TABLEAUX ON $k+1$-CORES, REDUCED WORDS FOR AFFINE PERMUTATIONS, AND k-SCHUR EXPANSIONS

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ABSTRACT. The k-Young lattice Y^k is a partial order on partitions with no part larger than k. This weak subposet of the Young lattice originated [9] from the study of the k-Schur functions $s_\lambda^{(k)}$, symmetric functions that form a natural basis of the space spanned by homogeneous functions indexed by k-bounded partitions. The chains in the k-Young lattice are induced by a Pieri-type rule experimentally satisfied by the k-Schur functions. Here, using a natural bijection between k-bounded partitions and $k+1$-cores, we establish an algorithm for identifying chains in the k-Young lattice with certain tableaux on $k+1$ cores. This algorithm reveals that the k-Young lattice is isomorphic to the weak order on the quotient of the affine symmetric group $\tilde{S}_{k+1}$ by a maximal parabolic subgroup. From this, the conjectured k-Pieri rule implies that the k-Kostka matrix connecting the homogeneous basis $\{h_\lambda\}_{\lambda\in Y^k}$ to $\{s_\lambda^{(k)}\}_{\lambda\in Y^k}$ may now be obtained by counting appropriate classes of tableaux on $k+1$-cores. This suggests that the conjecturally positive k-Schur expansion coefficients for Macdonald polynomials (reducing to q,t-Kostka polynomials for large k) could be described by a q,t-statistic on these tableaux, or equivalently on reduced words for affine permutations.

1. Introduction

1.1. **The k-Young lattice.** Recall that λ is a successor of a partition μ in the Young lattice when λ is obtained by adding an addable corner to μ where partitions are identified by their Ferrers diagrams, with rows weakly decreasing from bottom to top. This relation, which we denote "$\mu \to \lambda$", occurs naturally in the classical Pieri rule

$$h_1[X]\, s_\mu[X] \,=\, \sum_{\lambda:\, \mu\to\lambda} s_\lambda[X]\,, \tag{1.1}$$

and the partial order of the Young lattice may be defined as the transitive closure of $\mu \to \lambda$. It was experimentally observed that the k-Schur functions [9, 11] satisfy the rule

$$h_1[X]\, s_\mu^{(k)}[X] \,=\, \sum_{\lambda:\, \mu\to_k\lambda} s_\lambda^{(k)}[X]\,, \tag{1.2}$$

where "$\mu \to_k \lambda$" is a certain subrelation of "$\mu \to \lambda$". This given, the partial order of the k-Young lattice Y^k is defined as the transitive closure of $\mu \to_k \lambda$.

The precise definition of the relation $\mu \to_k \lambda$ stems from another "Schur" property of k-Schur functions. Computational evidence suggests that the usual ω-involution for symmetric functions acts on k-Schur functions according to the formula

$$\omega\, s_\mu^{(k)}[X] \,=\, s_{\mu^{\omega_k}}^{(k)}[X]\,, \tag{1.3}$$

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where the map $\mu \mapsto \mu^{\omega_k}$ is an involution on k-bounded partitions called "*k-conjugation*" that generalizes partition conjugation $\mu \mapsto \mu'$. Then viewing the covering relations on the Young lattice as

$$\mu \to \lambda \iff |\lambda| = |\mu| + 1 \quad \& \quad \mu \subseteq \lambda \quad \& \quad \mu' \subseteq \lambda' \,, \tag{1.4}$$

we accordingly, in our previous work [9], defined $\mu \to_k \lambda$ in terms of the involution $\mu \mapsto \mu^{\omega_k}$ by

$$\mu \to_k \lambda \iff |\lambda| = |\mu| + 1 \quad \& \quad \mu \subseteq \lambda \quad \& \quad \mu^{\omega_k} \subseteq \lambda^{\omega_k} \,. \tag{1.5}$$

Thus only certain addable corners may be added to a partition μ to obtain its successors in the k-Young lattice. We shall call such corners the "*k-addable corners*" of μ.

Here, we provide a direct characterization of k-addable corners. This characterization is obtained by first constructing a bijection between k-bounded partitions and $k+1$-cores. We then show that certain operators preserving the set of $k+1$-cores act on the k-Young lattice (through this bijection) by lowering or raising its elements according to the covering relations. Passing from an element to its successor by means of these operators, we thus obtain an algorithm for constructing any saturated chain in the k-Young lattice. Such a construction leads us to a bijection between chains in the k-Young lattice and a new family of tableaux called "k-tableaux" that share properties of usual semi-standard tableaux.

On the other hand, the weak order on the quotient of the affine symmetric group $\tilde{S}_{k+1}$ by a maximal parabolic subgroup can be characterized using the previously mentioned operators on cores. This enables us to show that the k-Young lattice is isomorphic to the weak order on this quotient. Consequences include a bijection between standard k-tableaux of a fixed shape and reduced words for a fixed affine permutation, as well as a new bijection between k-bounded partitions and affine permutations in the quotient.

To precisely summarize our results, first recall that a $k+1$-core is a partition with no $k+1$-hooks. For any $k+1$-core γ, we then define

$$\mathfrak{p}(\gamma) = (\lambda_1, \ldots, \lambda_\ell)$$

where λ_i is the number of cells with k-bounded hook length in row i of γ. It turns out that $\mathfrak{p}(\gamma)$ is a k-bounded partition and that the correspondence $\gamma \mapsto \mathfrak{p}(\gamma)$ bijectively maps $k+1$-cores onto k-bounded partitions. With $\lambda \mapsto \mathfrak{c}(\lambda)$ denoting the inverse of $\mathfrak{p}$, we define the k-conjugation of a k-bounded partition λ to be

$$\lambda^{\omega_k} \;=\; \mathfrak{p}\big(\mathfrak{c}(\lambda)'\big) \,. \tag{1.6}$$

That is, if γ is the $k+1$-core corresponding to λ, then λ^{ω_k} is the partition whose row lengths equal the number of k-bounded hooks in corresponding rows of γ'. This reveals that k-conjugation, which originally emerged from the action of the ω involution on k-Schur functions, is none other than the $\mathfrak{p}$-image of ordinary conjugation of $k+1$-cores.

The $\mathfrak{p}$-bijection then leads us to a characterization for k-addable corners that determine successors in the k-Young lattice. By labeling every square (i,j) in the i^{th} row and j^{th} column by its "*$k+1$-residue*", $j-i \mod k+1$, we find

(Theorem 23) *Let c be any addable corner of a k-bounded partition λ and c' (of $k+1$-residue i) be the addable corner of $\mathfrak{c}(\lambda)$ in the same row as c. c is k-addable if and only if c' is the highest addable corner of $\mathfrak{c}(\lambda)$ with $k+1$-residue i.*

This characterization of k-addability leads us to a notion of standard k-tableaux which we prove are in bijection with saturated chains in the k-Young lattice.

(**Definition 27)** *Let γ be a $k+1$-core and m be the number of k-bounded hooks of γ. A standard k-tableau of shape γ is a filling of the cells of γ with the letters $1, 2, \ldots, m$ which is strictly increasing in rows and columns and such that the cells filled with the same letter have the same $k+1$-residue.*

(Theorem 37) *The saturated chains in the k-Young lattice joining the empty partition $\emptyset$ to a given k-bounded partition λ are in bijection with the standard k-tableaux of shape $\mathfrak{c}(\lambda)$.*

We then consider the affine symmetric group $\tilde{S}_{k+1}$ modulo a maximal parabolic subgroup denoted by S_{k+1}. Bruhat order on the minimal coset representatives of $\tilde{S}_{k+1}/S_{k+1}$ can be defined by containment of $k+1$-core diagrams (this connection is stated by Lascoux in [8] and is equivalent to other characterizations such as in [15, 1]). From this, stronger relations among $k+1$-core diagrams can be used to describe the weak order on such coset representatives. We are thus able to prove that our new characterization of the k-Young lattice chains implies an isomorphism between the k-Young lattice and the weak order on the minimal coset representatives. Consequently, a bijection between the set of k-tableaux of a given shape $\mathfrak{c}(\lambda)$ and the set of reduced decompositions for a certain affine permutation $\sigma_\lambda \in \tilde{S}_{k+1}/S_{k+1}$ can be achieved by mapping:

$$\mathfrak{w} : T \mapsto s_{i_\ell} \cdots s_{i_2} s_{i_1} \,, \tag{1.7}$$

where i_a is the $k+1$-residue of letter a in the standard k-tableau T. A by-product of this result is a simple bijection between k-bounded partitions and affine permutations in $\tilde{S}_{k+1}/S_{k+1}$:

$$\phi : \lambda \mapsto \sigma_\lambda \,, \tag{1.8}$$

where σ_λ corresponds to the reduced decomposition obtained by reading the $k+1$-residues of λ from right to left and from top to bottom. It is shown in [13] that this bijection, although algorithmically distinct, is equivalent to the one presented by Björner and Brenti [1] using a notion of inversions on affine permutations. It follows from our results that Eq. (1.2) reduces simply to

$$h_1[X]\, s^{(k)}_{\phi^{-1}(\sigma)}[X] = \sum_{\sigma \lessdot_w \tau} s^{(k)}_{\phi^{-1}(\tau)}[X] \,, \tag{1.9}$$

where the sum is over all permutations that cover σ in the weak order on $\tilde{S}_{k+1}/S_{k+1}$.

As will be detailed in § 1.2, Theorem 37 also plays a role in the theory of Macdonald polynomials and the study of k-Schur functions, thus motivating a semi-standard extension of Definition 27:

(Definition 59) *Let m be the number of k-bounded hooks in a $k+1$-core γ and let $\alpha = (\alpha_1, \ldots, \alpha_r)$ be a composition of m. A semi-standard k-tableau of shape γ and evaluation α is a column strict filling of γ with the letters $1, 2, \ldots, r$ such that the collection of cells filled with letter i is labeled with exactly α_i distinct $k+1$-residues.*

As with the ordinary semi-standard tableaux, we show that there are no semi-standard k-tableaux under conditions relating to dominance order on the shape and evaluation. An analogue of Theorem 37 can then be used to show that this coincides with unitriangularity of coefficients in the k-Schur expansion of homogeneous symmetric functions and suggests that the k-tableaux should have statistics to combinatorially describe the k-Schur function expansion of the Hall-Littlewood polynomials. The analogue of Theorem 37 relies on the following definition: with the pair of k-bounded partitions λ, μ defined to be "*r-admissible* " if and only if λ/μ and $\lambda^{\omega_k}/\mu^{\omega_k}$ are respectively horizontal and vertical r-strips, we say a sequence of partitions

$$\emptyset = \lambda^{(0)} \longrightarrow \lambda^{(1)} \longrightarrow \lambda^{(2)} \longrightarrow \cdots \longrightarrow \lambda^{(\ell)}$$

is α-admissible when $\lambda^{(i)}, \lambda^{(i-1)}$ is a α_i-admissible pair for $i = 1, \ldots, \ell$. It turns out that all α-admissible sequences are in fact chains in the k-Young lattice and that Theorem 37 extends to:

(Theorem 71) *Let m be the number of k-bounded hooks in a $k+1$-core γ and let α be a composition of m. The collection of α-admissible chains joining $\emptyset$ to $\mathfrak{p}(\gamma)$ is in bijection with the semi-standard k-tableaux of shape γ and evaluation α.*

An affine permutation interpretation for the α-admissible chains that generalizes our $\mathfrak{w}$-bijection between standard k-tableaux and reduced words is given in [13] along with a more detailed discussion of the connection between the type-A affine Weyl group and the k-Schur functions. The reader is also referred to [12] for a study of principal order ideals in the k-Young lattice along with further properties of the lattice such as the fact that the covering relation is invariant under translation by rectangular shapes with hook-length equal to k. This is the underlying mechanism in the proof that the k-Young lattice corresponds to a cone in a tiling of $\mathbb{R}^k$ by permutahedrons [17].

As mentioned, the root of our work lies in the study of symmetric functions. We conclude our introduction with a summary of these ideas.

1.2. **Macdonald expansion coefficients.** The k-Young lattice emerged from the experimental Pieri rule (1.2) satisfied by k-Schur functions. In turn, k-Schur functions have arisen from a close study of Macdonald polynomials. To appreciate the role of our findings in the theory of Macdonald polynomials we shall briefly review this connection. To begin, we consider the Macdonald polynomial $H_\lambda[X;q,t]$ obtained from the Macdonald integral form [14] $J_\lambda[X;q,t]$ by the plethystic substitution

$$H_\mu[X;q,t\,] \;=\; J_\mu\big[\tfrac{X}{1-q};q,t\,\big]\,. \tag{1.10}$$

For $\mu\vdash n$, this yields the Schur function expansion

$$H_\mu[X;q,t\,] \;=\; \sum_{\lambda\vdash n} K_{\lambda\mu}(q,t)s_\lambda[X]\,, \tag{1.11}$$

where $K_{\lambda\mu}(q,t)\in\mathbb{N}[q,t]$ are known as the q,t-Kostka polynomials. Formula (1.11), when $q=t=1$, reduces to

$$h_1^n \;=\; \sum_{\lambda\vdash n} f_\lambda\, s_\lambda[X]\,, \tag{1.12}$$

where f_λ is the number of standard tableaux of shape λ. This given, one of the outstanding problems in algebraic combinatorics is to associate a pair of statistics $a_\mu(T), b_\mu(T)$ on standard tableaux to the partition μ so that

$$K_{\lambda\mu}(q,t) \;=\; \sum_{T\in ST(\lambda)} q^{a_\mu(T)}t^{b_\mu(T)}\,, \tag{1.13}$$

where "$ST(\lambda)$" denotes the collection of standard tableaux of shape λ.

In previous work [9, 11], we proposed a new approach to the study of the q,t-Kostka polynomials. This approach is based on the discovery of a certain family of symmetric functions $\{s_\lambda^{(k)}[X;t]\}_{\lambda\in Y^k}$ for each integer $k\geq 1$, which we have shown [11] to be a basis for the space $\Lambda_t^{(k)}$ spanned by the Macdonald polynomials $H_\mu[X;q,t\,]$ indexed by k-bounded partitions. This revealed a mechanism underlying the structure of the coefficients $K_{\lambda\mu}(q,t)$. To be precise, for $\mu,\nu\in Y^k$, consider

$$H_\mu[X;q,t\,] \;=\; \sum_{\nu\in Y^k} K_{\nu\mu}^{(k)}(q,t)\,s_\nu^{(k)}[X;t\,]\ ,\ \text{ and }\ s_\nu^{(k)}[X;t\,] \;=\; \sum_{\lambda} \pi_{\lambda\nu}(t)\,s_\lambda[X]\,. \tag{1.14}$$

We then we have the factorization

$$K_{\lambda\mu}(q,t) \;=\; \sum_{\nu\in Y^k} \pi_{\lambda\nu}(t)\,K_{\nu\mu}^{(k)}(q,t)\,. \tag{1.15}$$

It was experimentally observed (and proven for $k=2$ in [10, 11]) that $K_{\nu\mu}^{(k)}(q,t)\in\mathbb{N}[q,t]$ and $\pi_{\lambda\nu}(t)\in\mathbb{N}[t]$. This suggests that the problem of finding statistics for $K_{\lambda\mu}(q,t)$ may be decomposed into two separate analogous problems for $K_{\nu\mu}^{(k)}(q,t)$ and $\pi_{\lambda\nu}(t)$. We also have experimental evidence to support that $K_{\lambda\mu}(q,t)-K_{\nu\mu}^{(k)}(q,t)\in\mathbb{N}[q,t]$ which brings about the fact that $s_\lambda^{(k)}[X;t\,]$-expansions are formally simpler.

