QUANTUM COHOMOLOGY AND THE k-SCHUR BASIS

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Abstract. We prove that structure constants related to Hecke algebras at roots of unity are special cases of k-Littlewood-Richardson coefficients associated to a product of k-Schur functions. As a consequence, both the 3-point Gromov-Witten invariants appearing in the quantum cohomology of the Grassmannian, and the fusion coefficients for the WZW conformal field theories associated to $\widehat{su}(\ell)$ are shown to be k-Littlewood Richardson coefficients. From this, Mark Shimozono conjectured that the k-Schur functions form the Schubert basis for the homology of the loop Grassmannian, whereas k-Schur coproducts correspond to the integral cohomology of the loop Grassmannian. We introduce dual k-Schur functions defined on weights of k-tableaux that, given Shimozono's conjecture, form the Schubert basis for the cohomology of the loop Grassmannian. We derive several properties of these functions that extend those of skew Schur functions.

1. Introduction

The study of Macdonald polynomials led to the discovery of symmetric functions, $s_\lambda^{(k)}$, indexed by partitions whose first part is no larger than a fixed integer $k \geq 1$. Experimentation suggested that these functions play the fundamental combinatorial role of the Schur basis in the symmetric function subspace $\Lambda^k = \mathbb{Z}[h_1, \ldots, h_k]$; that is, they satisfy properties generalizing classical properties of Schur functions such as Pieri and Littlewood-Richardson rules. The study of the $s_\lambda^{(k)}$ led to several different characterizations [15, 16, 19] (conjecturally equivalent) and to the proof of many of these combinatorial conjectures. We thus generically call the functions k-Schur functions, but in this article consider only the definition presented in [19].

Although prior work with k-Schur functions concentrated on proving that they act as the "Schur basis" for Λ^k, the analogy was so striking that it seemed likely to extend beyond combinatorics to fields such as algebraic geometry and representation theory. Our main finding in this direction is that the k-Schur functions are connected to representations of Hecke algebras $H_\infty(q)$, where q is a root of unity, and they provide the natural basis for work in the quantum cohomology of the Grassmannian just as the Schur functions do for the usual cohomology. In particular, the 3-point Gromov-Witten invariants are none other than relevant cases of "k-Littlewood-Richardson coefficients", the expansion coefficients in

$$s_\lambda^{(k)} s_\mu^{(k)} = \sum_{\nu:\nu_1 \leq k} c_{\lambda\mu}^{\nu,k} s_\nu^{(k)} \,. \tag{1}$$

To be precise, in Schubert calculus, the cohomology ring of the Grassmannian $Gr_{\ell n}$ (the manifold of ℓ-dimensional subspaces of $\mathbb{C}^n$) has a basis given by Schubert cells σ_λ that are indexed by partitions $\lambda \in \mathcal{P}^{\ell n}$ that fit inside an $\ell \times (n-\ell)$ rectangle.

There is an isomorphism,

$$H^*(Gr_{\ell n}) \cong \Lambda^\ell / \langle e_{n-\ell+1}, \ldots, e_n \rangle \,,$$

where the Schur function s_λ maps to the Schubert class σ_λ when $\lambda \in \mathcal{P}^{\ell n}$. Since s_λ is zero modulo the ideal when $\lambda \notin \mathcal{P}^{\ell n}$, the structure constants of $H^*(Gr_{\ell n})$ in the basis of Schubert classes:

$$\sigma_\lambda \sigma_\mu = \sum_{\nu \in \mathcal{P}^{\ell n}} c_{\lambda\mu}^{\nu} \sigma_\nu \,,$$

can be obtained from the Littlewood-Richardson coefficients for Schur functions,

$$s_\lambda s_\mu = \sum_{\nu \in \mathcal{P}^{\ell n}} c_{\lambda\mu}^{\nu} s_\nu + \sum_{\nu \notin \mathcal{P}^{\ell n}} c_{\lambda\mu}^{\nu} s_\nu \,,$$

which have well known combinatorial interpretations.

The small quantum cohomology ring of the Grassmannian $QH^*(Gr_{\ell n})$ is a deformation of the usual cohomology that has become the object of much recent attention. As a linear space, this is the tensor product $H^*(Gr_{\ell n}) \otimes \mathbb{Z}[q]$ and the σ_λ with $\lambda \in \mathcal{P}^{\ell n}$ form a $\mathbb{Z}[q]$-linear basis of $QH^*(Gr_{\ell n})$. Multiplication is a q-deformation of the product in $H^*(Gr_{\ell n})$, defined by

$$\sigma_\lambda * \sigma_\mu = \sum_{\substack{\nu \in \mathcal{P}^{\ell n} \\ |\nu| = |\lambda| + |\mu| - dn}} q^d C_{\lambda\mu}^{\nu,d} \sigma_\nu \,.$$

The $C_{\lambda\mu}^{\nu,d}$ are the 3-point Gromov-Witten invariants, which count the number of certain rational curves of degree d. Finding a combinatorial interpretation for these constants is an interesting open problem that would have applications to many areas, including the study of the Verlinde fusion algebra [25] as well as the computation of certain knot invariants [27].

As with the usual cohomology, quantum cohomology can be connected to symmetric functions by:

$$QH^*(Gr_{\ell n}) \cong \left(\Lambda^\ell \otimes \mathbb{Z}[q] \right) / J_q^{\ell n} \,,$$

where $J_q^{\ell n} = \langle e_{n-\ell+1}, \ldots, e_{n-1}, e_n + (-1)^\ell q \rangle$. When $\lambda \in \mathcal{P}^{\ell n}$, the Schubert class σ_λ still maps to the Schur function s_λ, but unfortunately when $\lambda \notin \mathcal{P}^{\ell n}$, there exist s_λ that are not zero modulo the ideal. Thus, the Schur functions cannot be used to directly obtain the quantum structure constants. Instead, these Gromov-Witten invariants arise as the expansion coefficients in

$$s_\lambda \, s_\mu = \sum_{\substack{\nu \in \mathcal{P}^{\ell n} \\ |\nu| = |\lambda| + |\mu| - dn}} q^d C_{\lambda\mu}^{\nu,d} s_\nu \mod J_q^{\ell n} \,,$$

and to compute the coefficients, an algorithm involving negatives [28, 10, 8] must be used to reduce a Schur function modulo the ideal $J_q^{\ell n}$.

Remarkably, by first working with an ideal that arises in the context of Hecke algebras at roots of unity, we find that the k-Schur functions circumvent this problem: a k-Schur function maps to a single Schur function times a q power (with no negatives) or to zero, modulo the ideal. To be more precise, let $I^{\ell n}$ denote the ideal

$$I^{\ell n} = \left\langle s_\lambda \,\middle|\, \#\{ j \mid \lambda_j < \ell \} = n - \ell + 1 \right\rangle .$$

A basis for $\Lambda^\ell/I^{\ell n}$ is given by the Schur functions indexed by partitions in $\Pi^{\ell n}$, the set of partitions with no part larger than ℓ and no more than $n-\ell$ rows of length smaller than ℓ. In [9], certain structure constants associated to representations of Hecke algebras at roots of unity are shown to be the expansion coefficients in

$$s_\lambda\, s_\mu = \sum_{\nu\in\Pi^{\ell n}} a_{\lambda\mu}^{\nu} s_\nu \mod I^{\ell n}\,.$$

We prove that the $a_{\lambda\mu}^{\nu}$ are just special cases of k-Littlewood-Richardson coefficients by showing that when $\nu\in\Pi^{\ell n}$, the k-Schur function $s_\nu^{(k=n-1)}$ modulo the ideal $I^{\ell n}$ is simply s_ν, and is zero otherwise. Thus it is revealed that the $a_{\lambda\mu}^{\nu}$ are coefficients in the expansion:

$$s_\lambda^{(k)}\, s_\mu^{(k)} = \sum_{\nu\in\Pi^{\ell n}} a_{\lambda\mu}^{\nu}\, s_\nu^{(k)} + \sum_{\nu\notin\Pi^{\ell n}} c_{\lambda\mu}^{\nu,k} s_\nu^{(k)}\,.$$

We can then obtain the 3-point Gromov-Witten invariants from this result by simply computing s_ν modulo $J^q_{\ell n}$ for $\nu\in\Pi^{\ell n}$, since $I^{\ell n}$ is a subideal of $J^q_{\ell n}$. In this case, s_ν beautifully reduces to positive $s_{\mathfrak{r}(\nu)}$ times a q power, where $\mathfrak{r}(\nu)$ is the n-core of ν. Consequently, we prove that the 3-point Gromov-Witten invariants are none other than certain k-Schur function Littlewood Richardson coefficients. To be more specific,

$$C_{\lambda\mu}^{\nu,d} = c_{\lambda\mu}^{\hat\nu,n-1}\,,$$

where the value of d associates a unique element $\hat\nu\in\Pi^{\ell n}$ (given explicitly in Theorem 17) to each $\nu\in\mathcal{P}^{\ell n}$.

It also follows from our results that the k-Littlewood-Richardson coefficients include the fusion rules for the Wess-Zumino-Witten conformal field theories associated to $\widehat{su}(\ell)$ at level $n-\ell$, since the algorithm given by Kac [10] and Walton [28] for computing in the fusion algebra reduces to the one given by Goodman and Wenzl [9] for computing the Hecke algebra structure constants.

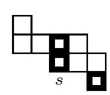
It is important to note that since the Gromov-Witten invariants under consideration are indexed by partitions fitting inside a rectangle, they are given by only a subset of the k-Littlewood-Richardson coefficients. We thus naturally sought the larger picture that would be explained by the complete set of k-Littlewood Richardson coefficients. In discussion with Mark Shimozono about this problem, he conjectured that the k-Schur functions form the Schubert basis for the homology of the loop Grassmannian, and that the k-Schur expansion coefficients of the k-Schur coproduct give the integral cohomology of the loop Grassmannian. Here we introduce a family of functions dual to the k-Schur functions, defined by the weight of certain "k-tableaux" related to the affine symmetric group [18]. Following the theory of skew Schur functions, we prove a number of results about these dual k-Schur functions including that their symmetry relies on a generalization [19] of the Bender-Knuth involution [5]. In particular, we show that the coefficients in a product of dual k-Schur functions are the structure constants in the k-Schur coproduct, implying from Shimozono's conjecture that the dual k-Schur functions form the Schubert basis for the cohomology of the loop Grassmannian.

