BIVARIATE KNOP-SAHI AND MACDONALD POLYNOMIALS RELATED TO q-ULTRASPHERICAL FUNCTIONS

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ABSTRACT: Knop and Sahi introduced a family of non-homogeneous and non-symmetric polynomials, $G_\alpha(x;q,t)$, indexed by compositions. An explicit formula for the bivariate Knop-Sahi polynomials reveals a connection between these polynomials and q-special functions. In particular, relations among the q-ultraspherical polynomials of Askey and Ismail, the two variable symmetric and non-symmetric Macdonald polynomials, and the bivariate Knop-Sahi polynomials are explicitly determined using the theory of basic hypergeometric series.

RÉSUMÉ: Knop et Sahi ont introduit une famille de polynômes non-homogènes et non-symétriques, $G_\alpha(x;q,t)$, indexés par des compositions. L'obtention d'une formule explicite pour les polynômes de Knop-Sahi en deux variables révèle une connexion entre ces polynômes et les q-fonctions spéciales. En particulier, des relations entre les polynômes q-ultrasphériques de Askey et Ismail, les polynômes en deux variables de Macdonald non-symétriques et symétriques, et les polynômes en deux variables de Knop-Sahi sont déterminées en utilisant la théorie des fonctions hypergéométriques.

Keywords: *Knop-Sahi polynomials, q-ultraspherical polynomials, Macdonald polynomials.*

1. Introduction

Important developments in the theory of symmetric functions rely on the use of the Macdonald polynomial basis, $\{P_\lambda(x_1, x_2, \ldots, x_n; q, t)\}_\lambda$. This basis for the symmetric function space specializes to several fundamental bases including the Schur, Hall-Littlewood, Zonal, and Jack. The Macdonald polynomials are eigenfunctions of a family of commuting difference operators with significance in many-body physics as they appear in the wave function of a system of relativistic particles on a circle [11]. It has also been conjectured that these polynomials occur naturally in representation theory of the symmetric group [3].

Difficulty encountered in the study of the Macdonald basis stems in part from the absence of explicit formulas for these polynomials in terms of a familiar basis. A major breakthrough in the study of Macdonald polynomials occurred when Knop [5],[6] and Sahi

[12] simultaneously discovered a family of non-symmetric, non-homogeneous polynomials, $G_\alpha(x_1, x_2, \ldots, x_n; q, t)$, whose top homogeneous components yield the non-symmetric version of the Macdonald polynomials, $E_\alpha(x_1, x_2, \ldots, x_n; q, t)$ [2],[7]. These non-symmetric polynomials are then related to the Macdonald polynomials, $P_\lambda(x_1, x_2, \ldots, x_n; q, t)$, by a Hecke algebra symmetrization [7].

The Knop-Sahi polynomials were originally characterized by elementary vanishing properties which enabled the verification of nontrivial properties of the Macdonald polynomials. The characterization also gave recursive relations allowing the polynomials to be constructed rather simply. Particular n-variable Knop-Sahi polynomials have been determined explicitly as well as the complete solution for the bivariate case [9],[10]. The two variable formula for the Knop-Sahi polynomials can be used to recover the bivariate Macdonald polynomials P_λ and to yield explicit expansions for the two variable case of the Macdonald polynomials E_α. We will establish the relation of the Knop-Sahi polynomials in two variables with the Askey-Ismail [1] q-ultraspherical polynomials and the bivariate symmetric and non-symmetric Macdonald polynomials.

The following notation will be of use in the presentation of our results; a partition, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, denotes a weakly increasing sequence of non-negative integers, while a composition is any vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with non-negative integral components.

2. Monomial expansion for Knop-Sahi polynomials

A precise relation among q-special functions and the bivariate Knop-Sahi polynomials requires that we first determine an explicit monomial expansion for these polynomials. As mentioned, the Knop-Sahi polynomials are indexed by compositions, α. An explicit formula for the two variable Knop-Sahi polynomials in the case $\alpha = (n, 0)$ is stated in [9] as follows:

$$G_{(n,0)}(x, y; q, t) = \frac{(-1)^n q^{\binom{n+1}{2}}}{(t;q)_{n+1}} \sum_{0 \le j+k \le n} \frac{(t;q)_{n-k}(t;q)_{n+1-j}(x;q)_k(y;q)_j}{t^{-(k+j)} q^{-j(k+1)} (q;q)_k (q;q)_j (q;q)_{n-k-j}} \tag{1}$$

where

$$(a;q)_N \quad = \quad (1-a)(1-aq)\cdots(1-aq^{N-1}) \,. \tag{2}$$

This explicit expression for $G_{(n,0)}(x, y; q, t)$ can be algebraically manipulated into a monomial expansion for the two variable Knop-Sahi polynomials using the theory of basic hypergeometric series.