These developments prompted a close study of the polynomials $s_\lambda^{(k)}[X;1\,]=s_\lambda^{(k)}[X]$. In addition to (1.2), it was also conjectured that these polynomials satisfy the more general rule

$$h_r[X]\,s_\mu^{(k)}[X] \;=\; \sum_{\substack{\lambda/\mu=\text{horizontal } r\text{-strip}\\ \lambda^{\omega_k}/\mu^{\omega_k}=\text{vertical } r\text{-strip}}} s_\lambda^{(k)}[X]\,. \tag{1.16}$$

Iteration of (1.2) starting from $s_\emptyset[X] = 1$ yields

$$h_1^n[X] \,=\, \sum_{\lambda\in Y^k} K^{(k)}_{\lambda,1^n}\, s^{(k)}_\lambda[X]\,, \tag{1.17}$$

while iterating (1.16) for suitable choices of r gives the k-Schur function expansion of an h-basis element indexed by any k-bounded partition μ . That is,

$$h_\mu[X] \,=\, \sum_{\lambda\in Y^k} K^{(k)}_{\lambda\mu}\, s^{(k)}_\lambda[X] \tag{1.18}$$

Since $s^{(k)}_\lambda[X] = s_\lambda[X]$ when all the hooks of λ are k-bounded, we see that (1.17) reduces to (1.12) for a sufficiently large k. For the same reason, the coefficient $K^{(k)}_{\lambda\mu}$ in (1.18) reduces to the classical Kostka number $K_{\lambda\mu}$ when k is large enough. Our definition of the k-Young lattice Y^k and admissible chains in Y^k, combined with the experimental Pieri rules (1.2) and (1.16), yield the following corollary of Theorems 37 and 71:

> *On the validity of* (1.16)*, the coefficient* $K^{(k)}_{\lambda,1^n}$ *is equal to the number of standard* k*-tableaux of shape* $\mathfrak{c}(\lambda)$*, or equivalently the number of reduced expressions for* σ_λ*, and the coefficient* $K^{(k)}_{\lambda\mu}$ *is equal to the number of semi-standard* k*-tableaux of shape* $\mathfrak{c}(\lambda)$ *and evaluation* μ*.*

Since (1.14) reduces to (1.17) when $q = t = 1$, this suggests that the positivity of $K^{(k)}_{\lambda\mu}(q,t)$ may be accounted for by q,t-counting standard k-tableaux of shape $\mathfrak{c}(\lambda)$, or reduced words of σ_λ, according to a suitable statistic depending on μ. More precisely, for $\mathcal{T}^k(\lambda)$ the set of k-tableaux of shape $\mathfrak{c}(\lambda)$ and $Red(\sigma)$ the reduced words for σ,

$$H_\mu[X;q,t\,] \;=\; \sum_{\lambda:\lambda_1\le k}\left(\sum_{T\in\mathcal{T}^k(\lambda)} q^{a_\mu(T)}\,t^{b_\mu(T)}\right) s^{(k)}_\lambda[X;t\,] \tag{1.19}$$

$$\;=\; \sum_{\sigma\in\tilde{S}_{k+1}/S_{k+1}}\left(\sum_{w\in Red(\sigma)} q^{a_{\sigma_\mu}(w)}\,t^{b_{\sigma_\mu}(w)}\right) s^{(k)}_{\phi^{-1}(\sigma)}[X;t\,]\,. \tag{1.20}$$

We should also mention that the relation in (1.18) was proven to be unitriangular [11] with respect to the dominance partial order "$\trianglerighteq$" as well as the t-analog of this relation, given by the Hall-Littlewood polynomials corresponding to the specialization $q = 0$ of the Macdonald polynomials:

$$H_\mu[X;0,t\,] \,=\, \sum_{\substack{\lambda\in Y^k\\ \lambda\trianglerighteq\mu}} K^{(k)}_{\lambda\mu}(t)\, s^{(k)}_\lambda[X;t\,]\,. \tag{1.21}$$

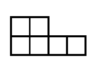
Our conjecture that $K^{(k)}_{\lambda\mu}(q,t) \in \mathbb{N}[q,t]$ implies $K^{(k)}_{\lambda\mu}(t)$ would also have positive integer coefficients. Our work here then suggests that this positivity may be accounted for by showing that the $K^{(k)}_{\lambda\mu}(t)$ can be obtained by t-counting semi-standard k-tableaux according to a suitable k-charge statistic.

In conclusion, since (1.18) is obtained by iterating (1.16) and the resulting matrix $\|K^{(k)}_{\lambda\mu}\|$ is unitriangular, the inversion of this matrix gives a well-defined family of functions that are conjectured to be the k-Schur functions. This provides a relatively simple algorithm for computing the "k-Schur functions" (at $t = 1$) for anyone who wishes to experiment. A study of the "k-Schur functions" obtained in this manner is carried out in [13] where it is shown, in particular, that they satisfy the k-Pieri rule (1.2).

Contents

2. Definitions

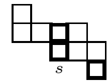
2.1. **Partitions.** A partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ is a non-increasing sequence of positive integers. The degree of λ is $|\lambda| = \lambda_1 + \cdots + \lambda_m$ and the length $\ell(\lambda)$ is the number of parts m. Each partition λ has an associated Ferrers diagram with λ_i lattice squares in the i^{th} row, from the bottom to top. For example,

$$\lambda = (4,2) = $$  . (2.1)

Given a partition λ, its conjugate λ' is the diagram obtained by reflecting λ about the diagonal. A partition λ is "*k-bounded*" if $\lambda_1 \leq k$. Any lattice square in the Ferrers diagram is called a cell, where the cell (i,j) is in the ith row and jth column of the diagram. We say that $\lambda \subseteq \mu$ when $\lambda_i \leq \mu_i$ for all i. The dominance order $\trianglerighteq$ on partitions is defined by $\lambda \trianglerighteq \mu$ when $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for all i, and $|\lambda| = |\mu|$.

More generally, for $\rho \subseteq \gamma$, the skew shape γ/ρ is identified with its diagram $\{(i,j) : \rho_i < j \leq \gamma_i\}$. Lattice squares that do not lie in γ/ρ will be simply called "*squares*". We shall say that any $c \in \rho$ lies "*below* " γ/ρ. The "*hook* " of any lattice square $s \in \gamma$ is defined as the collection of cells of

γ/ρ that lie inside the L with s as its corner. This is intended to apply to all $s \in \gamma$ including those below γ/ρ. In the example below the hook of $s = (1,3)$ is depicted by the framed cells

$$\gamma/\rho \;=\; (5,5,4,1)/(4,2) \;=$$ 

. (2.2)

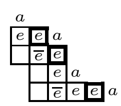
We then let $h_s(\gamma/\rho)$ denote the number of cells in the hook of s. Thus from the example above we have $h_{(1,3)}\big((5,5,4,1)/(4,2)\big) = 3$ and $h_{(3,2)}\big((5,5,4,1)/(4,2)\big) = 3$. We shall also say that the hook of a cell or a square is k-bounded if its length is not larger than k.

Remark 1. *It is important to note that when row and column lengths of γ/ρ weakly decrease from top to bottom and left to right, then the present notion of hook length satisfies some of the standard inequalities of hook lengths. In particular, $h_{s_1}(\gamma/\rho) \geq h_{s_2}(\gamma/\rho)$ when $s_1 = (i_1, j_1)$, $s_2 = (i_2, j_2)$ with $i_1 \leq i_2$ and $j_1 \leq j_2$ and the inequality is strict when $s_1, s_2 \in \gamma/\rho$ or $s_1 \in \rho$ and $s_2 \in \gamma/\rho$.*

Recall that a *"$k+1$-core* is a partition that does not contain any $k+1$-hooks (see [7] for more on cores and residues). The "$k+1$-*residue*" of square (i,j) is $j - i \mod k+1$. That is, the integer in this square when squares are periodically labeled with $0, 1, \ldots, k$, where zeros lie on the main diagonal. The 5-residues associated to the 5-core $(6,4,3,1,1,1)$ are

$$\begin{array}{lllllll}
4 \\
\boxed{0} \\
\boxed{1} \\
\boxed{2} & 3 \\
\boxed{3} & \boxed{4} & \boxed{0} & 1 \\
\boxed{4} & \boxed{0} & \boxed{1} & \boxed{2} & 3 \\
\boxed{0} & \boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} & \boxed{0} & 1
\end{array}$$

A cell (i,j) of a partition γ with $(i+1, j+1) \notin \gamma$ is called "*extremal*". An extremal cell which is neither at the end of its row nor at the top of its column is called "*corner extremal*". A "*removable*" corner of partition γ is a cell $(i,j) \in \gamma$ with $(i, j+1), (i+1, j) \notin \gamma$ and an "*addable*" corner of γ is a square $(i,j) \notin \gamma$ with $(i, j-1), (i-1, j) \in \gamma$. All removable corners are extremal. We should note that $(1, \gamma_1), (\ell(\gamma), 1)$ are removable corners and $(1, \gamma_1 + 1), (\ell(\gamma) + 1, 1)$ are addable. In the figure below we have labeled all addable corners with a, labeled extremal cells e (with the corner extremals overlined), and framed the removable corners.



(2.3)

Given any two squares, b south-east of a, "$a \wedge b$" will denote the square that is simultaneously directly south of a and directly west of b.

A composition α of an integer m is a vector of positive integers that sum to m. A "*tableau*" T of shape λ is a filling of T with integers that is weakly increasing in rows and strictly increasing in columns. The "*evaluation*" of T is given by a composition α where α_i is the multiplicity of i in T.

2.2. **Affine symmetric group.** The affine symmetric group $\tilde{S}_{k+1}$ is generated by the $k+1$ elements $\hat{s}_0, \ldots, \hat{s}_k$ satisfying the affine Coxeter relations:

$$\hat{s}_i^2 = id, \qquad \hat{s}_i \hat{s}_j = \hat{s}_j \hat{s}_i \quad (i - j \neq \pm 1 \mod k+1), \quad \text{and} \quad \hat{s}_i \hat{s}_{i+1} \hat{s}_i = \hat{s}_{i+1} \hat{s}_i \hat{s}_{i+1}\,. \tag{2.4}$$

Here, and in what follows, $\hat{s}_i$ is understood as $\hat{s}_{i \bmod k+1}$ if $i \geq k+1$. Elements of $\tilde{S}_{k+1}$ are called affine permutations, or simply permutations. A word $w = i_1 i_2 \cdots i_m$ in the alphabet $\{0, 1, \ldots, k\}$ corresponds to the permutation $\sigma \in \tilde{S}_{k+1}$ if $\sigma = \hat{s}_{i_1} \ldots \hat{s}_{i_m}$. The *"length"* of σ, denoted $\ell(\sigma)$, is the length of the shortest word corresponding to σ. Any word for σ with $\ell(\sigma)$ letters is said to be *"reduced"*. We denote by $Red(\sigma)$ the set of all reduced words of σ.

The weak order on $\tilde{S}_{k+1}$ is defined through the following covering relations:

$$\sigma \lessdot_w \tau \quad \Longleftrightarrow \quad \tau = \hat{s}_i\, \sigma \text{ for some } i \in \{0, \ldots, k\}\,, \text{ and } \ell(\tau) > \ell(\sigma)\,, \tag{2.5}$$

while the Bruhat order is such that $\sigma <_b \tau$ if there exist reduced words w and u, corresponding to σ and τ respectively, such that w is a subword of u.

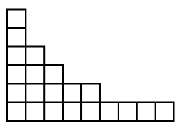
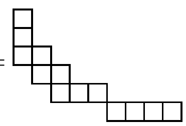
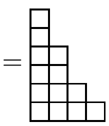
The subgroup of $\tilde{S}_{k+1}$ generated by the subset $\{\hat{s}_1, \ldots, \hat{s}_k\}$ is a maximal parabolic subgroup that is isomorphic to the symmetric group. We thus denote this subgroup by S_{k+1} and shall consider the set of minimal coset representatives of $\tilde{S}_{k+1}/S_{k+1}$. It is important to note that if σ is not the identity, then $\sigma \in \tilde{S}_{k+1}/S_{k+1}$ if and only if every $w \in Red(\sigma)$ ends in a zero. That is, the reduced expressions are all of the form $w = i_1 \ldots i_{m-1}\, 0$.

3. Bijection: $k+1$-cores and k-bounded partitions

Let $\mathcal{C}_{k+1}$ and $\mathcal{P}_k$ respectively denote the collections of $k+1$ cores and k-bounded partitions. We start by showing that a bijection between these sets can be defined by the map

$$\mathfrak{p} : \gamma \to (\lambda_1, \ldots, \lambda_\ell)$$

where λ_i is the number of cells with a k-bounded hook in row i of γ. If $\rho(\gamma)$ is the partition consisting only of the cells in γ whose hook lengths exceed k, then $\mathfrak{p}(\gamma) = \lambda$ is equivalently defined by letting λ_i denote the length of row i in the skew diagram $\gamma/\rho(\gamma)$. For example, with $k = 4$:

$\gamma =$  $\gamma/\rho(\gamma) =$  $\mathfrak{p}(\gamma) =$ 

Although it is not immediate that the codomain of $\mathfrak{p}$ is $\mathcal{P}_k$, we shall find that each diagram $\gamma/\rho(\gamma)$ can be uniquely associated to a skew diagram constructed from some k-bounded partition λ.

Definition 2. *For any $\lambda \in \mathcal{P}_k$, the "$k$-skew diagram of λ" is the diagram $\lambda/^k$ where*

(i) row i has length λ_i for $i = 1, \ldots, \ell(\lambda)$

(ii) no cell of $\lambda/^k$ has hook-length exceeding k

(iii) all squares below $\lambda/^k$ have hook-length exceeding k.

We shall thus find that $\mathfrak{p}$ is a bijection from $\mathcal{C}_{k+1}$ to $\mathcal{P}_k$ with inverse $\mathfrak{c}$ defined:

Definition 3. *For λ a k-bounded partition and $\lambda/^k = \gamma/\rho$, define $\mathfrak{c}(\lambda) = \gamma$.*

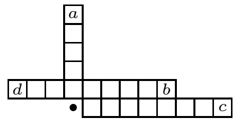
To this end, we start by characterizing the skew diagrams $\gamma/\rho(\gamma)$ by the following lemma, and consequently find that $\mathfrak{p}(\gamma) \in \mathcal{P}_k$.

Lemma 4. *Let $\rho \subseteq \gamma$ be partitions. Then γ is a $k+1$-core and $\rho = \rho(\gamma)$ if and only if the skew partition γ/ρ has the following four properties:*

(i) *the row lengths of γ/ρ weakly decrease from bottom to top,*
(ii) *the column lengths of γ/ρ weakly decrease from left to right,*
(iii) *the hooks of the cells of γ/ρ have at most k cells,*
(iv) *the squares below γ/ρ have hook-lengths exceeding k.*

Proof. We shall prove that conditions (iii) and (iv) are sufficient. Let $c \in \rho$. Condition (iv) asserts that the hook of c in γ/ρ contains at least $k+1$ cells. Since the hook of c in γ contains at least these cells and c itself, $h_c(\gamma) > k+1$. Moreover (iii) assures that all cells of γ/ρ have hook length $\leq k$. Thus γ has no $k+1$ hooks and therefore is a $k+1$-core. Since all the cells of γ whose hook length exceeds k lie in ρ, it follows that $\rho = \rho(\gamma)$ as desired.

To show that conditions (i)-(iv) are necessary, let γ be a $k+1$-core and $\rho = \rho(\gamma)$. Refer below to a typical case of two successive rows in γ, where b is the cell at the end of row $i+1$, c is the cell at the end of row i, and a labels the cell at the top of column ρ_i in γ. Thus $a \wedge c$ is the cell (i, ρ_i) in ρ (labeled with a "$\bullet$"). Let d be any cell of row $i+1$ that has at least $\gamma_i - \rho_i$ cells of γ to its right.



(3.1)

The definition of $\rho(\gamma)$ implies that $h_{a\wedge c}(\gamma) \geq k+2$ since γ is a $k+1$-core.

To show (i) we must prove that $\gamma_i - \rho_i \geq \gamma_{i+1} - \rho_{i+1}$, or equivalently that all such $d's$ are in ρ. For this, we observe that the hook length of d in γ is at least equal to the hook length of $a \wedge c$ minus one. Thus, $h_{a\wedge c}(\gamma) \geq k+2$ implies that the hook length of d is at least $k+1$ and therefore d belongs to ρ. Next note that since the conjugate of a $k+1$ core is also a $k+1$ core and $\rho(\gamma') = \rho(\gamma)'$, condition (ii) for $\gamma/\rho(\gamma)$ follows from (i) for $\gamma'/\rho(\gamma')$.

Condition (iii) is an immediate consequence of the definition of $\rho(\gamma)$. To prove (iv), consider $a \wedge c$ (the "$\bullet$" in our figure) as a typical removable corner of $\rho(\gamma)$. Since $h_{a\wedge c}(\gamma) \geq k+2$, at least $k+1$ of these cells lie in $\gamma/\rho(\gamma)$ implying $h_{a\wedge c}(\gamma/\rho) \geq k+1$. Since every square below $\gamma/\rho(\gamma)$ is weakly *south-west* of such a removable corner in $\rho(\gamma)$, condition (iv) follows Remark 1 given (i) and (ii). $\square$

We thus have that $\mathfrak{p}$ maps $\mathcal{C}_{k+1}$ into $\mathcal{P}_k$ since the parts of $\mathfrak{p}(\gamma)$ are weakly increasing by condition (i) and do not exceed k by condition (iii). To show that this map is a bijection, we will identify its inverse by considering the following auxiliary result:

Lemma 5. *For any k-bounded partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, there is a unique sequence of skew diagrams $\lambda^{(r)/k}, \lambda^{(r-1)/k}, \ldots, \lambda^{(1)/k}$ where $\lambda^{(r)/k} = (\lambda_r)$ and $\lambda^{(i)/k}$ is obtained by attaching a row of length λ_i to the bottom of $\lambda^{(i+1)/k}$ such that:*

(1) *the hook lengths of $\lambda^{(i)/k}$ do not exceed k*
(2) *all the lattice squares below $\lambda^{(i)/k}$ have hook lengths exceeding k.*

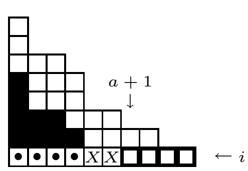
In particular, $\lambda^{/k} = \lambda^{(1)/k}$ is the unique skew partition γ/ρ such that

(a) *the row lengths of γ/ρ are the parts of λ*
(b) *γ is a $k+1$-core and $\rho = \rho(\gamma)$*

Proof. To prove that conditions (1) and (2) uniquely determine $\lambda^{(i)/k}$ from $\lambda^{(i+1)/k}$, let $\lambda^{(i+1)/k} = \gamma/\rho$ with $\gamma = (\gamma_{i+1}, \gamma_{i+2}, \ldots, \gamma_r)$ and $\rho = (\rho_{i+1}, \rho_{i+2}, \ldots, \rho_r)$. Inductively assume that all conditions have been met up to this point. By construction, we have $\lambda^{(i)/k} = \overline{\gamma}/\overline{\rho}$ with

$$\overline{\gamma} = (a + \lambda_i, \gamma_{i+1}, \gamma_{i+2}, \ldots, \gamma_r) \text{ and } \overline{\rho} = (a, \rho_{i+1}, \rho_{i+2}, \ldots, \rho_r) \tag{3.2}$$

for some $a \geq \rho_{i+1}$. We claim that conditions (1) and (2) uniquely determine a. From (3.2) we derive that the hook length of the first cell in the bottom row of $\lambda^{(i)/k}$ (the leftmost framed cell in the figure) is $b_{a+1} + \lambda_i$ where b_j is the length of the j^{th} column of $\lambda^{(i+1)/k}$.



