

A k -TABLEAU CHARACTERIZATION OF k -SCHUR FUNCTIONS

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ABSTRACT. We study k -Schur functions characterized by k -tableaux, proving combinatorial properties such as a k -Pieri rule and a k -conjugation. This new approach relies on developing the theory of k -tableaux, and includes the introduction of a weight-permuting involution on these tableaux that generalizes the Bender-Knuth involution. This work lays the groundwork needed to prove that the set of k -Schur Littlewood-Richardson coefficients contains the 3-point Gromov-Witten invariants; structure constants for the quantum cohomology ring.

1. INTRODUCTION

The Schur functions s_λ form a basis for the symmetric function space Λ which plays a fundamental role in combinatorics, representation theory and algebraic geometry. For instance, the Pieri formula for multiplying Schubert varieties in the intersection ring of a Grassmannian is equivalent to the formula for multiplying a Schur function and a homogeneous function h_ℓ in Λ :

$$h_\ell s_\mu = \sum_{\lambda=\mu+\text{horizontal strip}} s_\lambda. \quad (1)$$

A formula defined by vertical strips, rather than horizontal, describes the product of an elementary symmetric function $e_\ell s_\mu$. More generally, structure constants of the cohomology ring of the Grassmannian in the basis of Schubert classes are none other than the “*Littlewood-Richardson coefficients*”, occurring in the expansion

$$s_\nu s_\mu = \sum_{\lambda} c_{\nu\mu}^\lambda s_\lambda. \quad (2)$$

Combinatorics is deeply intertwined with the theory of Schur functions. The Littlewood-Richardson coefficients are characterized by certain skew tableaux and at a more fundamental level, the very definition of column-strict tableaux arises by iterating (1). That is,

$$h_\mu = \sum_{\lambda \supseteq \mu} K_{\lambda\mu} s_\lambda, \quad (3)$$

where the “*Kostka numbers*” $K_{\lambda\mu}$ count the number of tableaux of shape λ and weight μ , with the column-strict condition required by the Pieri rule. The role of Schur functions also ties into the combinatorial theory of partitions as can be seen

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when working with the algebra endomorphism defined by $\omega e_\ell = h_\ell$. This involution acts simply on a Schur function by

$$\omega s_\lambda = s_{\lambda'}, \quad (4)$$

where λ' is the partition conjugate to λ .

Recent developments in symmetric function theory involved the study of Macdonald polynomials. The Schur basis is again fundamental in this setting since the Macdonald expansion coefficients in this basis have a representation theoretic interpretation [6, 7, 11]. In work with Lascoux on the Macdonald polynomials [15], we discovered a new family of symmetric functions defined for each partition λ , where $\lambda_1 \leq k$, by

$$s_\lambda^{(k)} = \sum_{T \in \mathcal{S}_\lambda} s_{\text{shape}(T)},$$

for certain sets of tableaux \mathcal{S}_λ . Experimentation suggested that these functions play the fundamental role of the Schur functions in the subring $\Lambda^{(k)} = \mathbb{Z}[h_1, \dots, h_k]$; they form a basis for $\Lambda^{(k)}$ that satisfies generalizations of classical Schur function properties such as (1), (2), (3) and (4). As such, we coined the functions “*k-Schur functions*”. Unfortunately, while fertile for intuition and computer experimentation, the characterization of \mathcal{S}_λ lagged in mechanisms of proof. This led us to seek an alternative characterization and in [16], we introduced functions that are conjecturally equivalent. Although we were able to prove that these functions form a basis for $\Lambda^{(k)}$, the combinatorial conjectures remained open.

Continued empirical study of the *k-Schur* functions led us to tangential results in algebraic combinatorics. In particular, a new family of tableaux were drawn from the conjectured Pieri rule for *k-Schurs*:

$$h_\ell s_\mu^{(k)} = \sum_{\lambda \in H_{\mu, \ell}^k} s_\lambda^{(k)}, \quad (5)$$

where $H_{\mu, \ell}^k$ is a certain subset of the partitions obtained by adding a horizontal ℓ -strip to μ . Iterating this relation gives

$$h_\mu = \sum_{\lambda} K_{\lambda\mu}^{(k)} s_\lambda^{(k)}. \quad (6)$$

Using the Pieri rule as a guide, we defined the “*k-tableaux*” (see Definition 2) as certain fillings of $k+1$ -cores whose enumeration gives the “*k-Kostka numbers*” $K_{\lambda\mu}^{(k)}$. In [19], we proved that these tableaux directly connect to the type-A affine Weyl group, and explained their role in enumerating the monomial terms of coefficients in the *k-Schur* expansion of Macdonald polynomials.

The family of *k-tableaux* is the central object of study here. As with usual tableaux, these are associated to a shape λ and weight μ and satisfy [19]:

$$K_{\lambda\mu}^{(k)} = 0 \text{ when } \lambda \not\supseteq \mu \quad \text{and} \quad K_{\lambda\lambda}^{(k)} = 1. \quad (7)$$

Thus, (6) gives an invertible system that can be used to characterize the *k-Schur* functions. In this paper we investigate this third (conjecturally equivalent) characterization for the *k-Schur* functions. The very definition implies that these functions form a basis for $\Lambda^{(k)}$. Then, with an in-depth study of *k-tableaux*, we are able to prove that these polynomials satisfy several combinatorial properties including analogs of (1), (3), and (4). Moreover, our results strongly suggest that

k -tableaux are the objects to approach two long-standing open problems – finding a combinatorial interpretation for the 3-point Gromov-Witten invariants and for the Macdonald expansion coefficients.

To be more specific, by proving a number of results about the structure of k -tableaux, we discover an involution on the weight that reduces to the Bender-Knuth involution [2] on column-strict tableaux. Consequently, we can derive the following relation on k -Kostka numbers:

$$K_{\lambda\alpha}^{(k)} = K_{\lambda\mu}^{(k)}$$

for α any rearrangement of μ . From this, we are able to prove the k -Pieri rule for k -Schur functions (5). We also prove a formula for $e_\ell s_\mu^{(k)}$ that depends on a subset of the shapes obtained by adding vertical ℓ -strips to μ . From the e_ℓ and h_ℓ k -Pieri rules we can show that applying the ω -involution to $s_\lambda^{(k)}$ produces exactly one k -Schur function:

$$\omega s_\lambda^{(k)} = s_{\lambda^{\omega_k}}^{(k)}, \quad (8)$$

indexed by the “ k -conjugate” of λ . We show that $s_\lambda^{(k)}$ coincides with s_λ when the hook-length of λ is not larger than k , and thus that k -Schur functions reduce to Schur functions when k is large enough. This concurs with our assertion that the k -Schur functions are the “*Schur basis*” for $\Lambda^{(k)}$, since $\Lambda^{(k)} = \Lambda$ when $k \rightarrow \infty$.

As mentioned, the k -tableaux are connected to the affine symmetric group \tilde{S}_{k+1} . In particular, the k -Pieri rule induces an order that is isomorphic to the weak order on \tilde{S}_{k+1} modulo a maximal parabolic subgroup isomorphic to S_{k+1} . The interpretation of the weak order on \tilde{S}_{k+1}/S_{k+1} as the tiling of a cone in k -space by permutahedra can be seen on the level of symmetric functions by identifying vertices with k -Schur functions. The vectors of translation invariance in the tiling turn out simply to be usual Schur functions indexed by “ k -rectangles” – partitions of the form $(\ell^{k-\ell+1})$. This follows from our last property:

$$s_\square s_\lambda^{(k)} = s_{\lambda \cup \square}^{(k)}, \quad (9)$$

where \square is any k -rectangle. This result implies that there are $k!$ “ k -irreducible” k -Schur functions from which any other k -Schur can be constructed by multiplication with Schur functions indexed by k -rectangles. These k -irreducibles are indexed by partitions with no more than i parts equal to $k - i$. This property, as well as its t -generalization, also holds for the functions introduced in [16] (see [18]).

Although this article concentrates on proving that the k -Schur functions are the fundamental combinatorial analog for the Schur functions in the subspace $\Lambda^{(k)}$, this analogy extends beyond combinatorics. Results presented here are the tools needed to carry out the first step in this direction. In [20], we prove that the k -Schur functions provide the natural basis for the quantum cohomology of the Grassmannian [1, 26]. Consequently, the three point Gromov-Witten invariants are none other than “ k -Littlewood-Richardson coefficients” occurring in

$$s_\mu^{(k)} s_\nu^{(k)} = \sum_\lambda c_{\mu\nu}^{\lambda,k} s_\lambda^{(k)}. \quad (10)$$

This implies the positivity of $c_{\lambda\mu}^{\nu,k}$ in certain cases; conjectured to hold in general. Explicit connections are also made in [20] between k -Littlewood Richardson coefficients, fusion coefficients for the WZW-conformal field theories [24], and structure constants related to certain representations of Hecke algebras at roots of unity [8].

