RECURSION AND EXPLICIT FORMULAS FOR PARTICULAR N-VARIABLE KNOP-SAHI AND MACDONALD POLYNOMIALS

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ABSTRACT: Knop and Sahi simultaneously introduced a family of non-homogeneous, non-symmetric polynomials, $G_\alpha(x;q,t)$. The top homogeneous components of these polynomials are the non-symmetric Macdonald polynomials, $E_\alpha(x;q,t)$. An appropriate Hecke algebra symmetrization of E_α yields the Macdonald polynomials, $P_\lambda(x;q,t)$. A search for explicit formulas for the polynomials $G_\alpha(x;q,t)$ led to the main results of this paper. In particular, we give a complete solution for the case $G_{(k,a,\ldots,a)}(x;q,t)$. A remarkable by-product of our proofs is the discovery that these polynomials satisfy a recursion on the number of variables.

1. Introduction

The Macdonald polynomial basis $\{P_\lambda(x;q,t)\}_\lambda$ has recently become widely studied as the result of the many difficult conjectures that surround its ubiquitous appearance in various branches of mathematics. Important developments in the theory of symmetric functions rely on the Macdonald basis, which beautifully specializes to several fundamental bases such as the Schur, Hall-Littlewood, Zonal, and Jack. The Macdonald polynomials are known [11] to be connected to the theory of basic hypergeometric functions and further, it has been conjectured [2] that this basis occurs naturally in representation theory. More recently, because the Macdonald polynomials are eigenfunctions of an operator that describes a system of many particles, they have become an object of study [12] in research relating to particle mechanics.

The difficulty encountered in the study of the Macdonald polynomials stems in part from the absence of simple explicit formulas expressing $\{P_\lambda(x;q,t)\}_\lambda$ in terms of more familiar bases. Even determining that the coefficients of the Macdonald polynomials, expanded in terms of a modified Schur basis, are polynomials in q and t was an important breakthrough [3],[5],[6],[8],[13]. Among the proofs that these coefficients are polynomials is one which required the introduction of another family of polynomials. Knop [7] and Sahi [13] simultaneously introduced non-symmetric and non-homogenous polynomials, $G_\alpha(x_1,\ldots,x_n;q,t)$, of which the top component yields the non-symmetric version, $E_\alpha(x_1,\ldots,x_n;q,t)$, of the Macdonald polynomials [1],[10]. In turn, the polynomials $E_\alpha(x_1,\ldots,x_n;q,t)$ can be symmetrized to give the Macdonald polynomials $P_\lambda(x_1,\ldots,x_n;q,t)$. More precisely, it is shown in [10] that for α a composition that rearranges to λ,

$$P_\lambda \;=\; \sum_{\sigma\in S_n} t^{-length(\sigma)}\, T_\sigma\, E_\alpha$$

where T_σ is an appropriately defined Hecke algebra operator.

The Knop-Sahi polynomials are remarkable in that they are defined by very simple and elementary vanishing properties. This characterization yields a recursive algorithm for constructing the polynomials $G_\alpha(x_1,\ldots,x_n;q,t)$ and allows the derivation of several properties of the Macdonald polynomials. It is with this in mind that we have begun to search for explicit formulas for the Knop-Sahi polynomials. Our efforts have been motivated by the belief that the Knop-Sahi polynomials are a fundamental basis and a more intimate knowledge of this basis should be significant in any study of general polynomials in several variables .

For convenience, we let X_n denote the alphabet $\{x_1,x_2,\ldots,x_n\}$, and the symbol $(a;q)_n$ will be customarily defined as

$$(a;q)_n \;=\; (1-a)(1-qa)\cdots(1-q^{n-1}a)\,.$$

With this notation, our main results can be stated as follows;

Theorem 1.

$$G_{(k,0^{n-1})}(X_n;q,t) \;=\; \sum_{b=0}^{k} d_b(k,n)\,(q^{b+1}\,t^{n-1}\,x_n;q)_{k-b}\,G_{(b,0^{n-2})}(X_{n-1};q,t) \tag{1.1}$$

where $(k,0^{n-1})$, $(b,0^{n-2})$ represent n and $n-1$ tuples, respectively, with non negative integral components, and

$$d_b(k,n) \;=\; \frac{(-1)^{k+b}\,(q;q)_k\,(t;q)_{k-b}\,(t;q)_{b+1}}{q^{\binom{k}{2}-\binom{b}{2}}\,t^{(n-1)(k-b)}\,(t;q)_{k+1}\,(q;q)_b\,(q;q)_{k-b}} \tag{1.2}$$

Theorem 2. *With the convention $x_{n+1}\equiv x_1$, we have:*

$$G_{(k,r,\ldots,r)}(X_n;q,t) \;=\; \sum_{b_1+\cdots+b_n=k-r}\left(C_{b_1,\ldots,b_n}\prod_{i=1}^{n}(x_i;q)_r\,(q^{b_1+\cdots+b_{i-1}+r}t^{i-1}x_{i+1};q)_{b_i}\right) \tag{1.3}$$

where

$$C_{b_1,\ldots,b_n} \;=\; \frac{q^{(1-r)(k-r)-\binom{k-r}{2}-n\binom{r}{2}-b_n}\,(q;q)_{k-r}\,(t;q)_{b_1}\cdots(t;q)_{b_{n-1}}\,(qt;q)_{b_n}}{(-1)^{k+r(n-1)}\,t^{\sum_{j=1}^{n}(j-1)b_j}\,(qt;q)_{k-r}\,(q;q)_{b_1}\cdots(q;q)_{b_n}}\,. \tag{1.4}$$

We have mentioned that the non-symmetric Macdonald polynomials are obtained by taking the top component of the Knop-Sahi polynomials [7]. Theorem 2 thus yields as a corollary an explicit formula for $E_{(k,r,\ldots,r)}(x;1/q,1/t)$. Namely we have

Corollary 3.

$$E_{(k,r,\ldots,r)}(X_n;1/q,1/t) \;=\; \sum_{b_1+\cdots+b_n=k-r}\frac{(q;q)_{k-r}(t;q)_{b_1}\cdots(t;q)_{b_{n-1}}\,(qt;q)_{b_n}}{q^{b_n-k+r}(qt;q)_{k-r}\,(q;q)_{b_1}\cdots(q;q)_{b_n}}\,x_1^{b_n+r}x_2^{b_1+r}\cdots x_n^{b_{n-1}+r} \tag{1.5}$$

Remarkably, it will be shown that the recursion in (1.1) can be explicitly solved to yield the special case $r = 0$ of (1.3). This given, the general formula in (1.3) readily follows from a characterizing property of Knop-Sahi polynomials.

To see how all this comes about and to prove our results we need to review the definitions and some of the basic properties of Knop-Sahi polynomials.

2. Basic definitions and identities

We recall that a composition is a vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with non negative integral components. The parameter n will be referred to as the *length* of α. We shall also set

$$|\alpha| \;=\; \alpha_1 + \alpha_2 + \cdots + \alpha_n \;\;,$$

and refer to it as the *order* of α. The expression "*α is a composition of m*" will simply mean $|\alpha| = m$.