In addition to finding a combinatorial interpretation for the 3-point Gromov-Witten invariants using the k-Schur functions as a guide, there are a number of other open problems that arise from this work. For example, the conjecture that the (dual) k-Schur functions are the Schubert basis for the (cohomology) homology of the loop Grassmannian is intriguing. Furthermore, our results strongly support the idea that the (dual) k-Schur functions provide the symmetric Grassmannian component of a larger family of "affine Schubert polynomials", first suggested by Michelle Wachs. After discussion with Thomas Lam of the work presented here, he made a beautiful step in this direction by introducing a natural family of "affine Stanley symmetric functions" that reduce in special cases to the dual k-Schur functions (called "affine Schur functions" in [14]). Details of a connection between the dual k-Schur functions and the cylindric Schur functions of [21] is also carried out in [14].

2. DEFINITIONS

Let Λ denote the ring of symmetric functions, generated by the elementary symmetric functions $e_r = \sum_{i_1<\ldots<i_r} x_{i_1}\cdots x_{i_r}$, or equivalently by the complete symmetric functions $h_r = \sum_{i_1\le\ldots\le i_r} x_{i_1}\cdots x_{i_r}$, and let $\Lambda^k = \mathbb{Z}[h_1,\ldots,h_k]$. Bases for Λ are indexed by partitions $\lambda = (\lambda_1 \ge \cdots \ge \lambda_m > 0)$ whose degree λ is $|\lambda| = \lambda_1+\cdots+\lambda_m$ and whose length $\ell(\lambda)$ is the number of parts m. Each partition λ has an associated Ferrers diagram with λ_i lattice squares in the i^{th} row, from the bottom to top. Any lattice square in the Ferrers diagram is called a cell, where the cell (i,j) is in the ith row and jth column of the diagram. Given a partition λ, its conjugate λ' is the diagram obtained by reflecting λ about the main diagonal. A partition λ is "*k-bounded*" if $\lambda_1 \le k$ and the set of all such partitions is denoted $\mathcal{P}^k$. The set $\mathcal{P}^{\ell n}$ is the partitions fitting inside an $\ell \times (n-\ell)$ rectangle (with $n-\ell$ rows of size ℓ). We say that $\lambda \subseteq \mu$ when $\lambda_i \le \mu_i$ for all i. Dominance order $\trianglerighteq$ on partitions is defined by $\lambda \trianglerighteq \mu$ when $\lambda_1+\cdots+\lambda_i \ge \mu_1+\cdots+\mu_i$ for all i, and $|\lambda| = |\mu|$.

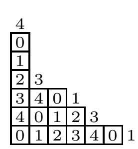
More generally, for $\rho \subseteq \gamma$, the skew shape γ/ρ is identified with its diagram $\{(i,j) : \rho_i < j \le \gamma_i\}$. Lattice squares that do not lie in γ/ρ will be simply called "*squares*". We say that any $c \in \rho$ lies "*below* " γ/ρ. The "*hook* " of any lattice square $s \in \gamma$ is defined as the collection of cells of γ/ρ that lie inside the L with s as its corner. This is intended to apply to all $s \in \gamma$ including those below γ/ρ. For example, the hook of $s = (1,3)$ is depicted by the framed cells:

$$\gamma/\rho \;=\; (5,5,4,1)/(4,2) \;=$$ 

. (2)

The *"hook-length"* of s, $h_s(\gamma/\rho)$, is the number of cells in the hook of s. In the preceding example, $h_{(1,3)}\big((5,5,4,1)/(4,2)\big) = 3$ and $h_{(3,2)}\big((5,5,4,1)/(4,2)\big) = 3$. A cell or square has a k-bounded hook if it's hook-length is no larger than k.

A *"p-core"* is a partition that does not contain any hooks of length p, and $\mathcal{C}^p$ will denote the set of all p-cores. The "*p-residue*" of square (i,j) is $j-i \mod p$; that is, the label of this square when squares are periodically labeled with $0,1,\ldots,p-1$, where zeros lie on the main diagonal (see [13] for more on cores and residues). The 5-residues associated to the 5-core $(6,4,3,1,1,1)$ are



A *"tableau"* is a filling of a Ferrers shape with integers that strictly increase in columns and weakly increase in rows. The *"weight"* of a given tableau is the composition α where α_i is the multiplicity of i in the tableau. A "*Schur function*" can be defined by

$$s_\lambda = \sum_T x^T \,, \tag{3}$$

where the sum is over all tableaux of shape λ, and where $x^T = x^{\mathrm{weight}(T)}$.

3. k-Schur functions

There are several conjecturally equivalent characterizations for the k-Schur functions. Here we use the definition explored in [19] that relies on a family of tableaux related to the affine symmetric group.

Definition 1. [18] *Let γ be a $k+1$-core, m be the number of k-bounded hooks of γ, and $\alpha = (\alpha_1, \ldots, \alpha_r)$ be a composition of m. A "k-tableau" of shape γ and "k-weight" α is a filling of γ with integers $1, 2, \ldots, r$ such that*
(i) rows are weakly increasing and columns are strictly increasing
(ii) the collection of cells filled with letter i are labeled with exactly α_i distinct $k+1$-residues.

Example 2. *The 3-tableaux of 3-weight $(1,3,1,2,1,1)$ and shape $(8,5,2,1)$ are:*

$$\begin{array}{llllllll} 5 \\ 4&6 \\ 2&3&4&4&6 \\ 1&2&2&2&3&4&4&6 \end{array} \qquad \begin{array}{llllllll} 6 \\ 4&5 \\ 2&3&4&4&5 \\ 1&2&2&2&3&4&4&5 \end{array} \qquad \begin{array}{llllllll} 4 \\ 3&6 \\ 2&4&4&5&6 \\ 1&2&2&2&4&4&5&6 \end{array} \tag{4}$$

More generally, the skew k-tableau of shape $\mathfrak{c}(\nu)/\mathfrak{c}(\mu)$ and k-weight that is a composition of $|\nu/\mu|$ is well-defined for any $\nu, \mu \in \mathcal{P}^k$ with $\mu \subseteq \nu$ since $\mu \subseteq \nu$ implies that $\mathfrak{c}(\mu) \subseteq \mathfrak{c}(\nu)$ (e.g.[18] Prop. 14).

Although a k-tableau is associated to a shape γ and weight α, in contrast to usual tableaux, $|\alpha|$ does not equal $|\gamma|$. Instead, $|\alpha|$ is the number of k-bounded hooks in γ. This distinction becomes natural through a correspondence between $k+1$-cores and k-bounded diagrams. This bijection between $\mathcal{C}^{k+1}$ and $\mathcal{P}^k$ was defined in [18] by the map

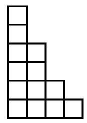
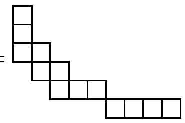
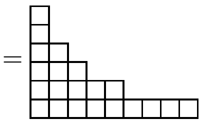
$$\mathfrak{c}^{-1}(\gamma) = (\lambda_1, \ldots, \lambda_\ell)$$

where λ_i is the number of cells with a k-bounded hook in row i of γ. Note that the number of k-bounded hooks in γ is $|\lambda|$. The inverse map relies on constructing a certain "k-skew diagram" $\lambda/^k = \gamma/\rho$ from λ, and setting $\mathfrak{c}(\lambda) = \gamma$. These special skew diagrams are defined:

Definition 3. *For $\lambda \in \mathcal{P}^k$, the "$k$-skew diagram of λ" is the diagram $\lambda/^k$ where*
(i) row i has length λ_i for $i = 1, \ldots, \ell(\lambda)$
(ii) no cell of $\lambda/^k$ has hook-length exceeding k
(iii) all squares below $\lambda/^k$ have hook-length exceeding k.

A convenient algorithm for constructing the diagram of $\lambda/^k$ is given by successively attaching a row of length λ_i to the bottom of $(\lambda_1, \ldots, \lambda_{i-1})/^k$ in the leftmost position so that no hook-lengths exceeding k are created.

Example 4. *Given* $\lambda = (4,3,2,2,1,1)$ *and* $k=4$,

$\lambda =$  $\Longrightarrow$ $\lambda/^4 =$  $\Longrightarrow$ $\mathfrak{c}(\lambda) =$ 

The analogy with usual tableaux is now more apparent, and we let $\mathcal{T}^k_\alpha(\mu)$ denote the set of all k-tableaux of shape $\mathfrak{c}(\mu)$ and k-weight α. When the k-weight is (1^n), a k-tableau is called *"standard"*. The "k-Kostka numbers" $K^{(k)}_{\mu\alpha} = |T^k_\alpha(\mu)|$ satisfy a triangularity property [18] similar to that of the Kostka numbers: for k-bounded partitions λ and μ,

$$K^{(k)}_{\mu\lambda} = 0 \quad \text{when} \quad \mu \not\trianglerighteq \lambda \quad \text{and} \quad K^{(k)}_{\mu\mu} = 1\,. \tag{5}$$

Given this triangularity, the inverse of $||K^{(k)}_{\mu\lambda}||_{\lambda,\mu\in\mathcal{P}^k}$ exists. Our main object of study can now be defined by $||K^{(k)}||^{-1}$, denoted $||\bar{K}^{(k)}||$.