(3.3)

To satisfy (1) we must have

$$b_s + \lambda_i \leq k\,, \qquad \text{for all } s \geq a+1\,. \tag{3.4}$$

To satisfy (2), the squares *west* of the added row must have hook lengths $\geq k+1$. That is,

$$b_s + \lambda_i \geq k+1\,, \qquad \text{for all } s \leq a\,. \tag{3.5}$$

Since $b_a+\lambda_i \geq b_a+\lambda_{i+1}$, the inductive hypothesis guarantees that (3.5) will be true for all $a \leq \rho_{i+1}$. That is, the squares marked with a "$\bullet$" in the figure will necessarily have hook lengths exceeding k. It follows from these observations that to obtain a skew shape that satisfies both (1) and (2), we are forced to take a as the smallest integer such that $b_a+\lambda_i \leq k$. In this case, (3.5) is automatically satisfied. And (3.4) follows because $b_s \geq b_{a+1}$ for all $s \geq a+1$, given that when considering only the columns of $\lambda^{(i+1)}/^k$ starting from column $a+1$, the diagram $\lambda^{(i+1)}/^k$ is that of a partition. This completes the induction.

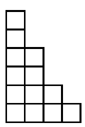
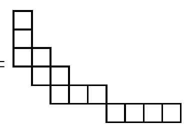
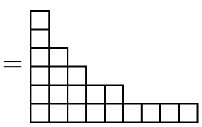
Now let $\lambda^{(1)}/^k = \gamma/\rho$. Our construction assures property (a). Also by construction, γ/ρ satisfies conditions (iii) and (iv) of Lemma 4, which were shown in the proof of that lemma to be sufficient to guarantee that γ is a $k+1$-core with $\rho=\rho(\gamma)$. Conversely, if a skew diagram $\overline{\gamma}/\overline{\rho}$ satisfies (a) and (b) then (a) implies $\overline{\gamma}=(\overline{\gamma}_1,\overline{\gamma}_2,\ldots\overline{\gamma}_r)$ and $\overline{\rho}=(\overline{\rho}_1,\overline{\rho}_2,\ldots\overline{\rho}_r)$ with $\overline{\gamma}_i-\overline{\rho}_i=\lambda_i$. Moreover property (b) assures that all the hook lengths of $\overline{\gamma}/\overline{\rho}$ do not exceed k and all the squares below ρ have lengths exceeding k. These two properties thus necessarily hold for all the skew diagrams

$$\overline{\lambda}^{(i)}/^k = (\overline{\gamma}_i,\overline{\gamma}_{i+1},\ldots\overline{\gamma}_r)/(\overline{\rho}_i,\overline{\rho}_{i+1},\ldots\overline{\rho}_r)\,.$$

Therefore, $\overline{\gamma}/\overline{\rho}=\lambda^{(1)}/^k$ by the uniqueness of such diagrams satisfying (1)-(2). $\square$

Note that the proof of Lemma 5 reveals that $\lambda^{(i)}/^k$ is the diagram obtained by attaching the row of length λ_i to the bottom of $\lambda^{(i+1)}/^k$ in the leftmost position so that no hook-lengths exceeding k are created. This algorithm gives a convenient method for constructing $\lambda/^k$.

Example 6. *Given* $\lambda=(4,3,2,2,1,1)$ *and* $k=4$,

$\lambda =$  $\Longrightarrow$ $\lambda/^4 =$  *revealing that* $\mathfrak{c}(\lambda) =$ 

We are now in the position to prove our bijection:

Theorem 7. $\mathfrak{p}$ *is a bijection from* $\mathcal{C}_{k+1}$ *onto* $\mathcal{P}_k$ *with inverse* $\mathfrak{c}$.

Proof. For $\lambda\in\mathcal{P}_k$, consider $\gamma=\mathfrak{c}(\lambda)\in\mathcal{C}_{k+1}$. Since $\lambda/^k=\gamma/\rho$ by definition of $\mathfrak{c}$, Lemma 5(a) and (b) implies that $\mathfrak{p}(\gamma)=\lambda$ and thus $\mathfrak{p}\left(\mathfrak{c}(\lambda)\right)=\lambda$. Now consider $\gamma\in\mathcal{C}_{k+1}$. Lemma 4 implies that $\mathfrak{p}(\gamma)=(\gamma_1-\rho(\gamma)_1,\ldots,\gamma_n-\rho(\gamma)_n)\in\mathcal{P}_k$, and thus by Lemma 5,

$$\gamma/\rho(\gamma)=\mathfrak{p}(\gamma)/^k\,. \tag{3.6}$$

Therefore, $\mathfrak{c}\left(\mathfrak{p}(\gamma)\right)=\gamma$ by definition of $\mathfrak{c}$. $\square$

4. The k-lattice

The notion of a k-skew diagram gives rise to an involution on $\mathcal{P}_k$, where λ is sent to the partition whose rows are obtained from the columns of $\lambda/^k$:

Definition 8. *For any* $\lambda\in\mathcal{P}_k$, *the "k-conjugate" of* λ *denoted* λ^{ω_k} *is the* k*-bounded partition given by the columns of* $\lambda/^k$.

Equivalently, we may define the k-conjugation as the partition given by the number of k-bounded cells in the columns of $\mathfrak{c}(\lambda)$, or simply $\lambda^{\omega_k}=\mathfrak{p}(\mathfrak{c}(\lambda)')$. This given, since $\mathfrak{p}=\mathfrak{c}^{-1}$, we see that k-conjugation is an involution by:

$$(\lambda^{\omega_k})^{\omega_k}=\mathfrak{p}\Big(\left[\mathfrak{c}\big(\mathfrak{p}\big(\mathfrak{c}(\lambda)'\big)\big)\right]'\Big)=\mathfrak{p}\big([\mathfrak{c}(\lambda)']'\big)=\mathfrak{p}\big(\mathfrak{c}(\lambda)\big)=\lambda\,. \tag{4.1}$$

Example 9. *With* λ *as in Example 6, the columns of* $\lambda/^4$ *give* $\lambda^{\omega_4}=(3,2,2,1,1,1,1,1,1)$.

Remark 10. *If $h_{(1,1)}(\lambda) \leq k$, all hooks of λ are k-bounded and thus $\lambda/^k = \lambda$. In this case, $\lambda^{\omega_k} = \lambda'$.*

Now, we can consider a partial order "$\preceq$" on the collection of k-bounded partitions stemming from k-conjugation.

Definition 11. *The "k-Young lattice" $\preceq$ on partitions in $\mathcal{P}_k$ is defined by the covering relation*

$$\lambda \to_k \mu \quad \text{when} \quad \lambda \subseteq \mu \quad \text{and} \quad \lambda^{\omega_k} \subseteq \mu^{\omega_k} \tag{4.2}$$

for $\mu, \lambda \in \mathcal{P}_k$ where $|\mu| - |\lambda| = 1$. Figure 1 gives the case $k = 2$.

While this poset on k-bounded partitions originally arose in connection to a rule for multiplying generalized Schur functions [9], we shall see in § 8 that this poset turns out to be isomorphic to the weak order on the quotient of the affine symmetric group by a maximal parabolic subgroup. Consequently, the k-Young lattice is in fact a lattice [18] (see [17] for a proof that follows from the identification of the k-Young lattice as a cone in a tiling of $\mathbb{R}^k$ by permutahedrons).

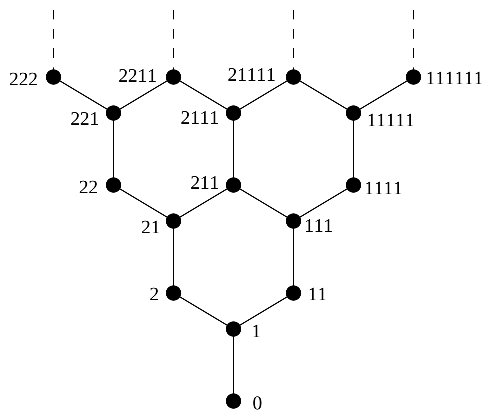


FIGURE 1. Hasse diagram of the k-Young lattice in the case $k = 2$.

The k-Young lattice generalizes the Young lattice since:

Property 12. *$\lambda \preceq \mu$ reduces to $\lambda \leq \mu$ when λ and μ are partitions such that $h_{(1,1)}(\mu) \leq k$.*

Proof. Since $\lambda \subseteq \mu$ when $\lambda \preceq \mu$, $h_{(1,1)}(\mu) \leq k$ implies that $h_{(1,1)}(\lambda) \leq k$. Remark 10 then implies that $\lambda^{\omega_k} = \lambda'$ and $\mu^{\omega_k} = \mu'$. Thus, the conditions that $\lambda \subseteq \mu$ and $\lambda^{\omega_k} \subseteq \mu^{\omega_k}$ reduce to $\lambda \subseteq \mu$ and $\lambda' \subseteq \mu'$, or simply $\lambda \subseteq \mu$. □

Although the ordering $\preceq$ is defined by the covering relation $\to_k$, the definition implies that

Property 13. *If $\lambda \preceq \mu$, then $\lambda \subseteq \mu$ and $\lambda^{\omega_k} \subseteq \mu^{\omega_k}$.*

It is important to notice that the converse of this statement does not hold. For example, with $k = 3$, $\lambda = (2, 2)$ and $\mu = (3, 2, 1, 1, 1, 1)$, we have $\lambda^{\omega_k} = \lambda$ and $\mu^{\omega_k} = \mu$ satisfying $\lambda \subseteq \mu$ and $\lambda^{\omega_k} \subseteq \mu^{\omega_k}$, but $\lambda \not\preceq \mu$. This is easily seen from [12, Th. 19] (since λ contains the 3-rectangle $(2, 2)$ while μ does not), or can be more tediously verified by constructing all chains using Theorem 23.

Similarly, although $\mathfrak{c}(\lambda) \subseteq \mathfrak{c}(\mu)$ does not necessarily imply $\lambda \subseteq \mu$, the converse is nevertheless true.

Property 14. *$\lambda \subseteq \mu$ implies $\mathfrak{c}(\lambda) \subseteq \mathfrak{c}(\mu)$.*

Proof. Let $\gamma_\lambda = \mathfrak{c}(\lambda)$ and $\gamma_\mu = \mathfrak{c}(\mu)$. Assume by contradiction that there is some row of γ_μ that is strictly shorter than a row of γ_λ and let row r be the highest such row. Let s be the last cell of row r of $\rho(\gamma_\lambda)$ and s' be the first cell of row r of $\gamma_\mu/\rho(\gamma_\mu)$. Since γ_λ has λ_r k-bounded hooks in row r and γ_μ has $\mu_r \geq \lambda_r$, the choice of r forces s' to be weakly to the left of s. Now let ℓ be the number of cells above s in γ_λ and ℓ' be the number of cells above s' in γ_μ. Since all rows of γ_λ above row r are weakly smaller than the corresponding rows of γ_μ and s' is weakly to the left of s, we must have $\ell' \geq \ell$. Thus the choice of s and s' gives

$$k \geq h_{s'}(\gamma_\mu) = \ell' + \mu_r \geq \ell + \lambda_r = h_s(\gamma_\lambda) - 1 > k\,,$$

where the last inequality holds since $h_s(\gamma_\lambda) > k$ and γ_λ has no $k+1$-hooks. The result follows by contradiction. □

In what follows, we shall develop an explicit description of the chains in this poset and provide a bijection with certain tableaux. These tableaux will then play a central role in the connection between the k-Young lattice and the weak order, and will also be discussed in our study of Macdonald polynomials (see § 11).

5. $k+1$-cores

Since the set of μ such that $\mu \supset \lambda$ and $|\mu| = |\lambda| + 1$ consists of all partitions obtained by adding a corner to λ, a subset of these partitions will be the elements that cover λ with respect to $\preceq$. The definition of $\preceq$ implies that to determine which corners can be added to give partitions that cover λ, we must find which corners can be added to λ so that the resulting diagram has a k-conjugate diagram that differs from λ^{ω_k} by only one box. Since a k-conjugate diagram is given by the number of k-bounded cells in the columns of a $k+1$-core, a close study of $k+1$-cores will enable us to characterize the allowable, or "*k-addable*", corners.

We begin with a number of basic properties of cores that rely on their associated residues. For the sake of completeness, we include all proofs although some may be known. For any integer d, we shall consider the diagonals of a partition, $D_d = \{(i,j) : j - i = d\}$. Note that a fixed $k+1$-residue $r = 0, 1, \ldots k$ occurs in successive diagonals $D_{r+\ell(k+1)}$ for any integer ℓ. A sequence of lattice cells $c_0, c_1, \ldots, c_n$ forms a "*$k+1$-string*" if the cells respectively lie in the successive diagonals:

$$D_{r+i(k+1)}, D_{r+(i+1)(k+1)}, D_{r+(i+2)(k+1)}, \ldots, D_{r+(i+n)(k+1)}\,.$$

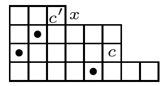
As such, all cells in a $k+1$-string have the same $k+1$-residue. For any $0 \leq r \leq k$, we shall also say that a square c' on the diagonal $D_{r+(\ell)(k+1)}$ is a "*$k+1$-predecessor*" of any square on $D_{r+(\ell+1)(k+1)}$. It is important to notice that if cell c' is a $k+1$-predecessor of a cell c in a partition λ, then $h_{c'\wedge c}(\lambda) = k+2$. This given, a $k+1$-string of cells $c_0, c_1, \ldots, c_n$ is simply a succession of cells where c_i is a $k+1$-predecessor of c_{i+1}.

Our point of departure here is the following known [5] basic result.

Property 15. *Let γ be a $k+1$-core.*

(1) *Let c and c' be extremal cells of γ with the same $k+1$-residue (c' weakly north-west of c).*
    (a) *If c is at the end of its row, then so is c'.*
    (b) *If c has a cell above it, then so does c'.*
(2) *Let c and c' be extremal cells of γ with the same $k+1$-residue (c' weakly south-east of c).*
    (a) *If c is at the top of its column, then so is c'.*
    (b) *If c has a cell to its right, then so does c'.*
(3) *Let c be a corner extremal cell and c' be an extremal cell of the same $k+1$-residue as c.*
    (a) *If c' is weakly south-east of c, then c' has a cell to its right.*
    (b) *If c' is weakly north-west of c, then c' has a cell above it.*

Proof. 1(a): Given that there is no cell to the right of c, it suffices to prove that there is no cell x to the right of the extremal cell c' that is a $k+1$-predecessor of c – by iteration the property will follow for non-predecessors c'. If $x \in \gamma$ then the hook-length of the cell determined by x and c is $k+1$ since no cell lies above x (it is to the right of an extremal cell). However, this contradicts that γ is a $k+1$-core implying that there is no cell to the right of c'.



2(a) follows from 1(a) since the transpose of a $k+1$-core is a $k+1$-core. Further, 1(b) and 2(b) are simply the contrapositive of 2(a) and 1(a) respectively with $c \leftrightarrow c'$. Finally, since a corner extremal cell has a cell to its right and above it, 3(a) and 3(b) follow respectively from 2(b) and 1(b). $\square$

Remark 16. *A $k+1$-core γ never has both a removable corner and an addable corner of the same $k+1$-residue. This follows by assuming there is an addable corner c of some $k+1$-residue i and using Property 15(3) with the corner extremal cell e immediately south-west of c. The proposition gives that all extremals of $k+1$-residue i either have a cell to their right or above. Therefore they are not removable corners.*

Property 17. *Let γ be a $k+1$-core*

(i) *If γ has a removable corner of $k+1$-residue i, then the collection of all removable corners of γ with $k+1$-residue i forms a $k+1$-string.*
(ii) *If γ has an addable corner with $k+1$-residue i, then the collection of all addable corners of γ with $k+1$-residue i forms a $k+1$-string.*

Proof. Let c and c' be the leftmost and rightmost removable corners of γ with $k+1$-residue i and let $c = c_0, c_1, c_2, \ldots, c_m = c'$ be the extremal cells (with c_j a $k+1$-predecessor of c_{j+1}) lying between c and c'. By Property 15(2a), each c_j lies at the top its column since it is south-east of c_0 and by (1a) lies at the end of its row since it is north-west of c_m. Therefore, each c_j is a removable corner.