Denote by α^* the partition obtained by rearranging the components of α in weakly decreasing order. The 'position vector', $k(\alpha) = \big(k_1(\alpha),\ k_2(\alpha), \ldots,\ k_n(\alpha)\big)$, is a crucial ingredient defined as follows: if α has distinct parts then each α_i occupies a well defined position $k_i = k_i(\alpha)$ in α^*. By this we mean that $\alpha_i = \alpha^*_{k_i}$. We may then extend the definition of $k(\alpha)$ to the case in which α has equal components, breaking ties by considering equal parts as decreasing from left to right. In other words, if we label the parts of α by decreasing size and from left to right then $k_i(\alpha)$ is taken to be the label of α_i.

This given, Knop and Sahi associate a vector of monomials $\overline{\alpha}$ to each composition α. The vector contains parts defined by

$$(\overline{\alpha})_i \;=\; q^{-\alpha_i} t^{-n+k_i(\alpha)}\ . \tag{2.1}$$

This notation allows us to present the Knop-Sahi results. To begin with, adhering to Sahi's notation, it is shown in [13] that if α is a composition of m then in the linear span of the monomials $\{x^\beta\}_{|\beta|\le m}$ there exists a unique polynomial $G_\alpha(x;q,t)$ which satisfies the following two conditions:

$$\begin{aligned} &(a) \quad G_\alpha(\overline{\beta};q,t) \;=\; 0 \qquad \text{for all } |\beta| \le |\alpha| \text{ and } \beta \ne \alpha\ , \\ &(b) \quad G_\alpha(x;q,t)\,|_{x^\alpha} \;=\; 1, \end{aligned} \tag{2.2}$$

where (2.2)b is to say that the coefficient of x^α in G_α is normalized to 1. The uniqueness part of the Knop-Sahi result is relatively easy to show, yet uniqueness permits the immediate derivation of a number of surprising identities and recursions. Some of these are given by Sahi in [13] and others only in Knop [6]. Their basic identities which we shall use are expressed by the following three properties:

Property 4. *If $\gamma = (\gamma_1, \ldots, \gamma_n)$ and $\alpha = (\gamma_1 + r, \ldots, \gamma_n + r)$, then*

$$G_\alpha(x;q,t) \;=\; (-1)^{rn}\, q^{n\binom{r+1}{2} - r|\alpha|} \prod_{i=1}^{n} (x_i;q)_r \;\; G_\gamma(q^r x;q,t) \tag{2.3}$$

Property 5. *If $\alpha_n > 0$, then*

$$G_\alpha(x;q,t) \;=\; q^{1-\alpha_n}(x_n-1)\,G_{(\alpha_n-1,\alpha_1,\alpha_2,\ldots,\alpha_{n-1})}(qx_n,x_1,x_2,\ldots,x_{n-1};q,t)\ . \tag{2.4}$$

For $1 \le i \le n-1$, let $s_i = (i,i+1)$ denote the transposition that interchanges x_i and x_{i+1} and set [9]

$$T_{s_i} \;=\; s_i\,\frac{(1-t)}{x_i - x_{i+1}}\,x_i\,(1-s_i)\ . \tag{2.5}$$

It is well known that the operators T_{s_i} generate a faithful representation of the Hecke algebra of S_n in the space of polynomials in $x_1,\ldots,x_n$. It can be verified, now using symbolic manipulation software, that we have

$$\begin{aligned} &a)\quad T_{s_i}T_{s_j} \;=\; T_{s_j}T_{s_i} && \text{for}\quad |i-j|>1\ ,\\ &b)\quad T_{s_i}T_{s_{i+1}}T_{s_i} \;=\; T_{s_{i+1}}T_{s_i}T_{s_{i+1}} && \text{for}\quad i=1\ldots n-1\ ,\\ &c)\quad t\,T_{s_i}^{-1} \;=\; T_{s_i}-(1-t)\ . \end{aligned} \tag{2.6}$$

This permits us to extend the definition of T to all permutations $\sigma \in S_n$ by setting for any reduced expression $\sigma = s_{i_1}s_{i_2}\cdots s_{i_k}$,

$$T_\sigma \;=\; T_{s_{i_1}}T_{s_{i_2}}\cdots T_{s_{i_k}}\ . \tag{2.7}$$

Property 6. *With equivalence up to a scalar multiple denoted $\doteq$, we have*

$$\begin{aligned} &(a)\quad G_\alpha(x;q,t) \;=\; T_{s_i}\,G_\alpha(x;q,t) \;=\; s_i\,G_\alpha(x;q,t) && \text{if } \alpha_i=\alpha_{i+1}\\ &(b)\quad G_{s_i\alpha}(x;q,t) \;\doteq\; \Big(1-\frac{\overline{\alpha}_{i+1}}{\overline{\alpha}_i}\Big)\,T_{s_i}G_\alpha(x;q,t)+(t-1)\,G_\alpha(x;q,t) && \text{if } \alpha_i\neq\alpha_{i+1}\ . \end{aligned} \tag{2.8}$$

The properties 4, 5, and 6 combine nicely into a recursive algorithm for computing the polynomials G_α given the initial condition $G_{(0,0,\ldots,0)}(x;q,t)=1$. This enables rapid computation of extensive tables. Our strategy has been to induct general identities from examination of special cases.

3. Solving the recursion.

Property 4 yields theorem 2 as an immediate corollary of the following special case

Theorem 7. *With the convention $x_{n+1} \equiv x_1$, we have*

$$G_{(k,0,\ldots,0)}(X_n;q,t) \;=\; \sum_{b_1+\cdots+b_n=k}\left(C_{b_1,\ldots,b_n}\prod_{i=1}^{n}(q^{b_1+\cdots+b_{i-1}}t^{i-1}x_{i+1};q)_{b_i}\right) \tag{3.1}$$

where

$$C_{b_1,\ldots,b_n} \;=\; \frac{(-1)^k\,q^{k-\binom{k}{2}-b_n}\,(q;q)_k\,(t;q)_{b_1}\cdots(t;q)_{b_{n-1}}\,(qt;q)_{b_n}}{t^{\sum_{i=1}^n(i-1)b_i}\,(qt;q)_k\,(q;q)_{b_1}\cdots(q;q)_{b_n}}\ . \tag{3.2}$$

The proof of this identity will be the ultimate consequence of a number of auxiliary results which should be of intrinsic interest. We shall begin by showing that the family of polynomials defined by setting

$$W_k(X_m) \;=\; \sum_{b_1+\cdots+b_m=k} \left(C_{b_1,\ldots,b_m} \prod_{i=1}^{m} (q^{b_1+\cdots+b_{i-1}} t^{i-1} x_{i+1}; q)_{b_i} \right) , \tag{3.3}$$

where $x_{m+1} \equiv x_1$ and $C_{b_1,\ldots,b_m}$ is as given in (3.2), satisfies the recursion stated by theorem 1. To be precise we show that