Given these developments on k -tableaux and k -Schur functions, there are many natural paths for future work. Most notable is to investigate a likely connection between the k -Schur functions and the affine (loop) Grassmannian. In particular, M. Shimozono conjectured that the k -Littlewood-Richardson coefficients give the integral homology of the loop Grassmannian. There is extensive computational evidence that the “*dual k -Schur functions*”, defined in [20] by summing over the monomial weights of k -tableaux, are the Schubert classes in the cohomology of the loop Grassmannian. Another topic to be explored is the problem of finding appropriate skew k -tableaux to combinatorially describe the k -Littlewood-Richardson coefficients (and consequently the 3-point Gromov-Witten invariants). A last example goes back to the Macdonald problem from where the k -Schur functions arose. (3) gives in particular that

$$h_{1^n} = \sum_{\lambda} K_{\lambda 1^n}^{(k)} s_{\lambda}^{(k)}, \quad (11)$$

where $K_{\lambda 1^n}$ enumerates the subset of “*standard*” k -tableaux. Thus, there should exist a pair of statistics on k -standard tableaux to explain the k -Schur expansion coefficients in a Macdonald polynomial $H_{\mu}[X; q, t]$ since $h_{1^n} = H_{\mu}[X; 1, 1]$. Exciting mathematics has sprung from the search for combinatorial interpretations of the Gromov-Witten invariants and the Macdonald coefficients, however the conjectures remain open. See [3, 4, 5, 13, 14, 22, 23, 25] and [9, 10] for examples of recent progress in these directions.

2. DEFINITIONS

Let Λ denote the ring of symmetric functions, generated by the elementary symmetric functions $e_r = \sum_{i_1 < \dots < i_r} x_{i_1} \cdots x_{i_r}$, or equivalently by the complete symmetric functions $h_r = \sum_{i_1 \leq \dots \leq i_r} x_{i_1} \cdots x_{i_r}$, and let $\Lambda^k = \mathbb{Z}[h_1, \dots, h_k]$. Bases for Λ are indexed by partitions $\lambda = (\lambda_1 \geq \dots \geq \lambda_m > 0)$ whose degree $|\lambda|$ is $|\lambda| = \lambda_1 + \dots + \lambda_m$ and whose length $\ell(\lambda) = m$. Each partition λ has an associated Ferrers diagram with λ_i lattice squares in the i^{th} row, from the bottom to top (French notation). Any lattice square (i, j) in the i th row and j th column of a Ferrers diagram is called a cell. The conjugate of λ , denoted λ' , is the reflection of λ about the main diagonal. λ is “ k -bounded” if $\lambda_1 \leq k$ and the set of all such partitions is denoted \mathcal{P}^k . The partition $\lambda \cup \mu$ is the non-decreasing rearrangement of the parts of λ and μ . We say that $\lambda \subseteq \mu$ when $\lambda_i \leq \mu_i$ for all i . Dominance order \succeq is defined on partitions by $\lambda \succeq \mu$ when $\lambda_1 + \dots + \lambda_i \geq \mu_1 + \dots + \mu_i$ for all i , and $|\lambda| = |\mu|$.

More generally, for $\rho \subseteq \gamma$, the skew shape γ/ρ is identified with its diagram $\{(i, j) : \rho_i < j \leq \gamma_i\}$. Lattice squares that do not lie in γ/ρ will be called “*squares*” instead of cells. We say that any $c \in \rho$ lies “*below*” γ/ρ . The hook of any lattice square $s \in \gamma$ is defined as the collection of cells of γ/ρ that lie inside the L with s as its corner. This applies to all $s \in \gamma$ including those below γ/ρ . For example, the hook of $s = (1, 3)$ is depicted by the framed cells:

$$\gamma/\rho = (5, 5, 4, 1)/(4, 2) = \begin{array}{c} \square \\ \square \square \square \\ \square \square \square \\ \square \end{array} \quad (12)$$

The hook-length of s , $h_s(\gamma/\rho)$, is the number of cells in the hook of s . In the example, $h_{(1,3)}((5, 5, 4, 1)/(4, 2)) = 3$ and $h_{(3,2)}((5, 5, 4, 1)/(4, 2)) = 3$. A cell or

A p -core is a partition that does not contain any hooks of length p , and \mathcal{C}^p will denote the set of all p -cores. The p -residue of square (i, j) is $j - i \bmod p$; the label of this square when squares are periodically labeled with $0, 1, \dots, p-1$, where zeros lie on the main diagonal (see [12] for more on cores and residues). The 5-residues associated to the 5-core $(6, 4, 3, 1, 1, 1)$ are

0							
1							
2	3						
3	4	0	1				
4	0	1	2	3			
0	1	2	3	4	0	1	

Remark 1. A given p -core never has a removable and an addable corner of the same residue (e.g. [19]).

3. DEFINITION OF k -SCHUR FUNCTIONS

Definition 2. Let γ be a $k+1$ -core with m k -bounded hooks and let $\alpha = (\alpha_1, \dots, \alpha_r)$ be a composition of m . A “ k -tableau” of shape γ and “ k -weight” α is a filling of γ with integers $1, 2, \dots, r$ such that

- (i) rows are weakly increasing and columns are strictly increasing
- (ii) the collection of cells filled with letter i are labeled by exactly α_i distinct $k+1$ -residues.

$$\begin{array}{ccccc} \begin{array}{cccccc} 5 & & & & & \\ 4 & 6 & & & & \\ 2 & 3 & 4 & 4 & 6 & \\ 1 & 2 & 2 & 2 & 3 & 4 & 4 & 6 \end{array} & \begin{array}{cccccc} 6 & & & & & \\ 4 & 5 & & & & \\ 2 & 3 & 4 & 4 & 5 & \\ 1 & 2 & 2 & 2 & 3 & 4 & 4 & 5 \end{array} & \begin{array}{cccccc} 4 & & & & & \\ 3 & 6 & & & & \\ 2 & 4 & 4 & 5 & 6 & \\ 1 & 2 & 2 & 2 & 4 & 4 & 5 & 6 \end{array} \end{array} \quad (13)$$

More generally a notion of skew k -tableaux follows naturally from k -tableaux, see Definition 30.

Remark 4. When $k \geq h(\gamma)$, a k -tableau T of shape γ and k -weight μ is a semi-standard tableau of weight μ since no two diagonals of T can have the same residue.

Although a k -tableau is associated to a shape γ and weight α , in contrast to usual tableaux, $|\alpha|$ does not equal $|\gamma|$ in general. Instead, $|\alpha|$ is the number of k -bounded hooks in γ . This distinction is natural in view of a bijective correspondence between $k+1$ -cores and k -bounded partitions that was defined in [19] by the map:

$$\mathbf{c}^{-1} : \mathcal{C}^{k+1} \rightarrow \mathcal{P}^k \quad \text{where} \quad \mathbf{c}^{-1}(\gamma) = (\lambda_1, \dots, \lambda_\ell),$$

with λ_i denoting the number of cells with a k -bounded hook in row i of γ . Note that the number of k -bounded hooks in γ is $|\lambda|$. The inverse map relies on constructing a certain skew diagram γ/ρ from λ , and setting $\mathbf{c}(\lambda) = \gamma$. These special skew diagrams are defined:

Definition 5. For $\lambda \in \mathcal{P}^k$, the “ k -skew diagram of λ ” is the diagram λ/k where

- (i) row i has length λ_i for $i = 1, \dots, \ell(\lambda)$
- (ii) no cell of λ/k has hook-length exceeding k
- (iii) all squares below λ/k have hook-length exceeding k .

A convenient algorithm for constructing the diagram of λ/k is given by successively attaching a row of length λ_i to the bottom of $(\lambda_1, \dots, \lambda_{i-1})/k$ in the leftmost position so that no hook-lengths exceeding k are created.