To prove (ii), now let c and c' be leftmost and rightmost addable corners with $k+1$-residue i, and let e and e' be the extremal cells immediately south-east of c and c' respectively. With $e = e_0, e_1, e_2, \ldots, e_m = e'$ the extremal cells (with e_j a $k+1$-predecessor of e_{j+1}) lying between e and e', we claim that each e_j is corner extremal. This follows from Property 15(3). Indeed, each e_j must have a cell to its right because it is south-east of e and must have a cell above because it is north-west of e'. This forces the square c_j that is immediately north-east of e_j to be an addable corner of γ. Since each c_j has $k+1$-residue i and is a $k+1$-predecessor of c_{j+1} it follows that $c = c_0, \; c_1, c_2, \ldots, c_m = c'$ forms a $k+1$-string with head c and tail c'. $\square$

Armed with these special properties of $k+1$-cores, we turn to the study of certain operators that help us characterize the k-addable corners in the k-order, and that enable us to identify the k-Young lattice with the weak order on $\tilde{S}_{k+1}/S_{k+1}$. Operators that add a corner of given residue to partitions arose in [2] and [15], and coincide with those introduced in [16]. In the case of a $k+1$-core, since there is never both a removable and addable corner with the same $k+1$-residue by Remark 16, we consider the operator [8] that deletes or adds all such corners from elements in $\mathcal{C}_{k+1}$. That is,

Definition 18. *The "operator s_i" acts on a $k+1$-core by:*

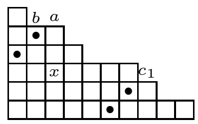
(a) *removing all removable corners with $k+1$-residue i if there is at least one removable corner of $k+1$-residue i*
(b) *adding all addable corners with $k+1$-residue i if there is at least one addable corner with $k+1$-residue i*
(c) *leaving it invariant when there are no addable or removable corners of $k+1$-residue i.*

We now give a number of properties that concern the s_i operators, beginning with the observation that they preserve the set $\mathcal{C}_{k+1}$. Note, some properties given here are implied in [8], but we shall include all proofs for the sake of completeness.

Property 19. *Let γ be a $k+1$-core.*

(i) *If γ has an addable corner of $k+1$-residue i, then $s_i(\gamma)$ is a $k+1$-core whose shape is obtained by adding all addable corners of $k+1$-residue i to γ.*
(ii) *If γ has a removable corner of $k+1$-residue i, then $s_i(\gamma)$ is a $k+1$-core obtained by deleting all removable corners of $k+1$-residue i from γ.*

Proof. Let $c_1, \ldots, c_n$ denote the collection of addable corners of γ with $k+1$-residue i where c_j is a $k+1$-predecessor of c_{j+1} for $j = 1, \ldots, n-1$. By Definition 18, the diagram of $s_i(\gamma)$ is obtained by adding $c_1, \ldots, c_n$ to γ. Since γ is a $k+1$ core and no hook of γ is affected by the action of s_i unless it corresponds to a cell that lies in a column or row containing some c_j, it suffices to check that there are no $k+1$-hooks in the rows of $s_i(\gamma)$ containing c_j (the columns will have no $k+1$-hooks by the transpose of our argument).



First, there can only be a $k+1$-hook in the row of $s_i(\gamma)$ containing c_1 if γ has a k-hook in this row. Assume by contradiction that a cell x in the row with c_1 has a k-hook in γ (see the figure). Let $a \notin \gamma$ denote the lowest square at the top of the column containing x. Since a is not an addable corner (c_1 is the highest addable corner of $k+1$-residue i), no cell lies in the square b to the left of a. Thus, the hook of $b \wedge c_1$ is $k+1$ contradicting that γ is a $k+1$-core. Therefore x is not a k-hook. For rows corresponding to c_j for $j > 1$, the cells $x = c_{j-1} \wedge c_j$ have hook-length k in γ by Property 17 while cells to the right (left) of x are strictly smaller (larger) than k by Remark 1. However, since $s_i(\gamma)$ is obtained by adding c_{j-1} and c_j to γ, the hook of x is $k+2$ in $s_i(\gamma)$ while the hooks to the right and left of x increase by one and are thus not $k+1$.

The proof when there is a removable corner of $k+1$-residue i in γ follows similarly. $\square$

Property 20. *If $\gamma \in \mathcal{C}_{k+1}$ then $s_i^2(\gamma) = \gamma$ for all $i \in \{0, \ldots, k\}$.*

Proof. When there are no removable or addable corners of $k+1$-residue i, s_i is clearly an involution. If γ has at least one removable corner of $k+1$-residue i then by Property 19(ii), $s_i(\gamma) = \delta$ is the $k+1$-core where all removable corners $c_1, \ldots, c_n$ of $k+1$-residue i have been removed from γ. Since Remark 16 implies that there can be no addable corners of $k+1$-residue i in γ, $c_1, \ldots, c_n$ are exactly the addable corners of δ and $s_i(\delta) = \gamma$ by Property 19(i). Similar reasoning proves that s_i is an involution if γ has at least one addable corner of $k+1$-residue i. $\square$

In fact, the s_i operators satisfy the affine Coxeter relations (see § 8). We now conclude this section with one last property.

Property 21. *For any $i = 0, \ldots, k$ and $k+1$-core γ, $s_i(\gamma)$ is a $k+1$-core such that:*

(i) *if $c_1, \ldots, c_n$ is the $k+1$-string of removable corners with $k+1$-residue i in γ, then the cells $c_1 \wedge c_2, \ldots, c_{n-1} \wedge c_n$ are the only cells whose hook exceeds k in γ but is k-bounded in $s_i(\gamma)$.*
(ii) *if $c_1, \ldots, c_n$ is the $k+1$-string of addable corners with $k+1$-residue i in γ, then the cells $c_1 \wedge c_2, \ldots, c_{n-1} \wedge c_n$ are the only cells whose hook is k-bounded in γ and exceeds k in $s_i(\gamma)$.*

Proof. In the case (ii) that γ has addable corners $c_1, \ldots, c_n$ of $k+1$-residue i, let A denote the set of cells with hooks exceeding k in γ and let B denote such cells in $s_i(\gamma)$. Thus, $s \in B - A$ satisfies $h_s(\gamma) \leq k$ and $h_s(s_i(\gamma)) > k$. However, since $s_i(\gamma)$ is a $k+1$-core by Property 19 and thus has no

$k+1$-hooks, we have $h_s(\gamma)) \leq k$ and $h_s\big(s_i(\gamma)\big) > k+1$. Since $s_i(\gamma)$ is obtained from γ by adding corners c_i, the hook of any cell x in $s_i(\gamma)$ has two more cells than the hook of x in γ only if $x = c_j \wedge c_\ell$ for some j and ℓ. However, of these, only $c_1 \wedge c_2, \ldots, c_{n-1} \wedge c_n$ have a k-bounded hook in γ since the c_i are separated by hooks of length $\ell(k+1)+k$. This proves Case (ii) and Case (i) follows by replacing $\gamma \leftrightarrow s_i(\gamma)$ and using $s_i^2 = id$. $\square$

6. k-Young lattice and $k+1$-cores

Recall that the set of elements covered by λ with respect to $\preceq$ is a subset of the partitions obtained by removing a corner box from λ. These partitions must also satisfy an additional condition that concerns the number of k-bounded hooks in the $\mathfrak{c}(\lambda)$. Equipped with the previous discussion of cores and their properties, we are now in the position to precisely understand how the number of k-bounded hooks in a $k+1$-core changes under the action of s_i. This then enables us to characterize the k-addable corners and consequently, the saturated chains in the k-Young lattice.

Proposition 22. *Given any k-bounded partition λ and $\gamma = \mathfrak{c}(\lambda)$,*

$$s_i(\gamma) = \begin{cases} \mathfrak{c}(\lambda - e_r) & \text{where } r \text{ is the highest row of } \gamma \text{ containing a removable corner of residue } i \\ \mathfrak{c}(\lambda + e_r) & \text{where } r \text{ is the highest row of } \gamma \text{ containing an addable corner of residue } i \\ \gamma & \text{when } \gamma \text{ has no removable or addable corner of residue } i \end{cases} \tag{6.1}$$

Further, when s_i does not act as the identity, it acts on γ by removing/adding corners so that every row and column of γ and $s_i(\gamma)$ has the same number of k-bounded cells except in one row (and column) where $s_i(\gamma)$ has one fewer/more k-bounded cell then γ. In particular, the total number of k-bounded cells in $s_i(\gamma)$ is exactly one more/fewer than in γ when γ contains an addable/removable corner of $k+1$-residue i.

Proof. Let $c_1, \ldots, c_n$ be the $k+1$-string of removable corners with $k+1$-residue i in γ. Property 21(i) reveals that the n k-bounded cells $c_1, \ldots, c_n$ in γ are not k-bounded in $s_i(\gamma)$ while the $n-1$ cells $c_1 \wedge c_2, \ldots, c_{n-1} \wedge c_n$ are k-bounded in $s_i(\gamma)$ but not in γ. Therefore, s_i acts on γ by decreasing the number of k-bounded cells only in the row containing c_1 and in the column containing c_n since $c_j \wedge c_{j+1}$ and c_{j+1} lie in the same row while $c_j \wedge c_{j+1}$ and c_j lie in the same column for $i = 1, \ldots, n-1$. Therefore, since $\lambda = \mathfrak{c}^{-1}(\gamma)$ indicates the number of k-bounded hooks in rows of γ, we have $\lambda - e_r = \mathfrak{c}^{-1}(s_i(\gamma))$ where r is the highest row of γ containing a removable corner (c_1) of $k+1$-residue i.

Replacing $\gamma \leftrightarrow s_i(\gamma)$ and using $s_i^2 = id$ proves the case with $c_1, \ldots, c_n$ the addable corners. $\square$

This given, we can characterize the k-addable corners.

Theorem 23. *The order $\preceq$ can be characterized by the covering relation*

$$\lambda \to_k \mu \iff \lambda = \mu - e_r \tag{6.2}$$

where r is any row of $\mathfrak{c}(\mu)$ with a removable corner whose $k+1$-residue i does not occur in a higher removable corner, in which case $s_i\big(\mathfrak{c}(\mu)\big) = \mathfrak{c}(\lambda)$. Equivalently, r can be characterized as a row of $\mathfrak{c}(\lambda)$ with an addable corner whose $k+1$-residue i does not occur in a higher addable corner.

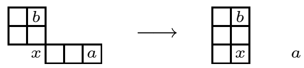
Example 24. *With $k = 4$ and $\lambda = (4,2,1,1)$,*

$$\mathfrak{c}(4,2,1,1) = \begin{array}{lllllll} 1 & & & & & & \\ \boxed{2} & & & & & & \\ \boxed{3} & 4 & & & & & \\ \boxed{4} & \boxed{0} & 1 & & & & \\ \boxed{0} & \boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} & \boxed{0} & 1 \end{array} \quad , \tag{6.3}$$

and thus the partitions that are covered by λ are $(4,1,1,1)$, and $(4,2,1)$, while those that cover it are $(4,2,1,1,1)$ and $(4,2,2,1)$.

Proof. Assume that r is a row of $\mathfrak{c}(\mu)$ with a removable corner a of $k+1$-residue i. If no removable corner of $\mathfrak{c}(\mu)$ with $k+1$-residue i lies higher than a, then Proposition 22 implies $s_i\big(\mathfrak{c}(\mu)\big) = \mathfrak{c}(\mu - e_r)$, and that the number of k-bounded cells of $\mathfrak{c}(\mu - e_r)$ differs from $\mathfrak{c}(\mu)$ in only one column where it is shorter by one. Therefore, by the definition of k-conjugation, $(\mu - e_r)^{\omega_k} \subseteq \mu^{\omega_k}$, implying $\mu - e_r \to_k \mu$.

On the other hand, assume there is a removable corner b of $k+1$-residue i higher than a. To prove $(\mu - e_r)^{\omega_k} \not\subseteq \mu^{\omega_k}$ (implying by definition that $\mu - e_r \not\to_k \mu$), it suffices to assume b is a $k+1$-predecessor of a since Property 17 implies the removable corners form a $k+1$-string. Now we shall show that a column of $\mu/^k$ is shorter than the same column of $(\mu - e_r)/^k$. The diagrams of $\mu/^k$ and $(\mu - e_r)/^k$ coincide strictly above row r by the recursive method of constructing a k-skew diagram presented in Lemma 5 (i.e. Example 6). In row r, the square $x = b \wedge a$ (see (6.4)) must satisfy $k < h_x(\mu/^k) \leq k+1$ (or $h_x(\mu/^k) = k+1$) since b is a $k+1$-predecessor of a, both removable corners. Therefore, deleting a cell in row r allows $x \in (\mu - e_r)/^k$ without producing a hook exceeding k. Thus the column with x in $\mu - e_r$ is longer than the corresponding column in $\mu/^k$.



(6.4)

Finally, r can be equivalently characterized as the highest row in $\mathfrak{c}(\lambda)$ with an addable corner of given $k+1$-residue since the addable corners of $\mathfrak{c}(\lambda)$ are exactly the removable corners of $\mathfrak{c}(\mu)$, given that $\mathfrak{c}(\mu) = s_i\big(\mathfrak{c}(\lambda)\big)$. □

Thus, we combine this result with Proposition 22 to derive the following consequences:

Corollary 25. *Given k-bounded partitions λ and μ,*

$$\lambda \to_k \mu \iff \mathfrak{c}(\lambda) \subset \mathfrak{c}(\mu) \text{ and } s_i\big(\mathfrak{c}(\lambda)\big) = \mathfrak{c}(\mu) \text{ for some } i \in \{0, \ldots, k\}\,. \tag{6.5}$$

This given, we are now able to provide a core-characterization of the saturated chains from the empty partition (hereafter $\emptyset = \lambda^{(0)}$) to any k-bounded partition $\lambda \vdash n$:

$$\mathcal{D}^k(\lambda) = \Big\{(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(n)} = \lambda) \;:\; \lambda^{(j)} \to_k \lambda^{(j+1)}\Big\}\,. \tag{6.6}$$

Corollary 26. *The saturated chains to the vertex $\lambda \vdash n$ in the k-lattice are given by*

$$\mathcal{D}^k(\lambda) = \Big\{(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(n)} = \lambda) \,:\, \mathfrak{c}(\lambda^{(j)}) \subset \mathfrak{c}(\lambda^{(j+1)}) \text{ and } \mathfrak{c}(\lambda^{(j+1)}) = s_i\Big(\mathfrak{c}(\lambda^{(j)})\Big) \text{ for some } i\Big\}$$

7. Standard k-tableaux

Motivated by the proposed role of k-lattice chains in the study of certain Macdonald polynomial expansion coefficients, we pursue a tableaux interpretation for these chains. In this section we shall provide a bijection between the set of chains $\mathcal{D}^k(\lambda)$ and a new family of tableaux defined on cores. Following our discussion in § 8 of the connection between the k-lattice and weak order on affine permutations, a bijection from these tableaux to certain reduced expressions will also be revealed.

7.1. **Definition.**

Definition 27. *A k-tableau T of shape $\gamma \in \mathcal{C}_{k+1}$ with n k-bounded hooks is a filling of γ with integers $\{1, \ldots, n\}$ such that*

(i) rows and columns are strictly increasing

(ii) repeated letters have the same $k+1$-residue

The set of all k-tableaux of shape $\mathfrak{c}(\lambda)$ is denoted by $\mathcal{T}^k(\lambda)$.

Example 28. *$\mathcal{T}^3(3,2,1,1)$, or the set of 3-tableaux of shape $(6,3,1,1)$, is*

$$
\begin{array}{|c|c|c|c|c|c|}
\hline 7 \\
\hline 5 \\
\hline 4 & 6 & 7 \\
\hline 1 & 2 & 3 & 4 & 6 & 7 \\
\hline
\end{array}
\qquad
\begin{array}{|c|c|c|c|c|c|}
\hline 7 \\
\hline 6 \\
\hline 4 & 5 & 7 \\
\hline 1 & 2 & 3 & 4 & 5 & 7 \\
\hline
\end{array}
\qquad
\begin{array}{|c|c|c|c|c|c|}
\hline 7 \\
\hline 4 \\
\hline 3 & 6 & 7 \\
\hline 1 & 2 & 4 & 5 & 6 & 7 \\
\hline
\end{array}
\qquad
\begin{array}{|c|c|c|c|c|c|}
\hline 7 \\
\hline 4 \\
\hline 2 & 6 & 7 \\
\hline 1 & 3 & 4 & 5 & 6 & 7 \\
\hline
\end{array}
\tag{7.1}
$$

Our first task is to show that deleting all occurrences of the largest letter from a given k-tableau produces a new k-tableau. For this we shall need yet another property about cores.

