

SCHUR FUNCTION ANALOGS FOR A FILTRATION OF THE SYMMETRIC FUNCTION SPACE

L. LAPOINTE AND J. MORSE

ABSTRACT. We work here with the linear span $\Lambda_t^{(k)}$ of Hall-Littlewood polynomials indexed by partitions whose first part is no larger than k . The sequence of spaces $\Lambda_t^{(k)}$ yield a filtration of the space Λ of symmetric functions in an infinite alphabet X . In joint work with Lascoux [5] we gave a combinatorial construction of a family of symmetric polynomials $\{A_\lambda^{(k)}[X; t]\}_{\lambda_1 \leq k}$, with $\mathbb{N}[t]$ -integral Schur function expansions, which we conjectured to yield a basis for $\Lambda_t^{(k)}$. Our primary motivation for this construction is to provide a positive integral factorization of the Macdonald q, t -Kostka matrix. More precisely, we conjecture that the connection coefficients expressing the Hall-Littlewood or Macdonald polynomials belonging to $\Lambda_t^{(k)}$ in terms the basis $\{A_\lambda^{(k)}[X; t]\}_{\lambda_1 \leq k}$ are polynomials in $\mathbb{N}[q, t]$. We give here a purely algebraic construction of a new family $\{s_\lambda^{(k)}[X; t]\}_{\lambda_1 \leq k}$ of polynomials in $\Lambda_t^{(k)}$ which we conjecture is identical to $\{A_\lambda^{(k)}[X; t]\}_{\lambda_1 \leq k}$. We prove that $\{s_\lambda^{(k)}[X; t]\}_{\lambda_1 \leq k}$ is in fact a basis of $\Lambda_t^{(k)}$ and derive several further properties including that $s_\lambda^{(k)}[X; t]$ reduces to the Schur function $s_\lambda[X]$ for sufficiently large k . We also state a number of conjectures which reveal that the polynomials $\{s_\lambda^{(k)}[X; t]\}_{\lambda_1 \leq k}$ are in fact the natural analogues of Schur functions for the space $\Lambda_t^{(k)}$.

1. INTRODUCTION

Let Λ be the ring of symmetric functions in the variables x_1, x_2, \dots , with coefficients in $\mathbb{Q}(q, t)$, for parameters q and t . The Schur functions, $s_\lambda[X]$, form a fundamental basis of Λ , with central roles in fields such as representation theory and algebraic geometry. For example, the Schur functions can be identified with the characters of irreducible representations of the symmetric group, and their products are equivalent to the Pieri formulas for multiplying Schubert varieties in the intersection ring of a Grassmannian. Furthermore, the connection coefficients of the Schur function basis with various bases such as the homogeneous symmetric functions, the Hall-Littlewood polynomials, and the Macdonald polynomials, are positive and have representation theoretic interpretations. In the case of the Macdonald polynomials, $H_\lambda[X; q, t]$, this expansion takes the form

$$H_\lambda[X; q, t] = \sum_{\mu} K_{\mu\lambda}(q, t) s_\mu[X], \quad K_{\mu\lambda}(q, t) \in \mathbb{N}[q, t], \quad (1.1)$$

where $K_{\mu\lambda}(q, t)$ are known as the q, t -Kostka polynomials. The representation theoretic interpretation for these polynomials is given in [1, 3].

Here, we consider the filtration $\Lambda_t^{(1)} \subseteq \Lambda_t^{(2)} \subseteq \dots \subseteq \Lambda_t^{(\infty)} = \Lambda$, given by linear spans of Hall-Littlewood polynomials indexed by k -bounded partitions. That is,

$$\Lambda_t^{(k)} = \mathcal{L}\{H_\lambda[X; t]\}_{\lambda; \lambda_1 \leq k}, \quad k = 1, 2, 3, \dots$$

1991 *Mathematics Subject Classification*. Primary 05E05.
Research supported in part by NSF Grant #0100179.

In [5], jointly with Lascoux, we gave a purely combinatorial construction of a family of symmetric polynomials $\{A_\lambda^{(k)}[X; t]\}_{\lambda_1 \leq k}$ with the Schur function expansion

$$A_\lambda^{(k)}[X; t] = \sum_{\mu \geq \lambda} v_{\mu\lambda}^{(k)}(t) s_\mu[X], \quad v_{\mu\lambda}^{(k)}(t) \in \mathbb{N}[t]. \quad (1.2)$$

which we conjectured formed a basis of $\Lambda_t^{(k)}$ yielding the following remarkable decomposition of a Macdonald polynomial indexed by a k -bounded partition

$$H_\lambda[X; q, t] = \sum_{\mu; \mu_1 \leq k} K_{\mu\lambda}^{(k)}(q, t) A_\mu^{(k)}[X; t], \quad K_{\mu\lambda}^{(k)}(q, t) \in \mathbb{N}[q, t]. \quad (1.3)$$

Such a refinement of the decomposition in (1.1) has been the primary goal of our investigations. We were led to the construction of the polynomials $A_\lambda^{(k)}[X; t]$ by a close analysis of tableaux combinatorics under “*katabolism*” [5]. We discovered that, in a sense that can be made precise, the polynomials $A_\lambda^{(k)}[X; t]$, represent an “*atomic*” decomposition of the space $\Lambda_t^{(k)}$ and are characterized by the additional requirement that $A_\lambda^{(k)}[X; t]$, when embedded in $\Lambda_t^{(k')}$ for $k' > k$, has the decomposition

$$A_\lambda^{(k)}[X; t] = A_\lambda^{(k')}[X; t] + \sum_{\mu > \lambda} v_{\mu\lambda}^{(k \rightarrow k')}(t) A_\mu^{(k')}[X; t], \quad \text{with } v_{\mu\lambda}^{(k \rightarrow k')}(t) \in \mathbb{N}[t]. \quad (1.4)$$

In [5] the combinatorics of the construction led to a number of conjectures revealing that the “*Atoms*” $A_\lambda^{(k)}[X; t]$ have a remarkable kinship with classical Schur functions extending some of their fundamental properties. Yet the combinatorial setting, while fertile for intuition and computer experimentation, lagged somewhat in mechanisms of proof. In this paper we give a purely algebraic construction of a family $\{s_\lambda[X; t]\}_{\lambda_1 \leq k}$ we call “*k-Schur functions*” which we conjecture is identical to the family of atoms $\{A_\lambda[X; t]\}_{\lambda_1 \leq k}$ and prove some of the properties which we were only conjectures for the atoms. We show here that our k -Schur functions $\{s_\lambda[X; t]\}_{\lambda_1 \leq k}$ are triangularly related to the Hall-Littlewood basis $\{H_\lambda[X; t]\}_{\lambda_1 \leq k}$ and in particular they form a basis for the space $\Lambda_t^{(k)}$. We also show here that the expansions (1.2), (1.3), and (1.4) with $A_\lambda^{(k)}[X; t]$ replaced by $s_\lambda^{(k)}[X; t]$, do hold for $k = 2$ (Section 7) and in the general case, with $\mathbb{N}[t]$ and $\mathbb{N}[q, t]$ replaced by $\mathbb{Z}[t]$ and $\mathbb{Z}[q, t]$.

Formally, the definition of $s_\lambda^{(k)}[X; t]$ is quite simple. Indeed, for a k -bounded partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$, we let:

$$s_\lambda^{(k)}[X; t] = \bar{T}_{\lambda_1}^{(k)} B_{\lambda_1}^t \cdots \bar{T}_{\lambda_\ell}^{(k)} B_{\lambda_\ell}^t \cdot 1. \quad (1.5)$$

where the operators B_a^t were introduced in [4] to recursively build the Hall-Littlewood polynomials. More precisely,

$$H_\lambda^{(k)}[X; t] = B_{\lambda_1}^t \cdots B_{\lambda_\ell}^t \cdot 1.$$

The complexity of our k -Schur functions is encoded in the definition of the operators $\bar{T}_m^{(k)}$ which requires the construction of an auxiliary basis $\{G_\lambda[X; t]\}_{\lambda_1 \leq k}$ for $\Lambda_t^{(k)}$. This done, $\bar{T}_m^{(k)}$ is simply defined by setting

$$\bar{T}_m^{(k)} G_\lambda^{(k)}[X; t] = \begin{cases} G_\lambda^{(k)}[X; t] & \text{if } \lambda_1 = m, \\ 0 & \text{otherwise.} \end{cases} \quad (1.6)$$

Our construction of $G_\lambda^{(k)}[X; t]$ is best understood by working first under the specialization $t = 1$. To this end, we associate to any k -bounded partition λ a sequence of partitions, $\lambda^{\rightarrow k} = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(r)})$, called the k -*split* of λ . The sequence $\lambda^{\rightarrow k}$ is obtained by partitioning λ (without rearranging the entries) into partitions $\lambda^{(i)}$ with main hook length equal to k , for all $i < r$. The last partition in $\lambda^{\rightarrow k}$ may have main hook-length less than k . It is important to note that $\lambda^{\rightarrow k} = (\lambda)$

when the main hook-length of λ is $\leq k$. This given, we let $G^{(k)}[X; 1]$ be the ordinary Schur function product

$$G^{(k)}[X; 1] = s_{\lambda^{(1)}}[X]s_{\lambda^{(2)}}[X] \cdots s_{\lambda^{(r)}}[X]. \quad (1.11)$$

We call these “*k-split*” polynomials. The construction of our *k-split* polynomials for $t \neq 1$ relies on a *t-extension* of Schur function products. We simply set

$$G^{(k)}[X; t] = B_{\lambda^{(1)}}^t B_{\lambda^{(2)}}^t \cdots B_{\lambda^{(r)}}^t \cdot 1. \quad (1.12)$$

where the B_{λ}^t ’s are certain “*vertex*” operators introduced in [8, 10, 12]. It happens, remarkably, that not all of the *k-Schur* functions need to be constructed using (1.5). We proved in [6] that for each *k*, there is a subset of $s_{\lambda}^{(k)}[X; t]$, called the irreducible *k-Schur* functions, from which all other *k-Schur* functions may be constructed by simply applying a succession of operators. The elements of this set are the *k-Schur* functions indexed by a partition with no more than *i* parts equal to $k - i$, we call such a partition “*k-irreducible*”. To construct $s_{\lambda}^{(k)}[X; t]$, we simply split the given partition λ as the concatenation

$$\lambda = R_1 R_2 \cdots R_{\ell} \mu,$$

where the R_i are rectangularly shaped partitions of the form (i^{k+1-i}) and μ is *k-irreducible*. This given we have

$$s_{\lambda}^{(k)}[X; t] = t^c B_{R_1} B_{R_2} \cdots B_{R_{\ell}} s_{\mu}^{(k)}[X; t] \quad \text{with } c \in \mathbb{N}, \quad (1.7)$$

with the B_{R_i} vertex operators [13] associated to partitions of rectangular shape.

We should note that the case $t = 1$ is quite interesting in itself. In fact, the Hall-Littlewood polynomials at $t = 1$ reduce to the complete symmetric functions

$$H_{\lambda}[X; 1] = h_{\lambda_1}[X] h_{\lambda_2}[X] \cdots h_{\lambda_{\ell}}[X],$$

and $\Lambda_t^{(k)}$ reduces to the polynomial ring $\Lambda^{(k)} = \mathbb{Q}[h_1, \dots, h_k]$. Each of the properties held by the *k-Schur* functions has a specialization in this subring. In particular, since the vertex operator B_{R_i} is simply multiplication by the Schur function s_{R_i} (1.7) reduces to

$$s_{\lambda}^{(k)}[X] = s_{R_1}[X] s_{R_2}[X] \cdots s_{R_{\ell}}[X] s_{\mu}^{(k)}[X]. \quad (1.13)$$

The irreducible *k-Schur* functions thus constitute a natural basis for the quotient ring $\Lambda_t^{(k)}/\mathcal{I}_k$, where \mathcal{I}_k is the ideal generated by Schur functions indexed by partitions of the form (i^{k+1-i}) .

We should emphasize that the conjectured equality $A^{(k)}[X; t] = s^{(k)}[X; t]$ implicitly transfers all the properties and conjectures from atoms to *k-Schur* functions and vice versa. As a note of caution the reader should be aware of notational differences in [5], in particular V_k in [5] is $\Lambda_t^{(k)}$ here and $S_{\lambda}[X]$ there is $s_{\lambda}[X]$ here. Tables of the coefficients $v_{\lambda, \mu}^{(k)}(t)$ and $K_{\lambda, \mu}^{(k)}(q, t)$ are included in Section 9. They may be computed by the algorithms given in [5] as well as the formulas given here. They not only beautifully illustrate our conjectures but also reveal the further inequalities

$$0 \subseteq K_{\lambda, \mu}^{(k)}(q, t) \subseteq K_{\lambda, \mu}(k')(q, t)$$

which are implicitly implied by (1.3) and (1.4). Here for two polynomials $P, Q \in \mathbb{Z}[q, t]$, $P \subseteq Q$ means $Q - P \in \mathbb{N}[q, t]$.

Our assertion that the *k-Schur* functions are natural extensions of classical Schur functions is based on findings that go well beyond the simple identity $s_{\lambda}^{(\infty)}[X; t] = s_{\lambda}[X]$. In extending to *k-Schur* functions well known properties of Schur functions we have discovered analogues of Pieri and Littlewood Richardson rules and a *k-analogue* of partition conjugation. In particular, the multiplicative action of h_1 led us to a *k-analogue* of the Young Lattice. We have also observed that $s_{\lambda}^{(k)}[X + Y; t]$ when expanded in terms of the bases $\{s_{\lambda}^{(k)}[X; t]\}_{\lambda_1 \leq k}$ and $\{s_{\lambda}^{(k)}[Y; t]\}_{\lambda_1 \leq k}$ the coefficients are in $\mathbb{N}[t]$. This is a special property of Schur functions that is not shared by Hall-Littlewood or Macdonald polynomials.

The article proceeds as follows: after the introduction of basic definitions in Section 2, the case $t = 1$ is addressed in Sections 3, 4 and 5. Although some of what appears in these sections can be obtained from the results in Sections 6 and 7, owing to its greater simplicity, we find it pedagogical to treat the case $t = 1$ independently. In Section 6, an important property of the generalized Schur products is used. However, the proof has been relegated to the Appendix, where known properties of these products including a Morris-type recursion can also be found.