Example 6. Given $\lambda = (4, 3, 2, 2, 1, 1)$ and $k = 4$,

$$\lambda = \begin{array}{|c|c|c|c|} \hline \square \\ \square \\ \square \\ \square \\ \hline \end{array} \implies \lambda/k = \begin{array}{|c|c|c|c|c|c|} \hline \square & & & & & \\ \square & \square & & & & \\ \square & \square & \square & & & \\ \square & \square & \square & \square & & \\ \square & \square & \square & \square & \square & \\ \square & \square & \square & \square & \square & \square \\ \hline \end{array} \implies \mathbf{c}(\lambda) = \begin{array}{|c|c|c|c|c|c|c|c|} \hline \square & & & & & & & \\ \square & \square & & & & & & \\ \square & \square & \square & & & & & \\ \square & \square & \square & \square & & & & \\ \square & \square & \square & \square & \square & & & \\ \square & \square & \square & \square & \square & \square & & \\ \square & \square & \square & \square & \square & \square & \square & \\ \hline \end{array}$$

The bijection between $k+1$ -cores and k -bounded partitions gives rise to a natural *involution* on the set of k -bounded partitions that refines partition conjugation.

Definition 7. [19] The “ k -conjugate” of a k -bounded partition λ is

$$\lambda^{\omega_k} := \mathbf{c}^{-1}(\mathbf{c}(\lambda)')$$

Remark 8. $\lambda^{\omega_k} = \lambda'$ when $h(\lambda) \leq k$ since $\mathbf{c}(\lambda) = \lambda$ in this case.

The analogy with usual tableaux is now more apparent. We denote the set of all k -tableaux of shape $\mathbf{c}(\mu)$ and k -weight α by $\mathcal{T}_\alpha^k(\mu)$, and call a k -tableau “*standard*” when its k -weight is (1^n) . Here, we will study properties of the “ k -Kostka numbers”:

$$K_{\mu\alpha}^{(k)} = |\mathcal{T}_\alpha^k(\mu)|. \quad (14)$$

For example, they satisfy a triangularity property similar to the Kostka numbers.

Property 9. [19] For any k -bounded partitions λ and μ ,

$$K_{\mu\lambda}^{(k)} = 0 \quad \text{when} \quad \mu \not\geq \lambda \quad \text{and} \quad K_{\mu\mu}^{(k)} = 1. \quad (15)$$

Thus the matrix $||K^{(k)}||_{\lambda, \mu \in \mathcal{P}^k}$ is invertible, naturally giving rise to a family of functions defined by:

Definition 10. [19] The “ k -Schur functions”, indexed by k -bounded partitions, are defined by inverting the unitriangular system:

$$h_\lambda = s_\lambda^{(k)} + \sum_{\mu: \mu \triangleright \lambda} K_{\mu\lambda}^{(k)} s_\mu^{(k)} \quad \text{for all } \lambda_1 \leq k. \quad (16)$$

This characterization has the advantage that properties of k -Schur functions can be derived from the study of k -tableaux. The next sections will thus be devoted to understanding the k -tableaux and the proofs of k -Schur function properties will follow in §6.

4. PROPERTIES OF k -TABLEAUX

Since the homogeneous symmetric functions commute, Definition 10 suggests that $K_{\lambda\alpha}^{(k)} = K_{\lambda\mu}^{(k)}$ for any rearrangement μ of the parts of α . In [19], it was conjectured that this could be explained by finding an involution on the set of k -tableaux sending $\mathcal{T}_\alpha^k(\lambda)$ to $\mathcal{T}_{\hat{\alpha}}^k(\lambda)$, where $\hat{\alpha}$ is obtained by transposing two adjacent components of α . The construction of such an involution is the spring board to proving properties of the k -Schur functions. To this end, we now explore characteristics of k -tableaux that are necessary to construct and prove the involution.

We first earmark certain cells of a partition λ . A cell (i, j) where $(i+1, j+1) \notin \lambda$ is called “*extremal*”. The squares $(0, \lambda_1)$ and $(\ell(\lambda), 0)$, those below and to the left of the diagram of λ , will also be called extremal. The cell immediately to the left (or right) of the cell c is “*left-adj*” (or “*right-adj*”) to c , while the cell $(i-1, j-1)$ is “*south-west*” of (i, j) . We will repeatedly use the following property of cores:

Remark 11. In a p -core γ , if an extremal lies at the top of its column in some row r , then in all rows weakly lower than r , extremal of the same residue must lie at the top of their column. Similarly, if an extremal lies at the end of its row in some row r , then in all rows weakly higher than r , all extremals of the same residue lie at the end of their row. Note that this argument applies to all extremals, including those that are not cells $-(0, \gamma_1)$ or $(\ell(\gamma), 0)$.

With the goal of producing an involution that switches the weight of consecutive letters a and $b = a + 1$ in a k -tableau, the behavior of these letters is our main concern. We consider entries a and b to be “*married*” if they occur in the same column, an entry a (resp. b) is “*divorced*” if it has the same residue as some married a (resp. b), and “*single*” otherwise. When the letter x occupies a cell in row r that is labeled with residue j , we say this cell contains an $x_r(j)$, or simply an $x(j)$. $\text{Res}_r(x)$ will be the set of all residues that label cells occupied by a letter x in row r , while $\text{URes}_r(x)$ will be only the residues labeling unmarried x ’s in row r . We also consider $\text{URes}_r(a, b) = \text{URes}_r(a) \cup \text{URes}_r(b)$.

The Bender-Knuth involution for semi-standard tableau is based on the following simple observation, also needed for our purposes:

Remark 12. The married b ’s in row r lie at the end of the sequence of b ’s in that row since an unmarried b must have an entry smaller than a lying below it. Similarly, married a ’s in any given row lie at the beginning of the sequence of a ’s in that row.

The extension of their involution to k -tableaux requires several intricate properties whose proofs rely heavily on the fact that deleting a letter from a k -tableau gives a k -tableau. To be precise, Proposition 32 in [19] states that deleting the largest letter from a standard k -tableau produces a new standard k -tableau, and thus by iteration, deleting all letters larger than any letter x produces a k -tableau. This idea can be generalized to the non-standard k -tableau case by introducing a total order on the letters/residues. In particular, consider the list of residues (j_1, \dots, j_r) occupied by x , where j_1 is the residue of the lowest, rightmost x in T ,

j_2 is the residue of the lowest, rightmost x in \hat{T} (obtained by deleting $x(j_1)$ from T), and so forth. It was shown in §10.1 of [19] that replacing $x(j_1)$ with n , $x(j_2)$ with $n-1$, and so forth produces a standard k -tableau. Therefore, using the order $z > x(j_1) > x(j_2) > \cdots > x(j_r) > y$ for $z > x > y$, a k -tableau $T_{\leq x(i)}$ (resp. $T_{< x(i)}$) is obtained by deleting all letters larger (weakly larger) than $x(i)$ from T .

Remark 13. It is important to note that $x(i) < x(i+1)$ when both $x(i), x(i+1)$ are in a k -tableau T . Otherwise, $x(i) > x(i+1)$ would imply that $x(i), x(i+1) \in T_{\leq x(i)}$, where $T_{\leq x(i)}$ is a k -tableau with an extremal $x_r(i)$ at the end of some row r and an extremal of residue i left-adj to an $x_m(i+1)$ for $m > r$, contradicting Remark 11.

Property 14.

- (i) *Given an unmarried $b(j)$ in a k -tableau T , any $a(j) \in T$ is married and lies weakly higher than the highest unmarried $b(j)$. Further, $a(j)$ occurs in T if and only if there is a divorced $b(j-1)$ left-adj to the unmarried $b(j)$.*
- (ii) *Given an unmarried $a(j)$ in a k -tableau T , any $b(j) \in T$ is married and lies strictly higher than the highest unmarried $a(j)$. Further, $b(j)$ occurs in T if and only if there is a divorced $a(j+1)$ right-adj to the unmarried $a(j)$.*

Proof. We prove case (i) and note that the other case follows similarly. Given $a_r(j) \in T$, we first claim it does not lie lower than any unmarried $b(j)$. Suppose there is an unmarried $b_m(j)$ for some $m > r$. In $T_{\leq a(j)}$, $a_r(j)$ lies at the end of its row by Remark 13 implying that all extremals above row r lie at the end of their row by Remark 11. However, the extremal south-west of $b_m(j)$ lies at the end of its row in $T_{\leq a(j)}$ only if an $a(j+1)$ lies below the $b_m(j)$ in T , contradicting that $b_m(j)$ is unmarried. Thus, given an unmarried $b_m(j) \in T$ and an $a_r(j)$ with $r \geq m$, it remains to show that $a_r(j)$ is married. In $T_{< b(j)}$, the cell of residue $j-1$ left-adj to $b_m(j)$ lies at the end of its row. Thus since $a_r(j) \in T_{< b(j)}$, $a_r(j)$ must be married to prevent its left-adj cell of residue $j-1$ from being extremal in $T_{< b(j)}$.