Proposition 8. *For all $k \geq 0$ and $m \geq 1$ we have*

$$W_k(X_m) \;=\; \sum_{b_m=0}^{k} \frac{(q;q)_k \, (t;q)_{k-b_m} \, (t;q)_{b_m+1} \, (q^{b_m+1} \, t^{m-1} \, x_m; q)_{k-b_m}}{(-1)^{k+b_m} \, q^{\binom{k}{2}-\binom{b_m}{2}} \, t^{(m-1)(k-b_m)} \, (t;q)_{k+1} \, (q;q)_{b_m} \, (q;q)_{k-b_m}} \, W_{b_m}(X_{m-1}) \, . \tag{3.4}$$

Proof

Let R denote the right hand side of (3.4). The proof consists of making the replacement of $W_{b_m}(X_{m-1})$ in R with the polynomial given by the case $m-1$ of (3.3). The resulting expression can then be shown to sum to $W_k(X_m)$, again given by (3.3). To begin, note that making this substitution expresses R in the form

$$R \;=\; \sum_{b_m=0}^{k} \left(\frac{(-1)^{k-b_m} \, (q;q)_k \, (t;q)_{k-b_m} \, (t;q)_{b_m+1} \, (q^{b_m+1} t^{m-1} x_m; q)_{k-b_m}}{q^{\binom{k}{2}-\binom{b_m}{2}} \, t^{(m-1)(k-b_m)} \, (t;q)_{k+1} \, (q;q)_{b_m} \, (q;q)_{k-b_m}} \right.$$

$$\left. \times \sum_{b_1+\cdots+b_{m-1}=b_m} \frac{(q;q)_{b_m} \, (t;q)_{b_1} \cdots (t;q)_{b_{m-2}} \, (qt;q)_{b_{m-1}} \, \prod_{i=1}^{m-1} \, (q^{b_1+\cdots+b_{i-1}} t^{i-1} x_{i+1}; q)_{b_i}}{(-1)^{b_m} \, q^{b_{m-1}-b_m+\binom{b_m}{2}} \, t^{\sum_{i=1}^{m-1}(i-1)b_i} \, (qt;q)_{b_m} \, (q;q)_{b_1} \cdots (q;q)_{b_{m-1}}} \right) .$$

Direct substitution of $b_m = b_{m-1} + b_{m-2} + \cdots + b_1$ yields

$$R \;=\; \sum_{b_1+\cdots+b_{m-1}=0}^{k} \left(\frac{(-1)^k \, (q;q)_k \, (t;q)_{k-b_{m-1}-\cdots-b_1} \, (t;q)_{b_1} \cdots (t;q)_{b_{m-2}} \, (qt;q)_{b_{m-1}}}{q^{\binom{k}{2}-b_{m-2}-\cdots-b_1} \, t^{(m-1)(k-b_{m-1}-\cdots-b_1)} \, t^{\sum_{i=1}^{m-1}(i-1)b_i}} \right.$$

$$\left. \times \frac{(q^{b_{m-1}+\cdots+b_1+1} t^{m-1} x_m; q)_{k-b_{m-1}-\cdots-b_1}}{(qt;q)_k \, (q;q)_{b_1} \cdots (q;q)_{b_{m-1}} \, (q;q)_{k-b_{m-1}-\cdots-b_1}} \prod_{i=1}^{m-1} \, (q^{b_1+\cdots+b_{i-1}} t^{i-1} x_{i+1}; q)_{b_i} \right) .$$

Denote $b = b_1 + \cdots + b_{m-2}$ and apply the following property of q-shifted factorials to obtain an alternate expression for R.

$$(a;q)_{n-k} \;=\; \frac{(a;q)_n}{(q^{1-n}/a;q)_k} \left(-\frac{q}{a}\right)^k q^{\binom{k}{2}-nk} \tag{3.5}$$

$$R = \sum_{b=0}^{k} \left(\frac{(-1)^k\, q^{b-\binom{k}{2}}\, (q;q)_k\, (t;q)_{b_1} \cdots (t;q)_{b_{m-2}}\, (t;q)_{k-b}}{t^{(m-1)k-\sum_{i=1}^{m-2}(m-i)b_i} (qt;q)_k\, (q;q)_{b_1} \cdots (q;q)_{b_{m-2}} (q;q)_{k-b}} \prod_{i=1}^{m-2} (q^{b_1+\cdots+b_{i-1}} t^{i-1} x_{i+1};q)_{b_i} \right.$$

$$\left. \times \sum_{b_{m-1}=0}^{k-b} \frac{(qt;q)_{b_{m-1}}\, (q^{b-k};q)_{b_{m-1}}\, (q^{b_{m-1}+b+1} t^{m-1} x_m;q)_{k-b_{m-1}-b}\, (q^b t^{m-2} x_1;q)_{b_{m-1}}}{q^{-b_{m-1}}\, (q;q)_{b_{m-1}}\, (q^{1-k+b}/t;q)_{b_{m-1}}} \right)$$

Now the useful identity

$$(aq^k;q)_{n-k} \;=\; \frac{(a;q)_n}{(a;q)_k}\,, \tag{3.6}$$

specializes to the transformation,

$$(q^{b_{m-1}+b+1} t^{m-1} x_m;q)_{k-b-b_{m-1}} \;=\; \frac{(q^{b+1} t^{m-1} x_m;q)_{k-b}}{(q^{b+1} t^{m-1} x_m;q)_{b_{m-1}}}$$

and allows us to rewrite R in the form

$$R \;=\; \sum_{b=0}^{k} \left(\frac{q^{b-\binom{k}{2}}\, (q;q)_k\, (t;q)_{b_1} \cdots (t;q)_{b_{m-2}}\, (t;q)_{k-b}\, (q^{b+1} t^{m-1} x_m;q)_{k-b}}{(-1)^k\, t^{(m-1)k-\sum_{i=1}^{m-2}(m-i)b_i} (qt;q)_k\, (q;q)_{b_1} \cdots (q;q)_{b_{m-2}} (q;q)_{k-b}} \right.$$

$$\left. \times \prod_{i=1}^{m-2} (q^{b_1+\cdots+b_{i-1}} t^{i-1} x_{i+1};q)_{b_i} \sum_{b_{m-1}=0}^{k-b} \frac{q^{b_{m-1}}\, (qt;q)_{b_{m-1}}\, (q^{b-k};q)_{b_{m-1}}\, (q^b t^{m-2} x_1;q)_{b_{m-1}}}{(q;q)_{b_{m-1}}\, (q^{1-k+b}/t;q)_{b_{m-1}} (q^{b+1} t^{m-1} x_m;q)_{b_{m-1}}} \right). \tag{3.7}$$