Acknowledgments. *The enthusiasm from A. Garsia and A. Lascoux greatly contributed to this work and we are thankful to M. Zabrocki for helping us with [12]. L. Lapointe thanks L. Vinet for his support. J. Morse held an NSF grant for part of the period devoted to this research. ACE [14] was instrumental towards this work.*

2. DEFINITIONS

2.1. Partitions. Symmetric polynomials are indexed by partitions, sequences of non-negative integers $\lambda = (\lambda_1, \lambda_2, \dots)$ with $\lambda_1 \geq \lambda_2 \geq \dots$. The number of non-zero parts in λ is denoted $\ell(\lambda)$ and the degree of λ is $|\lambda| = \lambda_1 + \dots + \lambda_{\ell(\lambda)}$. We say that λ is a partition of n , denoted $\lambda \vdash n$, if $|\lambda| = n$. We use the dominance order on partitions with $|\lambda| = |\mu|$, where $\lambda \leq \mu$ when $\lambda_1 + \dots + \lambda_i \leq \mu_1 + \dots + \mu_i$ for all i . Given two partitions λ and μ , $\lambda \cup \mu$ stands for the partition rearrangement of the parts of λ and μ . Note that if $\lambda \leq \mu$ and $\nu \leq \omega$, then $\lambda \cup \nu \leq \mu \cup \omega$.

Any partition λ has an associated Ferrers diagram with λ_i lattice squares in the i^{th} row, from the bottom to top. For example,

$$\lambda = (4, 2) = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & & \\ \hline \end{array}. \quad (2.1)$$

For each cell $s = (i, j)$ in the diagram of λ , let $\ell'(s)$, $\ell(s)$, $a(s)$ and $a'(s)$ be respectively the number of cells in the diagram of λ to the south, north, east and west of the cell s . The hook-length of any cell in λ , is $h_s(\lambda) = \ell(s) + a(s) + 1$. In the example, $h_{(1,2)}(4, 2) = 2 + 1 + 1$. The *main hook-length* of λ , $h_M(\lambda)$, is the hook-length of the cell $s = (1, 1)$ in the diagram of λ . Therefore, $h_M((4, 2)) = 5$.

The conjugate λ' of a partition λ is defined by the reflection of the Ferrers diagram about the main diagonal. For example, the conjugate of $(4, 2)$ is

$$\lambda' = \begin{array}{|c|} \hline \square \\ \square \\ \square \\ \square \\ \hline \end{array} = (2, 2, 1, 1). \quad (2.2)$$

A skew shape λ/μ is the diagram obtained by deleting the diagram of μ from λ .

$$(6, 5, 4, 3, 1)/(4, 2) = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \hline \end{array}. \quad (2.3)$$

A partition λ is said to be k -*bounded* if its first part is not larger than k , i.e, if $\lambda_1 \leq k$. We associate to any k -bounded partition λ a sequence of partitions, $\lambda^{\rightarrow k} = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(r)})$, called the k -*split* of λ . $\lambda^{\rightarrow k}$ is obtained by partitioning λ (without rearranging the entries) into partitions $\lambda^{(i)}$ where $h_M(\lambda^{(i)}) = k$, for all $i < r$. For example, $(3, 2, 2, 2, 1, 1)^{\rightarrow 3} = ((3), (2, 2), (2, 1), (1))$ and $(3, 2, 2, 2, 1, 1)^{\rightarrow 4} = ((3, 2), (2, 2, 1), (1))$. Equivalently, the diagram of λ is cut horizontally into partitions with main hook-length k .

$$\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \square \\ \square \\ \hline \end{array} \xrightarrow{(3)} \begin{array}{|c|c|} \hline \square \\ \square & \square \\ \square & \square \\ \square & \square \\ \square & \square \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|} \hline \square \\ \square \\ \square \\ \square \\ \square \\ \hline \end{array} \xrightarrow{(4)} \begin{array}{|c|c|} \hline \square \\ \square & \square \\ \square & \square \\ \square & \square \\ \square & \square \\ \hline \end{array}. \quad (2.4)$$

The last partition in the sequence $\lambda^{\rightarrow k}$ may have main hook-length less than k . It is important to note that $\lambda^{\rightarrow k} = (\lambda)$ when $h_M(\lambda) \leq k$.

2.2. Symmetric functions. The power sum $p_i(x_1, x_2, \dots)$ is

$$p_i(x_1, x_2, \dots) = x_1^i + x_2^i + \dots, \quad (2.5)$$

and for a partition $\lambda = (\lambda_1, \lambda_2, \dots)$,

$$p_\lambda(x_1, x_2, \dots) = p_{\lambda_1}(x_1, x_2, \dots) p_{\lambda_2}(x_1, x_2, \dots) \cdots. \quad (2.6)$$

We employ the notation of λ -rings, needing only the formal ring of symmetric functions Λ to act on the ring of rational functions in x_1, \dots, x_N, q, t , with coefficients in \mathbb{R} . The action of a power sum p_i on a rational function is, by definition,

$$p_i \left[\frac{\sum_\alpha c_\alpha u_\alpha}{\sum_\beta d_\beta v_\beta} \right] = \frac{\sum_\alpha c_\alpha u_\alpha^i}{\sum_\beta d_\beta v_\beta^i}, \quad (2.7)$$

with $c_\alpha, d_\beta \in \mathbb{R}$ and u_α, v_β monomials in x_1, \dots, x_N, q, t . Since the power sums form a basis of the ring Λ , any symmetric function has a unique expression in terms of power sums, and (2.7) extends to an action of Λ on rational functions. In particular $f[X]$, the action of a symmetric function f on the monomial $X = x_1 + \dots + x_N$, is simply $f(x_1, \dots, x_N)$. In the remainder of the article, we will always consider the number of variables N to be infinite, unless otherwise specified.

The complete symmetric function $h_r[X]$ is

$$h_r[X] = \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}, \quad (2.8)$$

and $h_\lambda[X]$ stands for the homogeneous symmetric function

$$h_\lambda[X] = h_{\lambda_1}[X] h_{\lambda_2}[X] \cdots. \quad (2.9)$$

In the same way, the elementary symmetric function $e_r[X]$ is

$$e_r[X] = \sum_{1 \leq i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r}, \quad (2.10)$$

and $e_\lambda[X]$ stands for

$$e_\lambda[X] = e_{\lambda_1}[X] e_{\lambda_2}[X] \cdots. \quad (2.11)$$

Although the Schur functions may be characterized in many ways, here it will be convenient to use the Jacobi-Trudi determinantal expression:

$$s_\lambda[X] = \det \left| h_{\lambda_i + j - 1}[X] \right|_{1 \leq i, j \leq \ell(\lambda)} = \det \left| e_{\lambda'_i + j - 1}[X] \right|_{1 \leq i, j \leq \ell(\lambda)}, \quad (2.12)$$

where $h_r[X] = e_r[X] = 0$ if $r < 0$. Note, in particular, $s_r[X] = h_r[X]$ and $s_{1^r}[X] = e_r[X]$.

The homomorphism ω , which is an involution on Λ , is defined by

$$\omega(h_r[X]) = e_r[X], \quad (2.13)$$

and is such that $\omega(s_\lambda[X]) = s_{\lambda'}[X]$.

We recall that the Macdonald scalar product, $\langle \cdot, \cdot \rangle_{q,t}$, on $\Lambda \otimes \mathbb{Q}(q, t)$ is defined by setting

$$\langle p_\lambda[X], p_\mu[X] \rangle_{q,t} = \delta_{\lambda\mu} z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}, \quad (2.14)$$

where for a partition λ with $m_i(\lambda)$ parts equal to i , we associate the number

$$z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \cdots \quad (2.15)$$

When $q = t$, this scalar product does not depend on a parameter and then satisfies

$$\langle s_\lambda[X], s_\mu[X] \rangle = \delta_{\lambda\mu}. \quad (2.16)$$

The Macdonald integral forms $J_\lambda[X; q, t]$ are uniquely characterized [9] by

$$(i) \langle J_\lambda, J_\mu \rangle_{q,t} = 0, \quad \text{if } \lambda \neq \mu, \quad (2.17)$$

$$(ii) J_\lambda[X; q, t] = \sum_{\mu \leq \lambda} v_{\lambda\mu}(q, t) s_\mu[X], \quad \text{with } v_{\lambda\mu}(q, t) \in \mathbb{Q}(q, t), \quad (2.18)$$

$$(iii) v_{\lambda\lambda}(q, t) = \prod_{s \in \lambda} (1 - q^{a(s)} t^{\ell(s)+1}). \quad (2.19)$$

Here, we use a modification of the Macdonald integral forms that is obtained by setting

$$H_\lambda[X; q, t] = J_\lambda[X/(1-t); q, t] = \sum_{\mu} K_{\mu\lambda}(q, t) s_\mu[X], \quad (2.20)$$

with the coefficients $K_{\mu\lambda}(q, t) \in \mathbb{N}[q, t]$ known as the q, t -Kostka polynomials. In the case $q = 0$, $J_\lambda[X; q, t]$ reduces to the Hall-Littlewood polynomial, $J_\lambda[X; q, t] = Q_\lambda[X; t]$. Again, we shall use a modification of the Hall-Littlewood polynomials,

$$H_\lambda[X; t] = H_\lambda[X; 0, t] = Q_\lambda[X/(1-t); t] = s_\lambda[X] + \sum_{\mu > \lambda} K_{\mu\lambda}(t) s_\mu[X], \quad (2.21)$$

with the coefficients $K_{\mu\lambda}(t) \in \mathbb{N}[t]$ known as the Kostka-Foulkes polynomials. Then, in the limit $t = 1$, we have $H_\lambda[X; 1] = h_\lambda[X]$, giving

$$h_\lambda[X] = H_\lambda[X; 1] = s_\lambda[X] + \sum_{\mu > \lambda} K_{\mu\lambda} s_\mu[X], \quad (2.22)$$

with the coefficients $K_{\mu\lambda} \in \mathbb{N}$ known as the Kostka numbers.

For partitions λ and μ , we have

$$s_\lambda[X] s_\mu[X] = \sum_{\rho: |\rho| = |\lambda| + |\mu|} c_{\lambda\mu}^\rho s_\rho[X], \quad (2.23)$$

where the coefficients $c_{\lambda\mu}^\rho \in \mathbb{N}$ are the Littlewood-Richardson coefficients. They satisfy

Property 1. $c_{\lambda\mu}^{\lambda \cup \mu} = 1$, and $c_{\lambda\mu}^\rho = 0$ unless $\lambda \cup \mu \leq \rho$ and $(\lambda' \cup \mu')' \geq \rho$.

Proof. The unitriangular relation (2.22) gives

$$s_\lambda[X] = h_\lambda[X] + \sum_{\mu > \lambda} K_{\mu\lambda}^{-1} h_\mu[X], \quad (2.24)$$

which implies

$$s_\lambda[X] s_\mu[X] = \sum_{\nu \geq \lambda; \gamma \geq \mu} K_{\nu\lambda}^{-1} K_{\gamma\mu}^{-1} h_{\nu \cup \gamma}[X] = h_{\lambda \cup \mu} + \sum_{\rho > \lambda \cup \mu} d_{\lambda\mu}^\rho h_\rho, \quad (2.25)$$

since $\nu \geq \lambda$ and $\gamma \geq \mu$ imply $\nu \cup \gamma \geq \lambda \cup \mu$. Then, from (2.22), $c_{\lambda\mu}^{\lambda \cup \mu} = 1$ and $c_{\lambda\mu}^\rho = 0$ unless $\lambda \cup \mu \leq \rho$. Similarly, it can be shown that $(\lambda' \cup \mu')' \geq \rho$ by applying the involution ω to (2.22).

The Pieri rules are the cases $\mu = (r)$ and $\mu = (1^r)$ of expression (2.23). Then

$$h_r[X] s_\lambda[X] = \sum_{\mu} s_\mu[X] \quad \text{and} \quad e_r[X] s_\lambda[X] = \sum_{\nu} s_\nu[X], \quad (2.26)$$

where the sums run over all μ 's and ν 's such that μ/λ and ν/λ are respectively a horizontal r -strip and a vertical r -strip.

3. THE k -SCHUR FUNCTIONS

Here, and in the following section, we study the space $\Lambda_t^{(k)}$ when $t = 1$. In this case, the subspace reduces to a subring of Λ , defined by

$$\Lambda^{(k)} = \Lambda_{t=1}^{(k)} = \mathcal{L} \{h_\lambda[X]\}_{\lambda; \lambda_1 \leq k} = \mathcal{L} \{e_\lambda[X]\}_{\lambda; \lambda_1 \leq k}. \quad (3.1)$$

The last equality holds since the determinantal expression (2.12) of $s_r[X] = h_r[X]$, in terms of elementary functions, gives only terms of the type $e_i[X]$, where $i \leq r$ and vice versa.

3.1. k -split polynomials. Our construction relies on the introduction of a family of symmetric polynomials called the k -split polynomials. If $\lambda \rightarrow k = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(n)})$ is the k -split of a k -bounded partition λ , then we define the k -split polynomial by

$$G_\lambda^{(k)}[X] = s_{\lambda^{(1)}}[X] s_{\lambda^{(2)}}[X] \cdots s_{\lambda^{(n)}}[X]. \quad (3.2)$$

A number of properties can be derived from this definition. For example, because $\lambda \rightarrow k = (\lambda)$ when $h_M(\lambda) \leq k$, it follows immediately that

Property 2.

$$G_\lambda^{(k)}[X] = s_\lambda[X], \quad \text{for } h_M(\lambda) \leq k. \quad (3.3)$$

Moreover, since $G_\lambda^{(k)}[X]$ is the product $s_{\lambda^{(1)}}[X] \cdots s_{\lambda^{(n)}}[X]$, Property 1 of the Littlewood-Richardson rule applied repeatedly implies that $\lambda = \lambda^{(1)} \cup \cdots \cup \lambda^{(n)}$ must index the minimal element in the Schur function expansion of $G_\lambda^{(k)}[X]$. That is,

Property 3. For any k -bounded partition λ ,

$$G_\lambda^{(k)}[X] = s_\lambda[X] + \sum_{\mu > \lambda} a_{\lambda\mu}^{(k)} s_\mu[X], \quad a_{\lambda\mu}^{(k)} \in \mathbb{N}. \quad (3.4)$$

Finally, since $(k, \lambda) \rightarrow k = ((k), \lambda \rightarrow k)$ where $(k, \lambda) = (k, \lambda_1, \lambda_2, \dots)$, we have

Property 4. For any k -bounded partition λ ,

$$s_k[X] G_\lambda^{(k)}[X] = G_{(k, \lambda)}^{(k)}[X]. \quad (3.5)$$

In addition to having these properties, the k -split polynomials form a basis for the subring $\Lambda^{(k)}$.