Definition 5. *For any* $\lambda \in \mathcal{P}^k$*, the "k-Schur function" is defined*

$$s^{(k)}_\lambda = \sum_{\mu \trianglerighteq \lambda} \bar{K}^{(k)}_{\mu\lambda} h_\mu\,. \tag{6}$$

A number of properties held by k-Schur functions suggest that these elements play the role of the Schur functions in the subspace Λ^k. First, the definition implies that the set $\left\{s^{(k)}_\lambda\right\}_{\lambda_1\le k}$ forms a basis of Λ^k, and that for any $\lambda \in \mathcal{P}^k$,

$$h_\lambda = \sum_{\mu \trianglerighteq \lambda} K^{(k)}_{\mu\lambda} s^{(k)}_\mu\,. \tag{7}$$

In [19] it was shown that these functions satisfy the "k-Pieri formula": for $\nu_1, \ell \le k$,

$$h_\ell\, s^{(k)}_\nu = \sum_{\lambda \in H^{(k)}_{\nu,\ell}} s^{(k)}_\lambda \tag{8}$$

where the sum is over partitions of the form:

$$H^{(k)}_{\nu,\ell} = \left\{\lambda \,\middle|\, \lambda/\nu = \text{horizontal } \ell\text{-strip} \quad \text{and} \quad \lambda^{\omega_k}/\nu^{\omega_k} = \text{vertical } \ell\text{-strip}\right\}\,.$$

More generally, if $K^{(k)}_{\nu/\mu,\lambda}$ is the number of skew tableaux of shape $\mathfrak{c}(\nu)/\mathfrak{c}(\mu)$ and k-weight λ, then

$$h_\lambda s^{(k)}_\mu = \sum_\nu K^{(k)}_{\nu/\mu,\lambda}\, s^{(k)}_\nu\,. \tag{9}$$

The k-Schur functions also naturally play the Schur function role when acted on by the ω-involution, defined as the homomorphism $\omega(h_i) = e_i$. In particular, ω maps a Schur function s_λ to it's conjugate $s_{\lambda'}$. Using a refinement of partition conjugation that arose in [15, 16], it was shown in [19] that

$$\omega s^{(k)}_\lambda = s^{(k)}_{\lambda^{\omega_k}}\,, \tag{10}$$

where $\lambda^{\omega_k} = \mathfrak{c}^{-1}\big(\mathfrak{c}(\lambda)'\big)$ is the "k-conjugate" of λ. From (10), it can be shown that

$$s^{(k)}_\lambda = s_\lambda \text{ when } h(\lambda) \le k\,. \tag{11}$$

In the spirit of Schur function theory, it is conjectured [15, 16, 19] that the "k-Littlewood-Richardson coefficients" in

$$s_\lambda^{(k)} s_\mu^{(k)} = \sum_{\nu:\nu_1\le k} c_{\lambda\mu}^{\nu,k} s_\nu^{(k)}\,, \tag{12}$$

are positive numbers. Our development here will prove that in certain cases, these coefficients are the Gromov-Witten invariants thus proving positivity in these cases. Note that given the action of the ω involution on k-Schur functions, the k-Littlewood-Richardson coefficients satisfy

$$c_{\lambda\mu}^{\nu,k} = c_{\lambda^{\omega_k}\mu^{\omega_k}}^{\nu^{\omega_k},k}\,. \tag{13}$$

4. Hecke algebras, fusion rules, and the k-Schur functions

The generalized Littlewood-Richardson coefficients for (ℓ,n)-representations of the Hecke algebras $H_\infty(q)$ when q is an n-th root of unity [9] are equivalent to the structure constants for the Verlinde (fusion) algebra associated to the $\widehat{su}(\ell)$-Wess-Zumino-Witten conformal field theories at level $n-\ell$. In this section, we will use the k-Pieri rule to establish that for $k=n-1$, the k-Littlewood-Richardson coefficients contain these constants as special cases.

4.1. **The connection.** From [9], we recall a simple interpretation for these "(ℓ,n)-Littlewood-Richardson coefficients" given in the language of symmetric functions. For $n>\ell\ge 1$, consider the quotient $R^{\ell n}=\Lambda^\ell/I^{\ell n}$ where $I^{\ell n}$ is the ideal generated by Schur functions that have exactly $n-\ell+1$ rows of length smaller than ℓ:

$$I^{\ell n} = \Big\langle s_\lambda \,\Big|\, \#\{j \mid \lambda_j<\ell\} = n-\ell+1 \Big\rangle\,.$$

A basis for $R^{\ell n}$ is given by the set $\{s_\lambda\}_{\lambda\in\Pi^{\ell n}}$ where the indices are partitions in:

$$\Pi^{\ell n} = \{\lambda\in\mathcal{P} : \lambda_1\le \ell \text{ and } \#\{j\mid\lambda_j<\ell\}\le n-\ell\}\,.$$

The (ℓ,n)-Littlewood-Richardson coefficients of interest here are simply $a_{\lambda\mu}^\nu$ in

$$s_\lambda s_\mu = \sum_\nu a_{\lambda\mu}^\nu\, s_\nu \mod I^{\ell n}\,,\quad \text{where } \lambda,\mu,\nu\in\Pi^{\ell n}. \tag{14}$$

It is in this context that we prove the coefficients $a_{\lambda\mu}^\nu$ are none other than k-Littlewood-Richardson coefficients when $k=n-1$.

Remark 6. The results of [9] are presented in a transposed form, where they instead work with the ideal $\langle s_\lambda \mid \lambda_1-\lambda_\ell = n-\ell+1\rangle$ in $\mathbb{Z}[e_1,\ldots,e_l]$. Their (ℓ,n)-Littlewood-Richardson coefficients $d_{\lambda\mu}^\nu$ are our $a_{\lambda'\mu'}^{\nu'}$, for $\lambda',\mu',\nu'\in\Pi^{\ell n}$.

To provide some insight into how this connection arose, consider the special case of Eq. (14) with $\lambda=(1)$:

$$s_1 s_\mu = \sum_{\substack{\nu\,:\,\mu\subset\nu\in\Pi^{\ell n}\\ |\nu|=|\mu|+1}} s_\nu \mod I^{\ell n}\,, \tag{15}$$

and define a poset by letting $\mu \prec\!\cdot\, \nu$ for all ν in the summand. Frank Sottile brought this poset to our attention and asked if it was related to our study [18] of the k-Young lattice Y^k. Y^k is defined by the k-Pieri rule, where $\mu <\!\cdot\, \nu$ when $\nu\in H_{\mu,1}^{(k)}$. Investigating his question, we discovered the posets can be connected through the principal order ideal $L^k(\ell,m)$ generated by an $\ell\times m$ rectangle in Y^k. In [17], we found that the vertices of $L^k(\ell,m)$ are the partitions contained in an $\ell\times m$ rectangle with no more than $k-\ell+1$ rows shorter than k, and that μ covers λ in this poset if and only if $\lambda\subseteq\mu$ and $|\lambda|+1=|\mu|$. Therefore, the elements of $L^k(\ell,\infty)$ are precisely those of $\Pi^{\ell n}$ (given $k=n-1$). Since the k-Young lattice was defined by multiplication by s_1, we have

$$s_1 s_\lambda^{(k)} = \sum_{\substack{\mu\,:\,\lambda\subset\mu\in\Pi^{\ell n}\\ |\mu|=|\lambda|+1}} s_\mu^{(k)} + \text{ other terms}\,, \tag{16}$$

where "other terms" are k-Schur functions indexed by $\mu\notin\Pi^{\ell n}$. The likeness of (15) and (16) led us to surmise the following result:

Theorem 7. *For any partition $\lambda\in\mathcal{P}^{n-1}$,*

$$s_\lambda^{(n-1)} \mod I^{\ell n} = \begin{cases} s_\lambda & \text{if } \lambda\in\Pi^{\ell n}\\ 0 & \text{otherwise}\end{cases} \tag{17}$$

Before proving this theorem, we mention several implications. Since all partitions in $\Pi^{\ell n}$ are $(n-1)$-bounded ($\lambda_1\le\ell\le n-1$), the set of k-Schur functions indexed by partitions in $\Pi^{\ell n}$ forms a natural basis for the quotient $R^{\ell n}$. Computation modulo the ideal $I^{\ell n}$ is trivial in this basis. In particular, the structure constants under consideration are simply certain k-Littlewood Richardson coefficients.

Corollary 8. *For all $\lambda,\mu,\nu\in\Pi^{\ell n}$,*

$$a_{\lambda\mu}^{\nu} = c_{\lambda\mu}^{\nu,n-1}\,.$$

Another consequence of our theorem produces a tableau interpretation for the dimension of the representations $\pi_\lambda^{(\ell,n)}$, for $\lambda'\in\Pi^{\ell n}$, of the Hecke algebras $H_\infty(q)$, when q is an n-th root of unity (see [9] for details on these representations).

Corollary 9. *For $\lambda'\in\Pi^{\ell n}$, the dimension of the representation $\pi_\lambda^{(\ell,n)}$ is the number of standard $(n-1)$-tableaux of shape $\mathfrak{c}(\lambda')$*[1].

Proof. Let $m=|\lambda|$, and $k=n-1$. In [9], it is shown that the dimension of $\pi_\lambda^{(\ell,n)}$ is the coefficient of $s_{\lambda'}$ in $s_1^m \mod I^{\ell n}$. By Theorem 7, this is the coefficient of $s_{\lambda'}^{(k)}$ in the k-Schur expansion of $s_1^m=h_{1^m}$. Using Definition 5 for k-Schur functions, this coefficient is $K_{\lambda' 1^m}^{(k)}$, or the number of standard k-tableaux of shape $\mathfrak{c}(\lambda')$. □

The Verlinde (fusion) algebra of the Wess-Zumino-Witten model associated to $\widehat{sl}(\ell)$ at level $n-\ell$ is isomorphic to the quotient of $R^{\ell n}$ modulo the single relation $s_\ell\equiv 1$ [10, 28, 9]. The fusion coefficient $\mathcal{N}_{\lambda\mu}^{\nu}$ is defined for $\lambda',\mu',\nu'\in\mathcal{P}^{\ell-1,n-1}$ by

$$L(\lambda)\otimes_{n-\ell} L(\mu) = \oplus\mathcal{N}_{\lambda\mu}^{\nu} L(\nu)\,,$$

where the fusion product $\otimes_{n-\ell}$ is the reduction of the tensor product of integrable representations with highest weight λ and μ via the representation at level $n-\ell$ of $\widehat{sl}(\ell)$. Thus, our results imply that

Corollary 10. *For all λ,μ,ν inside an $(n-\ell)\times(\ell-1)$ rectangle,*

$$\mathcal{N}_{\lambda\mu}^{\nu} = c_{\lambda'\mu'}^{\hat\nu,n-1}\,,$$

where $\hat\nu=(\ell^{(|\lambda|+|\mu|-|\nu|)/\ell},\nu')$.