For the second part of the assertion, the \Leftarrow implication holds since a divorced $b(j-1) \in T$ implies there is a married $b(j-1) \in T$, lying above an $a(j)$. For the \Rightarrow implication, consider T with an unmarried $b_r(j)$ and some $a_m(j)$. The previous paragraph explains that $m \geq r$ and $a_m(j)$ is married. In $T_{< b(j-1)}$, since $a_m(j)$ lies at the top of its column, the extremal south-west of $b_r(j)$ lies at the top of its column. The only way the cell left-adj to $b_r(j)$ is not in $T_{< b(j-1)}$ is if it is $b_r(j-1)$. This $b_r(j-1)$ is not single since $a_m(j)$ is married to $b(j-1)$. Further, it is not married since no $a(j)$ lies lower than row $m \geq r$. \square

Lemma 15. *In a k -tableau T with an $x_r(i)$ and an $x_m(i)$ for $r < m$, $\text{Res}_m(x) \subseteq \text{Res}_r(x)$.*

Proof. If an $x_m(i+1) \in T$, then it lies at the top of its column in $T_{\leq x(i+1)}$. Thus, there must be an $x_r(i+1)$ right-adj to $x_r(i)$ to prevent the entry of residue $i+1$ below $x_r(i)$ from being extremal in $T_{\leq x(i+1)}$. If an $x_m(i-1) \in T$, then the cell of residue i below $x_m(i-1)$ lies at the top of its column in $T_{< x(i-1)}$. Therefore, an $x_r(i-1)$ must be left-adj to $x_r(i)$ to ensure that the extremal south-west of $x_r(i)$ lies at the top of its column in $T_{< x(i-1)}$. By iteration, $\text{Res}_m(x) \subseteq \text{Res}_r(x)$. \square

Property 16. *Let $U\text{Res}_m(a, b) \cap U\text{Res}_r(a, b) \neq \emptyset$ for some $r < m$. Then*

$$U\text{Res}_m(a) \subseteq U\text{Res}_r(a) \quad \text{and} \quad U\text{Res}_m(b) \subseteq U\text{Res}_r(b).$$

In particular, this implies $URes_m(a, b) \subseteq URes_r(a, b)$.

Proof. Since an unmarried a and b of the same residue cannot lie in T by Property 14, $URes_m(a, b) \cap URes_r(a, b) \neq \emptyset$ implies $URes_m(a) \cap URes_r(a) \neq \emptyset$ or $URes_m(b) \cap URes_r(b) \neq \emptyset$. Thus $Res_m(a) \subseteq Res_r(a)$ or $Res_m(b) \subseteq Res_r(b)$ by Lemma 15. Note that $Res_m(a) \subseteq Res_r(a)$ implies $URes_m(a) \subseteq URes_r(a)$ since an unmarried $a_m(i) \in T$ lies at the top of its column in $T_{\leq b(i-1)}$ forcing $a_r(i)$ also to lie at the top of its column. Similarly we have $URes_m(b) \subseteq URes_r(b)$. Therefore, $URes_m(a) \subseteq URes_r(a)$ or $URes_m(b) \subseteq URes_r(b)$ and it remains to show that both in fact are true.

It suffices to prove that $\emptyset \neq URes_m(a) \subseteq URes_r(a)$ implies $URes_m(b) \subseteq URes_r(b)$ and to note by a similar argument that $\emptyset \neq URes_m(b) \subseteq URes_r(b)$ implies $URes_m(a) \subseteq URes_r(a)$. Let $a_m(i)$ denote the rightmost a in row m . If any unmarried b lies in row m , then there is an unmarried $b_m(i+1)$ right-adj to $a_m(i)$ by Remark 12. Since $b_m(i+1)$ lies at the top of its column in $T_{\leq b(i+1)}$, there must be an entry $x \leq b(i+1)$ right-adj to $a_r(i)$ to prevent the entry of residue $i+1$ below $a_r(i)$ from being extremal in $T_{\leq b(i+1)}$. Property 14(i) ensures that $x \neq a_r(i+1)$ since there is an unmarried $b_m(i+1)$. Therefore $x = b_r(i+1) \in T$. Thus we have $b_m(i+1), b_r(i+1) \in T$ and can use Lemma 15 to obtain $Res_m(b) \subseteq Res_r(b)$. \square

Our involution will be defined on certain rows of a k -tableau that are characterized by the following equivalence relation.

Definition 17. Rows r_1 and r_2 in a k -tableau are equivalent, “ $r_1 \sim_a r_2$ ”, when they satisfy the following conditions:

- $URes_{r_1}(a, b) \neq \emptyset$ and $URes_{r_2}(a, b) \neq \emptyset$
- $URes_{r_1}(a, b) \subseteq URes_r(a, b)$ and $URes_{r_2}(a, b) \subseteq URes_r(a, b)$ for some r .

Proposition 18. \sim_a is an equivalence relation on the set of rows in a k -tableau containing an unmarried a or b .

Proof. The only non-trivial part is to show transitivity. With $r_1 \sim_a r_2$ and $r_2 \sim_a r_3$, $\emptyset \neq URes_{r_1}(a, b) \subseteq URes_r(a, b)$ and $\emptyset \neq URes_{r_2}(a, b) \subseteq URes_r(a, b)$ for some r $\emptyset \neq URes_{r_2}(a, b) \subseteq URes_t(a, b)$ and $\emptyset \neq URes_{r_3}(a, b) \subseteq URes_t(a, b)$ for some t . Thus, in particular, $URes_r(a, b) \cap URes_t(a, b) \neq \emptyset$ giving that $URes_{\max(r, t)}(a, b) \subseteq URes_{\min(r, t)}(a, b)$ by Property 16. Therefore

$$URes_{r_1}(a, b) \subseteq URes_{\min(r, t)}(a, b) \quad \text{and} \quad URes_{r_3}(a, b) \subseteq URes_{\min(r, t)}(a, b),$$

implying that $r_1 \sim_a r_3$. \square

We can take the lowest row in each equivalence class for a set of representatives. Property 16 implies that these representatives can equivalently be defined by:

Definition 19. A “fundamental row” of a tableau is a row m where $URes_m(a, b)$ is not contained in $URes_r(a, b)$ for any $r < m$.

5. AN INVOLUTION ON k -TABLEAUX

We are now ready to construct an involution on the set of k -tableaux sending $T_\alpha^k(\lambda)$ to $T_{\hat{\alpha}}^k(\lambda)$, where $\hat{\alpha}$ is obtained by transposing two adjacent components of α . Recall that the Bender-Knuth involution [2] is defined on semi-standard tableau by sending the string $a^r b^s$ of single a ’s and b ’s in each row to the string $a^s b^r$, thus

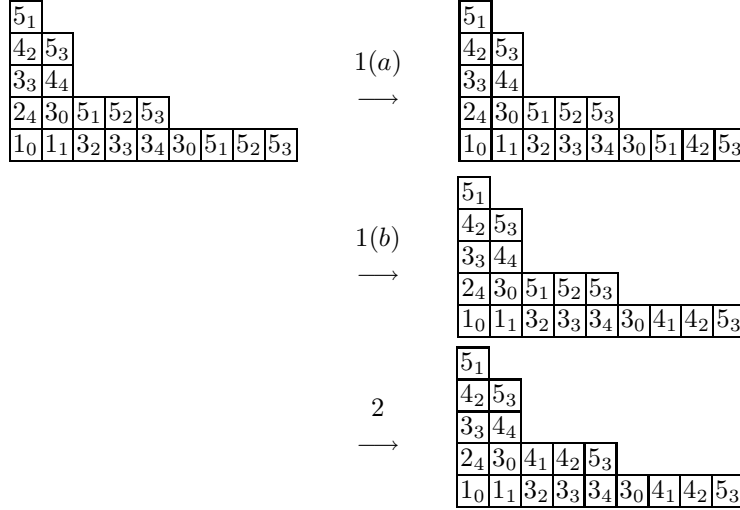
permuting the weight of the tableau. In our case, we perform a similar operation but must take into consideration the added notion of divorced entries. Our algorithm boils down to applying the BK involution to fundamental rows and “correcting”. It reduces to the BK involution for large k .