We have arrived at an expression in which the variable b_{m-1} has been isolated. We shall next use the following summation identity [4]

$$_2\phi_1(a, q^{-n}; c; q, cq^n/a) \;=\; \sum_j \frac{(cq^n/a)^j\, (a;q)_j\, (q^{-n};q)_j}{(c;q)_j\, (q;q)_j} \;=\; \frac{(c/a;q)_n}{(c;q)_n} \tag{3.8}$$

to add a sum to the expression (3.7). To be precise, by setting

$$a = q^{1-k}/t^{m-1} x_1 \qquad \text{and} \qquad c = q^{1-k+b}/t$$

in (3.8), we can express the simple term,

$$\frac{(q^b t^{m-2} x_1;q)_{b_{m-1}}}{(q^{1-k+b}/t;q)_{b_{m-1}}}\,,$$

as a sum, obtaining

$$R = \sum_{b=0}^{k}\left(\frac{(-1)^k\, q^{b-\binom{k}{2}}\,(q;q)_k\,(t;q)_{b_1}\cdots(t;q)_{b_{m-2}}\,(t;q)_{k-b}\,(q^{b+1}t^{m-1}x_m;q)_{k-b}}{t^{(m-1)k+\sum_{i=1}^{m-2}(m-i)b_i}\,(qt;q)_k\,(q;q)_{b_1}\cdots(q;q)_{b_{m-2}}(q;q)_{k-b}}\right.$$

$$\times\prod_{i=1}^{m-2}(q^{b_1+\cdots+b_{i-1}}t^{i-1}x_{i+1};q)_{b_i}\sum_{b_{m-1}=0}^{k-b}\frac{q^{b_{m-1}}\,(qt;q)_{b_{m-1}}\,(q^{-k+b};q)_{b_{m-1}}}{(q;q)_{b_{m-1}}\,(q^{b+1}t^{m-1}x_m;q)_{b_{m-1}}}$$

$$\left.\times\sum_{a=0}^{b_{m-1}}\frac{(q^{b+b_{m-1}}t^{m-1}x_1)^a\,(q^{1-k}/t^{m-1}x_1;q)_a\,(q^{-b_{m-1}};q)_a}{(q^{1-k+b}/t;q)_a\,(q;q)_a}\right).$$

We now make the change of variables $b_{m-1}\to b_{m-1}+a$ to derive a formula that will make possible the elimination of the variable b_{m-1}:

$$R = \sum_{b=0}^{k}\left(\frac{(q;q)_k\,(t;q)_{b_1}\cdots(t;q)_{b_{m-2}}\,(t;q)_{k-b}\,(q^{b+1}t^{m-1}x_m;q)_{k-b}\,\prod_{i=1}^{m-2}(q^{b_1+\cdots+b_{i-1}}t^{i-1}x_{i+1};q)_{b_i}}{(-1)^k\,q^{\binom{k}{2}-b}\,t^{(m-1)k-\sum_{i=1}^{m-2}(m-i)b_i}\,(qt;q)_k\,(q;q)_{b_1}\cdots(q;q)_{b_{m-2}}(q;q)_{k-b}}\right.$$

$$\left.\times\sum_{b_{m-1}+a=0}^{k-b}\frac{(q^{1+b+a}t^{m-2}x_1)^a\,(qt;q)_{b_{m-1}+a}\,(q^{-k+b};q)_{b_{m-1}+a}\,(q^{1-k}/t^{m-1}x_1;q)_a\,(q^{-b_{m-1}-a};q)_a}{q^{-(1+a)b_{m-1}}\,(q;q)_{b_{m-1}+a}\,(q^{b+1}t^{m-1}x_m;q)_{b_{m-1}+a}\,(q^{1-k+b}/t;q)_a\,(q;q)_a}\right)$$

The isolation of b_{m-1} is again possible with the help of the properties

$$(aq^{-n};q)_n = (q/a;q)_n\left(-\frac{a}{q}\right)^n q^{-\binom{n}{2}}, \tag{3.9}$$

and

$$(a;q)_{n+k} = (a;q)_n\,(aq^n;q)_k\,. \tag{3.10}$$

This gives

$$R = \sum_{b=0}^{k}\left(\frac{(q;q)_k\,(t;q)_{b_1}\cdots(t;q)_{b_{m-2}}\,(t;q)_{k-b}\,(q^{b+1}t^{m-1}x_m;q)_{k-b}\,\prod_{i=1}^{m-2}(q^{b_1+\cdots+b_{i-1}}t^{i-1}x_{i+1};q)_{b_i}}{(-1)^k\,t^{(m-1)k-\sum_{i=1}^{m-2}(m-i)b_i}\,q^{\binom{k}{2}-b}\,(qt;q)_k\,(q;q)_{b_1}\cdots(q;q)_{b_{m-2}}(q;q)_{k-b}}\right.$$

$$\times\sum_{a=0}^{k-b}\frac{q^{a(1+b+a)}\,(t^{m-2}x_1)^a\,(qt;q)_a\,(q^{-k+b};q)_a\,(q^{-a};q)_a\,(q^{1-k}/t^{m-1}x_1;q)_a}{(q;q)_a\,(q^{b+1}t^{m-1}x_m;q)_a\,(q^{1-k+b}/t;q)_a\,(q;q)_a}$$

$$\left.\times\sum_{b_{m-1}=0}^{k-b-a}\frac{q^{b_{m-1}}\,(q^{1+a}t;q)_{b_{m-1}}\,(q^{-k+b+a};q)_{b_{m-1}}}{(q^{b+1+a}t^{m-1}x_m;q)_{b_{m-1}}(q;q)_{b_{m-1}}}\right).$$

A slight variation [4] of identity (3.8),

$${}_2\phi_1(a,q^{-n};c;q,q) = \frac{(c/a;q)_n}{(c;q)_n}\,a^n \tag{3.11}$$

finally allows us to eliminate b_{m-1} and reduce R to the expression

$$R = \sum_{b=0}^{k} \left(\frac{(q;q)_k\,(t;q)_{b_1}\cdots(t;q)_{b_{m-2}}\,(t;q)_{k-b}\,(q^{b+1}t^{m-1}x_m;q)_{k-b}\,\prod_{i=1}^{m-2}(q^{b_1+\cdots+b_{i-1}}t^{i-1}x_{i+1};q)_{b_i}}{(-1)^k\,q^{\binom{k}{2}-b}\,t^{(m-1)k-\sum_{i=1}^{m-2}(m-i)b_i}(qt;q)_k\,(q;q)_{b_1}\cdots(q;q)_{b_{m-2}}(q;q)_{k-b}} \right.$$

$$\left. \times \sum_{a=0}^{k-b} \frac{q^{k+ak-ab}\,x_1^a\,(qt;q)_a\,(q^{-k+b};q)_a\,(q^{-a};q)_a\,(q^{1-k}/t^{m-1}x_1;q)_a\,(q^bt^{m-2}x_m;q)_{k-b-a}}{t^{-(m-3)a-k+b}(q;q)_a\,(q^{b+1}t^{m-1}x_m;q)_a\,(q^{1-k+b}/t;q)_a\,(q;q)_a\,(q^{b+a+1}t^{m-1}x_m;q)_{k-b-a}} \right).$$