Theorem 5. $\{G_\lambda^{(k)}[X]\}_{\lambda_1 \leq k}$ forms a basis for $\Lambda^{(k)}$.

Proof. $\Lambda^{(k)}$ is the linear span of homogeneous symmetric functions over all k -bounded partitions. Since the elements $G_\lambda^{(k)}[X]$ are also indexed by k -bounded partitions and by Property 3, are linearly independent, it suffices to show that these elements lie in $\Lambda^{(k)}$. That is, for a k -bounded partition λ , we must be able to expand $G_\lambda^{(k)}[X]$ in terms of $h_{\mu_1}[X] h_{\mu_2}[X] \cdots$ where $\mu_i \leq k$ for all i . Observe that any entry $h_j[X]$ in the determinantal expression (2.12) for s_ν can be indexed by j no larger than $\nu_1 + \ell(\nu) - 1 = h_M(\nu)$. Since $G_\lambda^{(k)}[X] = s_{\lambda^{(1)}}[X] s_{\lambda^{(2)}}[X] \cdots$ with $h_M(\lambda^{(i)}) \leq k$ for all i , $G_\lambda^{(k)}[X]$ is a product of determinants having entries $h_j[X]$, with $j \leq k$. \square

We now have that $G_\lambda^{(k)} \in \Lambda^{(k)}$ for all k -bounded partitions λ . Therefore, these polynomials can be expanded in terms of h_μ with $\mu_1 \leq k$. In fact, since $G_\lambda^{(k)}$ is unitriangularly related to s_λ by Property 3, and s_λ is unitriangularly related to h_λ (2.22), we have

Property 6.

$$G_\lambda^{(k)}[X] = h_\lambda[X] + \sum_{\substack{\mu > \lambda \\ \mu_1 \leq k}} g_{\lambda\mu}^{(k)} h_\mu[X], \quad g_{\lambda\mu}^{(k)} \in \mathbb{Z}. \quad (3.6)$$

3.2. k -Schur functions. Although the k -split polynomials form a basis for $\Lambda^{(k)}$, they do not have the fundamental role for $\Lambda^{(k)}$ that the Schur functions have for Λ . For example, since

$$s_2[X] G_{3,1,1}^{(3)}[X] = G_{3,2,1,1}^{(3)}[X] - G_{3,2,2}^{(3)}[X]$$

reveals a negative coefficient, the k -split basis does not satisfy a refinement of the Pieri Rule. Additionally, there are no partitions μ such that the involution ω sends $G_\lambda^{(k)}[X]$ to a single element $G_\mu^{(k)}[X]$ as with the Schur functions. However, these polynomials do play a key role in the construction of our Schur analog, $s_\lambda^{(k)}[X]$. The intricacy of our definition stems from the use of a projection operator, $T_j^{(k)}$, acting linearly on $\Lambda_t^{(k)}$ and defined by

$$T_j^{(k)} G_\lambda^{(k)}[X] = \begin{cases} G_\lambda^{(k)}[X] & \text{if } \lambda_1 = j \\ 0 & \text{otherwise} \end{cases}. \quad (3.7)$$

Note that this projection operator sends an element of $\Lambda^{(k)}$ into an element belonging to the linear span of k -split polynomials whose first parts all equal to j . We can now introduce the k -Schur functions, our fundamental objects in $\Lambda^{(k)}$.

Definition 7. For a k -bounded partition λ , the k -Schur function is defined recursively by

$$s_\lambda^{(k)}[X] = T_{\lambda_1}^{(k)} s_{\lambda_1}[X] s_{\lambda_2, \lambda_3, \dots}^{(k)}[X], \quad \text{with } s_\emptyset^{(k)}[X] = 1. \quad (3.8)$$

Explicit examples of the expansion of k -Schur functions in terms of Schur functions are given, letting $t = 1$, in the tables of Subsection 9.1.

Since the space $\Lambda^{(k)}$ is simply Λ when $k \rightarrow \infty$, we expect to recover s_λ from $s_\lambda^{(k)}$ in this case. The following property supports our assertion that the k -Schur functions generalize the Schur functions.

Property 8. For any k -bounded partition λ ,

$$s_\lambda^{(k)}[X] = s_\lambda[X], \quad \text{for } h_M(\lambda) \leq k. \quad (3.9)$$

Proof. Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition with $\lambda_1 \leq k$ and $h_M(\lambda) \leq k$. In the case that $\lambda = ()$, by definition, $s_\emptyset^{(k)}[X] = 1 = s_\emptyset[X]$. Let $\hat{\lambda} = (\lambda_2, \lambda_3, \dots)$ and assume by induction on $\ell(\lambda)$, that $s_{\hat{\lambda}}^{(k)}[X] = s_{\hat{\lambda}}[X]$. We proceed to show $s_\lambda^{(k)}[X] = s_\lambda[X]$. From (2.12), we have that

$$s_\lambda[X] = \det \begin{vmatrix} h_{\lambda_1}[X] & h_{\lambda_1+1}[X] & \cdots & h_{\lambda_1+\ell-1}[X] \\ h_{\lambda_2-1}[X] & h_{\lambda_2}[X] & \cdots & h_{\lambda_2+\ell-2}[X] \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_\ell-\ell+1}[X] & h_{\lambda_\ell-\ell+2}[X] & \cdots & h_{\lambda_\ell}[X] \end{vmatrix}. \quad (3.10)$$

The expansion of this determinant about the first row gives that the coefficient of $h_{\lambda_1}[X]$ is exactly $s_{\hat{\lambda}}[X]$. Further, since all other terms in the first row are of the form $h_i[X]$, with $i > \lambda_1$, we have

$$s_\lambda[X] = h_{\lambda_1}[X] s_{\hat{\lambda}}[X] + \sum_{\mu; \mu_1 > \lambda_1} c_\mu h_\mu[X]. \quad (3.11)$$

Property 2 gives that $s_\lambda[X] = G_\lambda^{(k)}[X]$ since $h_M(\lambda) \leq k$. Noting also that $s_{\lambda_1} = h_{\lambda_1}$, (3.11) becomes

$$s_{\lambda_1} s_{\hat{\lambda}}[X] = G_\lambda^{(k)}[X] - \sum_{\mu; \mu_1 > \lambda_1} c_\mu h_\mu[X]. \quad (3.12)$$

Since $s_{\lambda_1}[X]s_{\hat{\lambda}}[X]$ and $G_{\hat{\lambda}}^{(k)}$ both belong to $\Lambda^{(k)}$, $\sum_{\mu; \mu_1 > \lambda_1} c_{\mu} h_{\mu}[X]$ is in $\Lambda^{(k)}$ and can therefore be expanded in terms of $G_{\mu}^{(k)}$. This expansion, by the unitriangularity property (3.6), yields

$$s_{\lambda_1}[X]s_{\hat{\lambda}}[X] = G_{\hat{\lambda}}^{(k)}[X] - \sum_{\mu; \mu_1 > \lambda_1} d_{\mu} G_{\mu}^{(k)}[X]. \quad (3.13)$$

Since $s_{\hat{\lambda}}^{(k)}[X] = T_{\lambda_1}^{(k)} s_{\lambda_1}[X]s_{\hat{\lambda}}[X]$ by the induction hypothesis, applying T_{λ_1} to this expression gives

$$s_{\hat{\lambda}}^{(k)}[X] = T_{\lambda_1}^{(k)} \left(G_{\hat{\lambda}}^{(k)}[X] - \sum_{\mu; \mu_1 > \lambda_1} d_{\mu} G_{\mu}^{(k)}[X] \right) = G_{\hat{\lambda}}^{(k)}[X] = s_{\hat{\lambda}}[X], \quad (3.14)$$

thus proving the property. \square

Remarkably, the k -Schur functions satisfy a unitriangularity property.

Property 9. For any k -bounded partition λ ,

$$s_{\lambda}^{(k)}[X] = h_{\lambda}[X] + \sum_{\substack{\mu > \lambda \\ \mu_1 \leq k}} d_{\lambda\mu}^{(k)} h_{\mu}[X], \quad \text{with } d_{\lambda\mu}^{(k)} \in \mathbb{Z}. \quad (3.15)$$

Proof. Let λ be a k -bounded partition. If $\lambda = (r)$ then $h_M(\lambda) \leq k$ and Property 8 gives

$$s_r^{(k)}[X] = s_r[X] = h_r[X]. \quad (3.16)$$

Let $\hat{\lambda}$ denote the partition λ without its first part and assume by induction on $\ell(\lambda)$ that

$$s_{\hat{\lambda}}^{(k)}[X] = h_{\hat{\lambda}}[X] + \sum_{\substack{\gamma > \hat{\lambda} \\ \gamma_1 \leq k}} d_{\gamma} h_{\gamma}[X], \quad \text{with } d_{\gamma} \in \mathbb{Z}. \quad (3.17)$$

Substituting this expression into the definition $s_{\lambda}^{(k)}[X] = T_{\lambda_1} s_{\lambda_1} s_{\hat{\lambda}}^{(k)}$ gives

$$\begin{aligned} s_{\lambda}^{(k)}[X] &= T_{\lambda_1}^{(k)} \left(h_{\lambda}[X] + \sum_{\gamma > \hat{\lambda}; \gamma_1 \leq k} d_{\gamma} h_{(\lambda_1) \cup \gamma}[X] \right) \\ &= T_{\lambda_1}^{(k)} \left(h_{\lambda}[X] + \sum_{\nu > \lambda; \nu_1 \leq k} c_{\nu} h_{\nu}[X] \right), \quad \text{with } c_{\nu} \in \mathbb{Z}. \end{aligned} \quad (3.18)$$

since $\gamma > \hat{\lambda} \implies (\lambda_1) \cup \gamma > (\lambda_1) \cup \hat{\lambda} = \lambda$. The unitriangularity relation (3.6) then gives

$$s_{\lambda}^{(k)}[X] = T_{\lambda_1} \left(G_{\hat{\lambda}}^{(k)}[X] + \sum_{\mu > \hat{\lambda}} v_{\mu} G_{\mu}^{(k)}[X] \right) \quad (3.19)$$

$$= G_{\lambda}^{(k)}[X] + \sum_{\mu > \lambda; \mu_1 = \lambda_1} v_{\mu} G_{\mu}^{(k)}[X], \quad \text{with } v_{\mu} \in \mathbb{Z}. \quad (3.20)$$

Using formula (3.6) again then proves the claim. \square

Since $\Lambda^{(k)} = \mathcal{L}\{h_{\lambda}[X]\}_{\lambda; \lambda_1 \leq k}$, this unitriangularity relation immediately implies that the k -Schur functions form a basis.

Theorem 10. The k -Schur functions form a basis for $\Lambda^{(k)}$. That is

$$\Lambda^{(k)} = \mathcal{L} \left\{ s_{\lambda}^{(k)}[X] \right\}_{\lambda_1 \leq k}. \quad (3.21)$$

The unitriangularity relation also allows us to prove that the k -Schur functions decompose into an integral sum of $k+1$ -Schur functions.

Property 11. For any k -bounded partition λ ,

$$s_\lambda^{(k)}[X] = s_\lambda^{(k+1)}[X] + \sum_{\substack{\mu > \lambda \\ \mu_1 \leq k+1}} b_{\mu\lambda}^{(k+1)} s_\mu^{(k+1)}[X] \quad \text{where } b_{\mu\lambda}^{(k+1)} \in \mathbb{Z} \quad (3.22)$$

Proof. For $\gamma_1 \leq k+1$, Property 9 gives the expansion

$$h_\gamma[X] = s_\gamma^{(k+1)}[X] + \sum_{\substack{\mu > \gamma \\ \mu_1 \leq k+1}} \hat{d}_{\mu\gamma}^{(k+1)} s_\mu^{(k+1)}[X] \quad \text{where } \hat{d}_{\mu\lambda}^{(k+1)} \in \mathbb{Z}. \quad (3.23)$$

However, substituting this expression into (3.15) gives, for $\lambda_1 \leq k$,

$$s_\lambda^{(k)}[X] = h_\lambda[X] + \sum_{\substack{\mu > \lambda \\ \mu_1 \leq k}} d_{\mu\lambda}^{(k)} h_\mu[X] \quad \text{where } d_{\mu\lambda}^{(k)} \in \mathbb{Z} \quad (3.24)$$

$$= s_\lambda^{(k+1)}[X] + \sum_{\substack{\mu > \lambda \\ \mu_1 \leq k+1}} \hat{b}_{\mu\lambda}^{(k+1)} s_\mu^{(k+1)}[X] \quad \text{where } \hat{b}_{\mu\lambda}^{(k+1)} \in \mathbb{Z}. \quad (3.25)$$

□

4. IRREDUCIBILITY

Here we study the product of certain Schur functions with a k -Schur function. In particular, we are concerned with partitions of the form $(\ell^{k+1-\ell})$ for $\ell = 1, \dots, k$. Hereafter, such a partition is referred to as a k -rectangle. A k -bounded partition with no more than i parts equal to $k-i$, for $i = 0, \dots, k-1$, is called an irreducible partition. Otherwise, the partition is reducible.

Definition 12. A k -Schur function indexed by an irreducible partition is said to be a irreducible. Otherwise, the k -Schur function is called reducible.