Property 29. *Let γ and δ be $k+1$-cores. If $\gamma \subset \delta$, then the number of k-bounded hooks of γ is smaller than that of δ.*

Proof. Let n_γ and n_δ denote the number of k-bounded hooks of the $k+1$-cores γ and δ respectively. If $|\delta| = 1$ then $\gamma = \emptyset$ and we have $n_\delta = 1$ while $n_\gamma = 0$. Now, assume the result holds for all $k+1$-cores δ with $n_\delta < N$, and consider a pair of $k+1$-cores γ and δ such that $n_\delta = N$ and $\gamma \subset \delta$. For i the $k+1$-residue of a removable corner of δ, $s_i(\delta)$ is a $k+1$-core whose number of k-bounded hooks is less than N since $n_{s_i(\delta)} = n_\delta - 1$ by Proposition 22. Thus, if $\gamma \subset s_i(\delta)$ then $n_\gamma < n_{s_i(\delta)} < N$ by induction. Further, if $\gamma = s_i(\delta)$ then $n_\gamma = n_{s_i(\delta)} < N$. Finally, in the case that $\gamma \not\subset s_i(\delta)$, we have $s_i(\gamma) \subseteq s_i(\delta)$ since any cell of γ not in $s_i(\delta)$ is a removable corner of γ of $k+1$-residue i. However, $s_i^2 = id$ implies that $s_i(\gamma) \subset s_i(\delta)$ and thus by induction, $n_{s_i(\gamma)} < n_{s_i(\delta)} \implies n_\gamma < n_\delta$ by Proposition 22. □

Definition 30. *For any partition ν, let $\#(\nu)$ denote the smallest number such that there exists a $k+1$-core γ with $\#(\nu)$ k-bounded hooks and $\nu \subseteq \gamma$.*

Lemma 31. *For any partition ν, $\#(\nu)$ is the smallest number of letters needed to fill the shape ν in such a way that*

(i) rows and columns are strictly increasing

(ii) repeated letters have the same $k+1$-residue

Proof. Let γ be a $k+1$-core with $\#(\nu)$ k-bounded hooks such that $\nu \subseteq \gamma$ and let n be the smallest number of letters needed to fill ν properly (i.e. satisfying conditions (i) and (ii)).

To show that $n \le \#(\nu)$, it suffices to find a proper filling of γ using $\#(\nu)$ letters. For i_1 the $k+1$-residue of a removable corner of γ, put letter $N = \#(\nu)$ in all removable corners of γ with residue i_1. For i_2 the $k+1$-residue of a removable corner in $s_{i_1}(\gamma)$, put letter $N-1$ in all cells of γ corresponding to corners of $s_{i_1}(\gamma)$ with residue i_2. By iteration, we obtain a proper filling of γ (and consequently of its subshape ν) with N letters since Proposition 22 implies that each s_j decreases the number of k-bounded hooks by one.

To prove that $n \ge N$, let T be a proper filling of the shape $\nu = \nu^{(n+1)}$ and consider the tableaux of shape $\nu^{(i)}$ obtained by deleting the letters $i, \dots, n$ from T (for $i = n, \dots, 1$). The lemma will follow by showing that $\#(\nu^{(i)}) \ge \#(\nu^{(i+1)}) - 1$. That is, starting with $\#(\nu) = N$, this would imply that $\#(\nu^{(n)}) \ge N-1$ and then $\#(\nu^{(n-1)}) \ge N-2$, and by iteration that the empty partition $\nu^{(1)}$ satisfies $\#(\nu^{(1)}) \ge N-n$. Therefore, $0 \ge N-n$.

Let $a_1, \dots, a_m$ denote the positions of the letter n in ν and let $\bar\nu$ be the partition ν minus these cells. It remains to show that $\#(\bar\nu) \ge \#(\nu) - 1$. Let $\bar\gamma$ and γ denote $k+1$-cores with $\#(\bar\nu)$ and $\#(\nu)$ k-bounded hooks, respectively, and where $\bar\nu \subseteq \bar\gamma$ and $\nu \subseteq \gamma$. Note that since $a_1, \dots, a_m$ are removable corners of some $k+1$-residue i in ν, they are addable corners of $\bar\nu$. Thus, these are either addable corners of $\bar\gamma$ or lie in $\bar\gamma$. If all $a_1, \dots, a_m \in \bar\gamma$ then $\nu \subseteq \bar\gamma$ implies $\#(\bar\nu) \ge \#(\nu)$ by definition of $\#(\nu)$. Otherwise, given a_j is an addable corner of $\bar\gamma$, the number of k-bounded hooks M of $s_i(\bar\gamma)$ is $\#(\bar\nu)+1$ by Proposition 22. However, since $\nu \subseteq s_i(\bar\gamma)$, $M \ge \#(\nu)$ and we reach our claim. □

Proposition 32. *Deleting all cells filled with the letter $n = |\lambda|$ from $T \in \mathcal{T}^k(\lambda)$ gives a k-tableau $\bar T \in \mathcal{T}^k(\mu)$, where $\mathfrak{c}(\mu) = s_i\big(\mathfrak{c}(\lambda)\big)$ for i the $k+1$-residue of the cells containing the letter n.*

Proof. Let γ be the shape of T, and let $\bar{T}$ be T without letter n. To prove that $\bar{T}$ is a k-tableau, it suffices to show that the shape ν of $\bar{T}$ is that of a $k+1$-core. If i is the $k+1$-residue of some removable corner containing the letter n in T, then $s_i(\gamma)$ is a $k+1$-core with $n-1$ k-bounded hooks by Proposition 22 and $s_i(\gamma) \subseteq \nu$. If we assume by contradiction that ν is not a $k+1$-core then $s_i(\gamma) \subset \nu$. Thus, any $k+1$-core δ containing ν also satisfies $s_i(\gamma) \subset \delta$. Therefore, Property 29 implies that δ has more k-bounded hooks than $s_i(\gamma)$ and thus, $\#(\nu) > n-1$ by definition. Lemma 31 then leads to the contradiction saying that $\bar{T}$ of shape ν cannot be properly filled with $n-1$ letters. Finally, since ν is a $k+1$-core, it cannot have a removable and addable corner of the same $k+1$-residue by Remark 16. Therefore, given that i is the $k+1$-residue of a removable corner in γ containing n (thus an addable corner in ν), *every* removable corner of residue i in γ contains n, implying $\nu = s_i(\gamma)$. $\square$

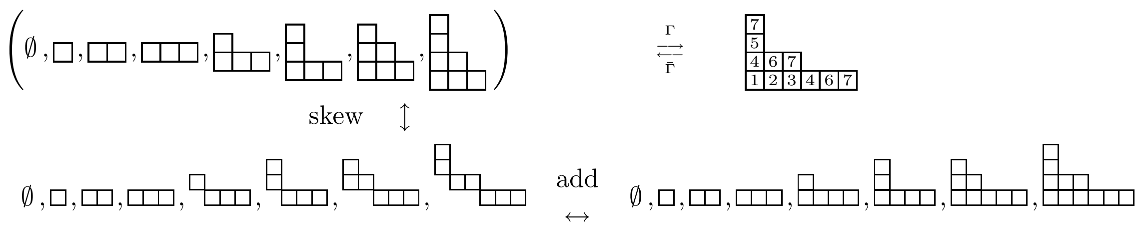
7.2. **Bijection: k-tableaux and saturated chains.** We now introduce two maps that lead to our bijection between chains $\mathcal{D}^k(\lambda)$ in the k-lattice and k-tableaux $\mathcal{T}^k(\lambda)$.

Definition 33. *For any path $P = (\lambda^{(0)}, \ldots, \lambda^{(n)}) \in \mathcal{D}^k(\lambda)$, let $\Gamma(P)$ be the tableau constructed by putting letter j in positions $\mathfrak{c}(\lambda^{(j)})/\mathfrak{c}(\lambda^{(j-1)})$ for $j = 1, \ldots, n$.*

Given $T \in \mathcal{T}^k(\lambda)$, let $\bar{\Gamma}(T) = (\lambda^{(0)}, \ldots, \lambda^{(n)})$ where $\mathfrak{c}(\lambda^{(j)})$ is the shape of the tableau obtained by deleting letters $j+1, \ldots, n$ from T.

To compute the action of Γ on a path, we view the action of $\mathfrak{c}$ as a composition of maps on a partition – first skew the diagram and then add the squares below the skew to obtain a core.

Example 34. *With $k = 3$:*



The example suggests that $\Gamma^{-1} = \bar{\Gamma}$. This will indeed follow from the following lemmas:

Lemma 35. *If $P \in \mathcal{D}^k(\lambda)$, then $\Gamma(P) \in \mathcal{T}^k(\lambda)$.*

Proof. Since the only path in $\mathcal{D}^k(\square)$ is $P = (\emptyset, \square)$, and $\Gamma(P) = \boxed{1} \in \mathcal{T}^k(\square)$, we proceed by induction on $|\lambda|$. Assume that Γ sends any path of length $n-1$ to a k-tableau on $n-1$ letters and let $P = (\lambda^{(0)}, \ldots, \lambda^{(n)}) \in \mathcal{D}^k(\lambda)$. The definition of Γ implies that $\Gamma(P)$ is obtained by adding letter n to $T^{n-1} = \Gamma(\lambda^{(0)}, \ldots, \lambda^{(n-1)})$ in positions $\mathfrak{c}\left(\lambda^{(n)}\right)/\mathfrak{c}\left(\lambda^{(n-1)}\right)$. Since $\mathfrak{c}\left(\lambda^{(n)}\right) = s_i\left(\mathfrak{c}(\lambda^{(n-1)})\right)$ for some i by Corollary 26, these positions are the addable corners of $\mathfrak{c}(\lambda^{(n-1)})$ with $k+1$-residue i. Therefore, the letters n in $\Gamma(P)$ all have the same $k+1$-residue and no two can occur in the same row or column. Thus $\Gamma(P)$ is a k-tableau of shape $\mathfrak{c}(\lambda^{(n)})$ given that the subtableau T^{n-1} is a k-tableau by induction. $\square$

Lemma 36. *If $T \in \mathcal{T}^k(\lambda)$, then $\bar{\Gamma}(T) \in \mathcal{D}^k(\lambda)$.*

Proof. Consider $T \in \mathcal{T}^k(\lambda)$ and let $\bar{\Gamma}(T) = (\lambda^{(0)}, \ldots, \lambda^{(n)})$. For all j, if $\mathfrak{c}(\lambda^{(j)}) \subset \mathfrak{c}(\lambda^{(j+1)})$ and $s_i(\mathfrak{c}(\lambda^{(j+1)})) = \mathfrak{c}(\lambda^{(j)})$ for some i, then $\bar{\Gamma}(T) \in \mathcal{D}^k(\lambda)$ by Corollary 26. The definition of $\bar{\Gamma}$ implies that $\mathfrak{c}(\lambda^{(n)})$ is the shape of T, and further that $\mathfrak{c}(\lambda^{(n-1)})$ is the shape of T minus all occurrences of n. Clearly $\mathfrak{c}(\lambda^{(j)}) \subset \mathfrak{c}(\lambda^{(j+1)})$, and further $s_i\big(\mathfrak{c}(\lambda^{(n)})\big) = \mathfrak{c}(\lambda^{(n-1)})$ where i is the $k+1$-residue of the cells containing n by Proposition 32. Thus, the codomain of $\bar{\Gamma}$ is $\mathcal{D}^k(\lambda)$ by iteration. $\square$

We are now set to prove that Γ is a bijection between saturated chains and k-tableaux.

Theorem 37. *Γ is a bijection between $\mathcal{D}^k(\lambda)$ and $\mathcal{T}^k(\lambda)$ with $\Gamma^{-1} = \bar{\Gamma}$.*

Proof. From Lemmas 35 and 36, it suffices to prove that Γ and $\bar{\Gamma}$ are inverses. We start by showing that $\bar{\Gamma}\Gamma(P) = P$. Given $\Gamma(\lambda^{(0)}, \ldots, \lambda^{(n)}) = T$, we must show that if $\bar{\Gamma}(T) = (\mu^{(0)}, \ldots, \mu^{(n)})$ then $\mathfrak{c}(\mu^{(\ell)}) = \mathfrak{c}(\lambda^{(\ell)})$ for $\ell = 0, \ldots, n$. The definition of Γ implies that shape$(T) = \mathfrak{c}(\lambda^{(n)})$ and that the letter n lies in $\mathfrak{c}(\lambda^{(n)})/\mathfrak{c}(\lambda^{(n-1)})$. At the same time, the definition of $\bar{\Gamma}$ implies that shape$(T) = \mathfrak{c}(\mu^{(n)})$ and $\mathfrak{c}(\mu^{(n-1)})$ is the shape of the tableau obtained by deleting all occurrences of the letter n from T. Therefore, $\mathfrak{c}(\mu^{(n-1)}) = \mathfrak{c}(\lambda^{(n-1)})$. By iteration, $\bar{\Gamma}\Gamma(P) = P$.

On the other hand, given $\bar{\Gamma}(T) = (\lambda^{(0)}, \ldots, \lambda^{(n)})$, we must show that if $\Gamma(\lambda^{(0)}, \ldots, \lambda^{(n)}) = \bar{T}$ then $\bar{T} = T$. The definition of Γ implies that $\bar{T}$ is the tableau obtained by filling the cells of $\mathfrak{c}(\lambda^{(j+1)})/\mathfrak{c}(\lambda^{(j)})$ with letter $j+1$. However, by definition of $\bar{\Gamma}$, $\mathfrak{c}(\lambda^{(j)})$ is the shape obtained by deleting the letters $j+1, \ldots, N$ from T, and $\mathfrak{c}(\lambda^{(j+1)})$ is the shape obtained by deleting $j+2, \ldots, N$ from T. Therefore, the cells $\mathfrak{c}(\lambda^{(j+1)})/\mathfrak{c}(\lambda^{(j)})$ in T are filled with letter $j+1$. Thus $T = \bar{T}$. ◻

8. The k-Young lattice and the weak order on $\tilde{S}_{k+1}/S_{k+1}$

In this section we shall see how the $k+1$-core characterization of the k-Young lattice covering relations given in Corollary 25 leads to the identification of the k-Young lattice as the weak order on $\tilde{S}_{k+1}/S_{k+1}$. A by-product of this result is a simple bijection between reduced words and k-tableaux and one between k-bounded partitions and affine permutations in $\tilde{S}_{k+1}/S_{k+1}$.

8.1. **The isomorphism.** To establish that the k-Young lattice is isomorphic to weak order on the set of minimal coset representatives of $\tilde{S}_{k+1}/S_{k+1}$, we rely foremost on the fact [8] that the s_i operators satisfy the affine Coxeter relations (2.4), and thus provide a realization of the affine symmetric group on $k+1$-cores.

Property 38. *The s_i operators satisfy*

$$s_i^2 = id, \qquad s_i s_j = s_j s_i \quad (i - j \neq \pm 1 \mod k+1), \quad \textit{and} \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}\,. \tag{8.1}$$

The following map is then well defined:

Definition 39. *For $\sigma \in \tilde{S}_{k+1}$, let $\mathfrak{s}$ send σ to a $k+1$-core by*

$$\mathfrak{s} : \sigma = s_{i_1} \cdots s_{i_\ell} \cdot \emptyset\,, \tag{8.2}$$

where $i_1 \cdots i_\ell$ is any reduced word for σ and $\emptyset$ is the empty $k+1$-core.

A characterization for Bruhat order in terms of the containment of cores stemming from this map is provided by Lascoux in [8]. To be precise,

Proposition 40. *The map $\mathfrak{s} : \tilde{S}_{k+1}/S_{k+1} \rightarrow \mathcal{C}_{k+1}$ is an isomorphism from Bruhat order on $\tilde{S}_{k+1}/S_{k+1}$ to Young order ($\subseteq$) on $\mathcal{C}_{k+1}$.*

We are thus able to obtain from our $k+1$-core characterization of the chains in the k-lattice that this lattice is isomorphic to the weak order on $\tilde{S}_{k+1}/S_{k+1}$:

Corollary 41. *Let $\sigma, \tau \in \tilde{S}_{k+1}/S_{k+1}$, and let $\lambda = \mathfrak{p}(\mathfrak{s}(\sigma))$ and $\mu = \mathfrak{p}(\mathfrak{s}(\tau))$. Then*

$$\sigma \lessdot_w \tau \iff \lambda \rightarrow_k \mu\,. \tag{8.3}$$

Proof. Proposition 40 implies a characterization of the covering relations for weak order on $\tilde{S}_{k+1}/S_{k+1}$. That is, since $\mathfrak{s}$ is a bijection and the weak order is a suborder of the Bruhat order, we have for $\sigma, \tau \in \tilde{S}_{k+1}/S_{k+1}$

$$\sigma \lessdot_w \tau \iff \mathfrak{s}(\sigma) \subset \mathfrak{s}(\tau) \quad \text{and} \quad s_i\,\mathfrak{s}(\sigma) = \mathfrak{s}(\tau) \text{ for some } i\,. \tag{8.4}$$

The result thus follows from the characterization of $\rightarrow_k$ given in Corollary 25. ◻

8.2. **Bijection: k-tableaux and reduced words.** We have seen in Theorem 37 that the saturated chains to shape λ in the k-lattice are in bijection with k-tableaux of shape $\mathfrak{p}(\gamma)$. On the other hand, the reduced words for $\sigma \in \tilde{S}_{k+1}/S_{k+1}$ encode the chains to σ. Corollary 41 thus implies there is a bijection between k-tableaux of shape γ and the reduced words for $\mathfrak{s}^{-1}(\gamma)$.