Definition 20. *The operator τ_a on a k -tableau T is defined as follows on the equivalence classes $\mathcal{C}_i = \{r \mid r \sim_a r_i\}$, for the set of fundamental rows $r_1, \dots, r_n \in T$:*

- (1) *In row r_i :*
 - (a) *Replace the entries $a^t b^s$ of single a 's and b 's by $a^s b^t$.*
 - (b) *If $t > s$, relabel any a lying to the right of some b by a b . Otherwise, relabel any b lying to the left of an a with an a .*
- (2) *In rows above r_i : for \mathcal{S}_i the set of residues of a 's (or b 's) that were relabeled in step 1, correspondingly relabel every unmarried a (or b) that has residue in \mathcal{S}_i .*

Note by definition of \sim_a that step 2 only involves rows in the class \mathcal{C}_i implying that no row is involved in this step for two distinct values of i .

Example 21. *Given a 4-tableau of weight $(2, 1, 4, 2, 3)$, we act with τ_4 to permute the number of residues occupied by letters 4 and 5.*



Although it is not immediately clear that the number of residues occupied by 4's and 5's in the k -tableaux has been switched, we will use properties from the previous section to prove that τ_a does in fact change k -tableaux as desired.

Proposition 22. *For any $T \in \mathcal{T}_\alpha^k(\lambda)$, the tableau $\tau_a(T)$ belongs to $\mathcal{T}_{\hat{\alpha}}^k(\lambda)$, where $\hat{\alpha} = (\dots, \alpha_{a+1}, \alpha_a, \dots)$ is obtained by transposing α_a and α_{a+1} in α .*

Proof. We start by showing that $\hat{T} = \tau_a(T)$ is a column-strict tableau. Then proving that the weight changes in the specified manner will imply it is a k -tableau. By Remark 12, T has a non-decreasing contiguous sequence of unmarried letters a and b . Thus in Step 1, only unmarried a and b in rows r_i are changed, and the definition of τ_a implies that these rows of \hat{T} are non-decreasing. In Step 2, rows of T in the classes \mathcal{C}_i are changed according to entries that were changed in Step 1.

Since the unmarried a 's and b 's in such rows form a (contiguous) subsequence of the unmarried letters in row r_i by Property 16, these rows are also non-decreasing in \hat{T} . Further, since an unmarried a lies below an entry strictly larger than b while an unmarried b lies above an entry strictly smaller than a , changing an unmarried a to b or b to a retains the property of strictly increasing columns.

This given, it remains to show that $\hat{\alpha}_a = \alpha_b$ and $\hat{\alpha}_b = \alpha_a$. Let α_a^s and α_a^m denote the number of single (resp. married) residues occupied by letter a in T , and observe that $\alpha_a = \alpha_a^s + \alpha_a^m$. Similarly for the letter b . Since a married a or b remains as such under the action of τ_a , we have that $\hat{\alpha}_a^m = \alpha_a^m = \alpha_b^m = \hat{\alpha}_b^m$. Thus, we need only show that $\hat{\alpha}_a^s = \alpha_b^s$ and $\hat{\alpha}_b^s = \alpha_a^s$. However, by Property 16,

$$\alpha_a^s = \sum_i \alpha_a^s(r_i) \quad \text{and} \quad \alpha_b^s = \sum_i \alpha_b^s(r_i), \quad (17)$$

for $\alpha_a^s(r_i)$ the number of single a residues in row r_i of T . Since the definition of τ_a implies that each single $a(j)$ or $b(j)$ occurs in exactly one fundamental row r_i of \hat{T} , we can further reduce our problem to showing

$$\hat{\alpha}_a^s(r_i) = \alpha_b^s(r_i) \quad \text{and} \quad \hat{\alpha}_b^s(r_i) = \alpha_a^s(r_i). \quad (18)$$

To show that the number of single a and b residues is permuted in a fundamental row r_i , first note in Step 1(a) that when single b 's are relabeled by a 's, the number of single b -residues is decreased by $\alpha_b^s(r_i) - \alpha_a^s(r_i)$ (considering the case $\alpha_b^s(r_i) > \alpha_a^s(r_i)$). Since Step 1(b) involves only divorced entries, no further single b -residues are lost implying that $\hat{\alpha}_b^s(r_i) = \alpha_a^s(r_i)$.

To prove $\hat{\alpha}_a^s(r_i) = \alpha_b^s(r_i)$, we must verify that precisely $\alpha_b^s(r_i) - \alpha_a^s(r_i)$ b 's are sent to *single* a 's. Each relabeled b goes to either a single or divorced $a \in \hat{T}$. To be precise, a $b(j)$ goes to a divorced $a(j)$ only if there is an $a(j) \in T$, and Property 14(i) tells us $a(j) \in T$ iff there is a divorced $b(j-1)$ left-adj to $b(j)$. Therefore, of the $\alpha_b^s(r_i) - \alpha_a^s(r_i)$ b 's relabeled in Step 1(a), each $b(j)$ that is right-adj to a divorced $b(j-1)$ does not give rise to a single a . However, each of these $b(j)$'s can be matched with a $b(i)$ that is not right-adj to a divorced $b(i-1)$ and thus goes to a single a in Step 1(b). For example,

$$\begin{array}{cccccccc} b_s/a & b_d & b_d & b_d & b_d & b_s & b_s & b_s/b \\ \downarrow & & & & \swarrow & & \downarrow & \downarrow \\ \downarrow & & & \swarrow & \swarrow & & \downarrow & \downarrow \\ \downarrow & & & \swarrow & \swarrow & & \downarrow & \downarrow \\ a_s/a & a_s & a_d & a_d & a_d & a_d & a_s & a_s/b \end{array} \quad (19)$$

Therefore, exactly $\alpha_b^s(r_i) - \alpha_a^s(r_i)$ new single a residues arise and we have proven our assertion. The case $\alpha_a^s(r_i) > \alpha_b^s(r_i)$ is similar. \square

Proposition 23. *The operator τ_a is an involution on $\mathcal{T}_\alpha^k(\lambda)$, for all $1 \leq a < \ell(\alpha)$.*

Proof. Since τ_a acts by sending certain unmarried a to unmarried b or vice-versa, the sets $URes_i(a, b)$ are fixed by τ_a . In particular, the fundamental rows of T are those of \hat{T} . Property 14(ii) and an argument similar to the one given in the previous proposition implies further that $(\tau_a)^2$ fixes the entries in the fundamental rows of

T. For example, applying τ_a to our previous illustration:

$$\begin{array}{cccccccc}
 a_s/a & a_s & a_d & a_d & a_d & a_d & a_s & a_s/b \\
 \downarrow & \searrow & & & & & \downarrow & \downarrow \\
 & & & \searrow & & & \downarrow & \downarrow \\
 & & & & \searrow & & \downarrow & \downarrow \\
 b_s/a & b_d & b_d & b_d & b_d & b_s & b_s & b_s/b
 \end{array} \tag{20}$$

By definition of τ_a , the equivalence classes $\mathcal{C}_i = \{r \mid r \sim_a r_i\}$ are then also left unchanged by $(\tau_a)^2$ and the claim follows. \square

The two previous propositions immediately imply:

Theorem 24. *Given $\lambda \in \mathcal{P}^k$, α a composition of $|\lambda|$, and any $1 \leq a < \ell(\alpha)$,*

$$\tau_a : \mathcal{T}_\alpha^k(\lambda) \rightarrow \mathcal{T}_{\hat{\alpha}}^k(\lambda)$$

is a bijection, where $\hat{\alpha} = (\dots, \alpha_{a+1}, \alpha_a, \dots)$.

Given that $K_{\lambda\alpha}^{(k)} = |\mathcal{T}_\alpha^k(\lambda)|$, the theorem has the following corollary.

Corollary 25. *For $\lambda \in \mathcal{P}^k$, and a composition α of $|\lambda|$,*

$$K_{\lambda\alpha}^{(k)} = K_{\lambda\mu}^{(k)}, \tag{21}$$

where μ is the weakly decreasing rearrangement of α .