For example, the irreducible k -Schur functions for $k = 1, 2, 3$ are

$$\begin{aligned} k = 1 : & \quad s_0^{(1)}, \\ k = 2 : & \quad s_0^{(2)}, \quad s_1^{(2)}, \\ k = 3 : & \quad s_0^{(3)}, \quad s_1^{(3)}, \quad s_2^{(3)}, \quad s_{1,1}^{(3)}, \quad s_{2,1}^{(3)}, \quad s_{2,1,1}^{(3)}. \end{aligned} \quad (4.1)$$

From our definition it follows that

Property 13. [5] There are $k!$ distinct k -irreducible partitions.

The concept of irreducibility arose in our study of the multiplication of a k -Schur function with a Schur function indexed by a k -rectangle. It happens that this produces a single k -Schur function. More precisely, it was shown that

Theorem 14. [6] If λ is a k -bounded partition, then

$$s_{(\ell^{k-\ell+1})}[X] s_\lambda^{(k)}[X] = s_{\lambda \cup (\ell^{k-\ell+1})}^{(k)}[X]. \quad (4.2)$$

Therefore, given the $k!$ irreducible k -Schur functions, any reducible $s_\lambda^{(k)}$ is obtained simply by the multiplication of a sequence of Schur functions indexed by k -rectangles on the proper irreducible k -Schur function. The case that the k -rectangle is the partition (k) relies only on properties of the k -split polynomials. We include the proof here.

Property 15. For λ any k -bounded partition,

$$s_k[X] s_\lambda^{(k)}[X] = s_{k,\lambda}^{(k)}[X]. \quad (4.3)$$

Proof. The unitriangular relation between the Schur functions and $G_\lambda^{(k)}[X]$ given in (3.20) implies

$$s_k[X] s_\lambda^{(k)}[X] = s_k \left(G_\lambda^{(k)}[X] + \sum_{\mu > \lambda; \mu_1 = \lambda_1} v_{\mu\lambda}^{(k)} G_\mu^{(k)}[X] \right). \quad (4.4)$$

Property 4 then gives the action of $s_k[X]$ on a k -split polynomial, and we have

$$s_k[X] s_\lambda^{(k)}[X] = G_{k,\lambda}^{(k)}[X] + \sum_{\mu > \lambda; \mu_1 = \lambda_1} v_{\mu\lambda}^{(k)} G_{k,\mu}^{(k)}[X]. \quad (4.5)$$

Since each polynomial in the right hand side of this expression is indexed by a partition with first component k , the expression is invariant under the action of the projection operator $T_k^{(k)}$. That is,

$$T_k^{(k)} s_k[X] s_\lambda^{(k)}[X] = T_k^{(k)} \left(G_{k,\lambda}^{(k)}[X] + \sum_{\mu > \lambda; \mu_1 = \lambda_1} v_{\mu\lambda}^{(k)} G_{k,\mu}^{(k)}[X] \right) = G_{k,\lambda}^{(k)}[X] + \sum_{\mu > \lambda; \mu_1 = \lambda_1} v_{\mu\lambda}^{(k)} G_{k,\mu}^{(k)}[X]$$

The right hand side in this expression is the same as that of (4.5) and we thus have that by definition $T_k^{(k)} s_k[X] s_\lambda^{(k)}[X] = s_k[X] s_\lambda^{(k)}[X] = s_{k,\lambda}^{(k)}$. \square

The role of Schur functions indexed by k -rectangles in the subring $\Lambda^{(k)}$ leads naturally to the study of the quotient ring $\Lambda^{(k)}/\mathcal{I}_k$, where \mathcal{I}_k denotes the ideal generated by $s_{(\ell^{k+1-\ell})}[X]$. It is shown in [5] that

Proposition 16. *The homogeneous functions indexed by k -irreducible partitions form a basis of the quotient ring $\Lambda^{(k)}/\mathcal{I}_k$.*

Thus, the dimension of the quotient ring $\Lambda^{(k)}/\mathcal{I}_k$ is $k!$. Since we have shown that the k -Schur functions form a basis for $\Lambda^{(k)}$, Theorem 14 implies

Theorem 17. *The irreducible k -Schur functions form a basis of the quotient ring $\Lambda^{(k)}/\mathcal{I}_k$.*

In fact, the irreducible k -Schur function basis offers a very beautiful way to carry out operations in this quotient ring: first work in $\Lambda^{(k)}$ using k -Schur functions and then replace by zero all the k -Schur functions indexed by partitions which are not k -irreducible.

5. ANALOGS OF SCHUR FUNCTION PROPERTIES

Computer experimentation reveals that many of the fundamental properties of Schur functions are shared by the k -Schur functions. We now state several of these properties.

5.1. The k -conjugation of a partition. We give a generalization of partition conjugation that is an involution on k -bounded partitions, and reduces to usual conjugation of partitions for large k .

A skew diagram D has hook-lengths bounded by k if the hook-length of any cell in D is not larger than k . For a positive integer $m \leq k$, the k -multiplication $m \times^{(k)} D$ is the skew diagram \overline{D} obtained by prepending a column of length m to D such that the number of rows of \overline{D} is as small as possible while ensuring that its hook-lengths are bounded by k . For example,

$$\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \times^{(5)} \begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square \\ \hline \end{array}. \quad (5.1)$$

Definition 18. The k -conjugate of a k -bounded partition $\lambda = (\lambda_1, \dots, \lambda_n)$, denoted λ^{ω_k} , is the vector obtained by reading the number of boxes in each row of the skew diagram,

$$D = \lambda_1 \times^{(k)} \dots \times^{(k)} \lambda_n, \quad (5.2)$$

arising by k -multiplying the entries of λ from right to left.

When $k \rightarrow \infty$, $\lambda^{\omega_k} = \lambda' = D$ since each k -multiplication step reduces to concatenating a column of height λ_i . Further, the k -conjugate is an involution on k -bounded partitions:

Theorem 19. [5] ω_k is an involution on partitions bounded by k . That is, for λ with $\lambda_1 \leq k$,

$$(\lambda^{\omega_k})^{\omega_k} = \lambda. \quad (5.3)$$

We have observed that the k -conjugation of a partition plays a natural role in the generalization of classical Schur function properties. We now give two examples.

5.2. Pieri Rules. Beautiful combinatorial algorithms are known for the Littlewood-Richardson coefficients that appear in a product of Schur functions;

$$s_\lambda[X] s_\mu[X] = \sum_{\nu} c_{\lambda\mu}^{\nu} s_{\nu}[X] \quad \text{where } c_{\lambda\mu}^{\nu} \in \mathbb{N}. \quad (5.4)$$

Since $\Lambda^{(k)}$ is a ring, and we have shown that $s_{\lambda}^{(k)}$ forms a basis for this space, a similar expression holds for the product of two k -Schur functions. That is, for k -bounded partitions λ and μ ,

$$s_{\lambda}^{(k)}[X] s_{\mu}^{(k)}[X] = \sum_{\nu} c_{\lambda\mu}^{\nu(k)} s_{\nu}^{(k)}[X] \quad \text{where } c_{\lambda\mu}^{\nu(k)} \in \mathbb{Z}, \quad (5.5)$$

where the integrality of $c_{\lambda\mu}^{\nu(k)}$ follows from Property 9. Further, Property 8 says that the k -Schur functions are simply the Schur functions when k is large, and therefore $c_{\lambda\mu}^{\nu(k)} = c_{\lambda\mu}^{\nu}$ for $k \geq |\nu|$. In fact, we believe the coefficients are nonnegative for all k . That is,

Conjecture 20. For all k -bounded partitions λ, μ, ν , we have $0 \leq c_{\lambda\mu}^{\nu(k)} \leq c_{\lambda\mu}^{\nu}$.

In particular, (5.5) reduces to a k -generalization of the Pieri rule when λ is a row (resp. column) of length $\ell \leq k$ since $s_{\lambda}^{(k)}[X]$ reduces to $h_{\ell}[X]$ (resp. $e_{\ell}[X]$). That is, for $\ell \leq k$,

$$h_{\ell}[X] s_{\lambda}^{(k)}[X] = \sum_{\mu \in E_{\lambda, \ell}^{(k)}} s_{\mu}^{(k)}[X] \quad \text{and} \quad e_{\ell}[X] s_{\lambda}^{(k)}[X] = \sum_{\mu \in \bar{E}_{\lambda, \ell}^{(k)}} s_{\mu}^{(k)}[X], \quad (5.6)$$

for some sets of partitions $E_{\lambda, \ell}^{(k)}$ and $\bar{E}_{\lambda, \ell}^{(k)}$, which we believe naturally extend the Pieri rules by:

Conjecture 21. For any positive integer $\ell \leq k$,

$$\begin{aligned} E_{\lambda, \ell}^{(k)} &= \{\mu \mid \mu/\lambda \text{ is a horizontal } \ell\text{-strip and } \mu^{\omega_k}/\lambda^{\omega_k} \text{ is a vertical } \ell\text{-strip}\}, \\ \bar{E}_{\lambda, \ell}^{(k)} &= \{\mu \mid \mu/\lambda \text{ is a vertical } \ell\text{-strip and } \mu^{\omega_k}/\lambda^{\omega_k} \text{ is a horizontal } \ell\text{-strip}\}. \end{aligned} \quad (5.7)$$

For example, to obtain the indices of elements occurring in $e_2 s_{3,2,1}^{(4)}$, we find $(3, 2, 1)^{\omega_4} = (2, 2, 1, 1)$ by definition. Adding a horizontal 2-strip to $(2, 2, 1, 1)$ in all ways, we obtain $(2, 2, 2, 1, 1), (3, 2, 1, 1, 1), (3, 2, 2, 1)$ and $(4, 2, 1, 1)$ of which all are 4-bounded. Our set then consists of all the 4-conjugates of these partitions that leave a vertical 2-strip when $(3, 2, 1)$ is extracted from them. The 4-conjugates are

$$(2, 2, 2, 1, 1)^{\omega_4} = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}, \quad (3, 2, 1, 1, 1)^{\omega_4} = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}, \quad (3, 2, 2, 1)^{\omega_4} = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}, \quad (4, 2, 1, 1)^{\omega_4} = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}, \quad (5.8)$$

and of these, the first three are such that a vertical 2-strip remains when $(3, 2, 1)$ is extracted. Thus

$$e_2[X] s_{3,2,1}^{(4)}[X] = s_{3,3,2}^{(4)}[X] + s_{3,2,2,1}^{(4)}[X] + s_{3,2,1,1,1}^{(4)}[X]. \quad (5.9)$$

5.3. **The involution ω .** It is known that the involution in (2.13) acts on a Schur function by

$$\omega s_\lambda[X] = s_{\lambda'}[X]. \quad (5.10)$$

We thus examine the action of ω on a k -Schur function since ω preserves the space $\Lambda^{(k)}$ by (3.1). The following natural generalization of (5.10) follows from our conjectured k -Pieri rules:

Property 22. For any k -bounded partition λ ,

$$\omega s_\lambda^{(k)}[X] = s_{\lambda^{\omega_k}}^{(k)}[X]. \quad (5.11)$$

Proof. Let $C_{\lambda\mu}^{(k)}$ denote the coefficients in the iteration of (5.6),

$$h_\lambda[X] = h_{\lambda_1}[X] \cdots h_{\lambda_\ell}[X] s_0[X] = \sum_{\mu: \mu_1 \leq k} C_{\lambda\mu}^{(k)} s_\mu^{(k)}[X]. \quad (5.12)$$

Since this expression uniquely determines $s_\lambda^{(k)}[X]$ for any k -bounded partition λ , any function F_μ that satisfies

$$h_\lambda[X] = \sum_{\mu: \mu_1 \leq k} C_{\lambda\mu}^k F_\mu, \quad (5.13)$$

must be exactly $s_\mu^{(k)}[X]$. Note that Eq. (2.13) and Conjecture 21 imply

$$h_i[X] \omega(s_{\lambda^{\omega_k}}[X]) = \omega(e_i[X] s_{\lambda^{\omega_k}}[X]) = \sum_{\mu \in \bar{E}_{\lambda^{\omega_k}, i}^{(k)}} \omega s_\mu[X]. \quad (5.14)$$

Letting $F_\lambda = \omega s_{\lambda^{\omega_k}}[X]$ and $F_0 = 1$, we have

$$h_i[X] F_\lambda = \sum_{\mu \in \bar{E}_{\lambda^{\omega_k}, i}^{(k)}} F_{\mu^{\omega_k}} = \sum_{\mu \in \bar{E}_{\lambda, i}^{(k)}} F_\mu \quad (5.15)$$

and more generally,

$$h_\lambda[X] = \sum_{\mu: \mu_1 \leq k} C_{\lambda\mu}^{(k)} F_\mu. \quad (5.16)$$

Therefore, $F_\mu = s_\mu^{(k)}[X]$ and since $F_\mu = \omega s_{\mu^{\omega_k}}^{(k)}[X]$ our claim follows. \square

6. t -EXTENSION

The ring of symmetric polynomials over rational functions in an extra parameter t has proven to be of interest in many fields of mathematics and physics. One natural basis of this space is given by the Hall-Littlewood polynomials, $H_\lambda[X; t]$, which provide t -analogs of the homogeneous symmetric functions $h_\lambda[X]$. Our approach employs vertex operators that arise in the recursive construction for the Hall-Littlewood polynomials [4]. These operators can be defined [13] for $\ell \in \mathbb{Z}$, by

$$B_\ell = \sum_{i=0}^{\infty} s_{i+\ell}[X] s_i[X(t-1)]^\perp, \quad (6.1)$$

where for f, g and h arbitrary symmetric functions, f^\perp is such that on the scalar product (2.16),

$$\langle f^\perp g, h \rangle = \langle g, fh \rangle. \quad (6.2)$$

The operators add an entry to the Hall-Littlewood polynomials, that is,

$$H_\lambda[X; t] = B_{\lambda_1} H_{\lambda_2, \dots, \lambda_\ell}[X; t], \quad \text{for } \lambda_1 \geq \lambda_2. \quad (6.3)$$

Further, since they satisfy the relation

$$B_m B_n = t B_n B_m + t B_{m+1} B_{n-1} - B_{n-1} B_{m+1}, \quad m, n \in \mathbb{Z}, \quad (6.4)$$

and $B_\ell \cdot 1 = 0$ if $\ell < 0$, their action on Λ can be computed algebraically.

