w, we get

$$\sum_{\sigma,w\in S_n}\epsilon(w)\,x^{\sigma(\lambda)+w(\delta)}=\sum_{\sigma,w\in S_n}\epsilon(w)\,x^{w(\sigma(\lambda)+\delta)}\ . \tag{6.14}$$

Now, letting $w\sigma=\sigma'$ in the RHS of this equation, we obtain

$$\sum_{\sigma,w\in S_n}\epsilon(w)x^{\sigma(\lambda)+w(\delta)}=\sum_{\sigma',w\in S_n}\epsilon(w)x^{\sigma'(\lambda)+w(\delta)}\ , \tag{6.15}$$

which proves the formula. □

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