Manipulation of the right hand side using formulas (3.5), (3.9), and

$$(aq^k;q)_{n-k} = \frac{(a;q)_n}{(a;q)_k}\,, \tag{3.12}$$

yields the identity

$$R = \sum_{a+b=0}^{k} \left(\frac{(-1)^k\,(q;q)_k\,(t;q)_{b_1}\cdots(t;q)_{b_{m-2}}\,(qt;q)_a\,\prod_{i=1}^{m-2}(q^{b_1+\cdots+b_{i-1}}t^{i-1}x_{i+1};q)_{b_i}}{q^{\binom{k}{2}-k-b+ab+a}\,t^{(m-1)k-\sum_{i=1}^{m-2}(m-i)b_i}(qt;q)_k\,(q;q)_{b_1}\cdots(q;q)_{b_{m-2}}\,(q;q)_a} \right.$$

$$\left. \times \frac{(q^{k-a}t^{m-1}x_1;q)_a\,(q^bt^{m-2}x_m;q)_{k-b-a}\,(t;q)_{k-a-b}}{t^{2a-k+b-a}\,(q;q)_{k-a-b}} \right).$$

Denote $k-b-a = b_{m-1}$ and then let $a \to b_m$ to deduce that this expression is exactly $W_k(X_m)$ as given by (3.3). This completes our proof of proposition 8. □

4. Process of induction and implications

The proof that W_k are in fact Knop-Sahi polynomials (theorem 7) relies on an inductive argument on the number of variables. We begin with the base case.

Claim 9.

$$G_{(k)}(x_1;q,t) = W_k(x_1)$$

Proof

The Knop-Sahi polynomials are defined uniquely by properties (2.2a) and (2.2b). The beauty of this characterization is that if we establish that $W_k(x_1)$ satisfies these properties then it must be exactly $G_{(k)}(x_1;q,t)$. To first verify that $W_k(x_1)$ satisfies (2.2a), observe that definition (3.3) gives

$$W_k(x_1) = (-1)^k\,q^{-\binom{k}{2}}\,(x_1;q)_k\,. \tag{4.1}$$

Consider $\beta = (\beta_1)$ such that $\beta_1 < k$. The corresponding $\overline{\beta}$ thus become $\overline{\beta} = q^{-\beta_1}$. These are all the $\overline{\beta}$ on which $W_k(x_1)$ must vanish to satisfy condition (2.2a). Trivially we have

$$W_k(\overline{\beta}) = (-1)^k\,q^{-\binom{k}{2}}\,(q^{-\beta_1};q)_k = 0$$

since $(q^{-\beta_1};q)_k = 0$ for all $\beta_1 < k$. The validity of condition of (2.2b) is seen by taking the coefficient of x_1^k in (4.1); it is clearly 1. $\square$

The induction hypothesis may now be stated as

$$G_{(k,0,\ldots,0)}(X_{n-1};q,t) \;=\; W_k(X_{n-1})\,. \tag{4.2}$$

This hypothesis in conjunction with the recursive nature of W_k will help verify that the Knop-Sahi polynomials can be expressed as W_k. Let $m = n$ and $b_m = b$ in (3.4) to obtain the relation

$$W_k(X_n) \;=\; \sum_{b=0}^{k} \frac{(q;q)_k\,(t;q)_{k-b}\,(t;q)_{b+1}\,(q^{b+1}\,t^{n-1}\,x_n;q)_{k-b}}{(-1)^{k+b}\,q^{\binom{k}{2}-\binom{b}{2}}\,t^{(n-1)(k-b)}\,(t;q)_{k+1}\,(q;q)_b\,(q;q)_{k-b}}\,W_b(X_{n-1})\,.$$

The induction hypothesis (4.2) for $k = b$ allows us to rewrite this relation as a useful identity:

$$W_k(X_n) \;=\; \sum_{b=0}^{k} \frac{(q;q)_k\,(t;q)_{k-b}\,(t;q)_{b+1}\,(q^{b+1}\,t^{n-1}\,x_n;q)_{k-b}}{(-1)^{k+b}\,q^{\binom{k}{2}-\binom{b}{2}}\,t^{(n-1)(k-b)}\,(t;q)_{k+1}\,(q;q)_b\,(q;q)_{k-b}}\,G_{(b,0,\ldots,0)}(X_{n-1};q,t)\,. \tag{4.3}$$

The proof of theorem 7 now depends on verifying that $G_{(k,0,\ldots,0)}(X_n;q,t)$ can be expressed as $W_k(X_n)$ thereby completing the induction argument. The following implication of (4.3) must be shown preliminarily.

Lemma 10. *$W_k(X_n)$ is symmetric in the variables $x_2,\ldots,x_n$.*

Proof

Any permutation of the variables x_2 through x_n can be written as the product of the simple transpositions, s_i where $2 \le i \le n-1$. Thus the lemma may be equivalently stated as;

$$W_k(X_n) \;=\; s_i\,W_k(X_n) \qquad \text{for all} \quad 2 \le i \le n-1 \tag{4.4}$$

Sahi proves in [13] that $s_iG_\alpha(X_{n-1};q,t) = G_\alpha(X_{n-1};q,t)$ for all α such that $\alpha_i = \alpha_{i+1}$. This result directly implies that $s_iG_{(b,0,\ldots,0)}(X_{n-1};q,t) = G_{(b,0,\ldots,0)}(X_{n-1};q,t)$ for all $2 \le i \le n-2$. It thus follows using (4.3) that for all such i, $s_iW_k(X_n) = W_k(X_n)$. Therefore to prove lemma 10, it remains only to show

$$s_{n-1}W_k(X_n) = W_k(X_n)\,. \tag{4.5}$$

Proposition 8 expresses $W_k(X_n)$ in terms of $W_{b_n}(X_{n-1})$. Applying this proposition again, to $W_{b_n}(X_{n-1})$, we transform (3.4) into the form

$$
\begin{aligned}
W_k(X_n) &= \sum_{b_n=0}^{k} \frac{(q;q)_k\,(t;q)_{k-b_n}\,(t;q)_{b_n+1}\,(q^{b_n+1}\,t^{n-1}\,x_n;q)_{k-b_n}\,W_{b_n}(X_{n-1})}{(-1)^{k+b_n}\,q^{\binom{k}{2}-\binom{b_n}{2}}\,t^{(n-1)(k-b_n)}\,(t;q)_{k+1}\,(q;q)_{b_n}\,(q;q)_{k-b_n}} \\
&= \sum_{b_n=0}^{k} \left(\frac{(q;q)_k\,(t;q)_{k-b_n}\,(q^{b_n+1}\,t^{n-1}\,x_n;q)_{k-b_n}}{(-1)^{k+b_n}\,q^{\binom{k}{2}}\,t^{(n-1)(k-b_n)}\,(t;q)_{k+1}\,(q;q)_{k-b_n}} \right. \\
&\qquad \left. \times \sum_{b_{n-1}=0}^{b_n} \frac{(t;q)_{b_n-b_{n-1}}\,(t;q)_{b_{n-1}+1}\,(q^{b_{n-1}+1}\,t^{n-2}\,x_{n-1};q)_{b_n-b_{n-1}}\,W_{b_{n-1}}(X_{n-2})}{(-1)^{b_n+b_{n-1}}\,q^{-\binom{b_{n-1}}{2}}\,t^{(n-2)(b_n-b_{n-1})}\,(q;q)_{b_{n-1}}\,(q;q)_{b_n-b_{n-1}}} \right).
\end{aligned}
\tag{4.6}
$$