We now consider an extension of the subspace $\Lambda^{(k)}$, given by

$$\Lambda_t^{(k)} = \mathcal{L} \{H_\lambda[X; t]\}_{\lambda; \lambda_1 \leq k}. \quad (6.5)$$

It is clear that $\Lambda_t^{(1)} \subseteq \Lambda_t^{(2)} \subseteq \dots \subseteq \Lambda_t^{(\infty)} = \Lambda$ and thus that these subspaces provide a filtration for Λ . Note that $\Lambda_t^{(k)}$ can be equivalently defined as

$$\Lambda_t^{(k)} = \mathcal{L} \{s_\lambda[X/(1-t)]\}_{\lambda; \lambda_1 \leq k}, \quad \text{or} \quad \Lambda_t^{(k)} = \mathcal{L} \{H_\lambda[X; q, t]\}_{\lambda; \lambda_1 \leq k}. \quad (6.6)$$

Again, we want elements that play the role in $\Lambda_t^{(k)}$ that the Schur functions play in Λ . In particular, since $\Lambda_t^{(\infty)} = \Lambda$, we want a basis for $\Lambda_t^{(k)}$ that reduces to the Schur functions when k is large.

Our construction of these elements will naturally extend Definition 7 for the k -Schur functions in the case $t = 1$. First, multiplication by $s_{\lambda_1}[X]$ is replaced by the action of the operator B_{λ_1} . Then, the projection operator $T_{\lambda_1}^{(k)}$ is replaced by a t -analogous operator which, to define, requires an appropriate extension of the k -split polynomials. Recent developments in the theory of symmetric functions aid us with this task.

6.1. k -split polynomials. Important in our work with the k -Schur functions is a family of polynomials, studied in many recent papers such as [8, 10, 11, 12, 13], that give a t -analog of the product of Schur functions. These functions, indexed by a sequence of partitions, can be built recursively using vertex operators [13]. For a partition λ of length m , define

$$B_\lambda \equiv \prod_{1 \leq i < j \leq m} (1 - te_{ij}) B_{\lambda_1} \cdots B_{\lambda_m}, \quad (6.7)$$

where e_{ij} acts by

$$e_{ij}(B_{\lambda_1} \cdots B_{\lambda_m}) = B_{\lambda_1} \cdots B_{\lambda_i+1} \cdots B_{\lambda_j-1} \cdots B_{\lambda_m}. \quad (6.8)$$

For any sequence of partitions $(\lambda^{(1)}, \lambda^{(2)}, \dots)$, the generalized Schur function product is then defined recursively by

$$\mathcal{H}_{(\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \dots)}[X; t] = B_{\lambda^{(1)}} \mathcal{H}_{(\lambda^{(2)}, \lambda^{(3)}, \dots)}[X; t], \quad (6.9)$$

starting with $\mathcal{H}_\emptyset = 1$. Note that since $B_\lambda \cdot 1 = s_\lambda[X]$, we have that

$$\mathcal{H}_{(\lambda)}[X; t] = s_\lambda[X]. \quad (6.10)$$

Appendix 8.1 gives an earlier formulation [12] for $\mathcal{H}_{(\lambda^{(1)}, \lambda^{(2)}, \dots)}[X; t]$. Note in [13], B_λ is denoted H_λ^t and is given in terms of generating series. Formula (6.7) can be extracted from their formula (17).

It was shown [12] that for a sequence of partitions, $S = (\lambda^{(1)}, \lambda^{(2)}, \dots)$,

$$\mathcal{H}_S[X; t] = \sum_{\mu \vdash |\lambda^{(1)}| + |\lambda^{(2)}| + \dots} K_{\mu; S}(t) s_\mu[X], \quad \text{where} \quad K_{\mu; S}(t) \in \mathbb{Z}[t]. \quad (6.11)$$

Since when $t = 1$ the action of B_λ on an arbitrary function f reduces to multiplication by $s_\lambda[X]$,

$$B_\lambda f = s_\lambda[X] f. \quad (6.12)$$

The $K_{\mu; S}(t)$ are known as generalized Kostka polynomials since, in this case, they satisfy

$$K_{\mu; S}(1) = \langle s_\mu[X], s_{\lambda^{(1)}}[X] s_{\lambda^{(2)}}[X] \cdots \rangle. \quad (6.13)$$

For our purposes, we consider only the $\mathcal{H}_S[X; t]$ indexed by a dominant sequence S . That is, sequences of partitions $S = (\lambda^{(1)}, \lambda^{(2)}, \dots)$ such that the concatenation of $\lambda^{(1)}, \lambda^{(2)}, \dots$, denoted \tilde{S} , forms a partition. In this case, we prove that $\mathcal{H}_S[X; t]$ obeys important unitriangular relations.

Property 23. *If S is dominant, with $\bar{S} = \lambda$, then*

$$\mathcal{H}_S[X; t] = s_\lambda[X] + \sum_{\mu > \lambda} K_{\mu; S}(t) s_\mu[X], \quad \text{where } K_{\mu; S}(t) \in \mathbb{Z}[t], \quad (6.14)$$

$$\mathcal{H}_S[X; t] = H_\lambda[X] + \sum_{\mu > \lambda} C_{\mu; S}(t) H_\mu[X], \quad \text{where } C_{\mu; S}(t) \in \mathbb{Z}[t]. \quad (6.15)$$

Proof. See Appendix 8.2 and note that Proposition 45, with Proposition 48, prove relation (6.14). The second identity then follows from unitriangularity (2.21) of $s_\lambda[X]$ in terms of $H_\mu[X; t]$. \square

We have discovered that a particular subset of the \mathcal{H}_S not only form a basis for $\Lambda_t^{(k)}$, but are essential in the construction of the k -Schur functions.

Definition 24. *The k -split polynomials are defined, for a k -bounded partition λ , by*

$$G_\lambda^{(k)}[X; t] = \mathcal{H}_S[X; t] \quad \text{where } S = \lambda^{\rightarrow k} \text{ is the } k\text{-split of } \lambda. \quad (6.16)$$

Since $\mathcal{H}_S[X; t]$ reduces to the product of Schur functions indexed by elements of S when $t = 1$,

$$G_\lambda^{(k)}[X; 1] = G_\lambda^{(k)}[X], \quad (6.17)$$

and therefore, $G_\lambda^{(k)}[X; t]$ is a proper t -generalization of the $G_\lambda^{(k)}[X]$ introduced in Section 3.

It develops that the k -split polynomials satisfy properties analogous to those held by the k -split polynomials at $t = 1$. We first show that the $G_\lambda^{(k)}[X; t]$ actually lie in the space $\Lambda_t^{(k)}$. To this end, we start by showing that operators B_i preserve this space.

Proposition 25. *If $f \in \Lambda_t^{(k)}$ then $B_i f \in \Lambda_t^{(k)}$ for all $i \in \mathbb{Z}$ with $i \leq k$.*

This claim follows from a preliminary result on subspaces of $\Lambda_t^{(k)}$. More precisely, for $a \in \mathbb{Z}$ and $b \in \mathbb{N}$, with $a \leq b$, we define

$$\Lambda_t^{(a, b)} = \mathcal{L}\{H_\lambda[X; t]\}_{a \leq \lambda_1 \leq b}, \quad (6.18)$$

and $\Lambda_t^{(a, b)} = \Lambda_t^{(0, b)}$ for $a < 0$. In particular, $\Lambda_t^{(k)} = \Lambda_t^{(0, k)}$ and $\Lambda_t^{(a, b)} \subseteq \Lambda_t^{(k)}$ if $b \leq k$. Thus, Proposition 25 is an immediate consequence of

Lemma 26. *If $f \in \Lambda_t^{(i, k)}$ then $B_i f \in \Lambda_t^{(i, k)}$ for all integers $i \leq k$.*

Proof. The assertion holds for $i = k$ since $B_k H_\lambda[X; t] = H_{(k, \lambda_1, \lambda_2, \dots)}[X; t] \in \Lambda_t^{(k, k)}$. Assume by induction that for all $f \in \Lambda_t^{(i, k)}$,

$$B_i f \in \Lambda_t^{(i, k)} \quad j < i \leq k. \quad (6.19)$$

Thus, it suffices to show that $B_j H_\lambda[X; t] \in \Lambda_t^{(j, k)}$.

We prove this claim by induction on $\ell(\lambda)$. First, $B_j H_0[X; t] = H_{(j)}[X; t] \in \Lambda_t^{(j, k)}$ for $j \geq 0$ and otherwise $B_j H_0[X; t] = 0 \in \Lambda_t^{(0, k)} = \Lambda_t^{(j, k)}$. Now, let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a k -bounded partition and assume $B_j H_{\hat{\lambda}}[X; t] \in \Lambda_t^{(j, k)}$ for all $\hat{\lambda} = (\lambda_2, \dots, \lambda_n)$. If $\lambda_1 \leq j$ then $B_j H_\lambda[X; t] = H_{(j, \lambda_1, \lambda_2, \dots)} \in \Lambda_t^{(j, k)}$ by (6.3). Now consider $\lambda_1 = j + 1$. The commutation relation (6.4) reduces to $B_j H_\lambda = B_j B_{j+1} H_{\hat{\lambda}} = t B_{j+1} B_j H_{\hat{\lambda}}$. Since $f = B_j H_{\hat{\lambda}} \in \Lambda_t^{(j, k)} \subseteq \Lambda_t^{(k)}$ by assumption, $B_j H_\lambda = t B_{j+1} f \in \Lambda_t^{(j+1, k)} \subseteq \Lambda_t^{(j, k)}$ by (6.19). Finally, for $\lambda_1 > j + 1$ we again use (6.4) to obtain $B_j H_\lambda = B_j B_{\lambda_1} H_{\hat{\lambda}} = t B_{\lambda_1} B_j H_{\hat{\lambda}} + t B_{j+1} B_{\lambda_1 - 1} H_{\hat{\lambda}} - B_{\lambda_1 - 1} B_{j+1} H_{\hat{\lambda}}$. Regarding the first term in the right hand side, our assumption gives that $B_j H_{\hat{\lambda}} \in \Lambda_t^{(j, k)} \subseteq \Lambda_t^{(k)}$ and then using (6.19), $t B_{\lambda_1} B_j H_{\hat{\lambda}} \in \Lambda_t^{(\lambda_1, k)} \subseteq \Lambda_t^{(j, k)}$ since $\lambda_1 > j$. Similar reasoning applies to the second and third term, and we thus have $B_j H_\lambda \in \Lambda_t^{(j, k)}$, proving our claim. \square

Now, given that $\Lambda_t^{(k)}$ is invariant under B_i , we can prove a more general statement that will imply the k -split polynomials lie in $\Lambda_t^{(k)}$.

Proposition 27. *If λ is a partition with $h_M(\lambda) \leq k$, then $B_\lambda f \in \Lambda_t^{(k)}$ for any $f \in \Lambda_t^{(k)}$.*

Proof. An operator version of the Jacobi-Trudi determinant (2.12) is given by (6.7). Thus, the expansion of B_λ in terms of products of the operators B_i yields only terms B_i with $i \leq k$. Proposition 25 then implies the result. \square

Property 28. *For any k -bounded partition λ , we have that*

$$G_\lambda^{(k)}[X; t] \in \Lambda_t^{(k)}. \quad (6.20)$$

Proof. Recursion (6.9) for \mathcal{H}_S allows us to restate Definition 24 as,

$$G_\lambda^{(k)}[X; t] = B_{\lambda^{(1)}} \cdots B_{\lambda^{(n)}} \cdot 1, \quad \text{where } \lambda^{\rightarrow k} = (\lambda^{(1)}, \dots, \lambda^{(n)}). \quad (6.21)$$

Since all $h_M(\lambda^{(i)}) \leq k$, Proposition 27 implies $B_{\lambda^{(n)}} \cdot 1 \in \Lambda_t^{(k)}$ given $1 \in \Lambda_t^{(k)}$. By induction, assuming $B_{\lambda^{(2)}} \cdots B_{\lambda^{(n)}} \cdot 1 \in \Lambda_t^{(k)}$, Proposition 27 verifies our claim. \square

Property 29. *We have*

$$G_\lambda^{(k)}[X; t] = H_\lambda[X; t] + \sum_{\mu > \lambda; \mu_1 \leq k} g_{\lambda\mu}^{(k)}(t) H_\mu[X; t], \quad \text{where } g_{\lambda\mu}^{(k)}(t) \in \mathbb{Z}[t]. \quad (6.22)$$

Proof. The k -split polynomials can be expanded in terms of Hall-Littlewood polynomials indexed by k -bounded partitions since Property 28 proves they lie in $\Lambda_t^{(k)}$. Then, by the unitriangular expansion of \mathcal{H}_S given in (6.15), $G_\lambda^{(k)}[X; t] = \mathcal{H}_{\lambda^{\rightarrow k}}[X; t]$ has the asserted unitriangularity. \square

Property 30. *Let λ be such that $h_M(\lambda) \leq k$. Then,*

$$G_\lambda^{(k)}[X; t] = s_\lambda[X]. \quad (6.23)$$

Proof. $G_\lambda^{(k)}[X; t] = \mathcal{H}_{(\lambda)}[X; t]$ since $\lambda^{\rightarrow k} = (\lambda)$ for $h_M(\lambda) \leq k$. The claim then follows from (6.10), which states that $\mathcal{H}_{(\lambda)}[X; t] = s_\lambda[X]$ for all λ . \square

Theorem 31. *The k -split polynomials form a basis of $\Lambda_t^{(k)}$.*

Proof. Since $\Lambda_t^{(k)}$ is the span of Hall-Littlewood polynomials indexed by k -bounded partitions and $G_\lambda^{(k)}$ are also indexed by k -bounded partitions, the theorem follows from Property 29. \square

6.2. k -Schur functions. As with the $t = 1$ case, although the k -split polynomials form a basis for $\Lambda_t^{(k)}$, they do not play the fundamental role that the Schur functions do for Λ . However, these polynomials are needed in the construction of our Schur analog, $s_\lambda^{(k)}[X; t]$. We use a projection operator, $\bar{T}_j^{(k)}$, for $j \leq k$, that acts linearly on $\Lambda_t^{(k)}$ by

$$\bar{T}_j^{(k)} G_\lambda^{(k)}[X; t] = \begin{cases} G_\lambda^{(k)}[X; t] & \text{if } \lambda_1 = j \\ 0 & \text{otherwise} \end{cases}. \quad (6.24)$$

Definition 32. *For k -bounded partition λ , the k -Schur functions are recursively defined*

$$s_\lambda^{(k)}[X; t] = \bar{T}_{\lambda_1}^{(k)} B_{\lambda_1} s_{(\lambda_2, \lambda_3, \dots)}^{(k)}[X; t], \quad \text{where } s_{()}^{(k)}[X; t] = 1. \quad (6.25)$$

Tables of k -Schur functions in terms of Schur functions can be found in Subsection 9.1.