[1] Equivalently, this is the number of reduced words for a certain affine permutation $\sigma_{\lambda'}\in\hat{S}_n/S_n$. See [18] for the precise correspondence.

4.2. **Proof of the connection.** To prove Theorem 7, we use two preliminary properties. For simplicity, since $\Lambda^\ell = \Lambda/\langle h_{\ell+1}, h_{\ell+2}, \dots \rangle$, we will instead work with the ideal $\mathcal{I}$ in Λ, where

$$\mathcal{I} = \big\langle s_\lambda \,\big|\, \#\{j \,|\, \lambda_j < \ell\} = n - \ell + 1 \big\rangle \cup \big\langle h_i \,\big|\, i > \ell \big\rangle ,$$

and in the remainder of this section, k will always stand for $n-1$.

Property 11. *For any k-bounded partition λ and $\ell \le k$, $s_\lambda^{(k)} \equiv_{\mathcal{I}} 0$ when $\lambda_1 > \ell$.*

Proof. Since $\mu \ge \lambda$ implies that $\mu_1 \ge \lambda_1$, the unitriangular relation between $\{s_\lambda^{(k)}\}$ and $\{h_\lambda\}$ implies

$$s_\lambda^{(k)} = \sum_{\mu : \mu_1 > \ell} * \, h_\mu \,.$$

The claim thus follows since $h_\mu \in \mathcal{I}$ when $\mu_1 > \ell$. □

Property 12. *For any k-bounded partition λ with $\lambda_1 \le \ell$,*

$$s_\lambda^{(k)} \equiv_{\mathcal{I}} 0 \implies s_{(\ell^m,\lambda)}^{(k)} \equiv_{\mathcal{I}} 0 \quad \textit{for all } \; m \ge 0 \,. \tag{18}$$

Proof. The k-Pieri rule (8) implies in particular, that any k-Schur occurring in the expansion of $h_\ell s_\nu^{(k)}$ is indexed by a partition obtained by adding a horizontal ℓ-strip to ν. Thus, when $\ell \ge \nu_1$, we have

$$h_\ell s_\nu^{(k)} = s_{(\ell,\nu)}^{(k)} \; + \sum_{\substack{\mu : \mu_1 > \ell \\ \mu \in H^k_{\nu,\ell}}} s_\mu^{(k)} \,. \tag{19}$$

Starting from $s_\lambda^{(k)} \equiv_{\mathcal{I}} 0$, and assuming by induction that $s_{(\ell^{m-1},\lambda)}^{(k)} \equiv_{\mathcal{I}} 0$, the claim follows from Property 11 and the previous expression (19),

$$0 \; \equiv_{\mathcal{I}} \; h_\ell \, s_{(\ell^{m-1},\lambda)}^{(k)} \; = \; s_{(\ell^m,\lambda)}^{(k)} + \sum_{\gamma \,:\, \gamma_1 > \ell} * \, s_\gamma^{(k)} \; \equiv_{\mathcal{I}} \; s_{(\ell^m,\lambda)}^{(k)} \,.$$

□

4.3. **Proof of Theorem 7.** Recall $n = k+1$, and that $\lambda \in \Pi^{\ell,k+1}$ has the form $\lambda = (\ell^m, \mu)$ for some $\mu \in \mathcal{P}^{\ell-1\, k}$. First, by induction on m we prove that $s_\lambda^{(k)} \equiv_{\mathcal{I}} s_\lambda$ for each such λ. Since $h(\lambda) \le k$ when $m = 0$, $s_\lambda^{(k)} = s_\lambda$ by (11). By induction, assuming $s_{(\ell^m,\mu)}^{(k)} \equiv_{\mathcal{I}} s_{(\ell^m,\mu)}$, we have $h_\ell \, s_{(\ell^m,\mu)}^{(k)} \equiv_{\mathcal{I}} h_\ell \, s_{(\ell^m,\mu)}$. On the other hand, since $s_\gamma \equiv_{\mathcal{I}} 0$ when $\gamma_1 > \ell$, Identity (19) implies $h_\ell \, s_{(\ell^m,\mu)}^{(k)} \equiv_{\mathcal{I}} s_{(\ell^{m+1},\mu)}^{(k)}$. Therefore,

$$h_\ell \, s_{(\ell^m,\mu)} \equiv_{\mathcal{I}} s_{(\ell^{m+1},\mu)}^{(k)} \,.$$

The claim then follows by noting that the Pieri rule gives an expansion similar to (19) for $h_\ell \, s_{\ell^m,\mu}$, implying that $h_\ell \, s_{(\ell^m,\mu)} \equiv_{\mathcal{I}} s_{(\ell^{m+1},\mu)}$.

It remains to prove that $s_\eta^{(k)} \equiv_{\mathcal{I}} 0$ when $\eta \notin \Pi^{\ell,k+1}$. Since Property 11 proves the case when $\eta_1 > \ell$, we must show $s_\eta^{(k)} \equiv_{\mathcal{I}} 0$ for any η in the set:

$$\mathcal{Q} = \big\{ (\ell^m, \beta) \in \mathcal{P} : \beta_1 < \ell \text{ and } \ell(\beta) \ge k - \ell + 2 \big\} \,.$$

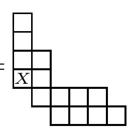
Our proof is inductive, using an order defined on $\mathcal{Q}$ as follows: $\eta = (\ell^a, \beta) \preceq (\ell^b, \alpha) = \mu$ if $\ell(\beta) < \ell(\alpha)$ or if $\ell(\beta) = \ell(\alpha)$ and $\eta \trianglerighteq \mu$ (this is a well-ordering if we restrict ourselves to $|\mu| = |\eta|$). Our base case includes partitions $\eta = (\ell^a, \beta)$ with $\beta_1 < \ell$ and $\ell(\beta) = k - \ell + 2$. In this case, $h(\beta) \le k$ implies $s_\beta^{(k)} = s_\beta$ from (11), and since $s_\beta \in \mathcal{I}$ when β has $k-\ell+2$ parts smaller than ℓ, we have $s_\beta^{(k)} \equiv_{\mathcal{I}} 0$. Property 12 then proves $s_\eta^{(k)} \equiv_{\mathcal{I}} 0$ in this case.

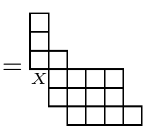
Now assume by induction that $s_\eta^{(k)} \equiv_{\mathcal{I}} 0$ for all $\eta \in \mathcal{Q}$ such that $\eta \prec \mu$, where $\mu = (\ell^b, \alpha)$ with $\alpha_1 < \ell$ and $\ell(\alpha) > k-\ell+2$. With $r < \ell$ denoting the last part of μ (and thus also the last part of α), let $\mu = (\hat\mu, r) = (\ell^b, \hat\alpha, r)$ and note that $\hat\mu \prec \mu$. Thus, using the induction hypothesis and the k-Pieri rule, we have

$$0 \equiv_{\mathcal{I}} \; s_r s_{\hat\mu}^{(k)} = s_\mu^{(k)} + \sum_{\nu \in H_{\hat\mu,r}^{(k)} \setminus \{\mu\}} s_\nu^{(k)} \,,$$

and it suffices to show that $s_\nu^{(k)} \equiv_{\mathcal{I}} 0$ for all $\nu \in H_{\hat\mu,r}^{(k)} \setminus \{\mu\}$. Property 11 proves this immediately for any ν with $\nu_1 > \ell$, and thus we shall consider only ℓ-bounded ν. Two properties of such ν follow since ν is obtained by adding a horizontal r-strip to $\hat\mu = (\ell^b, \hat\alpha)$: $\nu \rhd \mu$, and $\nu = (\ell^b, \beta)$, where $\ell(\beta) \leq \ell(\hat\alpha)+1 = \ell(\alpha)$. Thus, if these ν lie in $\mathcal{Q}$, then $\nu \prec \mu$ and our claim follows from the induction hypothesis. Since each such ν is obtained by adding a horizontal strip to $\hat\mu = (\ell^b, \hat\alpha)$, and $\ell(\alpha) > k-\ell+2$, we have $\ell(\beta) \geq \ell(\hat\alpha) \geq k-\ell+2$. Thus, these $\nu = (\ell^b, \beta)$ all lie in $\mathcal{Q}$ except in the case that $\ell(\beta) = \ell(\hat\alpha) = k-\ell+2$ and $\beta_1 = \ell$. The following paragraph explains why, in this case, $\nu \notin H_{\hat\mu,r}^{(k)}$, and thus never arise.

Given $\ell(\beta) = k-\ell+2$ and $\beta_1 = \ell$, $h(\beta) > k$ implies there is no cell in position $X = (1, \ell(\nu) - \ell(\beta))$ of $\nu/^k$. Assume by contradiction that $\nu \in H_{\hat\mu,r}^{(k)}$ – hence, in particular, that $\nu^{\omega_k}/\hat\mu^{\omega_k}$ is a vertical strip. Since $\ell(\hat\alpha) = \ell(\beta) = k-\ell+2$ and $\hat\alpha_1 < \ell$ imply $h(\hat\alpha) \leq k$, there is a cell in $\hat\mu/^k$ in position $(1, \ell(\mu) - \ell(\hat\alpha)) = X$. Since the height of $\nu/^k$ and $\hat\mu/^k$ are equal, but position X is empty in $\nu/^k$ and filled in $\hat\mu/^k$, the first column of $\nu/^k$ is shorter than that of $\hat\mu/^k$, implying $\nu^{\omega_k}/\hat\mu^{\omega_k}$ is not a vertical strip. By contradiction, $\nu \notin H_{\hat\mu,r}^{(k)}$ as claimed. Here is an example with $\hat\mu = (4,4,2,2,1,1)$, $\nu = (4,4,4,2,1,1)$, $n=4$ and $k=6$:

$\hat\mu/^k =$ 

$\nu/^k =$ 

□

5. Quantum cohomology

Witten [29] proved that the Verlinde algebra of $\widehat{u}(\ell)$ at level $n-\ell$ and the quantum cohomology of the Grassmannian $Gr_{\ell n}$ are isomorphic (see also [1]). Since $u(\ell) = su(\ell) \times u(1)$, the connection between k-Schur functions and the fusion coefficients of $\widehat{su}(\ell)$ at level $n-\ell$ given in the last section implies that there is also a connection between k-Schur functions and the quantum cohomology of the Grassmannian. We now set out to make this connection explicit.