We are also able to derive a recursive formula for the k -Kostka numbers using a correspondence between k -tableaux and certain chains of partitions. Following the notation of [19], μ, ν are “ ℓ -admissible” when μ/ν and $\mu^{\omega_k}/\nu^{\omega_k}$ are respectively horizontal and vertical ℓ -strips. More generally, for any composition $\alpha = (\alpha_1, \dots, \alpha_r)$, a sequence of partitions $(\mu^{(0)}, \mu^{(1)}, \dots, \mu^{(r)})$ is “ α -admissible” if $\mu^{(j)}, \mu^{(j-1)}$ are α_j -admissible for all j . A bijection was established between the sets:

$$\mathcal{T}_\alpha^k(\mu) \leftrightarrow \mathcal{D}_\alpha^k(\mu) := \left\{ (\emptyset = \mu^{(0)}, \dots, \mu^{(r)} = \mu) \text{ that are } \alpha\text{-admissible} \right\}, \tag{22}$$

implying in particular that

$$\mu, \nu \text{ are } \ell\text{-admissible iff } \mathfrak{c}(\mu)/\mathfrak{c}(\nu) = \text{horizontal strip with } \ell \text{ distinct residues.} \tag{23}$$

See [19] for the construction and details of this correspondence.

Corollary 26. *For k -bounded partitions μ and λ , and $0 < r \leq k$,*

$$K_{\mu(\ell, \lambda)}^{(k)} = \sum_{\substack{\mu/\nu = \text{horizontal } \ell\text{-strip} \\ \mu^{\omega_k}/\nu^{\omega_k} = \text{vertical } \ell\text{-strip}}} K_{\nu\lambda}^{(k)}. \tag{24}$$

Proof. Since every sequence $(\emptyset = \mu^{(0)}, \dots, \mu^{(r)} = \mu) \in \mathcal{D}_\alpha^k(\mu)$ has the property that $\mu, \mu^{(r-1)}$ are α_r -admissible, the cardinality of $\mathcal{D}_\alpha^k(\mu)$ satisfies the recursion:

$$|\mathcal{D}_\alpha^k(\mu)| = \sum_{\nu: \mu, \nu \text{ are } \alpha_r\text{-admissible}} |\mathcal{D}_{(\alpha_1, \dots, \alpha_{r-1})}^k(\nu)|.$$

The bijection (22) implies that $|\mathcal{D}_\alpha^k(\mu)| = K_{\mu\alpha}^{(k)}$ for all α , and thus by Corollary 25, $|\mathcal{D}_\alpha^k(\mu)| = K_{\mu\nu}^{(k)}$ for ν any rearrangement of α . Therefore,

$$K_{\mu(\ell, \lambda)}^{(k)} = |\mathcal{D}_{(\ell, \lambda)}^k(\mu)| = \sum_{\nu: \mu, \nu \text{ are } \ell\text{-admissible}} |\mathcal{D}_\lambda^k(\nu)| = \sum_{\nu: \mu, \nu \text{ are } \ell\text{-admissible}} K_{\nu\lambda}^{(k)}.$$

\square

6. PROPERTIES OF k -SCHUR FUNCTIONS

In the introduction, we discussed that the k -Schur functions have been thought to play the role of Schur functions in the spaces spanned by homogenous symmetric functions indexed by k -bounded partitions:

$$\Lambda^{(k)} = \mathcal{L} \{h_\lambda\}_{\lambda_1 \leq k}. \quad (25)$$

This belief was supported by strong computational evidence that the k -Schur functions obey refinements of the combinatorial properties of Schur functions. We can now capitalize on our knowledge of k -tableaux to prove that such beautiful combinatorial properties are held by the k -Schur functions introduced in Definition 10 by inverting the system

$$h_\lambda = s_\lambda^{(k)} + \sum_{\mu: \mu \triangleright \lambda} K_{\mu\lambda}^{(k)} s_\mu^{(k)} \quad \text{for all } \lambda_1 \leq k. \quad (26)$$

Immediate from the definition, we have that

Property 27. *The set $\{s_\lambda^{(k)}\}_{\lambda_1 \leq k}$ forms a basis for $\Lambda^{(k)}$.*

The unitriangular expression for h_λ in terms of k -Schurs, as well as the unitriangular relation between the usual Schur functions and h_λ imply that

Property 28. *For any k -bounded partition λ ,*

$$s_\lambda^{(k)} = s_\lambda + \sum_{\mu: \mu \triangleright \lambda} d_{\lambda\mu}^{(k)} s_\mu \quad \text{for } d_{\lambda\mu}^{(k)} \in \mathbb{Z}. \quad (27)$$

Although at this point, we can only prove that the coefficients $d_{\lambda\mu}^{(k)}$ are integral, we believe that they are in fact positive. This would follow by proving that the k -Schur functions studied here are precisely the atoms of [15], since the atoms are positive sums of Schur functions by definition.

6.1. Pieri rules. Much of our prior work with the k -Schur functions was drawn from a conjecture that the k -Schur functions satisfy a refinement of the Pieri rule called the “ k -Pieri rule”. In fact, the characterization of the k -Schurs used in this article was motivated purely so that they would satisfy this rule.

Theorem 29. *For any k -bounded partition ν and $\ell \leq k$,*

$$h_\ell s_\nu^{(k)} = \sum_{\mu \in H_{\nu, \ell}^{(k)}} s_\mu^{(k)} \quad (28)$$

where $H_{\nu, \ell}^{(k)} = \left\{ \mu \mid \mu/\nu = \text{horizontal } \ell\text{-strip and } \mu^{\omega_k}/\nu^{\omega_k} = \text{vertical } \ell\text{-strip} \right\}$.

Proof. Since the k -Schur functions form a basis of $\Lambda^{(k)}$, there is an expansion

$$h_\ell s_\nu^{(k)} = \sum_{\mu} c_{\mu\nu} s_\mu^{(k)}, \quad (29)$$

for some coefficients $c_{\mu\nu}$. To determine the $c_{\mu\nu}$, we examine $h_\ell h_\lambda$. Using the k -Schur expansion (16) for h_λ , we find that

$$h_\ell h_\lambda = \sum_{\nu} K_{\nu\lambda}^{(k)} h_\ell s_\nu^{(k)} = \sum_{\nu} K_{\nu\lambda}^{(k)} \sum_{\mu} c_{\mu\nu} s_\mu^{(k)}. \quad (30)$$

On the other hand, we can use (16) to expand $h_\ell h_\lambda = h_{(\ell, \lambda)}$. Then applying Corollaries 25 and 26 we obtain

$$h_{(\ell, \lambda)} = \sum_{\mu} K_{\mu(\ell, \lambda)}^{(k)} s_{\mu}^{(k)} = \sum_{\mu} \sum_{\substack{\mu/\nu = \text{horizontal } \ell\text{-strip} \\ \mu \omega_k / \nu \omega_k = \text{vertical } \ell\text{-strip}}} K_{\nu\lambda}^{(k)} s_{\mu}^{(k)}. \quad (31)$$

We can equate the coefficient of $s_{\mu}^{(k)}$ in the right side of this expression to that of (30) to get the system:

$$\sum_{\substack{\mu/\nu = \text{horizontal } \ell\text{-strip} \\ \mu \omega_k / \nu \omega_k = \text{vertical } \ell\text{-strip}}} K_{\nu\lambda}^{(k)} = \sum_{\nu} K_{\nu\lambda}^{(k)} c_{\mu\nu}. \quad (32)$$

One obvious solution is the one we want:

$$c_{\mu\nu} = \begin{cases} 1 & \text{if } \mu \in H_{\nu, \ell}^{(k)} \\ 0 & \text{otherwise} \end{cases}.$$

In fact, this is the unique solution since another solution $c'_{\mu\nu}$ would satisfy

$$0 = \sum_{\nu} K_{\nu\lambda}^{(k)} (c'_{\mu\nu} - c_{\mu\nu}), \quad (33)$$

where the invertibility of the matrix $K_{\nu\lambda}^{(k)}$ implies $c'_{\mu\nu} = c_{\mu\nu}$. \square

Skew k -tableaux can be used to encode the iteration of the k -Pieri rule, generalizing Theorem 29.

Definition 30. Let $\delta \subseteq \gamma$ be $k+1$ -cores with m_1 and m_2 k -bounded hooks respectively, and let $\alpha = (\alpha_1, \dots, \alpha_r)$ be a composition of $m_1 - m_2$. A “skew k -tableau” of shape γ/δ and “ k -weight” α is a filling of γ/δ with integers $1, 2, \dots, r$ such that
(i) rows are weakly increasing and columns are strictly increasing
(ii) the collection of cells filled with letter i are labeled by exactly α_i distinct $k+1$ -residues.

Remark 31. Our results on k -tableaux easily extend to include skew k -tableaux. In particular, the discussion in §4 of how to obtain a k -tableau by deleting the largest letters from a given k -tableau explains more generally that deleting the largest letter from a skew k -tableau produces a valid skew k -tableau. Furthermore, although we have defined τ_a on k -tableaux, the results clearly hold for skew k -tableaux.