When $t = 1$, from (6.12) and (6.17), we recover the k -Schur functions of Definition 7, that is,

$$s_{\lambda}^{(k)}[X; 1] = s_{\lambda}^{(k)}[X]. \quad (6.26)$$

We now prove several other properties satisfied by the k -Schur functions.

Property 33. *For λ a k -bounded partition, we have*

$$s_{\lambda}^{(k)}[X; t] = G_{\lambda}^{(k)}[X; t] + \sum_{\substack{\mu > \lambda \\ \mu_1 = \lambda_1}} u_{\mu\lambda}^{(k)}(t) G_{\mu}^{(k)}[X; t], \quad \text{where } u_{\mu\lambda}^{(k)}(t) \in \mathbb{Z}[t]. \quad (6.27)$$

Proof. The assertion holds for $s_0^{(k)}[X; t] = 1 = G_0^{(k)}[X; t]$ by definition. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a k -bounded partition and assume by induction on n that the property is true for $\hat{\lambda} = (\lambda_2, \dots, \lambda_n)$. Definition 32 gives that $s_{\lambda}^{(k)}[X; t] = \bar{T}_{\lambda_1}^{(k)} B_{\lambda_1} s_{\hat{\lambda}}^{(k)}[X; t]$. By the induction hypothesis, we have

$$B_{\lambda_1} s_{\hat{\lambda}}^{(k)}[X; t] = B_{\lambda_1} \left(G_{\hat{\lambda}}^{(k)}[X; t] + \sum_{\mu > \hat{\lambda}; \mu_1 = \hat{\lambda}_1} u_{\mu\hat{\lambda}}(t) G_{\mu}^{(k)}[X; t] \right), \quad u_{\mu\hat{\lambda}}(t) \in \mathbb{Z}[t]. \quad (6.28)$$

Recall the k -split polynomial is defined (6.16) by $G_{\gamma}^{(k)} = \mathcal{H}_{\gamma \rightarrow k}$, and thus we have

$$\begin{aligned} B_{\lambda_1} s_{\hat{\lambda}}^{(k)}[X; t] &= B_{\lambda_1} \left(\mathcal{H}_{\hat{\lambda} \rightarrow k}[X; t] + \sum_{\mu > \hat{\lambda}; \mu_1 = \hat{\lambda}_1} u_{\mu\hat{\lambda}}(t) \mathcal{H}_{\mu \rightarrow k}[X; t] \right) \\ &= \mathcal{H}_{((\lambda_1), \hat{\lambda} \rightarrow k)}[X; t] + \sum_{\mu > \hat{\lambda}; \mu_1 = \hat{\lambda}_1} u_{\mu\hat{\lambda}}(t) \mathcal{H}_{((\lambda_1), \mu \rightarrow k)}[X; t]. \end{aligned} \quad (6.29)$$

$\mu_1 = \hat{\lambda}_1 = \lambda_2 \leq \lambda_1$ implies that $((\lambda_1), \hat{\lambda} \rightarrow k)$ and $((\lambda_1), \mu \rightarrow k)$ are dominant sequences. Further, $\hat{\lambda} < \mu$ implies that $\lambda = (\lambda_1) \cup \hat{\lambda} < (\lambda_1) \cup \mu$. Therefore, untriangularity of \mathcal{H}_S (6.15) further gives that

$$B_{\lambda_1} s_{\hat{\lambda}}^{(k)}[X; t] = H_{\lambda}[X; t] + \sum_{\mu > \lambda} v_{\mu\lambda}(t) H_{\mu}[X; t], \quad v_{\mu\lambda}(t) \in \mathbb{Z}[t]. \quad (6.30)$$

$s_{\hat{\lambda}}^{(k)}[X; t] \in \Lambda_t^{(k)}$ since the induction hypothesis gives that it can be expanded in terms of $G_{\mu}^{(k)}[X; t]$. Thus, $B_{\lambda_1} s_{\hat{\lambda}}^{(k)}[X; t] \in \Lambda_t^{(k)}$ by Proposition 25 and therefore the coefficients $v_{\mu\lambda}(t)$ are non-zero only for $\mu_1 \leq k$. (6.30) then becomes

$$B_{\lambda_1} s_{\hat{\lambda}}^{(k)}[X; t] = H_{\lambda}[X; t] + \sum_{\mu > \lambda; \mu_1 \leq k} v_{\mu\lambda}(t) H_{\mu}[X; t], \quad v_{\mu\lambda}(t) \in \mathbb{Z}[t] \quad (6.31)$$

Untriangularity between Hall-Littlewood polynomials and k -split polynomials gives,

$$B_{\lambda_1} s_{\hat{\lambda}}^{(k)}[X; t] = G_{\lambda}^{(k)}[X; t] + \sum_{\mu > \lambda; \mu_1 \leq k} u_{\mu\lambda}(t) G_{\mu}^{(k)}[X; t], \quad u_{\mu\lambda}(t) \in \mathbb{Z}[t]. \quad (6.32)$$

Applying $\bar{T}_{\lambda_1}^{(k)}$ to both sides of this expression, we arrive at

$$\bar{T}_{\lambda_1}^{(k)} B_{\lambda_1} s_{\hat{\lambda}}^{(k)}[X; t] = G_{\lambda}^{(k)}[X; t] + \sum_{\mu > \lambda; \mu_1 = \lambda_1} u_{\mu\lambda}(t) G_{\mu}^{(k)}[X; t], \quad u_{\mu\lambda}(t) \in \mathbb{Z}[t]. \quad (6.33)$$

Since $s_{\lambda}^{(k)} = \bar{T}_{\lambda_1}^{(k)} B_{\lambda_1} s_{\hat{\lambda}}^{(k)}$, we have our claim. \square

The following theorem is an immediate consequence of this property.

Theorem 34. *The k -Schur functions form a basis of $\Lambda_t^{(k)}$.*

We can thus refine the expansion of Hall-Littlewood polynomials in terms of Schur functions (2.21).

Property 35. For any k -bounded partition λ ,

$$H_\lambda[X; t] = s_\lambda^{(k)}[X; t] + \sum_{\mu > \lambda: \mu_1 \leq k} K_{\mu\lambda}^{(k)}(t) s_\mu^{(k)}[X; t], \quad \text{where } K_{\mu\lambda}^{(k)}(t) \in \mathbb{Z}[t]. \quad (6.34)$$

Proof. The integrality of $K_{\mu\lambda}^{(k)}(t)$ follows from the unitriangularity and integrality in Properties 29 and 33. \square

This unitriangular expansion helps us prove that the k -Schur functions decompose integrally into $k+1$ -Schur functions.

Property 36. For any k -bounded partitions λ ,

$$s_\lambda^{(k)}[X; t] = s_\lambda^{(k+1)}[X; t] + \sum_{\mu > \lambda: \mu_1 \leq k} d_{\mu\lambda}^{(k+1)}(t) s_\mu^{(k+1)}[X; t], \quad \text{where } d_{\mu\lambda}^{(k+1)}(t) \in \mathbb{Z}[t]. \quad (6.35)$$

Proof. Property 35 implies that, for any k -bounded partition λ ,

$$s_\lambda^{(k)}[X; t] = H_\lambda[X; t] + \sum_{\mu > \lambda: \mu_1 \leq k} \bar{K}_{\mu\lambda}^{(k)}(t) H_\mu[X; t], \quad \text{where } \bar{K}_{\mu\lambda}^{(k)}(t) \in \mathbb{Z}[t]. \quad (6.36)$$

Then, since $\lambda_1 \leq k < k+1$, we can substitute (6.34) (with $k \rightarrow k+1$) into this expansion to obtain

$$s_\lambda^{(k)}[X; t] = s_\lambda^{(k+1)}[X; t] + \sum_{\mu > \lambda: \mu_1 \leq k+1} b_{\mu\lambda}^{(k+1)}(t) s_\mu^{(k+1)}[X; t], \quad \text{where } b_{\mu\lambda}^{(k+1)}(t) \in \mathbb{Z}[t]. \quad (6.37)$$

\square

Moreover, by our triangularity and integrality properties and (2.21), we also have integrality of the coefficients in the Schur function expansion of a k -Schur function. That is,

Property 37. For any k -bounded partition λ ,

$$s_\lambda^{(k)}[X; t] = s_\lambda[X] + \sum_{\mu > \lambda} v_{\mu\lambda}^{(k)}(t) s_\mu[X], \quad \text{where } v_{\mu\lambda}^{(k)}(t) \in \mathbb{Z}[t]. \quad (6.38)$$

This unitriangularity property, given that the coefficients in the Schur function expansion of the Macdonald polynomials are polynomials in q and t with integral coefficients, implies

Property 38. For any k -bounded partition λ ,

$$H_\lambda[X; q, t] = \sum_{\mu: \mu_1 \leq k} K_{\mu\lambda}^{(k)}(q, t) s_\mu^{(k)}[X; t], \quad \text{where } K_{\mu\lambda}^{(k)}(q, t) \in \mathbb{Z}[q, t]. \quad (6.39)$$

Now, to further support the idea that the $s_\lambda^{(k)}[X; t]$ provide a refinement for Schur function theory, we must show that they reduce to the usual $s_\lambda[X]$ when $k \rightarrow \infty$. This result relies on a lemma.

Lemma 39. If λ is a partition with $h_M(\lambda) \leq k$ then $\bar{T}_{\lambda_1}^{(k)} B_\lambda f = \bar{T}_{\lambda_1}^{(k)} B_{\lambda_1} B_{\hat{\lambda}} f$ for all $f \in \Lambda_t^{(k)}$.

Proof. We need to show that, for $f \in \Lambda_t^{(k)}$, $\bar{T}_{\lambda_1}^{(k)} (B_\lambda - B_{\lambda_1} B_{\hat{\lambda}}) \cdot f = 0$. Definition (6.7) gives

$$\begin{aligned} B_\lambda &= \prod_{2 \leq j \leq \ell(\lambda)} (1 - te_{1j}) B_{\lambda_1} \prod_{2 \leq i < j \leq \ell(\lambda)} (1 - te_{ij}) B_{\lambda_2} \cdots B_{\lambda_n} \\ &= \prod_{2 \leq j \leq \ell(\lambda)} (1 - te_{1j}) B_{\lambda_1} B_{\hat{\lambda}} \end{aligned} \quad (6.40)$$

In the expansion of this product, with the exception of the term $B_{\lambda_1} B_{\hat{\lambda}}$, each term contains at least one e_{1j} , which increases the index of B_{λ_1} . Therefore, using the argument of Proposition 27, we have

$$B_{\lambda} = B_{\lambda_1} B_{\hat{\lambda}} + \sum_{i=1}^{\ell(\lambda)-1} c_i(t) B_{\lambda_1+i} O_i, \quad c_i(t) \in \mathbb{Z}[t], \quad (6.41)$$

where O_i is a product of B_j 's with $j \leq k$. Proposition 25 states that $B_i f \in \Lambda_t^{(k)}$ for $f \in \Lambda_t^{(k)}$ and $i \leq k$ and it is thus clear that $O_i \cdot f \in \Lambda_t^{(k)}$. Furthermore, since $\lambda_1 + i \leq \lambda_1 + \ell(\lambda) - 1 = h_M(\lambda) \leq k$, we have $B_{\lambda_1+i} O_i \cdot f \in \lambda_t^{(\lambda_1+i, k)} \subseteq \Lambda_t^{(\lambda_1+1, k)}$ by Lemma 26. Thus, from expression (6.41),

$$(B_{\lambda} - B_{\lambda_1} B_{\hat{\lambda}}) \cdot f = \sum_{i=1}^{\ell(\lambda)-1} c_i(t) B_{\lambda_1+i} O_i \cdot f \in \Lambda_t^{(\lambda_1+1, k)}. \quad (6.42)$$

The definition of $\Lambda_t^{(\lambda_1+1, k)}$, with unitriangularity in Proposition 29, then gives the expansions

$$(B_{\lambda} - B_{\lambda_1} B_{\hat{\lambda}}) \cdot f = \sum_{\mu: \lambda_1+1 \leq \mu_1 \leq k} b_{\mu}(t) H_{\mu}[X; t] = \sum_{\mu: \mu \geq \lambda_1+1} d_{\mu}(t) G_{\mu}^{(k)}[X; t]. \quad (6.43)$$

Finally, acting with $\bar{T}_{\lambda_1}^{(k)}$, we have $\bar{T}_{\lambda_1}^{(k)} (B_{\lambda_1} B_{\hat{\lambda}} \cdot f - B_{\lambda} \cdot f) = 0$. \square

Property 40. For any k -bounded partition λ ,

$$s_{\hat{\lambda}}^{(k)}[X; t] = s_{\lambda}[X] \quad \text{if} \quad h_M(\lambda) \leq k. \quad (6.44)$$

Proof. We proceed by induction on $\ell(\lambda)$. First, $s_{\emptyset}^{(k)}[X; t] = 1 = s_{\emptyset}[X]$. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a k -bounded partition with $h_M(\lambda) \leq k$. If $\hat{\lambda} = (\lambda_2, \dots, \lambda_n)$ then $h_M(\hat{\lambda}) \leq h_M(\lambda)$ and induction implies $s_{\hat{\lambda}}^{(k)}[X; t] = s_{\hat{\lambda}}[X]$. Thus, $s_{\lambda}^{(k)}[X; t] = \bar{T}_{\lambda_1}^{(k)} B_{\lambda_1} s_{\hat{\lambda}}[X]$ and it suffices to show $\bar{T}_{\lambda_1}^{(k)} B_{\lambda_1} s_{\hat{\lambda}}[X] = s_{\lambda}[X]$. Since (6.10) gives that $B_{\mu} \cdot 1 = s_{\mu}[X]$, we need only show $\bar{T}_{\lambda_1}^{(k)} B_{\lambda_1} B_{\hat{\lambda}} \cdot 1 = B_{\lambda} \cdot 1$, or $\bar{T}_{\lambda_1}^{(k)} (B_{\lambda} - B_{\lambda_1} B_{\hat{\lambda}}) \cdot 1 = 0$. This result follows from the special case $f = 1 \in \Lambda_t^{(k)}$ of the previous lemma. \square

7. IRREDUCIBILITY AND k -CONJUGATION

The concept of irreducibility introduced in Section 4 extends naturally in the space $\Lambda_t^{(k)}$. In fact, the following t -analog of Theorem 14 holds:

Theorem 41. [6]: For any k -bounded partition λ ,

$$B_{(\ell^k - \ell + 1)} s_{\lambda}^{(k)}[X; t] = t^* s_{\lambda \cup (\ell^k - \ell + 1)}^{(k)}[X; t], \quad (7.1)$$

where t^* stands for some power of t .