Recall from the introduction that the quantum structure constants, or 3-point Gromov-Witten invariants $C_{\lambda\mu}^{\nu,d}$, arise in the expansion, for $\lambda, \mu \in \mathcal{P}^{\ell n}$,

$$s_\lambda \, s_\mu = \sum_{\substack{d \geq 0,\, \nu \in \mathcal{P}^{\ell n} \\ |\nu| = |\lambda| + |\mu| - dn}} q^d \, C_{\lambda\mu}^{\nu,d} \, s_\nu \mod J_q^{\ell n} \,, \tag{20}$$

where

$$J_q^{\ell n} = \langle e_{n-\ell+1}, \ldots, e_{n-1}, e_n + (-1)^\ell q \rangle \,.$$

Our main goal is to prove that the k-Schur function basis gives a direct route to these constants. In particular, by determining the value of a k-Schur function modulo this ideal, we will see that the Gromov-Witten invariants arise as special cases of the k-Littlewood-Richardson coefficients.

Again for simplicity, we work in $\Lambda/\mathcal{J}_q$, where $\mathcal{J}_q$ is the ideal

$$\mathcal{J}_q = J_q^{\ell n} \cup \langle h_i \,|\, i > \ell \rangle \,.$$

Theorem 7 reveals that a k-Schur function modulo the ideal $\mathcal{I}$ is a Schur function when $\lambda \in \Pi^{\ell n}$ and is otherwise zero. By showing that $\mathcal{I}$ is a subideal of $\mathcal{J}_q$, our task to determine a k-Schur function mod $\mathcal{J}_q$ is thus reduced to examining what happens to a usual Schur function s_λ mod $\mathcal{J}_q$ in the special case that $\lambda \in \Pi^{\ell n}$.

Proposition 13. *If $f \in \mathcal{I}$, then $f \in \mathcal{J}_q$.*

Proof. It suffice to prove that $s_\lambda \in \mathcal{J}_q$ when $\lambda = (\ell^m, \alpha)$, for some m and partition α such that $\alpha_1 < \ell$ and $\ell(\alpha) = n - \ell + 1$. When $m = 0$, the result follows from the Jacobi-Trudi determinantal formula since the first row of the determinant of s_α has entries $e_{n-\ell+1}, \ldots, e_{n+\alpha_1-\ell} \in \mathcal{J}_q$ given $\alpha_1 < \ell$. Assuming by induction that $s_{(\ell^m,\alpha)} \in \mathcal{J}_q$, since the Pieri rule implies $h_\ell s_{(\ell^m,\alpha)} = s_{(\ell^{m+1},\alpha)}$ mod $\langle h_{\ell+1}, h_{\ell+2}, \ldots \rangle$, the result follows by induction. $\square$

Now to determine the value of a usual Schur function s_λ mod $\mathcal{J}_q$ for partitions in $\Pi^{\ell n}$, we shall use an important result from [4], where the theory of rim-hooks was used to study the Schur functions modulo $\mathcal{J}_q$. To state their result, we first recall the necessary definitions. An *"n-rim hook"* is a connected skew diagram of size n that contains no 2×2 rectangle. "$\mathfrak{r}(\lambda)$" denotes the n-core of λ, obtained by removing as many n-rim hooks as possible from the diagram of λ (this is well-defined since the order in which rims are removed is known to be irrelevant [13]). The width of a rim hook is the number of columns it occupies minus one. Given a partition λ, let "d_λ" be the number of n-rim hooks that are removed to obtain $\mathfrak{r}(\lambda)$. Also, let "ϵ_λ" equal $d_\lambda(\ell-1)$ minus the sum of the widths of these rim hooks. This given, in [4] Eq. (19) it is shown for $\lambda \in \mathcal{P}^\ell$, that

$$s_\lambda \mod J_q^{\ell n} = \begin{cases} (-1)^{\epsilon_\lambda} q^{d_\lambda} s_{\mathfrak{r}(\lambda)} & \text{if } \mathfrak{r}(\lambda) \in \mathcal{P}^{\ell n} \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

This result helps us prove that in the special case that $\lambda \in \Pi^{\ell n}$, $s_\lambda \equiv_{\mathcal{J}_q} q^{d_\lambda} s_\nu$ for a partition ν obtained using the following operators:

Definition 14. *For $\lambda \in \Pi^{\ell n}$, $\blacktriangle(\lambda)$ is the partition obtained by adding an n-rim hook to λ starting in column ℓ and ending in the first column. For $\lambda \in \Pi^{\ell n}$ that is not an n-core, $\blacktriangledown(\lambda)$ is the partition obtained by removing an n-rim hook from λ starting in the first column of λ.*

Note that $\blacktriangledown$ is well-defined since when $\lambda \in \Pi^{\ell n}$ is not an n-core, $\ell(\lambda) > n - \ell$. Thus for $r = \ell(\lambda) - (n-\ell)$, $\lambda_r = \ell$ and $h_{(r,1)}(\lambda) = n$ implying an n-rim hook can be removed starting in the first column of λ and ending in the last column ℓ. Since the difference between the heights of the starting point and the ending point is $n - \ell$,

$$\blacktriangledown : \left\{ \lambda \,|\, \lambda \in \Pi^{\ell n} \;\&\; \lambda \neq n\text{-core} \right\} \;\to\; \Pi^{\ell n} \,. \tag{22}$$

Similarly, for any $\lambda \in \Pi^{\ell n}$, the difference between the heights of the first column and column ℓ is at most $n-\ell$. Thus, an n-rim hook can be added to λ starting from column ℓ and ending in the first column. Since the difference in heights of the starting point and ending point of the added n-rim hook is $n-\ell$, we have that

$$\blacktriangle : \Pi^{\ell n} \to \Pi^{\ell n} \,. \tag{23}$$

By construction, as long as $\blacktriangledown(\lambda)$ is defined, we have

$$\blacktriangledown(\blacktriangle(\lambda)) = \lambda \quad \text{and} \quad \blacktriangle(\blacktriangledown(\lambda)) = \lambda \,. \tag{24}$$

Proposition 15. *For $\lambda \in \Pi^{\ell n}$,*

$$s_\lambda \equiv q^{d_\lambda}\, s_\nu \mod \mathcal{J}_q \,, \tag{25}$$

where $\nu = \blacktriangledown^{d_\lambda}(\lambda) \in \mathcal{P}^{\ell n}$.

Proof. We have $s_\lambda \equiv_{\mathcal{J}_q} (-1)^{\epsilon_\lambda} q^{d_\lambda}\, s_{\mathfrak{r}(\lambda)}$ by (21). When λ is an n-core, then $\lambda \in \Pi^{\ell n}$ implies $\lambda \in \mathcal{P}_{\ell n}$. Thus $\mathfrak{r}(\lambda) = \lambda$ and (25) holds with $d_\lambda = 0$. Otherwise, $\mathfrak{r}(\lambda)$ is obtained by removing d_λ n-rim hooks in any order. Thus, by successively applying $\blacktriangledown$, we obtain $\mathfrak{r}(\lambda) = \blacktriangledown^{d_\lambda}(\lambda)$. Since $\blacktriangledown$ preserves $\Pi^{\ell n}$ by (22), $\mathfrak{r}(\lambda) \in \mathcal{P}^{\ell n}$. Further, $\epsilon_\lambda = d_\lambda(\ell-1) - d_\lambda(\ell-1) = 0$ since each removed n-rim hook has width $\ell - 1$. □

In this notation, we can now determine the value of a k-Schur function mod $\mathcal{J}_q$.

Theorem 16. *For any k-bounded partition λ,*

$$s_\lambda^{(n-1)} \mod \mathcal{J}_q = \begin{cases} q^{d_\lambda} s_\nu & \text{if } \lambda \in \Pi^{\ell n} \\ 0 & \text{otherwise} \end{cases}$$

where $\nu = \mathfrak{r}(\lambda) = \blacktriangledown^{d_\lambda}(\lambda) \in \mathcal{P}^{\ell n}$.

Proof. Proposition 13 gives that $\mathcal{I}$ is a subideal of $\mathcal{J}_q$, implying

$$s_\lambda^{(n-1)} \mod \mathcal{J}_q = \left(s_\lambda^{(n-1)} \mod \mathcal{I} \right) \mod \mathcal{J}_q \,.$$

For $\lambda \notin \Pi^{\ell n}$, $s_\lambda^{(n-1)} \mod \mathcal{I} = 0$ by Theorem 7. For $\lambda \in \Pi^{\ell n}$, Theorem 7 implies that $s_\lambda^{(n-1)} \mod \mathcal{I} = s_\lambda$, and the claim then follows by further moding out by $\mathcal{J}_q$ according to Proposition 15. □

This theorem enables us to connect the quantum product to the product of k-Schur functions.

Theorem 17. *For $\lambda, \mu, \nu \in \mathcal{P}^{\ell n}$, the 3-point Gromov-Witten invariants $C_{\lambda\mu}^{\nu,d}$ are*

$$C_{\lambda\mu}^{\nu,d} = c_{\lambda\mu}^{\hat\nu,n-1} \,, \tag{26}$$

where $\hat\nu = \blacktriangle^{d}(\nu)$, and where $c_{\lambda\mu}^{\hat\nu,n-1}$ is a k-Littlewood-Richardson coefficient.