This bijection arises naturally by noting from Corollary 26 that the association between a k-tableau and chain $(\lambda^{(0)}, \ldots, \lambda^{(n)} = \lambda)$ in the k-Young lattice is determined by a sequence $i_n \cdots i_2\, i_1$ such that $s_{i_j}(\mathfrak{c}(\lambda^{(j-1)})) = \mathfrak{c}(\lambda^{(j)})$ for $j = 1, \ldots, n$. However, this sequence can also be viewed as a reduced word for the permutation σ where $\mathfrak{s}(\sigma) = \mathfrak{c}(\lambda)$ by Eq. (8.4). Therefore, the following map provides the desired bijection:

Definition 42. *For a k-tableau T with m letters where i_a is the $k+1$-residue of the letter a, define*

$$\mathfrak{w} : T \mapsto i_m \cdots i_1 \,.$$

For $w = i_m \cdots i_1 \in Red(\sigma)$, $\mathfrak{w}^{-1}(w)$ is the tableau with letter $\ell = 1, \ldots, m$ occupying the cells of $s_{i_\ell} \cdots s_{i_1} \cdot \emptyset / s_{i_{\ell-1}} \cdots s_{i_1} \cdot \emptyset$.

Example 43. *With $k = 3$:*

$$T = \begin{array}{|c|c|c|c|c|c|} \cline{1-1} 7 \\ \cline{1-1} 6 \\ \cline{1-3} 4 & 5 & 7 \\ \hline 1 & 2 & 3 & 4 & 5 & 7 \\ \hline \end{array} \qquad \overset{\mathfrak{w}}{\longleftrightarrow} \qquad 1\,2\,0\,3\,2\,1\,0 \quad \textit{since the 4-residues are} \quad \begin{array}{|c|c|c|c|c|c|} \cline{1-1} 1 \\ \cline{1-1} 2 \\ \cline{1-3} 3 & 0 & 1 \\ \hline 0 & 1 & 2 & 3 & 0 & 1 \\ \hline \end{array}$$

Proposition 44. *The map $\mathfrak{w} : \mathcal{T}^k(\lambda) \longrightarrow Red(\sigma)$ is a bijection, where $\sigma \in \tilde{S}_{k+1}/S_{k+1}$ is defined uniquely by $\mathfrak{c}(\lambda) = \mathfrak{s}(\sigma)$.*

We will now make use of canonical chains in the k-Young lattice to obtain a simple bijection between k-bounded partitions and permutations in $\tilde{S}_{k+1}/S_{k+1}$.

Definition 45. *For any partition λ, let "w_λ" be the word obtained by reading the $k+1$-residues in each row of λ, from right to left, starting with the highest removable corner and ending in the first cell of the first row. Further, let "σ_λ" be the affine permutation corresponding to w_λ.*

Example 46. *For $\lambda = (3, 2, 2, 1)$ and $k = 3$, $w_\lambda = 1\,3\,2\,0\,3\,2\,1\,0$ and $\sigma_\lambda = \hat{s}_1\hat{s}_3\hat{s}_2\hat{s}_0\hat{s}_3\hat{s}_2\hat{s}_1\hat{s}_0$ since:*

$$\lambda = \begin{array}{|c|c|c|} \cline{1-1} 1 \\ \cline{1-2} 2 & 3 \\ \cline{1-2} 3 & 0 \\ \hline 0 & 1 & 2 \\ \hline \end{array} \tag{8.5}$$

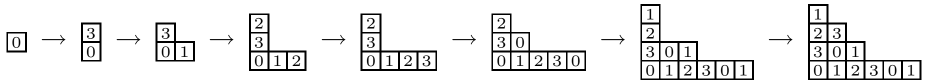
Proposition 47. *σ_λ belongs to $\tilde{S}_{k+1}/S_{k+1}$ and $\mathfrak{s}(\sigma_\lambda) = \mathfrak{c}(\lambda)$.*

Proof. Consider $\lambda \in \mathcal{P}_k$. In light of Proposition 44, it suffices to show that there is some k-tableau T of shape $\mathfrak{c}(\lambda)$ where $\mathfrak{w}(T) = w_\lambda$. Note that Corollaries 26 and 37 imply $\mathfrak{w}(T)$ (of shape $\mathfrak{c}(\lambda)$) is obtained from a certain chain $(\lambda^{(0)}, \ldots, \lambda^{(n)} = \lambda)$ in the k-Young lattice by taking the sequence $i_n \cdots i_2\, i_1$ such that $s_{i_j}(\mathfrak{c}(\lambda^{(j-1)})) = \mathfrak{c}(\lambda^{(j)})$ for $j = 1, \ldots, n$. Now, there exists a canonical saturated chain P_λ (and thus a canonical sequence $i_n \cdots i_2\, i_1$) such that $\lambda^{(j)}$ is obtained by removing the highest removable corner of $\lambda^{(j+1)}$. The existence of such a chain is ensured by Theorem 23 since the highest removable corner of a partition is always the highest of its $k+1$-residue. However, the highest removable corner of a partition λ coincides with the highest removable corner of $\mathfrak{c}(\lambda)$ and we therefore find that $i_n \cdots i_2\, i_1$ is exactly w_λ. □

Given the bijection between k-bounded partitions and $k+1$-cores, this immediately provides a bijection between k-bounded partitions and permutations in $\tilde{S}_{k+1}/S_{k+1}$.

Corollary 48. *The map $\phi : \mathcal{P}_k \to \tilde{S}_{k+1}/S_{k+1}$ where $\phi(\lambda) = \sigma_\lambda$ is a bijection whose inverse is $\phi^{-1} = \mathfrak{p} \circ \mathfrak{s}$.*

Example 49. *Given $\sigma \in \tilde{S}_4^I$ with a reduced expression $w = 3\,1\,0\,3\,2\,1\,3\,0$, we construct the shape:*

$s_3 s_1 s_0 s_3 s_2 s_1 s_3 s_0 \cdot \emptyset =$ 

from which we read the number of 3-bounded hooks to obtain $\phi^{-1}(\sigma) = (3,2,2,1)$. Conversely, σ can be recovered from $(3,2,2,1)$ by using Example 46 to find $\phi(3,2,2,1) = \hat{s}_1\hat{s}_3\hat{s}_2\hat{s}_0\hat{s}_3\hat{s}_2\hat{s}_1\hat{s}_0$ (one easily checks that $3\,1\,0\,3\,2\,1\,3\,0$ and $1\,3\,2\,0\,3\,2\,1\,0$ are reduced words for the same permutation).

9. Comparing elements differing by more than one box

Now that we have been able in § 7 to explicitly understand the covering relation for the k-order and to characterize the chains, it is natural to ask what can be said about the relation among vertices differing by more than one box. In this section we shall prove that

If μ/λ and $\mu^{\omega_k}/\lambda^{\omega_k}$ are horizontal and vertical strips respectively, then $\lambda \preceq \mu$.

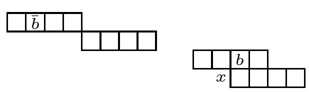
A number of somewhat technical properties will lead us to this result and shall also be used in our development of a semi-standard version of the k-tableaux corresponding to certain chains in the k-Young lattice. We begin by continuing the study of $k+1$-cores, concentrating on pairs $\gamma \subseteq \delta$.

Definition 50. *Let γ and δ be $k+1$-cores with $\gamma \subseteq \delta$. A "rowadder" is a cell $s \in \delta/\gamma$ such that there is no cell in δ/γ that is a $k+1$-predecessor of s.*

Two properties concerning the existence of rowadders are needed.

Property 51. *If γ and δ are $k+1$-cores with $\gamma \subseteq \delta$, then δ/γ has a rowadder at the top of the leftmost column that contains more than one cell.*

Proof. Let b (of $k+1$-residue i) denote the cell in δ/γ at the top of the leftmost column with more than one cell. Note that if $x \in \gamma$ lies immediately southwest of b, then no cell of γ lies to the right of x. Further, the diagram of δ/γ to the left of b is a series of horizontal rows since columns to the left of b have at most one cell.

 (9.1)

Suppose by contradiction that there is a cell $\bar{b} \in \delta/\gamma$ that is a $k+1$-predecessor of b. Then $h_{\bar{b}\wedge x}(\gamma) = k+1$, violating the assumption that γ is a $k+1$-core. $\square$

Remark 52. *For partitions λ and μ, μ/λ is a horizontal strip iff $\lambda \subseteq \mu$ and $\lambda_r \geq \mu_{r+1}$ for all r. Further, μ/λ is a vertical strip iff $\mu_r - \lambda_r \in \{0,1\}$ for all r.*

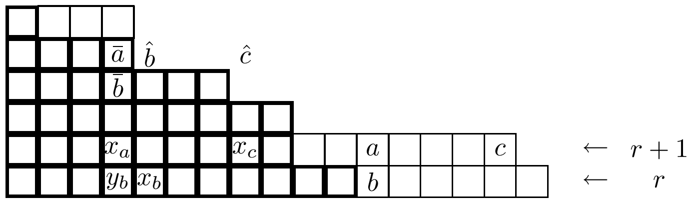
Property 53. *Consider $\gamma = \mathfrak{c}(\lambda)$ and $\delta = \mathfrak{c}(\mu)$ with $\gamma \subseteq \delta$. Let ℓ denote the leftmost column of δ/γ with more than one cell.*

(i) If there are rowadders in the top two cells of δ/γ in column ℓ, then μ/λ is not a horizontal strip.

(ii) If there is no rowadder in the second row of δ/γ in of column ℓ, then $\lambda^{\omega_k} \not\subseteq \mu^{\omega_k}$.

Proof. Case (i): the number of k-bounded hooks in row r of δ (resp. γ) is μ_r (resp. λ_r). Thus, by Remark 52, it suffices to prove that there are at least $\lambda_r + 1$ k-bounded hooks in row $r+1$ of δ for some r. We shall consider the rows $r+1$ and r containing rowadders $a, b \in \delta/\gamma$. If $\bar{a}$ denotes the extremal cell of γ that is a $k+1$-predecessor of a, then the extremal cell $\bar{b}$ of γ that $k+1$-precedes b either lies below or beside $\bar{a}$ since $\bar{a}$ is extremal. However, if $\bar{b}$ lies beside $\bar{a}$, the hook of $\bar{b} \wedge b$ is

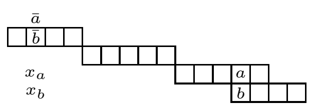
$k+1$ in the $k+1$-core γ implying this case does not occur. When $\bar{b}$ lies below $\bar{a}$, the square $\hat{b}$ to the right of $\bar{a}$ is not in δ since $\bar{b}$ is extremal in γ and b is a rowadder:



(9.2)

Notice that the hook of $x_b = \hat{b} \wedge b$ in γ is k-bounded while the hook of $y_b = \bar{b} \wedge b$ in γ exceeds k. Therefore, λ_r is the number of cells strictly between y_b and b (equivalently, x_a and a). To determine the number of k-bounded hooks of δ, let c denote the last cell in row $r+1$ of δ and $\hat{c}$ the square a $k+1$-predecessor of c in the row with $\hat{b}$. Since $\hat{c}$ does not belong to δ, the hook length of $x_c = \hat{c} \wedge c$ is at most $k+1$. But because δ is a $k+1$-core, $x_c = \hat{c} \wedge c$ thus has a k-bounded hook in δ as do all the cells of δ to the right of x_c. Given that the number of cells strictly between x_c and c equals the number of cells, λ_r, strictly between x_a and a, we have at least $\lambda_r + 1$ k-bounded hooks in row $r+1$ of δ as claimed.

Case (ii): Since the number of k-bounded hooks in a column of γ (resp. δ) corresponds to a row of λ^{ω_k} (resp. μ^{ω_k}), to prove $\lambda^{\omega_k} \not\subseteq \mu^{\omega_k}$, it suffices to show that there are more k-bounded hooks in some column of γ than in that column of δ. Let a (of $k+1$-residue i) denote the top cell in the first column ℓ_a of δ/γ containing more than one cell. By assumption, the cell b below a is not a rowadder and thus there is a cell $\bar{b} \in \delta/\gamma$ of the same $k+1$-residue as b to the left of column ℓ_a. Hence the square $\bar{a}$ above $\bar{b}$ has $k+1$-residue i and since the diagram of δ/γ to the left of column ℓ_a is a series of horizontal rows, $\bar{a} \notin \delta/\gamma$.



(9.3)

Since $\bar{a}$ and a have the same $k+1$-residue, the cell $x_a = \bar{a} \wedge a$ has hook-length bounded by k in γ and at least $k+1$ in δ. Similarly for the cell $x_b = \bar{b} \wedge b$. Therefore, in the column with x_a, $\bar{b} \in \delta$ is the only k-bounded hook in δ that is not in γ while x_a and x_b are k-bounded hooks in γ that are not k-bounded in δ. We reach our claim since γ has at least one more k-bounded hook than δ in this column. $\square$

We shall say that μ, λ are *admissible* iff μ/λ and $\mu^{\omega_k}/\lambda^{\omega_k}$ are respectively horizontal and vertical strips, *i.e.* iff μ, λ are r-admissible for some r.

Proposition 54. *If $\mu, \lambda \in \mathcal{P}_k$ forms an admissible pair, then $\mathfrak{c}(\mu)/\mathfrak{c}(\lambda)$ is a horizontal strip.*

Proof. Given μ, λ is an admissible pair, we have μ/λ is a horizontal strip and $\mu^{\omega_k}/\lambda^{\omega_k}$ is a vertical strip. In particular, $\lambda \subseteq \mu$ and by Property 14, $\mathfrak{c}(\lambda) \subseteq \mathfrak{c}(\mu)$. Now, assume by contradiction that $\mathfrak{c}(\mu)/\mathfrak{c}(\lambda)$ contains some column with more than one cell. The top cell c of the leftmost such column must be a rowadder by Property 51. If the cell $\bar{c}$ below c is a rowadder, then this column contains two rowadders implying by Property 53(i) that μ/λ is not a horizontal strip. On the other hand, if $\bar{c}$ is not a rowadder, then $\lambda^{\omega_k} \not\subseteq \mu^{\omega_k}$ by Property 53(ii) and thus $\mu^{\omega_k}/\lambda^{\omega_k}$ is not a vertical strip. Either case gives a contradiction. $\square$

Lemma 55. *Let γ and δ be $k+1$-cores where no column has more k-bounded hooks in γ than in δ, and where δ/γ is a horizontal strip. With i denoting the $k+1$-residue of the rightmost cell in δ/γ, the removable corners of $k+1$-residue i in δ are exactly the cells of $k+1$-residue i in δ/γ.*

Proof. Let a_1 (of $k+1$-residue i) denote the rightmost cell in δ/γ and note that a_1 is a removable corner since δ/γ is a horizontal strip. If a_1 is not a rowadder of δ/γ, then there is a cell $a_2 \in \delta/\gamma$ that is a $k+1$-predecessor of a_1. Similarly, if a_2 is not a rowadder then there is a cell $a_3 \in \delta/\gamma$ which is a $k+1$-predecessor of a_2. By iteration, we eventually reach a rowadder $a_m \in \delta/\gamma$, and have the $k+1$-string $a_1, a_2, \ldots, a_m$ of cells with $k+1$-residue i. Note that $a_1, \ldots, a_m$ are all extremal cells of δ since they lie in the horizontal strip δ/γ. Furthermore, no cell lies to the right of a_1 implying that no cell lies to the right of any extremal cell with $k+1$-residue i above a_1, by Property 15. Therefore, $a_1, \ldots, a_m$ are all removable corners of δ. It thus remains to show that any extremal cell of $k+1$-residue i in δ above a_m or below a_1 is not removable.

The diagrams of γ and δ coincide south-east of a_1, given a_1 is the rightmost element of δ/γ. If a_1 is a $k+1$-predecessor of an extremal cell d, then a cell must lie to the right of d since otherwise, $h_{a_1 \wedge d}(\gamma) = k+1$ in the $k+1$-core γ. Property 15 thus implies that all extremal cells of $k+1$-residue i lying south-east of d also have a cell to their right. Therefore there are no removable corners of $k+1$-residue i south-east of a_1.