This formula facilitates the understanding of the action of s_{n-1} on $W_k(X_n)$. The simple application of transposition s_{n-1} to (4.6) shows that

$$
\begin{aligned}
s_{n-1}\,W_k(X_n) &= \sum_{b_n=0}^{k} \left(\frac{(-1)^{k+b_n}\,(q;q)_k\,(t;q)_{k-b_n}\,(q^{b_n+1}\,t^{n-1}\,x_{n-1};q)_{k-b_n}}{q^{\binom{k}{2}}\,t^{(n-1)(k-b_n)}\,(t;q)_{k+1}\,(q;q)_{k-b_n}} \right. \\
&\qquad \left. \times \sum_{b_{n-1}=0}^{b_n} \frac{(t;q)_{b_n-b_{n-1}}\,(t;q)_{b_{n-1}+1}\,(q^{b_{n-1}+1}\,t^{n-2}\,x_n;q)_{b_n-b_{n-1}}\,W_{b_{n-1}}(X_{n-2})}{(-1)^{b_n+b_{n-1}}\,q^{-\binom{b_{n-1}}{2}}\,t^{(n-2)(b_n-b_{n-1})}\,(q;q)_{b_{n-1}}\,(q;q)_{b_n-b_{n-1}}} \right).
\end{aligned}
\tag{4.7}
$$

To derive the invariance of $W_k(X_n)$ under the action of s_{n-1}, it suffices to verify that the coefficients of $x_n^r x_{n-1}^s W_{k-r-s}(X_{n-2})$ in the right hand sides of (4.6) and (4.7) are equivalent. To obtain this coefficient in $W_k(X_n)$, we set $r = k - b_n$ and $s = b_n - b_{n-1}$ in the right hand side of (4.6).

$$
\begin{aligned}
W_k(X_n)|_{x_n^r x_{n-1}^s W_{k-r-s}(X_{n-2})} &= \left(\frac{q^{r(k-r+1)+\binom{r}{2}+\binom{k-r}{2}}\,(q;q)_k\,(t;q)_r\,(t;q)_{k-r+1}}{q^{\binom{k}{2}}\,(t;q)_{k+1}\,(q;q)_{k-r}\,(q;q)_r} \right. \\
&\qquad \left. \times \frac{q^{s(k-r-s+1)+\binom{s}{2}+\binom{k-r-s}{2}}\,(q;q)_{k-r}\,(t;q)_s\,(t;q)_{k-r-s+1}}{q^{\binom{k-r}{2}}\,(t;q)_{k-r+1}\,(q;q)_{k-r-s}\,(q;q)_s} \right) \\
&= \frac{q^{r+s}(q;q)_k\,(t;q)_r\,(t;q)_s\,(t;q)_{k-r-s+1}}{(t;q)_{k+1}\,(q;q)_r\,(q;q)_{k-r-s}\,(q;q)_s}
\end{aligned}
\tag{4.8}
$$

Alternatively, the coefficient of $s_{n-1}W_k(X_n)$ can be found by setting $r = b_n - b_{n-1}$ and $s = k - b_n$ in the right hand side of (4.7) This gives

$$
\begin{aligned}
s_{n-1}\,W_k(X_n)|_{x_n^r x_{n-1}^s W_{k-r-s}(X_{n-2})} &= \left(\frac{(-1)^{r+s} q^{\binom{r}{2}+r(k-r-s+1)}\,q^{\binom{k-s}{2}}\,(q;q)_k\,(t;q)_s\,(t;q)_{k-s+1}}{q^{\binom{k}{2}}\,(t;q)_{k+1}\,(q;q)_{k-s}\,(q;q)_s} \right. \\
&\qquad \left. \times \frac{q^{(k-s+1)s+\binom{s}{2}}\,q^{\binom{k-s-r}{2}}\,(q;q)_{k-s}\,(t;q)_r\,(t;q)_{k-s-r+1}}{(-1)^{r+s}\,q^{\binom{k-s}{2}}\,(t;q)_{k-s+1}\,(q;q)_{k-s-r}\,(q;q)_r} \right)
\end{aligned}
$$

This simplifies to the form;

$$s_{n-1}\, W_k(X_n)|_{x_n^r x_{n-1}^s G_{(k-r-s,0,\ldots,0)}(X_{n-2})} \;=\; \frac{q^{r+s}\,(q;q)_k\,(t;q)_s\,(t;q)_r\,(t;q)_{k-s-r+1}}{(t;q)_{k+1}\,(q;q)_s\,(q;q)_{k-s-r}\,(q;q)_r}\,.$$

This is identical to the right hand side of (4.8) giving us the desired invariance which, by the above discussion, implies that $W_k(X_n)$ must be symmetric in the variables $x_2,\ldots,x_n$ $\square$

The proof that $G_{(k,0,\ldots,0)}(X_n;q,t)$ can be expressed as $W_k(X_n)$ will proceed as in the base case. This requires that we verify $W_k(X_n)$ satisfies the characterizing properties, (2.2a) and (2.2b), of the Knop-Sahi polynomials. More precisely, we must show that $W_k(X_n)$ vanishes on all $\overline{\beta}$ where $|\beta|\le k$ and $\beta\neq(k,0,\ldots,0)$. To do this, we must separate these β into two groups: those in which there exists some component of β which is stricly larger than β_1, and the β of which β_1 is the weakly largest component. These two separate verifications will be carried out in the next two sections. We shall first consider the case in which there exist a component larger than β_1.

5. The vanishing on $\overline{\beta}$ when β_1 is not the largest component

The indicated vanishing may be expressed more precisely with the following claim:

Claim 11. *Let $\beta=(\beta_1,\ldots,\beta_n)$ be such that $|\beta|\le k$ and $\beta\neq(k,0,\ldots,0)$.*
If $\beta_1<\beta_i$ for some $2\le i\le n$, then $W_k(\overline{\beta}) \;=\; 0.$

This claim is the result of two lemmas; the first is a simple consequence of definition (2.1) for $\overline{\beta}$.