This theorem implies that when $k = 2$, any k -Schur function can be built using a sequence of operators B_2 and $B_{1,1}$ applied to either $s_1^{(2)}[X; t] = s_1[X]$ or $s_{\emptyset}^{(2)}[X; t] = s_{\emptyset}[X] = 1$. Thus, there is a connection between the k -Schur functions and the positive functions introduced in [7, 15] and, from [13], to the generalized Kostka polynomials of [12]. This connection implies that our refinement (1.1) of Macdonald's original positivity conjecture holds when $k = 2$.

With regard to extending Conjecture 22 to the general case, we first note that the involution ω is not well defined on $\Lambda_t^{(k)}$. That is, an element $f \in \Lambda_t^{(k)}$ may be such that $\omega f \notin \Lambda_t^{(k)}$. However, there is a simple generalization for ω that preserves $\Lambda_t^{(k)}$. Let ω_t be defined on an element $f \in \Lambda$, by

$$\omega_t f = (\omega f) \Big|_{t \rightarrow 1/t}. \quad (7.2)$$

For instance, if $f = \sum_{\mu} c_{\mu}(q, t) s_{\mu}[X]$, we have

$$\omega_t f = \sum_{\mu} c_{\mu}(q, 1/t) s_{\mu'}[X]. \quad (7.3)$$

Since ω is an involution, ω_t is also. In fact, it is well defined on $\Lambda_t^{(k)}$.

Proposition 42. *If $f \in \Lambda_t^{(k)}$, then $\omega_t f \in \Lambda_t^{(k)}$.*

Proof. Since $\Lambda_t^{(k)}$ can also be defined as $\Lambda_t^{(k)} = \mathcal{L}\{s_{\lambda}[X/(1-t)]\}_{\lambda; \lambda_1 \leq k}$, it suffices to show, for any partition λ ,

$$\omega_t s_{\lambda}[X/(1-t)] = (-t)^{|\lambda|} s_{\lambda}[X/(1-t)]. \quad (7.4)$$

Let \bar{X} denote the alphabet X where $x_i \rightarrow -x_i$ for all i . That is, $P[\bar{X}] = P(-x_1, -x_2, \dots)$ for an arbitrary symmetric function $P[X] = P(x_1, x_2, \dots)$. Note, this implies that if $P[X]$ is homogeneous of degree d , then $P[\bar{X}] = (-1)^d P[X]$. Since ω acts on $P[X]$ by $\omega P[X] = P[-\bar{X}]$, we thus have

$$\omega_t s_{\lambda}[X/(1-t)] = s_{\lambda}[-\bar{X}/(1-t)] = s_{\lambda}[t\bar{X}/(1-t)] = (-t)^{|\lambda|} s_{\lambda}[X/(1-t)], \quad (7.5)$$

which completes the proof. \square

The action of ω_t on a k -Schur function appears to take a simple form generalizing Conjecture 22.

Conjecture 43. *For any k -bounded partition λ ,*

$$\omega_t s_{\lambda}^{(k)}[X; t] = t^{-c(\lambda)} s_{\lambda_{\omega_k}^{(k)}}[X; t], \quad (7.6)$$

where $c(\lambda)$ is some nonnegative integer.

Another unique property of the Schur functions is the expansion,

$$s_{\lambda}[X + Y] = \sum_{|\mu| + |\rho| = |\lambda|} c_{\mu\rho}^{\lambda} s_{\mu}[X] s_{\rho}[Y] \quad \text{where } c_{\mu\rho}^{\lambda} \in \mathbb{N}. \quad (7.7)$$

We have found by experimentation that the k -Schur functions also satisfy a similar relation,

Conjecture 44. *For any k -bounded partition,*

$$s_{\lambda}^{(k)}[X + Y; t] = \sum_{|\mu| + |\rho| = |\lambda|} g_{\mu\rho}^{\lambda}(t) s_{\mu}^{(k)}[X; t] s_{\rho}^{(k)}[Y; t] \quad \text{where } g_{\mu\rho}^{\lambda} \in \mathbb{N}[t]. \quad (7.8)$$

8. APPENDIX

8.1. Definition of $\mathcal{H}_S[X; t]$ [12]. Consider a sequence of partitions $S = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \dots)$ with η_i parts (some of which may be zero) in $\lambda^{(i)}$. If $\eta = (\eta_1, \eta_2, \dots)$ and $\ell(S) \equiv |\eta| = n$, we set

$$\text{Roots}_{\eta} = \{(i, j) \mid 1 \leq i \leq \eta_1 + \dots + \eta_k < j \leq n \text{ for some } l\}. \quad (8.1)$$

The formal power series is then defined, for the alphabet $X_n = x_1 + \dots + x_n$, by

$$\mathcal{B}_{\eta}[X_n; t] = \prod_{(i, j) \in \text{Roots}_{\eta}} \frac{1}{1 - tx_i/x_j}. \quad (8.2)$$

Given a function $f(x_1, \dots, x_n)$, the action of a permutation σ of S_n is defined by

$$\sigma f(x_1, x_2, \dots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}). \quad (8.3)$$

If A is the antisymmetrizer, $\sum_{\sigma \in S_n} \text{sign}(\sigma) \sigma$, and $\delta = (n-1, n-2, \dots, 0)$, the operator π is defined as

$$\pi(f) = A(x^{\delta} f) / A(x^{\delta}). \quad (8.4)$$

Denoting the concatenation of the partitions in S by \bar{S} , we then have the generating series

$$\mathbb{H}_S[X_n; t] = \pi(x^{\bar{S}} \mathcal{B}_{\eta}[X; t]). \quad (8.5)$$

Since π sends a polynomial in the variables of X_n to a symmetric polynomial, \mathbb{H}_S is symmetric in X_n . Letting \mathcal{P}_n be the set of elements of \mathbb{Z}^n whose entries are weakly decreasing, we can thus write

$$\mathbb{H}_S[X_n; t] = \sum_{\omega \in \mathcal{P}_n} K_{\omega; S}(t) s_\omega[X_n]. \quad (8.6)$$

Considering only partitions of $|S| \equiv |\lambda^{(1)}| + |\lambda^{(2)}| + \dots$, we finally define the symmetric polynomial

$$\mathcal{H}_S[X_n; t] = \sum_{\mu \vdash |S|} K_{\mu; S}(t) s_\mu[X_n]. \quad (8.7)$$

Note that if S' denotes the sequence of partitions obtained by appending the partition (0^ℓ) to a sequence of partitions S , then $|S'| = |S|$. In this case, we can show that

$$\mathcal{H}_{S'}[X_{n+\ell}] = \sum_{\mu \vdash |S'|} K_{\mu; S'}(t) s_\mu[X_{n+\ell}] = \sum_{\mu \vdash |S|} K_{\mu; S}(t) s_\mu[X_{n+\ell}], \quad (8.8)$$

that is, $\mathcal{H}_{S'}[X_{n+\ell}]$ is equal to $\mathcal{H}_S[X_n \rightarrow X_{n+\ell}]$. The number of variables in $\mathcal{H}_S[X_n]$ is thus irrelevant (as long as it is large enough) and therefore, at times, we work with the infinite alphabet X .

8.2. Properties of the generalized Kostka polynomials. It is known that the coefficients $K_{\mu; S}(t)$ obey a Morris-type recurrence. Starting with a one-element sequence $S = (\lambda^{(1)})$, we have

$$K_{\mu; S}(t) = \begin{cases} 1 & \text{if } \mu = \lambda^{(1)} \\ 0 & \text{otherwise} \end{cases}. \quad (8.9)$$

Then for an arbitrary sequence $S = (\lambda^{(1)}, \lambda^{(2)}, \dots)$, letting $m = \ell(\lambda^{(1)})$ and $\hat{S} = (\lambda^{(2)}, \lambda^{(3)}, \dots)$, $K_{\mu; S}(t)$ satisfies the recurrence

$$K_{\mu; S}(t) = \sum_{w \in S_n / (S_m \times S_{n-m})} (-1)^{\ell(w)} t^{|\alpha(w)| - |\lambda^{(1)}|} \sum_{\nu \vdash |\hat{S}|} c_{\alpha(w)/\lambda^{(1)}, \beta(w)}^\nu K_{\nu; \hat{S}}(t), \quad (8.10)$$

where w runs over minimal length coset representatives, and where $\alpha(w)$ and $\beta(w)$ are the first m and last $n - m$ parts of the weight $w^{-1}(\mu + \delta) - \delta$, respectively. Note that $\beta(w)$ is always a partition, and that the w -th summand is understood to be zero unless all the parts of $\alpha(w)$ are nonnegative and $\alpha(w) \supseteq \lambda^{(1)}$. Further, $c_{\alpha(w)/\lambda^{(1)}, \beta(w)}^\nu$ is the Littlewood-Richardson coefficient

$$c_{\alpha(w)/\lambda^{(1)}, \beta(w)}^\nu = \langle s_{\alpha(w)/\lambda^{(1)}}[X] s_{\beta(w)}[X], s_\nu[X] \rangle. \quad (8.11)$$

Note that any element w of $S_n / (S_m \times S_{n-m})$ is of the form

$$[[i_1, \dots, i_m]] \equiv [i_1, \dots, i_m, 1, \dots, \hat{i}_1, \dots, \hat{i}_m, \dots, n] \quad 1 \leq i_1 < \dots < i_m \leq n, \quad (8.12)$$

where $1, \dots, \hat{i}_1, \dots, \hat{i}_m, \dots, n$ denotes the sequence $1, \dots, n$ with the elements i_1, \dots, i_m omitted. Since any permutation $w = [w_1, \dots, w_n]$ satisfies

$$w^{-1}(\mu + \delta) - \delta = (\mu_{w_1} - (w_1 - 1), \dots, \mu_{w_n} - (w_n - n)) \quad (8.13)$$

in the case that $w = [[i_1, \dots, i_m]]$, (8.12) implies that

$$(\mu_1, \dots, \mu_m) \supseteq \alpha(w) \quad (8.14)$$

and

$$\mu_j = (\beta(w))_{j-m} \quad \text{for all } j > i_m. \quad (8.15)$$

We now prove two properties held by the generalized Kostka polynomials, $K_{\mu; S}(t)$, when the sequence S is dominant, that is, when \bar{S} is a partition.

Proposition 45. *If S is a dominant sequence of partitions with $\bar{S} = \mu$, then $K_{\mu; S}(t) = 1$.*

Proof. For S with one element, the claim follows from (8.9). Let $S = (\lambda^{(1)}, \lambda^{(2)}, \dots)$, with $\ell(\lambda^{(1)}) = m$. Since $\bar{S} = \mu$, we have $K_{\hat{\mu}; \hat{S}} = 1$, where $\hat{\mu} = (\mu_{m+1}, \mu_{m+2}, \dots)$, by induction on the number of elements in S . Consider the w -th summand of (8.10). Since the only non-zero terms occur when $\alpha(w) \supseteq \lambda^{(1)} = (\mu_1, \dots, \mu_m)$, and $\alpha(w) \subseteq (\mu_1, \dots, \mu_m)$ by (8.14), the only term is when $\alpha(w) = \lambda^{(1)}$ and $\beta(w) = (\hat{S}) \equiv \hat{\mu}$. Thus, $K_{\mu; S}(t) = c_{0, \hat{\mu}}^\nu K_{\hat{\mu}; \hat{S}} = K_{\hat{\mu}; \hat{S}} = 1$, since $c_{0, \gamma}^\nu = 0$ unless $\nu = \gamma$. \square

We are also able to prove that many of the generalized Kostka polynomials vanish. To this end, we need the following two lemmas. The first lemma concerns the Littlewood-Richardson coefficients.