Proof. For $\lambda, \mu \in \mathcal{P}^{\ell n}$, (20) shows that $C_{\lambda\mu}^{\nu,d}$ arise in the expansion

$$s_\lambda\, s_\mu \equiv \sum_{\substack{d \geq 0,\, \nu \in \mathcal{P}^{\ell n} \\ |\nu| = |\lambda| + |\mu| - dn}} C_{\lambda\mu}^{\nu,d}\, q^d\, s_\nu \mod \mathcal{J}_q \,. \tag{27}$$

On the other hand, since $\lambda, \mu \in \mathcal{P}^{\ell n}$ have hook-length smaller than n, (11) implies that $s_\lambda^{(n-1)} s_\mu^{(n-1)} = s_\lambda s_\mu$. Therefore, applying Proposition 15 to the k-Schur expansion of this product gives

$$s_\lambda \, s_\mu = \sum_{\gamma : |\gamma| = |\lambda| + |\mu|} c_{\lambda\mu}^{\gamma, n-1} s_\gamma^{(n-1)} \equiv \sum_{\gamma \in \Pi^{\ell n}} c_{\lambda\mu}^{\gamma, n-1} \, q^{d_\gamma} \, s_\beta \mod \mathcal{J}_q \,, \tag{28}$$

where $\beta = \blacktriangledown^{d_\gamma}(\gamma)$. Taking the coefficient of $q^d s_\nu$ in (27) and (28) implies

$$C_{\lambda\mu}^{\nu,d} = \sum_{\substack{\gamma \in \Pi^{\ell n} \\ \gamma \,:\, \nu = \blacktriangledown^d(\gamma)}} c_{\lambda\mu}^{\gamma, n-1} \,.$$

Since $\nu = \blacktriangledown^d(\gamma) \in \mathcal{P}^{\ell n} \subseteq \Pi^{\ell n}$, we can apply $\blacktriangle$ to find there is a unique γ in the right summand. That is, $\blacktriangle^d(\nu) = \gamma$ by (24). $\square$

It is important to note that the quantum structure constants $C_{\lambda\mu}^{\nu,d}$ are indexed by $\lambda, \mu, \nu \in \mathcal{P}^{\ell n}$. We have now seen that these numbers are precisely k-Littlewood-Richardson coefficients in the relevant cases. However, since there are far more k-Littlewood-Richardson coefficients than Gromov-Witten invariants we naturally sought the larger geometric picture that would be explained by the complete set of k-Littlewood Richardson coefficients. In discussions with Mark Shimozono about this problem, he conjectured that the k-Schur functions form the Schubert basis for the homology of the loop Grassmannian and that the expansion coefficients of the coproduct of k-Schur functions in terms of k-Schur functions gives the integral cohomology of the loop Grassmannian (see e.g. [7, 11] for more studies of the loop Grassmannian). This conjecture is supported by extensive computer data. Corollary 17 provides further evidence for this assertion based on the existence [22] of a surjective ring homomorphism from the homology of the loop Grassmannian onto the quantum cohomology of the Grassmannian at $q = 1$.

6. Dual k-schur functions

While the homology of the loop Grassmannian is isomorphic to Λ^k, the cohomology is isomorphic to $\Lambda/\mathfrak{J}^{(k)}$ for the ideal

$$\mathfrak{J}^{(k)} = \langle m_\lambda : \lambda_1 > k \rangle \,.$$

The interplay between homology and cohomology suggests that there is a fundamental basis for $\Lambda/\mathfrak{J}^{(k)}$ that is closely tied to the k-Schur basis. Here, we introduce a family of functions defined by the k-weight of k-tableaux and derive a number of properties including a duality relation to the k-Schur functions. In particular, it will develop that if the coproduct of k-Schur functions in terms of k-Schur functions indeed gives the integral cohomology of the loop Grassmannian, then these functions are the Schubert basis for the cohomology of the loop Grassmannian.

Recall that a Schur function can be defined as

$$s_\lambda = \sum_T x^T \,, \tag{29}$$

where the sum is over all tableaux of shape λ. Several weight-permuting involutions have been defined on the set of tableaux such as the Bender-Knuth involution [5]. From this, the Schur functions are combinatorially proven to be symmetric. We extend these ideas by considering the family of functions that arises similarly from the set of k-tableaux (defined in § 3 with k-weight).

Definition 18. *For any $\lambda \in \mathcal{P}^k$, the "dual k-Schur function" is defined by*

$$\mathfrak{S}_\lambda^{(k)} = \sum_T x^T \,, \tag{30}$$

where the sum is over all k-tableaux of shape $\mathfrak{c}(\lambda)$, and $x^T = x^{k\text{-}weight(T)}$.

We will see that these functions are a basis that is dual to the k-Schur functions, but we first show that a generalization of the Bender-Knuth involution from [19] implies $\mathfrak{S}_\lambda^{(k)}$ is a symmetric function.

Proposition 19. *For any k-bounded partition λ, $\mathfrak{S}_\lambda^{(k)}$ is a symmetric function.*

Proof. For a composition γ (allowing zeros), the coefficient of x^γ in $\mathfrak{S}_\lambda^{(k)}$ is the number of k-tableaux of k-weight γ. The result then follows from the involution on k-tableaux defined in §7 (see Corollary 35). □

Since the k-Kostka number $K_{\lambda\alpha}^{(k)}$ denotes the number of k-tableaux of shape $\mathfrak{c}(\lambda)$ and k-weight α, the symmetry of dual k-Schur functions implies that

$$\mathfrak{S}_\lambda^{(k)} = \sum_\mu K_{\lambda\mu}^{(k)} m_\mu \,.$$

Further, by the unitriangularity of k-Kostka numbers (5), we have a monomial expansion of dual k-Schur functions that can be seen as an alternative definition for these functions:

Proposition 20. *For any k-bounded partition λ,*

$$\mathfrak{S}_\lambda^{(k)} = m_\lambda + \sum_{\mu \lhd \lambda} K_{\lambda\mu}^{(k)} m_\mu \,. \tag{31}$$

This proposition reveals that the dual k-Schur functions are a basis for the quotient of the symmetric function space by the ideal $\mathfrak{I}^{(k)}$:

Proposition 21. *The dual k-Schur functions form a basis of $\Lambda/\mathfrak{I}^{(k)}$.*

Recall that the k-Schur functions form a basis for $\Lambda/\langle h_i \,|\, i > k\rangle$. The ideal $\mathfrak{I}^{(k)}$ is dual to $\langle h_i \,|\, i > k\rangle$ with respect to the scalar product defined on Λ by

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu} \,.$$

Since the definition of k-Schur function, $s_\lambda^{(k)} = \sum_\nu \bar{K}_{\nu\lambda}^{(k)} \, h_\nu$, implies that

$$\langle s_\lambda^{(k)}, \mathfrak{S}_\mu^{(k)} \rangle = \Big\langle \sum_\alpha \bar{K}_{\alpha\lambda}^{(k)} h_\alpha, \sum_\beta K_{\mu\beta}^{(k)} m_\beta \Big\rangle = \sum_\alpha K_{\mu\alpha}^{(k)} \, \bar{K}_{\alpha\lambda}^{(k)} = \delta_{\lambda\mu} \,,$$

as suggested by their name, the dual k-Schur basis is dual to the k-Schur basis.

Proposition 22. *Let λ and μ be k-bounded partition. Then,*

$$\langle s_\lambda^{(k)}, \mathfrak{S}_\mu^{(k)} \rangle = \delta_{\lambda\mu}$$

We can extract several combinatorial properties for dual k-Schur functions from the k-Schur function properties using duality and the following lemma.

Lemma 23. *Let $f \in \Lambda^k$. Then, for $g \in \Lambda$, we have*

$$\langle f \,,\, g \rangle = \langle f \,,\, g \mod \mathfrak{J}^{(k)} \rangle \,.$$

Proof. It suffices to consider $f = h_\lambda$, with $\lambda \in \mathcal{P}^k$. If $A \in \mathfrak{J}^{(k)}$, then $A = \sum_\mu a_\mu \, m_\mu$ summing over $\mu \notin \mathcal{P}^k$. Thus, $\langle h_\lambda \,,\, A \rangle = 0$ and the claim follows. □

Since the ω-involution is an isometry with respect to $\langle \cdot, \cdot \rangle$, we discover from the action $\omega s_\mu^{(k)} = s_{\mu^{\omega_k}}^{(k)}$ (10) that ω acts naturally on the dual k-Schur functions.

Proposition 24. *Let λ be a k-bounded partition. Then*

$$\omega \left(\mathfrak{S}_\lambda^{(k)} \right) \mod \mathfrak{J}^{(k)} = \mathfrak{S}_{\lambda^{\omega_k}}^{(k)} \,.$$

From the k-Pieri formula (8), with $f^\perp$ defined for any $f \in \Lambda$ by $\langle g, f^\perp h \rangle = \langle f\, g, h \rangle$, we find

Proposition 25.

$$h_\ell^\perp \, \mathfrak{S}_\nu^{(k)} = \sum_{\lambda \in \bar{H}_{\nu,\ell}^{(k)}} \mathfrak{S}_\lambda^{(k)} \,, \tag{32}$$

where the sum is over partitions of the form:

$$\bar{H}_{\nu,\ell}^{(k)} = \left\{ \lambda \,\middle|\, \nu/\lambda = \text{horizontal } \ell\text{-strip} \quad \text{and} \quad \nu^{\omega_k}/\lambda^{\omega_k} = \text{vertical } \ell\text{-strip} \right\} \,.$$

As with the Schur functions, the definition of $\mathfrak{S}_\lambda^{(k)}$ makes sense if λ is replaced by a skew diagram.