Lemma 12. *Let $\beta=(\beta_1,\ldots,\beta_n)$ where β_j is the leftmost occurence of the largest component.*
Then for $\gamma=(\beta_1,\ldots,\beta_{j-1},\beta_{j+1},\ldots,\beta_n)$, we have $\overline{\gamma}=(\overline{\beta}_1,\ldots,\overline{\beta}_{j-1},\overline{\beta}_{j+1},\ldots,\overline{\beta}_n)$.

Proof

Since β_j is the leftmost occurence of the largest component of β, the construction of $k(\beta)$ gives $k_j(\beta)=1$. From this, we deduce

$$(\overline{\gamma})_l \;=\; \begin{cases} q^{-\gamma_l}t^{-(n-1)+k_l(\gamma)} \;=\; q^{-\beta_l}t^{-n+1+k_l(\beta)-1} \;=\; (\overline{\beta})_l & \text{for}\quad 1\le l\le j-1\\[2ex] q^{-\gamma_l}t^{-(n-1)+k_l(\gamma)} \;=\; q^{-\beta_{l+1}}t^{-n+1+k_{l+1}(\beta)-1} \;=\; (\overline{\beta})_{l+1} & \text{for}\quad n-1\ge l\ge j \end{cases}\;\square$$

The second lemma contributing to the proof of our claim depends on an extra vanishing property of the Knop-Sahi polynomials that is given by Knop. The following ordering,

Let α, γ be compositions with length n. Then $\alpha\le\gamma$ if there is a permutation $\pi\in S_n$ such that $\alpha_i<\gamma_{\pi(i)}$ for $i<\pi(i)$ and $\alpha_i\le\gamma_{\pi(i)}$ for $i\ge\pi(i)$,

is necessary to introduce this important result [7].

Property 13.

$G_\alpha(\overline{\gamma};q,t) \;=\; 0$ for all $\alpha\not\le\gamma$.

Lemma 14. *Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ and $\gamma = (\gamma_1, \ldots, \gamma_m)$.*
If $\gamma_1 < \alpha_1$ and $\gamma_i \le \alpha_1$ for all $2 \le i \le m$, then $G_\alpha(\overline{\gamma}; q, t) = 0$.

Proof

The extra vanishing condition yields that for $\alpha \not\le \gamma$, $G_\alpha(\overline{\gamma}; q, t) = 0$. Thus it suffices to show $\alpha \not\le \gamma$. Suppose $\alpha \le \gamma$. The definition of $\le$ implies in particular, that there must exist some $\pi \in S_m$ such that $\alpha_1 \le \gamma_{\pi(1)}$ if $1 \ge \pi(1)$ and $\alpha_1 < \gamma_{\pi(1)}$ if $1 < \pi(1)$. Consider the possibility that $\pi(1) = 1$, which includes all $\pi(1) \le 1$. Then we must have $\alpha_1 \le \gamma_1$. But our hypothesis requires that $\gamma_1 < \alpha_1$. We are left only with the possibility that $\pi(1) > 1$, where $\pi(1) = i$ for some $2 \le i \le m$. The ordering then implies $\alpha_1 < \gamma_i$ for some $2 \le i \le m$. This again contradicts the initial supposition that $\alpha_1 \ge \gamma_i$ for all $2 \le i \le m$ and we thus have lemma 14. □

Proof of claim 11.

The two lemmas proven, we are now in the position to prove claim 11. The symmetry held by $W_k(X_n)$ as shown with lemma 10 gives that

$$W_k(X_n) \;=\; (1, 2, \ldots, j-1, n, j+1, \ldots, n-1, j)\, W_k(X_n) \quad \forall\, j \ne 1\,. \tag{5.1}$$

Replacing the W_k in the right hand side with an equivalent expression given by (4.3), we obtain

$$W_k(X_n) \;= \sum_{b=0}^{k} \frac{(q;q)_k\,(t;q)_{k-b}\,(t;q)_{b+1}\,(q^{b+1}t^{n-1}x_j;q)_{k-b}\,G_{(b,0,\ldots,0)}(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n;q,t)}{(-1)^{k+b}\,q^{\binom{k}{2}-\binom{b}{2}}\,t^{(n-1)(k-b)}\,(t;q)_{k+1}\,(q;q)_b\,(q;q)_{k-b}}\,.$$

The evaluation of this expression at $\overline{\beta}$ yields

$$W_k(\overline{\beta}) \;= \sum_{b=0}^{k} \frac{(q;q)_k\,(t;q)_{k-b}\,(t;q)_{b+1}\,(q^{b+1}t^{n-1}\overline{\beta}_j;q)_{k-b}\,G_{(b,0,\ldots,0)}(\overline{\beta}_1,\ldots,\overline{\beta}_{j-1},\overline{\beta}_{j+1},\ldots,\overline{\beta}_n;q,t)}{(-1)^{k+b}\,q^{\binom{k}{2}-\binom{b}{2}}\,t^{(n-1)(k-b)}\,(t;q)_{k+1}\,(q;q)_b\,(q;q)_{k-b}}\,.$$

Denote $\gamma = (\beta_1, \ldots, \beta_{j-1}, \beta_{j+1}, \ldots, \beta_n)$ where again, as in Lemma 12, β_j is the largest component occuring first in β. Lemma 12 then gives that $\overline{\gamma} = (\overline{\beta}_1, \ldots, \overline{\beta}_{j-1}, \overline{\beta}_{j+1}, \ldots, \overline{\beta}_n)$. Furthermore, since the largest component occuring first in β will take position 1 in β^*, we shall have $k_j(\beta) = 1$ implying $\overline{\beta}_j = q^{-\beta_j} t^{1-n}$. Note also that because β_1 is strictly less than at least one component of β, it must be true that $2 \le j \le n$. With this in mind, the previous identity may now be expressed as

$$W_k(\overline{\beta}) \;=\; \sum_{b=0}^{k} \frac{(q;q)_k\,(t;q)_{k-b}\,(t;q)_{b+1}\,(q^{b+1-\beta_j};q)_{k-b}\,G_{(b,0,\ldots,0)}(\overline{\gamma};q,t)}{(-1)^{k+b}\,q^{\binom{k}{2}-\binom{b}{2}}\,t^{(n-1)(k-b)}\,(t;q)_{k+1}\,(q;q)_b\,(q;q)_{k-b}}\,. \tag{5.2}$$

The summand in (5.2) only vanishes if either $(q^{b+1-\beta_j};q)_{k-b} = 0$ or $G_{(b,0,\ldots,0)}(\overline{\gamma};q,t) = 0$. It develops that one of these cases occurs for *each* $0 \le b \le k$. We first consider the b for which $(q^{b+1-\beta_j};q)_{k-b} = 0$.

It happens that if we make the restriction $b + 1 < \beta_j$, we will have $(q^{b+1-\beta_j};q)_{k-b} = 0$ for all $\beta_j - b - 1 < k - b$. But it is clear that $\beta_j \le k$ since we are only considering $|\beta| \le k$, and thus the term vanishes on each of these restricted b . Observe further that if $b + 1 = \beta_j$ we have

$(q^{b+1-\beta_j};q)_{k-b} = (1;q)_{k-b} = 0$ unless $b = k$. Under the restriction on b, $b = k$ would imply that $k+1 = \beta_j$ contradicting $|\beta| \le k$. Thus again we have $(q^{b+1-\beta_j};q)_{k-b} = 0$ yielding a vanishing summand for all $b < \beta_j$.