Similarly by Property 15, our claim will follow by showing that there is a cell $b \in \delta$ above the extremal cell $a_{m+1} \in \delta$ that is a $k+1$-predecessor of a_m. Suppose $b \notin \delta$. Then the hook length of $a_{m+1} \wedge a_m$ is $k+2$ in δ since a_{m+1} and the removable corner a_m have the same $k+1$-residue, but is k-bounded in γ since $a_m \notin \gamma$ and γ has no $k+1$-hooks. Note that the column containing a_{m+1} is of the same length in γ as in δ since $b \notin \delta$ and $a_{m+1} \notin \delta/\gamma$. Therefore, γ has more k-bounded hooks in this column contradicting our assumption. $\square$

Theorem 56. *If μ, λ is n-admissible, then there are distinct integers $i_1, \ldots, i_n$ where*

$$\mathfrak{c}(\lambda) = s_{i_1} \cdots s_{i_n}\left(\mathfrak{c}(\mu)\right) .$$

Proof. Since $\mu_r^{\omega_k}$ is the number of k-bounded hooks in column r of $\delta = \mathfrak{c}(\mu)$, given that μ, λ is n-admissible, no column has more k-bounded hooks in $\gamma = \mathfrak{c}(\lambda)$ than in $\delta = \mathfrak{c}(\mu)$. Further, δ/γ is a horizontal strip by Proposition 54. Therefore, if i_N denotes the $k+1$-residue of the rightmost cell $a_N \in \delta/\gamma$, then Lemma 55 implies that the diagram $s_{i_N}(\delta)/\gamma$ can be obtained by deleting all cells of $k+1$-residue i_N from δ/γ and is thus a skew diagram with no more than one cell in each column.

We now claim that no column has more k-bounded hooks in γ than in $s_{i_N}(\delta)$. Proposition 22 gives that $s_{i_N}(\delta)$ has the same number of k-bounded hooks as δ in every column except the one containing the cell a_N, where it has one fewer. Since no column has more k-bounded hooks in γ than in δ, it suffices to show that in the column with a_N, γ does not have more k-bounded hooks than $s_{i_N}(\delta)$. This follows by noting that weakly to the right of the column with a_N, $s_{i_N}(\delta)$ and γ coincide.

Therefore we can use Lemma 55 to prove that $s_{i_{N-1}}(s_{i_N}(\delta))/\gamma$ can be obtained by deleting all cells of $k+1$-residue i_N and i_{N-1} from δ/γ. By iterating the preceding argument, there is some N where $s_{i_1} \cdots s_{i_N}(\delta)/\gamma$ is the empty partition implying that $\gamma = s_{i_1} \cdots s_{i_N}(\delta)$. Since each iteration causes the removal of all cells with a given $k+1$-residue from δ/γ, $i_1, \ldots, i_N$ are distinct. Further, since the number of k-bounded hooks in δ is lowered by one each time by Proposition 22, $N = |\mu| - |\lambda| = n$. $\square$

Using this result, Corollary 25 implies

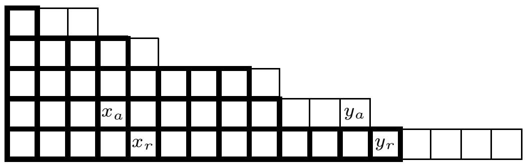
Corollary 57. *If μ, λ is an admissible pair, then $\lambda \preceq \mu$.*

We conclude this section with another set of conditions under which $\lambda \preceq \mu$.

Theorem 58. *If $\lambda \subseteq \mu$, $\lambda^{\omega_k} \subseteq \mu^{\omega_k}$, and $\mathfrak{c}(\mu)/\mathfrak{c}(\lambda)$ is a horizontal strip, then μ, λ is admissible.*

Proof. We start by showing that μ/λ is a horizontal strip, or equivalently by Remark 52, that the number of k-bounded hooks in row r of $\gamma = \mathfrak{c}(\lambda)$ is not smaller than the number of k-bounded hooks in row $r+1$ of $\delta = \mathfrak{c}(\mu)$.

In row r of γ, let y_r denote the last cell and let x_r be the rightmost cell with a hook exceeding k. Note that $h_{x_r}(\gamma) > k+1$ since γ is a $k+1$-core. If there are $d-1$ cells strictly between x_r and y_r, then γ has d k-bounded hooks in row r. It thus remains to prove that there are no more than d k-bounded hooks in row $r+1$ of δ. In row $r+1$ of δ, let y_a be the last cell and let x_a be the cell so that there are $d-1$ cells between x_a and y_a (if $x_a \notin \delta$ then $\delta_{r+1} \le d$ and the claim holds).



$\leftarrow$ row r

Note that x_a lies weakly to the left of x_r since y_a lies weakly to the left of y_r given δ/γ is a horizontal strip. Thus, the number of cells above x_a in γ is weakly greater than $\ell_r - 1$ for ℓ_r the number of cells above x_r in the partition γ. Since $\gamma \subseteq \delta$, the number of cells ℓ_a above x_a in δ satisfies $\ell_a \ge \ell_r - 1$. Hence, $h_{x_a}(\delta) = \ell_a + d + 1 \ge \ell_r + d = h_{x_r}(\gamma) - 1 > k$. That is, $h_{x_a}(\delta)$ exceeds k. Therefore the maximal number of k-bounded hooks in row $r+1$ of δ is d.

To see that $\mu^{\omega_k}/\lambda^{\omega_k}$ is a vertical strip, note that δ/γ has at most one box in every column. Thus, the number of k-bounded hooks in a column of δ cannot exceed the number of k-bounded hooks in that column of γ by more than one since $\gamma \subseteq \delta$ implies any hook exceeding k in γ must exceed k in δ. Now, recall that the number of k-bounded hooks in the columns of γ and δ are λ^{ω_k} and μ^{ω_k} respectively. Given $\lambda^{\omega_k} \subseteq \mu^{\omega_k}$, this leads to $\mu_r^{\omega_k} - \lambda_r^{\omega_k} \in \{0,1\}$ for all r – conditions that are equivalent to $\mu^{\omega_k}/\lambda^{\omega_k}$ being a vertical strip. $\square$

10. Generalized k-tableaux and the k-Young lattice

We now introduce a set of tableaux that serve as a semi-standard version of k-tableaux.

Definition 59. *Let γ be a $k+1$-core, m be the number of k-bounded hooks of γ, and $\alpha = (\alpha_1, \dots, \alpha_r)$ be a composition of m. A semi-standard k-tableau of shape γ and evaluation α is a filling of γ with integers $1, 2, \dots, r$ such that*

(i) rows are weakly increasing and columns are strictly increasing

(ii) the collection of cells filled with letter i are labeled with exactly α_i distinct $k+1$-residues.

We denote the set of all semi-standard k-tableaux of shape $\mathfrak{c}(\lambda)$ and evaluation α by $\mathcal{T}_\alpha^k(\lambda)$. When $\alpha = (1^m)$, we call the k-tableaux "*standard*". In this case, $\mathcal{T}_{(1^m)}^k(\lambda)$ is the set $\mathcal{T}^k(\lambda)$ of k-tableaux introduced in § 7. Hereafter, a semi-standard k-tableau will simply be referred to as a k-tableau.

Example 60. *For $k=3$, $\mathcal{T}^3_{(1,3,1,2,1,1)}(3,3,2,1)$ of shape $\mathfrak{c}((3,3,2,1)) = (8,5,2,1)$ is the set:*

$$\begin{array}{|c|c|c|c|c|c|c|c|} 5 \\ 4 & 6 \\ 2 & 3 & 4 & 4 & 6 \\ 1 & 2 & 2 & 2 & 3 & 4 & 4 & 6 \end{array} \qquad \begin{array}{|c|c|c|c|c|c|c|c|} 6 \\ 4 & 5 \\ 2 & 3 & 4 & 4 & 5 \\ 1 & 2 & 2 & 2 & 3 & 4 & 4 & 5 \end{array} \qquad \begin{array}{|c|c|c|c|c|c|c|c|} 4 \\ 3 & 6 \\ 2 & 4 & 4 & 5 & 6 \\ 1 & 2 & 2 & 2 & 4 & 4 & 5 & 6 \end{array} \tag{10.1}$$

10.1. **Standardizing and deleting a letter from k-tableaux.** As with the standard k-tableaux, we shall prove that deleting some letter from a k-tableau produces another k-tableau. To this end, we present a method for constructing a standard k-tableau from a given k-tableau of the same shape.

Definition 61. *For α a composition of m and $T \in \mathcal{T}_\alpha^k(\lambda)$, define $St(T)$ by the iterative process:*

If a is the biggest letter of T, let i denote the $k+1$-residue of the rightmost cell in T that contains a. Construct a tableau $\bar{T}$ by replacing each occurrence of letter a with $k+1$-residue i by the letter m. Now let a denote the biggest letter (smaller than m) in $\bar{T}$ and i the $k+1$-residue of the rightmost cell in $\bar{T}$ that contains a. Again construct a new tableau by replacing each occurrence of letter a with $k+1$-residue i by the letter $m-1$. $St(T)$ is the tableau obtained by iterating this process until each collection of repeated letters forms only one $k+1$-string. That is, $St(T) \in \mathcal{T}^k_{1^m}(\lambda)$.

Example 62. *Given a k-tableau $T \in \mathcal{T}^3_{(1,3,1,2,1,1)}(3,3,2,1)$ of shape $\mathfrak{c}(3,3,2,1) = (8,5,2,1)$:*

$$T = \begin{array}{|l} 5 \\ 4\,6 \\ 2\,3\,4\,4\,6 \\ 1\,2\,2\,2\,3\,4\,4\,6 \end{array} \qquad \text{4-residues} = \begin{array}{|l} 1 \\ 2\,3 \\ 3\,0\,1\,2\,3 \\ 0\,1\,2\,3\,0\,1\,2\,3 \end{array}$$

Every letter $a = 6$ of residue $i = 3$ is replaced by $m = 9$: $\begin{array}{|l} 5 \\ 4\,9 \\ 2\,3\,4\,4\,9 \\ 1\,2\,2\,2\,3\,4\,4\,9 \end{array}$

Then letters $a = 5$ of residue $i = 1$ are replaced by $m = 8$: $\begin{array}{|l} 8 \\ 4\,9 \\ 2\,3\,4\,4\,9 \\ 1\,2\,2\,2\,3\,4\,4\,9 \end{array}$

Then letters $a = 4$ of residue $i = 2$ are replaced by $m = 7$: $\begin{array}{|l} 8 \\ 7\,9 \\ 2\,3\,4\,7\,9 \\ 1\,2\,2\,2\,3\,4\,7\,9 \end{array}$

Then letters $a = 4$ of residue $i = 1$ are replaced by $m = 6$: $\begin{array}{|l} 8 \\ 7\,9 \\ 2\,3\,6\,7\,9 \\ 1\,2\,2\,2\,3\,6\,7\,9 \end{array}$

Similarly, $\xrightarrow[m=5]{a=3,i=0} \begin{array}{|l} 8 \\ 7\,9 \\ 2\,5\,6\,7\,9 \\ 1\,2\,2\,2\,5\,6\,7\,9 \end{array} \xrightarrow[m=4]{a=2,i=3} \begin{array}{|l} 8 \\ 7\,9 \\ 4\,5\,6\,7\,9 \\ 1\,2\,2\,4\,5\,6\,7\,9 \end{array} \xrightarrow[m=3]{a=2,i=2} \begin{array}{|l} 8 \\ 7\,9 \\ 4\,5\,6\,7\,9 \\ 1\,2\,3\,4\,5\,6\,7\,9 \end{array}$

Once the tableau is standard, the step $a = 2, i = 1, m = 2$ followed by $a = 1, i = 0, m = 1$ does not change the tableau.

Proposition 63. *Let $T \in \mathcal{T}^k_{(\alpha_1,\ldots,\alpha_m)}(\lambda)$. The tableau obtained by deleting the letter m from T belongs $\mathcal{T}^k_{(\alpha_1,\ldots,\alpha_{m-1})}(\mu)$ for some $\mu \preceq \lambda$ with $|\lambda| - |\mu| = \alpha_m$.*

Proof. Let $\hat{T}$ denote the tableau obtained by deleting the letter m from T. Since Conditions (i) and (ii) of a k-tableau clearly hold for $\hat{T}$, it suffices to show that the shape of $\hat{T}$ is given by $\mathfrak{c}(\mu)$ for some $\mu \preceq \lambda$. To this end, consider $St(T)$, the standard k-tableau of shape $\mathfrak{c}(\lambda)$ associated to T. Deleting the largest letter from $St(T)$ gives a k-standard tableau of shape $s_{i_{\alpha_m}}\big(\mathfrak{c}(\lambda)\big)$ by Proposition 32. By iteration, removing the largest α_m letters from $St(T)$ gives a standard k-tableau $\bar{T}$ of shape $s_{i_1}\cdots s_{i_{\alpha_m}}\big(\mathfrak{c}(\lambda)\big)$, where $i_1,\ldots,i_{\alpha_m}$ are respectively the $k+1$-residues of the α_m largest letters in $St(T)$. Since $\hat{T}$ has the same shape as $\bar{T}$, $\hat{T} \in \mathcal{T}^k_{(\alpha_1,\ldots,\alpha_{m-1})}(\mu)$ where $\mathfrak{c}(\mu) = s_{i_1}\cdots s_{i_{\alpha_m}}\big(\mathfrak{c}(\lambda)\big)$. Further, $\mu \preceq \lambda$ by Corollary 25 and $|\lambda| - |\mu| = \alpha_m$ by Proposition 22. $\square$

It is known (eg. [3]) that there are no semi-standard tableaux of shape λ and evaluation μ when $\lambda \not\trianglerighteq \mu$ in dominance order. We have found that this is also true for the k-tableaux.

Remark 64. *There are no k-tableaux in $\mathcal{T}^k_\mu(\lambda)$ when $\ell(\mu) < \ell(\lambda)$ since any element of $\mathcal{T}^k_\mu(\lambda)$ has height $\ell(\lambda)$, has only $\ell(\mu)$ distinct letters, and must be strictly increasing in columns.*

Theorem 65. *There are no semi-standard k-tableaux in $\mathcal{T}^k_\mu(\lambda)$ when $\lambda \not\trianglerighteq \mu$. Further, there is exactly one when $\lambda = \mu$.*

Proof. Consider $\lambda, \mu \in \mathcal{P}_k$ with $|\lambda| = |\mu|$. We shall proceed by induction on the length of μ. A k-tableau of evaluation $\mu = (\mu_1)$ must be of shape $\mathfrak{c}(\lambda)$ where $\ell(\lambda) \leq \ell(\mu)$ by Remark 64. Therefore, $\lambda = (\mu_1)$ and the claim holds. Assume the assertion holds when $\ell(\mu) < N$.

Consider $\mu = (\mu_1, \ldots, \mu_N)$ and $\lambda = (\lambda_1, \ldots, \lambda_{\bar{N}})$ with $\lambda \not\trianglerighteq \mu$. That is, $\mu_1 + \cdots + \mu_j > \lambda_1 + \cdots + \lambda_j$ for some $j \leq N$. Suppose by contradiction that there is some $T \in \mathcal{T}^k_\mu(\lambda)$. The previous proposition implies that removing the letter N from T results in a k-tableau $\bar{T} \in \mathcal{T}^k_{\bar{\mu}=(\mu_1,\ldots,\mu_{N-1})}(\bar{\lambda})$ where $\bar{\lambda} \preceq \lambda$. Thus, $\mathfrak{c}(\bar{\lambda}) = s_{i_1} \cdots s_{i_{\mu_N}}\big(\mathfrak{c}(\lambda)\big)$ for some $i_1, \ldots, i_{\mu_N}$ by Corollary 25. Since $\ell(\bar{\mu}) < N$, the induction hypothesis implies $\bar{\mu} \trianglelefteq \bar{\lambda}$. Therefore $\mu_1 + \cdots + \mu_r \leq \bar{\lambda}_1 + \ldots + \bar{\lambda}_r$ for all $r \leq N-1$. Further, $\bar{\lambda}_i \leq \lambda_i$ by Proposition 22 since the s_{i_j} act by deleting removable corners starting with $\mathfrak{c}(\lambda)$, and thus $\mu_1 + \cdots + \mu_r \leq \lambda_1 + \ldots + \lambda_r$ for all $r \leq N-1$. Therefore, $\mu_1 + \cdots + \mu_N > \lambda_1 + \cdots + \lambda_N$ given $\lambda \not\trianglerighteq \mu$. However, since $|\lambda| = |\mu|$, $|\mu| > \lambda_1 + \cdots + \lambda_N$ implies $\ell(\lambda) > \ell(\mu)$. We thus reach a contradiction by Remark 64.

To see that there is exactly one k-tableau $T \in \mathcal{T}^k_\lambda(\lambda)$, we shall first show by induction that there can be no more than one such tableau for $\lambda = (\lambda_1, \ldots, \lambda_N)$. Delete the letter N from T to obtain a k-tableau $\bar{T} \in \mathcal{T}^k_{\hat{\lambda}=(\lambda_1,\ldots,\lambda_{N-1})}(\bar{\lambda})$ where $\mathfrak{c}(\bar{\lambda}) = s_{i_1} \cdots s_{i_{\lambda_N}}\big(\mathfrak{c}(\lambda)\big)$. Remark 64 implies that $\ell(\bar{\lambda}) \leq \ell(\hat{\lambda}) = N-1$. Since exactly λ_N cells were removed from λ to obtain $\bar{\lambda}$, and the length of λ was decreased by at least one, the $i_1, \ldots, i_{\lambda_N}$ are uniquely determined and correspond to the $k+1$-residues in the top row of $\mathfrak{c}(\lambda)$. Thus, for two distinct k-tableaux in $\mathcal{T}^k_\lambda(\lambda)$ to exist, two distinct k-tableaux in $\mathcal{T}^k_{\hat{\lambda}}(\hat{\lambda})$ are necessary. By induction this is a contradiction.