Corollary 32. For any k -bounded partitions λ and μ ,

$$h_{\lambda} s_{\mu}^{(k)} = \sum_{\nu \in \mathcal{P}^k} K_{\nu/\mu, \lambda}^{(k)} s_{\nu}^{(k)}, \quad (34)$$

where $K_{\nu/\mu, \lambda}^{(k)}$ is the number of skew k -tableaux of shape $\mathfrak{c}(\nu)/\mathfrak{c}(\mu)$ and k -weight λ .

Proof. If $\nu \in H_{\mu, \ell}^k$ then $\mathfrak{c}(\nu)/\mathfrak{c}(\mu)$ is a horizontal strip with ℓ residues by (23). Thus, the k -Pieri rule implies our claim when $\lambda = (\ell)$ and we proceed by induction on $\ell(\lambda)$. Assuming (34) holds for λ with $\ell(\lambda) < n$, we have

$$h_{(\ell, \lambda)} s_{\mu}^{(k)} = h_{\ell} h_{\lambda} s_{\mu}^{(k)} = \sum_{\nu} K_{\nu/\mu, \lambda}^{(k)} h_{\ell} s_{\nu}^{(k)} = \sum_{\omega} \sum_{\nu} K_{\nu/\mu, \lambda}^{(k)} K_{\omega/\nu, (\ell)} s_{\omega}^{(k)}. \quad (35)$$

Since removing the highest letter from a skew k -tableau produces a skew k -tableau by Remark 31, we have that

$$\sum_{\nu} K_{\nu/\mu, \lambda}^{(k)} K_{\omega/\nu, (\ell)}^{(k)} = K_{\omega/\mu, (\ell, \lambda)}^{(k)}, \quad (36)$$

implying our claim. \square

As with the Schur functions, there is also a combinatorial rule to compute $e_{\ell} s_{\nu}^{(k)}$ in terms of k -Schurs by using vertical strips to ν rather than horizontal.

Theorem 33. *For any k -bounded partition ν and integer $\ell \leq k$,*

$$e_{\ell} s_{\nu}^{(k)} = \sum_{\lambda \in E_{\nu, \ell}^{(k)}} s_{\lambda}^{(k)}, \quad (37)$$

where $E_{\nu, \ell}^{(k)} = \left\{ \lambda \mid \lambda/\nu = \text{vertical } \ell\text{-strip and } \lambda^{\omega_k}/\nu^{\omega_k} = \text{horizontal } \ell\text{-strip} \right\}$.

In this case, $\lambda \in E_{\nu, \ell}$ implies $\mathbf{c}(\lambda^{\omega_k})/\mathbf{c}(\nu^{\omega_k})$ is a vertical strip with ℓ distinct residues by (23). We can thus apply the same argument used to derive Corollary 32 from Theorem 29 to prove the corollary:

Corollary 34. *For any k -bounded partitions λ and μ ,*

$$e_{\lambda} s_{\mu}^{(k)} = \sum_{\nu} \tilde{K}_{\nu/\mu, \lambda} s_{\nu}^{(k)}, \quad (38)$$

where $\tilde{K}_{\nu/\mu, \lambda}$ is the number of “transposed skew k -tableaux” of shape $\mathbf{c}(\nu)/\mathbf{c}(\mu)$ and k -weight λ . Such tableaux are defined by the same conditions as skew k -tableaux except that condition (i) is changed to: rows are strictly increasing and columns are weakly increasing.

Proof of Theorem 33. Since $e_1 = h_1$, Theorem 29 implies the case when $\ell = 1$ and we assume by induction that the action of e_r for all $r < \ell$ is given by (37). To prove our assertion for multiplication by e_{ℓ} , note that by applying the identity (e.g. [21]):

$$\sum_{r=0}^{\ell-1} (-1)^r h_{\ell-r} e_r + (-1)^{\ell} e_{\ell} = 0,$$

Eq. (37) follows from the expression

$$\sum_{r=0}^{\ell-1} (-1)^r h_{\ell-r} e_r s_{\nu}^{(k)} + (-1)^{\ell} e_{\ell} s_{\nu}^{(k)} = 0. \quad (39)$$

It thus suffices to show the coefficient of $s_{\mu}^{(k)}$ in the left side of this expression is zero.

By induction, Corollaries 32 and 34 tell us that for $r < \ell$, the coefficient of $s_{\mu}^{(k)}$ in $h_{\ell-r} e_r s_{\nu}^{(k)}$ is the number of fillings with weight $(r, \ell - r)$ in the following set:

Definition 35. *Let μ, ν be k -bounded partitions and fix $\ell \leq k$. An element $T \in A_{\nu, \ell}^{(k)}(\mu)$ of weight $(r, \ell - r)$, for any $0 \leq r \leq \ell$, has shape $\mathbf{c}(\mu)/\mathbf{c}(\nu)$ and is filled with letters $x < y$ such that*

- (i) $T_{\leq x}$ is a transposed skew k -tableau of k -weight (r) filled with letter x

(ii) $T/T_{\leq x}$ is a skew k -tableau of k -weight $(\ell - r)$ filled with letter y

Since the coefficient of $s_\mu^{(k)}$ in $\sum_{\lambda \in E_{\nu, \ell}^{(k)}} s_\lambda^{(k)}$ is the number of fillings in $\mathcal{A}_{\nu, \ell}^{(k)}(\mu)$ with weight $(\ell, 0)$, the coefficient of $s_\mu^{(k)}$ in the left side of (39) equals

$$\sum_{r=0}^{\ell-1} \sum_{\substack{T \in \mathcal{A}_{\nu, \ell}^k(\mu) \\ \text{weight}(T) = (r, \ell-r)}} (-1)^r + \sum_{\substack{T \in \mathcal{A}_{\nu, \ell}^k(\mu) \\ \text{weight}(T) = (\ell, 0)}} (-1)^\ell = \sum_{T \in \mathcal{A}_{\nu, \ell}^k(\mu)} (-1)^{\text{sgn}(T)},$$

where $\text{sgn}(T) = (-1)^r$ for $\text{weight}(T) = (r, \ell - r)$. If we can produce a sign reversing involution \mathbf{m} on $\mathcal{A}_{\nu, \ell}^k(\mu)$, then

$$\sum_{T \in \mathcal{A}_{\nu, \ell}^k(\mu)} (-1)^{\text{sgn}(T)} = \sum_{T \in \mathcal{A}_{\nu, \ell}^k(\mu)} (-1)^{\text{sgn}(\mathbf{m}(T))} = - \sum_{T \in \mathcal{A}_{\nu, \ell}^k(\mu)} (-1)^{\text{sgn}(T)},$$

implying the coefficient of $s_\mu^{(k)}$ is zero. Proposition 37 below gives the desired \mathbf{m} . \square

The involution \mathbf{m} acts on “free” entries of $T \in \mathcal{A}_{\nu, \ell}^k(\mu)$, where an $x(i)$ is free if every $x(i) \in T$ occurs at the top of its column, and $y(j)$ is free if no $y(j)$ is right-adj to an x or y .

Definition 36. The map \mathbf{m} acts on $T \in \mathcal{A}_{\nu, \ell}^k(\mu)$ by:

- 1) Let r_1 denote the lowest row containing a free x and i denote its residue (if there is no free x , set $r_1 = \infty$). Let r_2 be the lowest row containing a free y and j its residue (if there is no free y , set $r_2 = \infty$).
- 2) If $r_1 < r_2$, send every $x(i)$ to $y(i)$. Otherwise send every $y(j)$ to $x(j)$.

The definition of \mathbf{m} is well-defined since every $T \in \mathcal{A}_{\nu, \ell}^k(\mu)$ contains a free x or a free y . For example, $x(i)$ is not free in T implies there is an $x(i-1)$ or a $y(i-1)$ in T and $y(i)$ is not free implies there is an $x(i-1)$ or a $y(i-1)$ in T . By iteration, no letter is free implies that T contains

$$z(i), z(i-1), z(i-2), \dots, z(i+2), z(i+1),$$

with each $z(j) = x(j)$ or $y(j)$. This contradicts that T has weight $(r, \ell - r)$ for $\ell \leq k$.

Proposition 37. The map \mathbf{m} is an involution on $\mathcal{A}_{\nu, \ell}^{(k)}(\mu)$ such that $\text{weight}(\mathbf{m}(T)) = (n_1 \pm 1, n_2 \mp 1)$, given $\text{weight}(T) = (n_1, n_2)$.