The vanishing at the remaining $b \ge \beta_j$ is a result of verifying that we have the conditions of lemma 14, with $\alpha = (b,0,\ldots,0)$, to ascertain that the term $G_{(b,0,\ldots,0)}(\overline{\gamma};q,t)$ vanishes. Recalling the definition of γ, we have that $\gamma_1 = \beta_1 < \beta_j$ since β_j is the largest component of β and β_1 was assumed to be strictly less than the the largest component. Further, our restriction to the remaining $b \ge \beta_j$ yields $\gamma_1 < b = \alpha_1$. Now for $2 \le l \le n-1$, $\gamma_l = \beta_i$ for some $i \ne 1$ gives that $\gamma_l \le \beta_j \le b = \alpha_1$. These are exactly the conditions required by lemma 14 to yield the vanishing of $G_{(b,0,\ldots,0)}(\overline{\gamma};q,t)$ and thus we have proved claim 11.

6. Vanishing when β_1 is the largest component

To complete the verification that $W_k(X_n)$ satisfies property $(2.2a)$ we must prove:

Claim 15. *For $\beta = (\beta_1,\ldots,\beta_n)$ where $|\beta| \le k$ and $\beta \ne (k,0,\ldots,0)$,
If $\beta_1 \ge \beta_i$ for all $2 \le i \le n$, then $W_k(\overline{\beta}) = 0$.*

We first show the vanishing of $W_k(X_n)$ at partition β.

Lemma 16. *Let $\beta = (\beta_1,\ldots,\beta_n)$ where $|\beta| \le k$ and $\beta \ne (k,0,\ldots,0)$.
If β has the ordering, $\beta_1 \ge \beta_2 \ge \cdots \ge \beta_n$, then $W_k(\overline{\beta}) = 0$.*

Proof

$W_k(X_n)$ is symmetric in variables $x_2,\ldots,x_n$ implies that for $\sigma = (1,n,n-1,\ldots,2)$,

$$W_k(X_n) \;=\; \sigma\, W_k(X_n) \;=\; W_k(x_1,x_n,\ldots,x_2)\,.$$

The definition of $\overline{\beta}$ when β is a partition thus yields the following evaluation of $W_k(X_n)$:

$$W_k(\overline{\beta}) \;=\; W_k(\overline{\beta}_1,\overline{\beta}_n,\ldots,\overline{\beta}_2) \;=\; W_k(q^{-\beta_1}/t^{n-1},q^{-\beta_n},q^{-\beta_{n-1}}/t,\ldots,q^{-\beta_2}/t^{n-2})\,.$$

More explicitly, using formula (3.3), we have

$$W_k(\overline{\beta}) \;=\; \left(\sum_{b_1+\cdots+b_n=k} \frac{q^{k-\binom{k}{2}+b_n}\,(q;q)_k\,(t;q)_{b_1}\cdots(t;q)_{b_{n-1}}(qt;q)_{b_n}}{(-1)^k\,t^{\sum_{i=1}^n(i-1)b_i}(qt;q)_k\,(q;q)_{b_1}\cdots(q;q)_{b_n}} \prod_{i=1}^n (q^{b_1+\cdots+b_{i-1}-\beta_{n+1-i}};q)_{b_i} \right). \tag{6.1}$$

We will show that the summand vanishes for *each* $b_1+\ldots+b_n = k$ by exhibiting that some term in the product $\prod_{i=1}^n (q^{b_1+\cdots+b_{i-1}-\beta_{n+1-i}};q)_{b_i}$ vanishes on *all* such $\{b_1,\ldots,b_n\}$. Equivalently, we claim that both the following conditions hold for *some* i;

$$\begin{array}{ll} (i) & b_1+\cdots+b_{i-1} \le \beta_{n+1-i} \\ (ii) & \beta_{n+1-i}-b_1-\cdots-b_{i-1} < b_i\,. \end{array}$$

If we assume the converse, one of the conditions must fail for *all* i. Since $\beta_n \ge 0$ implies that condition (i) for $i=1$ holds, condition (ii) for $i=1$ must fail. Equivalently, $\beta_n \ge b_1$. Because β is a partition, we further have $\beta_{n-1} \geq \beta_n \geq b_1$, implying that condition (i) for $i = 2$ holds and thus condition (ii) for $i = 2$ must fail; $\beta_{n-1} \geq b_1 + b_2$. Iteration brings us to the situation in which condition (i) holds for $i = n$ implying condition (ii) fails for $i = n$. This is to say, $\beta_1 \geq b_1 + \cdots + b_n = k$. Since $\beta \neq (k, 0, \ldots, 0)$ and $|\beta| \leq k$, we have produced a contradiction. Thus we have proved the vanishing of $W_k(X_n)$ for β a partition. $\square$

We need now to extend this result to all β specified in our claim. To this end, because β^* is a partition, lemma 16 gives that

$$W_k(\overline{\beta^*}) \;=\; 0 \;.$$

Observe that β_1 is the largest component of β, giving $k_1(\beta) = 1$. Because $\sigma = k(\beta)$ is a permutation fixing 1 and W_k is invariant under all such permutations (lemma 10), we can further deduce that

$$W_k(\sigma\, \overline{\beta^*}) \;=\; 0\,. \tag{6.4}$$

The definition $(\overline{\beta^*})_i = q^{-\beta_i^*} t^{n-i}$ gives

$$(\sigma\overline{\beta^*})_i \;=\; (\overline{\beta^*})_{\sigma_i} \;=\; (\overline{\beta^*})_{k_i(\beta)} \;=\; q^{-\beta^*_{k_i(\beta)}}\; t^{n-k_i(\beta)} \;.$$

But because $\beta^*_{k_i(\beta)} = \beta_i$, we have

$$(\sigma\overline{\beta^*})_i \;=\; q^{-\beta_i}\; t^{n-k_i(\beta)} \;=\; (\overline{\beta})_i \;.$$

Now we can conveniently express $\overline{\beta}$ as

$$\overline{\beta} \;=\; \sigma\overline{\beta^*} \;,$$

and determine finally using (6.4) that

$$W_k(\overline{\beta}) \;=\; W_k(\sigma\overline{\beta^*}) \;=\; 0 \;.$$

Remark: The uniqueness of the Knop-Sahi result requires only that we ensure $W_k(X_n)$ has the proper normalization having just shown that these polynomials satisify the vanishing properties that characterize $G_{(k,0,\ldots,0)}(X_n; q, t)$. The desired normalization is given with condition $(2.2b)$, and it is easy to see that the coefficient of x_1^k in our formula for $W_k(X_n)$ is, in fact, 1.

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