Lemma 46. *Let ν , α , λ and β be partitions and let $N = |\alpha| - |\lambda|$. If $\beta^N \equiv (\beta_1 + N, \beta_2, \beta_3, \dots)$ then $c_{\alpha/\lambda, \beta}^\nu = 0$ when $\nu \not\leq \beta^N$.*

Proof. Given $s_{\alpha/\lambda} = \sum_{\rho \vdash N} d_\rho s_\rho$, we have, by definition, $c_{\alpha/\lambda, \beta}^\nu = \sum_{\rho \vdash N} d_\rho c_{\rho, \beta}^\nu$. Therefore, the lemma will hold if we can show that $c_{\rho, \beta}^\nu = 0$ for all $\rho \vdash N$ when $\nu \not\leq \beta^N$. Since $\rho \vdash N$ implies $\rho' \geq (N)'$, we have $(\rho' \cup \beta')' \leq ((N)' \cup \beta')' = \beta^N$. But Property 1 gives that $c_{\rho, \beta}^\nu = 0$ unless $\nu \leq (\rho' \cup \beta')'$. Thus, $c_{\rho, \beta}^\nu = 0$ unless $\nu \leq (\rho' \cup \beta')' \leq \beta^N$, that is $c_{\rho, \beta}^\nu = 0$ when $\nu \not\leq \beta^N$. \square

Lemma 47. *Let μ and λ be partitions such that $\mu \not\leq \lambda$, and let α and β denote respectively the first m and last $n - m$ parts of $w^{-1}(\mu + \delta) - \delta$ for $w \in S_n / (S_m \times S_{n-m})$. If $\alpha \supseteq (\lambda_1, \dots, \lambda_m)$ then $\beta^N \not\leq (\lambda_{m+1}, \dots, \lambda_n)$, where β^N is as defined in Lemma 46.*

Proof. Since $w = [[i_1, \dots, i_m]]$, (8.14) implies $(\mu_1, \dots, \mu_m) \supseteq \alpha$. Thus, given $\alpha \supseteq (\lambda_1, \dots, \lambda_m)$,

$$\lambda_r \leq \alpha_r \leq \mu_r \quad \text{for all } r \leq m. \quad (8.16)$$

Now, $\mu \not\leq \lambda$ implies that there exists some i that is the smallest integer such that

$$\mu_1 + \dots + \mu_i < \lambda_1 + \dots + \lambda_i. \quad (8.17)$$

This necessarily implies that $\mu_i < \lambda_i$, and therefore $i > m$ by (8.16). We now prove a result from which the lemma will then follow. That is, beyond the i^{th} entry, $\alpha \cup \beta$ is identical to μ ;

$$\alpha \cup \beta = (-, \dots, -, \beta_{i-m+1}, \dots, \beta_{n-m}) = (-, \dots, -, \mu_{i+1}, \dots, \mu_n). \quad (8.18)$$

The lemma follows since $|\alpha| + |\beta| = |\alpha \cup \beta| = |\mu|$ implies by (8.18) that $|\alpha| + \beta_1 + \dots + \beta_{i-m} = (\alpha \cup \beta)_1 + \dots + (\alpha \cup \beta)_i = \mu_1 + \dots + \mu_i$. Which is to say that $|\alpha| + \beta_1 + \dots + \beta_{i-m} < \lambda_1 + \dots + \lambda_m + \lambda_{m+1} + \dots + \lambda_i$, by (8.17). With $N = |\alpha| - (\lambda_1 + \dots + \lambda_m)$, we then have $N + \beta_1 + \dots + \beta_{i-m} < \lambda_{m+1} + \dots + \lambda_i$, which gives $\beta^N \not\leq (\lambda_{m+1}, \dots, \lambda_n)$.

Proof of (8.18): We first show that beyond the i^{th} entry, $w^{-1}(\mu + \delta) - \delta$ and μ are identical;

$$w^{-1}(\mu + \delta) - \delta = (-, \dots, -, \beta_{i-m+1}, \dots, \beta_{n-m}) = (-, \dots, -, \mu_{i+1}, \dots, \mu_n). \quad (8.19)$$

With $\mu_i < \lambda_i$ and $i > m$, we have $\mu_i < \lambda_i \leq \lambda_m$ and thus $\mu_i < \lambda_m \leq \alpha_m$ by (8.16). Further, using (8.13), we have that $\alpha_m = \mu_{i_m} - (i_m - m) \leq \mu_{i_m}$ since $i_m \geq m$. Thus, $\mu_i < \alpha_m \leq \mu_{i_m}$, which gives $i_m < i$. Hence, since $\beta_{j-m} = \mu_j$ for all $j > i_m$ by (8.15), this also holds for $j > i$, that is, $(\beta_{i-m+1}, \dots, \beta_{n-m}) = (\mu_{i+1}, \dots, \mu_n)$. Now, given (8.19), we will show (8.18). Note that $\alpha \cup \beta$ is the rearrangement of $w^{-1}(\mu + \delta) - \delta$. Since β is a partition, the rearrangement will not concern the entries of β . $\mu_i < \alpha_m$ implies that $\beta_{i-m+1} = \mu_{i+1} \leq \mu_i < \alpha_m$ (i.e. β_{i-m+1} is smaller than the smallest element of α), and we have $\alpha \cup \beta = (-, \dots, -, \beta_{i-m+1}, \dots, \beta_{n-m}) = (-, \dots, -, \mu_{i+1}, \dots, \mu_n)$. \square

We can now prove the following statement concerning the generalized Kostka polynomials.

Proposition 48. *If S is dominant with $\bar{S} = \lambda$, then $K_{\mu; S}(t) = 0$ for $\mu \not\leq \lambda$.*

Proof. If $S = (\lambda^{(1)})$ has only one element, then (8.9) gives that $K_{\mu; S}(t) = 0$ when $\mu \not\leq \lambda^{(1)}$. Assume by induction on the number of parts of S that $K_{\nu; \bar{S}}(t) = 0$ for all $\nu \not\leq \bar{S}$. We now show that the right hand side of (8.10) is zero for $\mu \not\leq \bar{S} = \lambda$. Since the w -th summand is non-zero only when

$\alpha(w) \supseteq (\lambda^{(1)})$, the conditions of Lemma 47 are satisfied. Therefore, $\beta(w)^N$ is such that $\beta(w)^N \not\geq \overline{(\hat{S})}$. Since Lemma 46 gives that $c_{\alpha(w)/\lambda^{(1)}, \beta(w)}^\nu = 0$ when $\nu \not\geq \beta(w)^N$, the only non-zero terms in the sum occur when $\nu \leq \beta(w)^N$, with $\hat{S} \not\geq \beta(w)^N$. If we assumed $\nu \geq \hat{S}$, then $\hat{S} \leq \nu \leq \beta(w)^N$ would imply $\hat{S} \leq \beta(w)^N$. By contradiction, we have that the non-zero terms occur when $\nu \not\geq \hat{S}$. Our induction hypothesis then proves all terms are zero. \square

9. TABLES

In the tables below, we have not included the cases when $k \geq |\lambda|$, which, from Property 40, simply correspond to the trivial cases $s_\lambda[X; t] = s_\lambda[X]$.

9.1. k -Schur functions in terms of Schur functions.

$k = 2$	1^3	21	3
1^3	1	t	
21		1	t

$k = 2$	1^4	21^2	2^2	31	4
1^4	1	t	t^2		
21^2		1		t	
2^2			1	t	t^2

$k = 2$	1^5	21^3	2^21	31^2	32	41	5
1^5	1	$t + t^2$	$t^2 + t^3$	t^3	t^4		
21^3		1	t	$t + t^2$	t^2	t^3	
2^21			1	t	$t + t^2$	$t^2 + t^3$	t^4

$k = 3$	1^4	21^2	2^2	31	4
1^4	1	t			
21^2		1		t	
2^2			1		
31				1	t

$k = 3$	1^5	21^3	2^21	31^2	32	41	5
1^5	1	t	t^2				
21^3		1		t			
2^21			1		t		
31^2				1		t	
32					1	t	t^2

$k = 3$	1^6	21^4	2^21^2	2^3	31^3	321	3^2	41^2	42	51	6
1^6	1	t	t^2	t^3							
21^4		1	t		t	t^2					
2^21^2			1			t	t^2				
2^3				1		t			t^2		
31^3					1			t			
321						1		t	t	t^2	
3^3							1		t	t^2	t^3

$k = 4$	1^5	21^3	2^21	31^2	32	41	5
1^5	1	t					
21^3		1		t			
2^21			1				
31^2				1		t	
32					1		
41						1	t

$k = 4$	1^6	21^4	2^21^2	2^3	31^3	321	3^2	41^2	42	51	6
1^6	1	t	t^2								
21^4		1			t						
2^21^2			1			t					
2^3				1							
31^3					1			t			
321						1			t		
3^3							1				
41^2								1		t	
42									1	t	t^2

9.2. Macdonald polynomials in terms of k -Schur functions.

$k = 2$	1^3	21
1^3	1	t^2
21	q	1

$k = 2$	1^4	21^2	2^2
1^4	1	$t^2 + t^3$	t^4
21^2	q	$1 + qt^2$	t
2^2	q^2	$q + qt$	1

$k = 2$	1^5	21^3	2^21
1^5	1	$t^3 + t^4$	t^6
21^3	q	$1 + qt^3$	t^2
2^21	q^2	$q + qt$	1

$k = 2$	1^6	21^4	2^21^2	2^3
1^6	1	$t^3 + t^4 + t^5$	$t^6 + t^7 + t^8$	t^9
21^4	q	$1 + qt^3 + qt^4$	$t^2 + t^3 + qt^6$	t^4
2^21^2	q^2	$q + qt + q^2t^3$	$1 + qt^2 + qt^3$	t
2^3	q^3	$q^2 + q^2t + q^2t^2$	$q + qt + qt^2$	1

$k = 3$	1^4	21^2	2^2	31
1^4	1	$t^2 + t^3$	$t^2 + t^4$	t^5
21^2	q	$1 + qt^2$	$t + qt^2$	t^2
2^2	q^2	$q + qt$	$1 + q^2t^2$	t
31	q^3	$q + q^2$	$q + q^2t$	1

$k = 3$	1^5	21^3	2^21	31^2	32
1^5	1	$t^2 + t^3 + t^4$	$t^3 + t^4 + t^5 + t^6$	$t^5 + t^6 + t^7$	t^8
21^3	q	$1 + qt^2 + qt^3$	$t + t^2 + qt^3 + qt^4$	$t^2 + t^3 + qt^5$	t^4
2^21	q^2	$q + qt + q^2t^2$	$1 + qt + qt^2 + q^2t^3$	$t + qt^2 + qt^3$	t^2
31^2	q^3	$q + q^2 + q^3t^2$	$q + qt + q^2t + q^2t^2$	$1 + qt^2 + q^2t^2$	t
32	q^4	$q^2 + q^3 + q^3t$	$q + q^2 + q^2t + q^3t$	$q + qt + q^2t$	1

$k = 4$	1^5	21^3	2^21	31^2	32	41
1^5	1	$t^2 + t^3 + t^4$	$t^2 + t^3 + t^4 + t^5 + t^6$	$t^5 + t^6 + t^7$	$t^4 + t^5 + t^6 + t^7 + t^8$	t^9
21^3	q	$1 + qt^2 + qt^3$	$t + t^2 + qt^2 + qt^3 + qt^4$	$t^2 + t^3 + qt^5$	$t^2 + t^3 + t^4 + qt^4 + qt^5$	t^5
2^21	q^2	$q + qt + q^2t^2$	$1 + qt + qt^2 + q^2t^2 + q^2t^3$	$t + qt^2 + qt^3$	$t + t^2 + qt^2 + qt^3 + q^2t^4$	t^3
31^2	q^3	$q + q^2 + q^3t^2$	$q + qt + q^2t + q^2t^2 + q^3t^2$	$1 + qt^2 + q^2t^2$	$t + qt + qt^2 + q^2t^2 + q^2t^3$	t^2
32	q^4	$q^2 + q^3 + q^3t$	$q + q^2 + q^2t + q^3t + q^4t^2$	$q + qt + q^2t$	$1 + qt + q^2t + q^2t^2 + q^3t^2$	t
41	q^6	$q^3 + q^4 + q^5$	$q^2 + q^3 + q^4 + q^4t + q^5t$	$q + q^2 + q^3$	$q + q^2 + q^2t + q^3t + q^4t$	1

REFERENCES

- [1] A. M. Garsia and M. Haiman, *A graded representation module for Macdonald's polynomials*, Proc. Natl. Acad. Sci. USA **90** (1993) 3607-3610.
- [2] A. M. Garsia and C. Procesi, *On certain graded S_n -modules and the q -Kostka polynomials*, Adv. Math. **87** (1992), 82-138.
- [3] M. Haiman, *Hilbert schemes, polygraphs, and the Macdonald positivity conjecture*, J. Am. Math. Soc. **14** (2001), 941-1006.
- [4] N. Jing, *Vertex operators and Hall-Littlewood symmetric functions*, Adv. Math. **87** (1991), 226-248.
- [5] L. Lapointe, A. Lascoux and J. Morse, *Tableau atoms and a new Macdonald positivity conjecture*, to be published in Duke Math. J., math.QA/0008073.
- [6] L. Lapointe and J. Morse, *Schur function identities, their t -analogs, and k -Schur irreducibility*, To appear in the Advance of Math.
- [7] L. Lapointe and J. Morse, *Tableaux statistics for two part Macdonald polynomials*, math.CO/9812001.
- [8] A. Lascoux, B. Leclerc and J.-Y. Thibon, *Ribbon tableaux, Hall-Littlewood functions, quantum affine algebras and unipotent varieties*, J. Math. Phys. **38** (1997), 1041-1068.
- [9] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd edition, Clarendon Press, Oxford, 1995.
- [10] A. Schilling and S. Warnaar, *Inhomogeneous lattice paths, generalized Kostka-Foulkes polynomials, and A_{n-1} -supernomials*, Comm. Math. Phys. **202** (1999), 359-401.
- [11] M. Shimozono, *A cyclage poset structure for Littlewood-Richardson tableaux*, European J Combin **22** (2001), 365-393.
- [12] M. Shimozono and J. Weyman, *Graded characters of modules supported in the closure of a nilpotent conjugacy class*, European J. Combin. **21** (2000), 257-288.
- [13] M. Shimozono and M. Zabrocki, *Hall-Littlewood vertex operators and generalized Kostka polynomials*, Adv. Math. **158** (2001), 66-85.
- [14] S. Veigneau, *ACE Version 3.0* (1998) <http://phalanstere.univ-mlv.fr/~ace/>.
- [15] M.A. Zabrocki, *A Macdonald vertex operator and standard tableaux statistics for the two-column (q,t) -Kostka coefficients*, Electron. J. Combinat. **5**, R45 (1998).

MC GILL UNIVERSITY, DEPARTMENT OF MATHEMATICS AND STATISTICS, MONTRÉAL, QUÉBEC, CANADA, H3A 2K6
E-mail address: `lapointe@math.mcgill.ca`

UNIVERSITY OF PENNSYLVANIA, DEPARTMENT OF MATHEMATICS, PHILADELPHIA, PA 19104
E-mail address: `morsej@math.upenn.edu`