Proof. Let $\hat{T} = \mathbf{m}(T)$ for some $T \in \mathcal{A}_{\nu, \ell}^{(k)}(\mu)$. First note that the definition of free implies the x ’s form a vertical strip and the y ’s a horizontal strip in \hat{T} .

To determine how the weight of T changes under \mathbf{m} , consider first the case that $r_1 < r_2$. Since every $x(i) \in T$ goes to $y(i) \in \hat{T}$, there are only $n_1 - 1$ residues of x in \hat{T} . To show that there are $n_2 + 1$ residues of y in \hat{T} , we must prove $y(i) \notin T$. Suppose there is a $y(i)$ in T . Since $T/T_{\leq x}$ is a skew k -tableau, $T_{< y(i)}/T_{\leq x}$ is a skew k -tableaux by Remark 31. Thus, $T_{< y(i)}$ has core shape and an addable corner $y(i)$ of residue i . Further, $x(i)$ is a removable corner in $T_{< y(i)}$ since $x(i)$ is at the top of its column and $y(i+1) \notin T_{< y(i)}$. We reach a contradiction by Remark 1 which tells us a core cannot have an addable and removable corner of the same residue. A similar argument works when $r_2 < r_1$.

Lastly, to see that \mathbf{m} is an involution, consider the case that $r_1 < r_2$. Since there is at most one x in each row, any row where $\mathbf{m} : x(i) \rightarrow y(i)$ contains only y 's in \hat{T} . Thus $y(i)$ is free in \hat{T} and r_1 is the lowest row with $y(i)$ since $y(i) \notin T$ by the previous paragraph. There are no lower free entries by definition of r_1 . Therefore, when \mathbf{m} is applied to \hat{T} , $y(i) \rightarrow x(i)$ and T is recovered. Similarly when $r_1 > r_2$. \square

6.2. Further properties. Recall the algebra endomorphism ω that provides an involution on Λ , defined by $\omega h_\ell = e_\ell$. This map has an especially simple action on the Schur functions: $\omega s_\lambda = s_{\lambda'}$. Since ω is also an involution on $\Lambda^{(k)}$, we can ask how it acts on a k -Schur function. Right on cue, we find:

Theorem 38. *The ω -involution acts on the k -Schur functions by*

$$\omega s_\lambda^{(k)} = s_{\lambda^{\omega_k}}^{(k)}. \quad (40)$$

Proof. Let $F_\mu = \omega s_{\mu^{\omega_k}}^{(k)}$. Since $h_\ell \omega \left(s_{\lambda^{\omega_k}}^{(k)} \right) = \omega \left(e_\ell s_{\lambda^{\omega_k}}^{(k)} \right)$, we can apply the k -Pieri rule (Theorem 33) to obtain

$$h_\ell F_\lambda = \omega \left(e_\ell s_{\lambda^{\omega_k}}^{(k)} \right) = \sum_{\mu \in E_{\lambda^{\omega_k}, \ell}^{(k)}} \omega s_\mu^{(k)} = \sum_{\mu^{\omega_k} \in E_{\lambda^{\omega_k}, \ell}^{(k)}} F_\mu = \sum_{\mu \in H_{\lambda, \ell}^{(k)}} F_\mu, \quad (41)$$

recalling that $(\mu^{\omega_k})^{\omega_k} = \mu$. Iteration of this expression from $F_0 = \omega s_0^{(k)} = 1$ matches iteration of the k -Pieri rule from $s_0^{(k)} = 1$. Thus, F_μ satisfies

$$h_\lambda = F_\lambda + \sum_{\mu: \mu \triangleright \lambda} K_{\mu\lambda}^{(k)} F_\mu \quad (42)$$

implying that $F_\mu = s_\mu^{(k)}$ by Definition 10 of the k -Schur functions. \square

From the action of ω on a k -Schur function, we are able to show that a k -Schur function reduces simply to a Schur function when k is large.

Property 39. *For any partition λ with main hook-length $h(\lambda) \leq k$, we have that $s_\lambda^{(k)} = s_\lambda$.*

Proof. Given the triangular form (27),

$$s_\lambda^{(k)} = s_\lambda + \text{higher terms}, \quad (43)$$

we can apply the ω -involution to obtain:

$$s_{\lambda^{\omega_k}}^{(k)} = s_{\lambda'} + \text{lower terms}. \quad (44)$$

However, since $\lambda^{\omega_k} = \lambda'$ when $h(\lambda) \leq k$ from Remark 8, the previous expression reduces to

$$s_{\lambda'}^{(k)} = s_{\lambda'} + \text{lower terms}. \quad (45)$$

Setting this equal to (43), with λ replaced by λ' , proves our claim. \square

We finish by deriving one last property from the action of the ω -involution. This property is one of the few that we were able to prove using a prior characterization (see [18]). In particular, there exists a subset of “irreducible” k -Schur functions from which all other $s_\lambda^{(k)}$ may be constructed with multiplication by usual Schur functions indexed by “ k -rectangles” – partitions of the form $(\ell^{k-\ell+1})$. The irreducibles consist of the special set of k -Schur functions indexed by irreducible partitions; k -bounded

partitions with no more than i parts equal to $k-i$, for $i = 0, \dots, k-1$. Remarkably, we can also prove this result using the characterization studied in this article.

Theorem 40. *For any k -rectangle \square and k -bounded partition μ , we have*

$$s_{\square} s_{\mu}^{(k)} = s_{\mu \cup \square}^{(k)}.$$

Proof. Consider the linear operator Θ_{\square} defined on $\Lambda^{(k)}$ by $\Theta_{\square} s_{\mu}^{(k)} = s_{\mu \cup \square}^{(k)}$. It suffices to show that $\Theta_{\square} s_{\mu}^{(k)} = s_{\mu}^{(k)} \Theta_{\square}$ since $\Theta_{\square} \cdot 1 = \Theta_{\square} s_{\emptyset}^{(k)} = s_{\square}^{(k)} = s_{\square}$ by Property 39. However, since the homogeneous functions generate $\Lambda^{(k)}$, we will instead prove that $\Theta_{\square} h_{\ell} = h_{\ell} \Theta_{\square}$. To this end, note that the k -Pieri rule (28) implies

$$\Theta_{\square} h_{\ell} s_{\mu}^{(k)} = \Theta_{\square} \sum_{\eta \in H_{\mu, \ell}^{(k)}} s_{\eta}^{(k)} = \sum_{\eta \in H_{\mu \cup \square, \ell}^{(k)}} s_{\eta \cup \square}^{(k)}, \quad (46)$$

and on the other hand,

$$h_{\ell} \Theta_{\square} s_{\mu}^{(k)} = h_{\ell} s_{\mu \cup \square}^{(k)} = \sum_{\gamma \in H_{\mu \cup \square, \ell}^{(k)}} s_{\gamma}^{(k)}. \quad (47)$$

It is known (Corollary 57 in [19]) that $\gamma \in H_{\mu \cup \square, \ell}^{(k)}$ implies $\mu \cup \square \preceq \gamma$, where $\alpha \preceq \beta$ is defined on k -bounded partitions by the covering relation: $\alpha \prec \beta$ when β, α are 1-admissible. Then, using Theorem 20 from [17]: $\mu \cup \square \preceq \gamma \iff \gamma = \eta \cup \square$ and $\mu \preceq \eta$ for some k -bounded η , we can transform (47) into

$$h_{\ell} \Theta_{\square} s_{\mu}^{(k)} = \sum_{\eta \cup \square \in H_{\mu \cup \square, \ell}^{(k)}} s_{\eta \cup \square}^{(k)}. \quad (48)$$

Since the k -Schur functions form a basis for $\Lambda^{(k)}$, it remains to show that the right side of Eq. (46) equals that of Eq. (48), or equivalently that

$$\eta \cup \square \in H_{\mu \cup \square, \ell}^{(k)} \iff \eta \in H_{\mu, \ell}^{(k)}.$$

Given $(\eta \cup \square)^{\omega_k} = \eta^{\omega_k} \cup \square^{\omega_k}$ by Theorem 10 of [17], we have $\eta \cup \square \in H_{\mu \cup \square, \ell}^{(k)}$ iff $\eta \cup \square / \mu \cup \square$ is a horizontal strip and $\eta^{\omega_k} \cup \square' / \mu^{\omega_k} \cup \square'$ is vertical strip. Thus, our claim follows by noting that $\alpha \cup \square / \beta \cup \square$ is a horizontal (resp. vertical) strip iff α / β is a horizontal (resp. vertical) strip. \square

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