Definition 26. *For any k-bounded partitions $\mu \subseteq \nu$, the "dual skew k-Schur function" is defined by*

$$\mathfrak{S}_{\nu/\mu}^{(k)} = \sum_{\mathcal{T}} x^{k\text{-}weight\,(\mathcal{T})} \,, \tag{33}$$

where the sum is over all skew k-tableaux of shape $\mathfrak{c}(\nu)/\mathfrak{c}(\mu)$.

Since a k-tableau can be obtained from a skew k-tableau $\mathcal{T}$ by filling the shape $\mathfrak{c}(\mu)$ below the skew diagram with a k-tableau, using letters smaller than those of $\mathcal{T}$, our involution given in §7 extends to skew k-tableau. Thus, $\mathfrak{S}_{\nu/\mu}^{(k)}$ is still a symmetric function by the same reasoning as Proposition 19. Since $K_{\nu/\mu,\lambda}^{(k)}$ denotes the number of skew k-tableaux of k-weight λ and shape $\mathfrak{c}(\nu)/\mathfrak{c}(\lambda)$, we have the expansion:

$$\mathfrak{S}_{\nu/\mu}^{(k)} = \sum_\lambda K_{\nu/\mu,\lambda}^{(k)} \, m_\lambda \,. \tag{34}$$

Notice $\mathfrak{S}_{\nu/\mu}^{(k)} \in \Lambda/\mathfrak{J}^{(k)}$ since the sum is over k-bounded partitions λ (a given letter cannot have k-weight larger than k). This form of the skew dual k-Schur function makes it clear that the "skew affine Schur functions" of [14] are the same functions.

Following the theory of usual skew Schur functions, we derive an implicit formula for the skew dual functions in terms of dual functions involving k-Littlewood-Richardson coefficients.

Theorem 27. *For any k-bounded partitions $\mu \subseteq \nu$,*

$$\mathfrak{S}^{(k)}_{\nu/\mu} = \sum_\lambda c^{\nu,k}_{\mu\lambda} \mathfrak{S}^{(k)}_\lambda \,.$$

Proof. Since $\mathfrak{S}^{(k)}_{\nu/\mu}$ lies in $\Lambda/\mathfrak{J}^{(k)}$, for which the dual k-Schur functions form a basis,

$$\mathfrak{S}^{(k)}_{\nu/\mu} = \sum_\lambda A^{\nu,k}_{\mu\lambda} \mathfrak{S}^{(k)}_\lambda \,,$$

for some $A^{\nu,k}_{\mu\lambda}$. On one hand consider:

$$\langle s^{(k)}_\lambda, \mathfrak{S}^{(k)}_{\nu/\mu} \rangle = \langle s^{(k)}_\lambda, \sum_\alpha A^{\nu,k}_{\mu\alpha} \mathfrak{S}^{(k)}_\alpha \rangle = A^{\nu,k}_{\mu\lambda}$$

and

$$\langle s^{(k)}_\lambda, \mathfrak{S}^{(k)}_{\nu/\mu} \rangle = \langle \sum_\alpha \bar{K}^{(k)}_{\alpha\lambda} h_\alpha, \sum_\beta K^{(k)}_{\nu/\mu,\beta} m_\beta \rangle = \sum_\alpha \bar{K}^{(k)}_{\alpha\lambda} K^{(k)}_{\nu/\mu,\alpha} \,.$$

On the other hand, since (9) tells us $h_\lambda \, s^{(k)}_\mu = \sum_\nu K^{(k)}_{\nu/\mu,\lambda} \, s^{(k)}_\nu$, we have

$$\begin{aligned}
\langle s^{(k)}_\mu s^{(k)}_\lambda, \mathfrak{S}^{(k)}_\nu \rangle &= \langle \sum_\alpha \bar{K}^{(k)}_{\alpha\lambda} h_\alpha s^{(k)}_\mu, \mathfrak{S}^{(k)}_\nu \rangle = \langle \sum_\alpha \bar{K}^{(k)}_{\alpha\lambda} \sum_\beta K^{(k)}_{\beta/\mu,\alpha} s^{(k)}_\beta, \mathfrak{S}^{(k)}_\nu \rangle \\
&= \sum_\alpha \bar{K}^{(k)}_{\alpha\lambda} K^{(k)}_{\nu/\mu,\alpha}
\end{aligned}$$

and

$$\langle s^{(k)}_\mu s^{(k)}_\lambda, \mathfrak{S}^{(k)}_\nu \rangle = \langle \sum_\alpha c^{\alpha,k}_{\lambda\mu} s^{(k)}_\alpha, \mathfrak{S}^{(k)}_\nu \rangle = c^{\nu,k}_{\lambda\mu} \,.$$

Therefore the result follows from

$$A^{\nu,k}_{\lambda\mu} = \sum_\alpha \bar{K}^{(k)}_{\alpha\lambda} K^{(k)}_{\nu/\mu,\alpha} = c^{\nu,k}_{\lambda\mu} \,.$$

□

Given the duality between Λ^k and $\Lambda/\mathfrak{J}^{(k)}$, it is natural also to consider a skew k-Schur function. The previous proposition, exposing k-Littlewood-Richardson coefficients as the expansion coefficients for a the dual skew k-Schur function in terms of dual k-Schur functions leads us to consider also the coefficients in

$$\mathfrak{S}^{(k)}_\lambda \, \mathfrak{S}^{(k)}_\mu = \sum_\nu \mathfrak{d}^{\nu,k}_{\lambda\mu} \, \mathfrak{S}^{(k)}_\nu \mod \mathfrak{J}^{(k)} \,.$$

Similar to the k-Littlewood-Richardson coefficients, Proposition 24 implies a symmetry satisfied by these coefficients:

$$\mathfrak{d}^{\nu,k}_{\lambda\mu} = \mathfrak{d}^{\nu^{\omega_k},k}_{\lambda^{\omega_k}\mu^{\omega_k}} \,. \tag{35}$$

We also note that:

$$\langle s^{(k)}_\nu, \mathfrak{S}^{(k)}_\lambda \, \mathfrak{S}^{(k)}_\mu \rangle = \langle s^{(k)}_\nu, \mathfrak{S}^{(k)}_\lambda \, \mathfrak{S}^{(k)}_\mu \mod \mathfrak{J}^{(k)} \rangle = \mathfrak{d}^{\nu,k}_{\lambda\mu} \,.$$

We can now introduce the skew k-Schur function and discuss several identities regarding the relations between these functions and their dual.

Definition 28. *For any k-bounded partitions $\mu \subseteq \nu$, the "skew k-Schur function" is defined by*

$$s^{(k)}_{\nu/\mu} = \sum_\lambda \mathfrak{d}^{\nu,k}_{\mu\lambda} s^{(k)}_\lambda \,.$$

This given, our first property is:

Proposition 29. *For any $f \in \Lambda$,*

$$\langle s^{(k)}_{\nu/\mu}, f\rangle = \langle s^{(k)}_\nu, f\, \mathfrak{S}^{(k)}_\mu\rangle \,,$$

and for any $f \in \Lambda^k$,

$$\langle f, \mathfrak{S}^{(k)}_{\nu/\mu}\rangle = \langle f\, s^{(k)}_\mu, \mathfrak{S}^{(k)}_\nu\rangle \,.$$

Proof. From Proposition 21 and Lemma 23, it suffices to consider $f = \mathfrak{S}^{(k)}_\lambda$. On one hand we have:

$$\langle s^{(k)}_{\nu/\mu}, \mathfrak{S}^{(k)}_\lambda\rangle = \langle \sum_\alpha \mathfrak{d}^{\nu,k}_{\mu\alpha} s^{(k)}_\alpha, \mathfrak{S}^{(k)}_\lambda\rangle = \mathfrak{d}^{\nu,k}_{\mu\lambda} \,,$$

and on the other,

$$\langle s^{(k)}_\nu, \mathfrak{S}^{(k)}_\lambda\, \mathfrak{S}^{(k)}_\mu\rangle = \langle s^{(k)}_\nu, \sum \mathfrak{d}^{\alpha,k}_{\mu\lambda} \mathfrak{S}^{(k)}_\alpha\rangle = \mathfrak{d}^{\nu,k}_{\mu\lambda} \,.$$

The second identity follows similarly. □

The ω-involution again has a natural role in our study. Given its action on k-Schur functions and their dual, with the symmetries (13) and (35), we find

Proposition 30.

$$\omega\left(\mathfrak{S}^{(k)}_{\nu/\mu}\right) \mod \mathfrak{J}^{(k)} = \mathfrak{S}^{(k)}_{\nu^{\omega_k}/\mu^{\omega_k}} \quad \text{and} \quad \omega\left(s^{(k)}_{\nu/\mu}\right) = s^{(k)}_{\nu^{\omega_k}/\mu^{\omega_k}} \,.$$

The next proposition explains why the coproduct of k-Schur functions in terms of k-Schur functions has the dual k-Littlewood-Richardson coefficients as expansion coefficients, and thus connects the dual k-Schur functions with the cohomology of the loop Grassmannian based on the conjecture of Shimozono. Recall (e.g. [20]) that from the coproduct, $\Delta : \Lambda \to \Lambda(x) \otimes \Lambda(y)$ by $\Delta f = f(x,y)$, a bialgebra structure is imposed:

$$\langle \Delta f, g(x)\, h(y)\rangle = \langle f, gh\rangle \,,$$

where the first scalar product is in $\Lambda(x) \otimes \Lambda(y)$.