We prove that there is in fact always a k-tableau $T \in \mathcal{T}^k_\lambda(\lambda)$ by construction: start with unique k-tableau of shape and evaluation (λ_1). For $j \geq 1$, let $\lambda^{(j)} = (\lambda_1, \ldots, \lambda_j)$ and consider the k-tableau of shape $\mathfrak{c}(\lambda^{(j)})$ and evaluation $\lambda^{(j)}$. Add the letter $j+1$ in all positions $s_{i_{\lambda_{j+1}}} \cdots s_{i_1}\big(\mathfrak{c}(\lambda^{(j)})\big)/\mathfrak{c}(\lambda^{(j)})$ where i_ℓ is the $k+1$-residue of the square $(j+1, \ell)$ of $\lambda^{(j)}$ for $\ell = 1, \ldots, \lambda_{j+1}$. □

10.2. **Bijection: generalized k-tableaux and chains in the k-lattice.** A rule for expanding the product of a k-Schur function with the homogeneous function h_ℓ (for $\ell \leq k$) in terms of k-Schur functions was conjectured in [11]. We introduce certain sequences of partitions based on this generalized Pieri rule and show their connection to the semi-standard k-tableaux. The connection with symmetric functions is then discussed in §11.

Recall from the introduction that a pair of k-bounded partitions λ, μ is "*r-admissible*" if and only if λ/μ and $\lambda^{\omega_k}/\mu^{\omega_k}$ are respectively horizontal and vertical r-strips. For composition α, a sequence of partitions $\big(\lambda^{(0)}, \lambda^{(1)}, \cdots, \lambda^{(r)}\big)$ is "α-admissible" if $\lambda^{(j)}, \lambda^{(j-1)}$ is a α_j-admissible pair for all j. This given, since Corollary 57 implies that if $\lambda^{(j)}, \lambda^{(j-1)}$ is α_j-admissible then $\lambda^{(j-1)} \preceq \lambda^{(j)}$, we have that any α-admissible sequence must be a chain in the k-Young lattice. We are interested in the set of chains:

Definition 66. *For any composition α, let*

$$\mathcal{D}^k_\alpha(\lambda) = \Big\{ (\emptyset = \lambda^{(0)}, \ldots, \lambda^{(r)} = \lambda) \text{ that are } \alpha\text{-admissible} \Big\} .$$

We now give a bijection between the set of chains in $\mathcal{D}^k_\alpha(\lambda)$ and the tableaux in $\mathcal{T}^k_\alpha(\lambda)$.

Definition 67. *For any $P = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(m)}) \in \mathcal{D}^k_\alpha(\lambda)$, let $\Gamma(P)$ be the tableau of shape $\mathfrak{c}(\lambda)$ where letter j fills cells in positions $\mathfrak{c}(\lambda^{(j)})/\mathfrak{c}(\lambda^{(j-1)})$, for $j = 1, \ldots, m$.*

Proposition 68. *If $P \in \mathcal{D}^k_\alpha(\lambda)$, then $\Gamma(P) \in \mathcal{T}^k_\alpha(\lambda)$.*

Proof. If $P = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(m)}) \in \mathcal{D}^k_\alpha(\lambda)$ then $\Gamma(P)$ has the shape of the $k+1$-core $\mathfrak{c}(\lambda)$. It thus suffices to prove that $\Gamma(P)$ is column strict and has α_j distinct $k+1$-residues that are filled with the letter j. Since $\lambda^{(j)}, \lambda^{(j-1)}$ is α_j-admissible by definition of $\mathcal{D}^k_\alpha(\lambda)$, Theorem 56 implies that $s_{i_1}\cdots s_{i_{\alpha_j}}\big(\mathfrak{c}(\lambda^{(j-1)})\big) = \mathfrak{c}(\lambda^{(j)})$ for some collection of distinct integers $i_1,\ldots,i_{\alpha_j}$ and Proposition 54 implies that $\mathfrak{c}(\lambda^{(j)})/\mathfrak{c}(\lambda^{(j-1)})$ is a horizontal strip. $\Gamma(P)$ is thus column strict since the letter j lies only in a horizontal strip. Further, given that each of the α_j operators s_{i_t} adds addable corners of residue i_t, the letter j occupies α_j distinct $k+1$-residues since $i_1,\ldots,i_{\alpha_j}$ are distinct. □

Definition 69. *For a k-tableau $T \in \mathcal{T}^k_\alpha(\lambda)$ with $\alpha = (\alpha_1,\ldots,\alpha_m)$, let $\bar\Gamma(T) = (\lambda^{(0)},\ldots,\lambda^{(m)})$, where $\mathfrak{c}(\lambda^{(i)})$ is the shape of the tableau obtained by deleting the letters $i+1,\ldots,m$ from T.*

Proposition 70. *If $T \in \mathcal{T}^k_\alpha(\lambda)$, then $\bar\Gamma(T) \in \mathcal{D}^k_\alpha(\lambda)$.*

Proof. Letting $\bar\Gamma(T) = (\lambda^{(0)},\ldots,\lambda^{(m)})$, the definition of $\bar\Gamma$ implies that the tableaux T^i obtained by deleting letters $i+1,\ldots,m$ from T have corresponding shapes $\mathfrak{c}(\lambda^{(i)})$. By Proposition 63, the T^i are k-tableaux. In particular, T^i has strictly increasing columns. Thus since T^{i-1} is obtained by deleting letter i from T^i, $\mathfrak{c}(\lambda^{(i)})/\mathfrak{c}(\lambda^{(i-1)})$ is a horizontal strip and further, by Proposition 63, $\lambda^{(i-1)} \preceq \lambda^{(i)}$ with $|\lambda^{(i)}| - |\lambda^{(i-1)}| = \alpha_i$. Property 13 then implies that $\lambda^{(i-1)} \subseteq \lambda^{(i)}$ and $(\lambda^{(i-1)})^{\omega_k} \subseteq (\lambda^{(i)})^{\omega_k}$. Therefore $\lambda^{(i)}, \lambda^{(i-1)}$ are α_i-admissible by Theorem 58 and we have that $\bar\Gamma(T) \in \mathcal{D}^k_\alpha(\lambda)$. □

Theorem 71. *Γ is a bijection between $\mathcal{T}^k_\alpha(\lambda)$ and $\mathcal{D}^k_\alpha(\lambda)$, with $\Gamma^{-1} = \bar\Gamma$.*

Proof. Given Propositions 68 and 70, we only have to show that if $P \in \mathcal{D}^k_\alpha(\lambda)$ and $T \in \mathcal{T}^k_\alpha(\lambda)$, then $\Gamma(\bar\Gamma(T)) = T$ and $\bar\Gamma(\Gamma(P)) = P$. This follows from the same deleting-filling letter argument given in the proof of Theorem 37. □

11. Symmetric functions and k-tableaux

Refer to [14] for details on symmetric functions and Macdonald polynomials. Here, we are interested in the study of the q,t-Kostka polynomials $K_{\mu\lambda}(q,t) \in \mathbb{N}[q,t]$. These polynomials arise as expansion coefficients for the Macdonald polynomials $J_\lambda[X;q,t]$ in terms of a basis dual to the monomial basis with respect to the Hall-Littlewood scalar product. As in the introduction, we use the modification of $J_\lambda[X;q,t]$ whose expansion coefficients in terms of Schur functions are the q,t-Kostka coefficients:

$$H_\lambda[X;q,t] = \sum_\mu K_{\mu\lambda}(q,t)\, s_\mu[X]\,. \tag{11.1}$$

The q,t-Kostka coefficients also have a representation theoretic interpretation [4], from which they were shown [6] to lie in $\mathbb{N}[q,t]$. Since $J_\lambda[X;q,t]$ reduces to the Hall-Littlewood polynomial $Q_\lambda[X;t]$ when $q=0$, we obtain a modification of the Hall-Littlewood polynomials by taking:

$$H_\lambda[X;t] = H_\lambda[X;0,t] = \sum_{\mu \trianglerighteq \lambda} K_{\mu\lambda}(t)\, s_\mu[X]\,, \tag{11.2}$$

with the coefficients $K_{\mu\lambda}(t) \in \mathbb{N}[t]$ known as Kostka-Foulkes polynomials. We can then obtain the homogeneous symmetric functions by letting $t=1$:

$$h_\lambda[X] = H_\lambda[X;1] = \sum_{\mu \trianglerighteq \lambda} K_{\mu\lambda}\, s_\mu[X]\,, \tag{11.3}$$

where $K_{\mu\lambda} \in \mathbb{N}$ are the Kostka numbers.

Recent work in the theory of symmetric functions has led to a new approach in the study of the q,t-Kostka polynomials. The underlying mechanism for this approach relies on a family of polynomials that appear to have a remarkable kinship with the classical Schur functions [9, 11, 12].

More precisely, consider the filtration $\Lambda_t^{(1)} \subseteq \Lambda_t^{(2)} \subseteq \cdots \subseteq \Lambda_t^{(\infty)} = \Lambda$, given by linear spans of Hall-Littlewood polynomials indexed by k-bounded partitions. That is,

$$\Lambda_t^{(k)} = \mathcal{L}\{H_\lambda[X;t]\}_{\lambda;\lambda_1\leq k}\,, \qquad k=1,2,3,\ldots\,.$$

A family of symmetric functions called the k-Schur functions, $s_\lambda^{(k)}[X;t]$, was introduced in [11] (these functions are conjectured to be precisely the polynomials defined using tableaux in [9]). It was shown that the k-Schur functions form a basis for $\Lambda_t^{(k)}$ and that, for λ a k-bounded partition,

$$H_\lambda[X;q,t\,] \;=\; \sum_{\mu;\mu_1\leq k} K_{\mu\lambda}^{(k)}(q,t)\, s_\mu^{(k)}[X;t\,]\,, \qquad K_{\mu\lambda}^{(k)}(q,t)\in\mathbb{Z}[q,t]\,, \tag{11.4}$$

and

$$H_\lambda[X;t\,] \;=\; s_\lambda^{(k)}[X;t\,] \;+\; \sum_{\substack{\mu;\mu_1\leq k\\ \mu>_D\lambda}} K_{\mu\lambda}^{(k)}(0,t)\, s_\mu^{(k)}[X;t\,]\,, \qquad K_{\mu\lambda}^{(k)}(0,t)\in\mathbb{Z}[t]\,. \tag{11.5}$$

The study of the k-Schur functions is motivated in part by the conjecture [9, 11] that the expansion coefficients actually lie in $\mathbb{N}[q,t]$. That is,

$$K_{\mu\lambda}^{(k)}(q,t)\in\mathbb{N}[q,t]\,. \tag{11.6}$$

Since it was shown that $s_\lambda^{(k)}[X;t]=s_\lambda[X]$ for k larger than the hook-length of λ, this conjecture generalizes Eq. (11.1). Also, there is evidence to support that $K_{\mu\lambda}(q,t)-K_{\mu\lambda}^{(k)}(q,t)\in\mathbb{N}[q,t]$, suggesting that the k-Schur expansion coefficients are simpler than the q,t-Kostka polynomials.

The preceding developments on the k-lattice can be applied to the study of the generalized q,t-Kostka coefficients as follows: the k-Schur functions appear to obey a generalization of the Pieri rule on Schur functions. To be precise, it was conjectured in [9, 11] that for the complete symmetric function $h_\ell[X]$,

$$h_\ell[X]\, s_\lambda^{(k)}[X;1] = \sum_{\mu\in E_{\lambda,\ell}^{(k)}} s_\mu^{(k)}[X;1]\,, \tag{11.7}$$

where

$$E_{\lambda,\ell}^{(k)} = \big\{\mu \,|\, \mu/\lambda \text{ is a horizontal } \ell\text{−strip and } \mu^{\omega_k}/\lambda^{\omega_k} \text{ is a vertical } \ell\text{−strip}\big\}\,. \tag{11.8}$$

Iteration, from $s_\emptyset^{(k)}[X;1]=1$, then yields that the expansion of $h_{\lambda_1}[X]h_{\lambda_2}[X]\cdots$ satisfies

$$h_\lambda[X] = \sum_\mu K_{\mu\lambda}^{(k)}\, s_\mu^{(k)}[X;1]\,, \tag{11.9}$$

where $K_{\mu\lambda}^{(k)}$ is a nonnegative integer reducing to the usual Kostka number $K_{\mu\lambda}$ when k is large since $s_\lambda^{(k)}[X;t]=s_\lambda[X]$ in this case. The definition of $E_{\lambda,\ell}^{(k)}$ in the k-Pieri expansion thus reveals the motivation behind the set of chains given in Definition 66. This connection implies that

$$K_{\mu\lambda}^{(k)} \;=\; \text{the number of chains of the } k\text{-lattice in } \mathcal{D}_\lambda^k(\mu)\,.$$

Equivalently, using the bijection between chains in $\mathcal{D}_\lambda^k(\mu)$ and $\mathcal{T}_\lambda^k(\mu)$ given in Theorem 71, we have

$$K_{\mu\lambda}^{(k)} \;=\; \text{ the number of } k\text{-tableaux of shape } \mathfrak{c}(\mu) \text{ and evaluation } \lambda\,.$$

Although this combinatorial interpretation relies on the conjectured Pieri rule (11.7), it was proven in [11] that the k-Schur functions are unitriangularly related to the homogeneous symmetric functions. That is, $K_{\lambda\mu}^{(k)}=0$ when $\mu\not\trianglerighteq\lambda$ and $K_{\lambda\lambda}^{(k)}=1$. Therefore, Theorem 65 implies that the number of k-tableaux does correspond to $K_{\lambda\mu}^{(k)}$ in these cases.

More generally, note that letting $q=0$ in Eq. (11.6) gives that the coefficients in Hall-Littlewood expansion Eq. (11.5) satisfy $K_{\mu\lambda}^{(k)}(0,t)\in\mathbb{N}[t]$. However, since $H_\lambda[X;1]=h_\lambda[X]$, we have that

$K^{(k)}_{\mu\lambda}(0,1) = K^{(k)}_{\mu\lambda}$ from Eq. (11.9). Therefore, since it appears that $K^{(k)}_{\mu\lambda}$ counts the number of semi-standard k-tableaux in $\mathcal{T}^k_\lambda(\mu)$, it is suggested that there exists a t-statistic on such k-tableaux giving a combinatorial interpretation for the generalized Kostka-Foulkes $K^{(k)}_{\mu\lambda}(0,t)$.

Alternatively, $H_\lambda[X;1,1] = h_{1^n}[X]$ for $\lambda \vdash n$ implies that $K^{(k)}_{\mu\lambda}(1,1) = K^{(k)}_{\mu\,1^n}$ by Eq. (11.9). This lends support to the idea that a q,t-statistic on the standard k-tableaux that would account for the apparently positive coefficients $K^{(k)}_{\mu\lambda}(q,t)$ in Eq. (11.6). That is,

$$K^{(k)}_{\mu\lambda}(1,1) = \text{ the number of standard } k\text{-tableaux of shape } \mathfrak{c}(\mu)\,.$$

Equivalently, our bijection between affine permutations and standard k-tableaux suggests there may be a q,t-statistic on reduced words that would account for the positivity:

$$K^{(k)}_{\mu\lambda}(1,1) = \text{ the number of reduced words of } \sigma_\mu \in \tilde{S}_{k+1}/S_{k+1}\quad.$$

We mention one final consequence of the k-Pieri rule. For λ a partition of length n, the product $h_{\lambda_1}\cdots h_{\lambda_n}$ giving h_λ can be written in any order since the functions commute. Therefore,

$$h_\alpha[X] = \sum_\mu K^{(k)}_{\mu\lambda}\, s^{(k)}_\mu[X;1]\,. \tag{11.10}$$

for any reordering α of the entries of λ. Therefore, $K^{(k)}_{\mu\lambda}$ is also the number of chains in $\mathcal{D}^k_\alpha(\mu)$. Equivalently, $K^{(k)}_{\mu\lambda}$ is the number of k-tableaux in $\mathcal{T}^k_\alpha(\mu)$. Thus, conjecture (11.7) implies:

> If α is a rearrangement of λ, then $\left|\mathcal{T}^k_\alpha(\mu)\right| = \left|\mathcal{T}^k_\lambda(\mu)\right|$. Equivalently, the number of k-tableaux in $\mathcal{T}^k_\alpha(\mu)$ equals the number of k-tableaux in $\mathcal{T}^k_\lambda(\mu)$.

This conjecture suggests that there is a generalization of the Bender-Knuth involution on semi-standard tableaux that permutes the evaluation of k-tableaux accounting for this phenomenon. See [13] for this new involution and thus the proof of this conjecture.

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