Proposition 31. *For any $\lambda \in \mathcal{P}^k$ and two sets of indeterminants, x and y,*

$$s^{(k)}_\lambda(x,y) = \sum_{\mu,\nu} \mathfrak{d}^{\lambda,k}_{\mu\nu} s^{(k)}_\mu(x)\, s^{(k)}_\nu(y) = \sum_\nu s^{(k)}_{\lambda/\nu}(x)\, s^{(k)}_\nu(y) \,,$$

and

$$\mathfrak{S}^{(k)}_\lambda(x,y) = \sum_{\mu,\nu} c^{\lambda,k}_{\mu\nu}\, \mathfrak{S}^{(k)}_\mu(x)\, \mathfrak{S}^{(k)}_\nu(y) = \sum_\nu \mathfrak{S}^{(k)}_{\lambda/\nu}(x)\, \mathfrak{S}^{(k)}_\nu(y) \,.$$

Proof. For the first identity, given $\left\{s^{(k)}_\lambda\right\}_{\lambda_1 \le k}$ forms a basis of Λ^k and $h_i(x,y) = \sum_{\ell=0}^i h_{i-\ell}(x) h_\ell(y)$, we can assume

$$s^{(k)}_\lambda(x,y) = \sum_{\gamma,\delta} e^\lambda_{\gamma\delta}\, s^{(k)}_\gamma(x)\, s^{(k)}_\delta(y) \,,$$

for some $e^{\lambda}_{\gamma\delta}$. We thus have

$$\langle s^{(k)}_\lambda(x,y), \mathfrak{S}^{(k)}_\nu(x)\,\mathfrak{S}^{(k)}_\mu(y)\,\rangle = \langle \sum_{\gamma,\delta} e^{\lambda}_{\gamma\delta} s^{(k)}_\gamma(x)\, s^{(k)}_\delta(y), \mathfrak{S}^{(k)}_\nu(x)\,\mathfrak{S}^{(k)}_\mu(y)\,\rangle = e^{\lambda}_{\nu\mu}\,,$$

with the scalar product being taken in $\Lambda(x)\otimes\Lambda(y)$. On the other hand, using the definition of coproduct,

$$\langle s^{(k)}_\lambda(x,y), \mathfrak{S}^{(k)}_\nu(x)\,\mathfrak{S}^{(k)}_\mu(y)\,\rangle = \langle s^{(k)}_\lambda, \mathfrak{S}^{(k)}_\nu\,\mathfrak{S}^{(k)}_\mu\rangle = \langle s^{(k)}_\lambda, \sum_\delta \mathfrak{d}^{\delta,k}_{\nu\mu}\mathfrak{S}^{(k)}_\delta\rangle = \mathfrak{d}^{\lambda,k}_{\nu\mu}\,,$$

where we made use of Lemma 23 in the second equality. Therefore,

$$s^{(k)}_\lambda(x,y) = \sum_{\gamma,\nu} \mathfrak{d}^{\lambda\,(k)}_{\gamma\nu}\, s^{(k)}_\gamma(x)\, s^{(k)}_\nu(y) = \sum_\nu s^{(k)}_{\lambda/\nu}(x)\, s^{(k)}_\nu(y)\,.$$

For the second identity, we follow the same argument, using the fact that if $\lambda_1\le k$, then $m_\lambda(x,y)=\sum_{\mu,\nu\,:\,\mu_1,\nu_1\le k} a_{\mu\nu}\, m_\mu(x)\, m_\nu(y)$. □

More generally, following the proof of of (5.10) in [20], the preceding results imply:

Proposition 32. *For any $\lambda\in\mathcal{P}^k$ and two sets of indeterminants, x and y,*

$$s^{(k)}_{\lambda/\mu}(x,y) = \sum_\nu s^{(k)}_{\lambda/\nu}(x)\, s^{(k)}_{\nu/\mu}(y)$$

and

$$\mathfrak{S}^{(k)}_{\lambda/\mu}(x,y) = \sum_\nu \mathfrak{S}^{(k)}_{\lambda/\nu}(x)\,\mathfrak{S}^{(k)}_{\nu/\mu}(y)\,.$$

We conclude our exploration of the k-Schurs and their dual by mentioning that they give rise to a refined Cauchy formula, relying on the ideal $R^{(k)}$, generated by the indeterminants y_i^{k+1} for $i=1,2,\ldots$. Note that moding out by $R^{(k)}$ simply amounts to setting $y_i^n=0$ in the Cauchy kernel whenever $n>k$.

Theorem 33. *Consider two bases of homogeneous symmetric functions, $\{a_\lambda\}_{\lambda\in\mathcal{P}^k}$ and $\{b_\lambda\}_{\lambda\in\mathcal{P}^k}$ for Λ^k and $\Lambda/\mathfrak{J}^{(k)}$, respectively.*

$$\prod_{i,j}(1-x_iy_j)^{-1}\quad \mathrm{mod}\ R^{(k)} = \sum_\lambda a_\lambda(x) b_\lambda(y)$$

iff $\langle a_\lambda, b_\mu\rangle=\delta_{\lambda\mu}$ for all k-bounded partitions λ and μ.

Proof. The proof is similar to that of [20], I (4.6). □

Corollary 34.

$$\prod_{i,j}(1-x_iy_j)^{-1}\quad \mathrm{mod}\ R^{(k)} = \sum_{\lambda_1\le k} h_\lambda(x) m_\lambda(y) = \sum_{\lambda_1\le k} s^{(k)}_\lambda(x)\mathfrak{S}^{(k)}_\lambda(y)\,.$$

7. An involution on k-tableaux

To prove the dual k-Schurs are symmetric functions, we use an involution presented in [19] that acts on the set of k-tableaux by sending $\mathcal{T}_\alpha(\lambda)$ to $\mathcal{T}_{\hat\alpha}(\lambda)$, where $\hat\alpha$ is obtained by transposing two adjacent components of α. The involution reduces to the Bender-Knuth involution [5] for large k. Given this involution, it is immediately clear that

Corollary 35. *For $\lambda \in \mathcal{P}^k$, and a composition α of $|\lambda|$,*

$$K^{(k)}_{\lambda,\alpha} = K^{(k)}_{\lambda,\mu}\,, \tag{36}$$

where μ is the weakly decreasing rearrangement of α.

Some terminology about consecutive letters a and $b = a+1$ in a k-tableau are needed to define this involution. Entries a and b are *"married"* if they occur in the same column, an entry a is *"divorced"* if it has the same residue as some married a, and *"single"* otherwise. When the letter x occupies a cell in row r, labeled with residue j, we say this cell contains an $x_r(j)$, or simply an $x(j)$. $Res_r(x)$ will be the set of all residues that label cells occupied by a letter x in row r, while $URes_r(x)$ will be only the residues labeling unmarried x's in row r. We also consider $URes_r(a,b) = URes_r(a) \cup URes_r(b)$.

Rows r_1 and r_2 in a k-tableau are equivalent, "$r_1 \sim_a r_2$", when $URes_{r_1}(a,b) \cap URes_r(a,b) \neq \emptyset$ and $URes_{r_2}(a,b) \cap URes_r(a,b) \neq \emptyset$ for some r. A "fundamental row" of a tableau is a row r where $URes_r(a,b)$ is not contained in $URes_m(a,b)$ for any $m < r$. It was shown in [19] that the set of fundamental rows $r_1, \ldots, r_n$ of a k-tableau are representatives of the equivalence classes by $\mathcal{C}_i = \{r \,|\, r \sim_a r_i\}$. This given, the involution of interest is defined by the operator $\tau^{(a)}$ that acts on $\mathcal{C}_i$ as follows:

(1) (a) In row r_i, replace entries in cells $c_1, \ldots, c_s$ with a's and entries in cells $c_{s+1}, \ldots, c_{s+t}$ with b's, where $c_1, \ldots, c_t$ denote cells of T with single a's and $c_{t+1}, \ldots, c_{t+s}$ those with single b's (where c_j lies west of c_{j+1}).
  (b) If $s > t$, relabel with a b, any a lying to the right of a b (or if $t > s$, relabel with an a, any b lying to the left of an a).
  Step 1 amounts to relabeling $(s-t)$ a's with b's (or $(t-s)$ b's with a's) and then relabeling any letter that is out-of-order.

(2) For $\mathcal{S}_i$ the set of residues of a's (or b's) that were relabeled in step 1, relabel correspondingly every unmarried a (or b) that lies above row r_i and has residue in $\mathcal{S}_i$.
  Note, by definition of $\sim_a$, this step only involves rows in the class $\mathcal{C}_i$ implying that no row is involved in this step for two distinct values of i.

Proposition 36. [19] *For any $T \in \mathcal{T}_\alpha(\lambda)$, the tableau $\tau^{(a)}(T)$ belongs to $\mathcal{T}_{\hat\alpha}(\lambda)$, where $\hat\alpha = (\ldots, \alpha_{a+1}, \alpha_a, \ldots,)$ is obtained by switching α_a and α_b in α. Further, $\tau^{(a)}$ is an involution.*

8. Further work

As detailed in the introduction, the Schur functions provide a vehicle to directly reach the structure constants for multiplication in the cohomology of the Grassmannian from the Littlewood-Richardson coefficients. More generally, Theorem 17 implies that the k-Schur functions provide the analogous link between the quantum structure constants (or 3-point Gromov-Witten invariants) and the k-Littlewood-Richardson coefficients. There are beautiful combinatorial methods for computing the Littlewood-Richardson coefficients that use, for example, skew tableaux or reduced words for permutations. Although there has been progress in certain cases [2, 3, 12, 23, 24, 26], a combinatorial interpretation for the 3-point Gromov-Witten invariants in complete generality remains an open problem. The theory of k-Schur functions suggests a number of natural approaches to this problem, with an extended notion of skew tableaux to k-skew tableaux, and the revelation [18] that affine permutations are the appropriate extended notion of permutations in this study.

The conjecture that the (dual) k-Schur functions are the Schubert basis for the (cohomology) homology of the loop Grassmannian remains to be proven. More generally, this supports the suggestion of Wachs that there exists an affine version of Schubert polynomials related to k-Schur functions. In particular, the (dual) k-Schur functions are indexed by k-bounded partitions, which are in bijection with affine permutations in the quotient [6]. Such affine permutations can be considered as the Grassmannian version of affine permutations. The results here, with the conjectures that the (dual) k-Schur basis is related to the Schubert basis for the (cohomology) homology of the loop Grassmannian, suggest that these functions provide the the symmetric component of (dual) affine Schubert polynomials. The first step in this direction is being developed by Thomas Lam in a forthcoming paper [14] on affine Stanley symmetric functions.

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