Theorem 1.

$$G_{(n,0)}(x, y; q, t) = \sum_{0 \le a+b \le n} \frac{(-t)^{a+b} q^{1/2((a+b)^2 - a + b)} (t^2;q)_{n+1} (t;q)_{a+1} (t;q)_b}{(t^2;q)_{a+b+1} (q;q)_{n-a-b} (q;q)_a (q;q)_b} x^a y^b \tag{3}$$

Proof The task of manipulating expression (1) into (3) begins by denoting the right hand side of (1) by K_n, and using the q-binomial expansion to convert the q-shifted factorial basis into monomials.

$$K_n = \sum_{0 \le j+k \le n} \frac{t^{k+j} q^{j(k+1)} (t;q)_{n-k}\,(t;q)_{n+1-j}}{(q;q)_k\,(q;q)_j\,(q;q)_{n-k-j}} \sum_{a=0}^{k} \frac{(q^{-k};q)_a\,(x\,q^k)^a}{(q;q)_a} \sum_{b=0}^{j} \frac{(q^{-j};q)_b\,(y\,q^j)^b}{(q;q)_b} \tag{4}$$

Two properties of q-shifted factorials,

$$(a;q)_{n-k} = \frac{(a;q)_n}{(q^{1-n}/a;q)_k} \left(-\frac{q}{a}\right)^k q^{\binom{k}{2}-nk} \tag{5}$$

and

$$(a;q)_{n+k} = (a;q)_n\,(aq^n;q)_k\ , \tag{6}$$

with the change of variables, $k \leftrightarrow k+a$ and $j \leftrightarrow j+b$, are necessary to transform expression (4) into the following form:

$$K_n = \sum_{\substack{0 \le j+k+a+b \le n \\ 0 \le k\ \&\ 0 \le j \\ 0 \le a\ \&\ 0 \le b}} \left(\frac{q^{\binom{a}{2}+\binom{b}{2}+b+ab+j+k}\ (t;q)_{n-a}\ (t;q)_{n+1-b}}{(-t)^{-a-b}\ (q;q)_{n-a-b}\ (q;q)_a\ (q;q)_b\ (q;q)_j} \times \frac{(q^{-n+a+b};q)_j (q^{j-n+a+b};q)_k\, x^a\, y^b}{(q^{b-n}/t;q)_j (q;q)_k (q^{1-n+a}/t;q)_k} \right) . \tag{7}$$

We may remove the restriction $j+k+a+b \le n$ since the term $(q^{-n+a+b+j};q)_k$ will vanish if $k > n-a-b-j$.

Adhering to the notation of [4], where

$${}_2\phi_1(a,b;c;q,z) = \sum_{i=0}^{\infty} \frac{(a;q)_i (b;q)_i}{(c;q)_i (q;q)_i} z^i\ ,$$

we use a particular case of the summation identity [4, (1.5.3), App.(II.6)]

$${}_2\phi_1(a,q^{-N};c;q,q) = \frac{(c/a;q)_N}{(c;q)_N}\, a^N\ . \tag{8}$$

The sum over k in expression (7) is eliminated by application of the case $a \to 0$, i.e.,

$${}_2\phi_1(0,q^{-N};c;q,q) = \frac{(-1)^N\, c^N\, q^{\binom{N}{2}}}{(c;q)_N}\ , \tag{9}$$

with $N = n-a-b-j$ and $c = q^{1-n+a}/t$.

$$K_n = \sum_{\substack{0\le a \,\&\, 0\le b\\ 0\le j}} \left(\frac{t^{2a+2b-n+j}\, q^{-\binom{n}{2}+a(n+b-1)+b(b+j)+\binom{j+1}{2}}\, (t;q)_{n-a}}{(q^{b-n}/t;q)_j\,(q^{1-n+a}/t;q)_{n-a-j-b}} \times \frac{(t;q)_{n+1-b}\,(q^{-n+a+b};q)_j\; x^a\, y^b}{(-1)^{n-j}\,(q;q)_{n-a-b}(q;q)_a\,(q;q)_b\,(q;q)_j} \right). \tag{10}$$

Transformation of this expression using another property of q-shifted factorials,

$$(aq^{-n};q)_{n-k} = \frac{(q/a;q)_n}{(q/a;q)_k} \left(\frac{a}{q}\right)^{n-k} q^{\binom{k}{2}-\binom{n}{2}}, \tag{11}$$

allows further that the sum over j be eliminated by applying summation identity (8) again. We finally obtain an expression that is easily seen to be the right hand side of expression (3),

$$K_n = \sum_{\substack{0\le a\\ 0\le b}} \frac{(-t)^{a+b} q^{(a^2+2ab+b^2+b-a)/2}\,(t^2;q)_{n+1}(t;q)_a\,(t;q)_b}{(q;q)_{n-a-b}(q;q)_a\,(q;q)_b\,(t^2;q)_{a+b+1}}\, x^a\, y^b, \tag{12}$$

completing the proof of Theorem 1. □

3. Knop-Sahi polynomials related to q-special functions

The monomial expansion of the Knop-Sahi polynomials enables us to give their relation to the q-ultraspherical polynomials and the non-symmetric and symmetric Macdonald polynomials. Askey and Ismail introduced in [1] a generalization of ultraspherical polynomials called q-ultraspherical polynomials. The explicit representation of these polynomials is

$$C_n(\cos\theta;\beta|q) = \sum_{k=0}^{n} \frac{(\beta;q)_k\,(\beta;q)_{n-k}}{(q;q)_k\,(q;q)_{n-k}}\, e^{i(n-2k)\theta}. \tag{13}$$

Macdonald gives [8] an explicit formula for the polynomials P_λ when λ is a one part partition.

$$P_n(x,y;q,t) = \frac{(q;q)_n}{(t;q)_n} \sum_{k=0}^{n} \frac{(t;q)_k\,(t;q)_{n-k}}{(q;q)_k\,(q;q)_{n-k}}\, x^{n-k}\, y^k \tag{14}$$

A preliminary result relating the Macdonald polynomials to the q-ultraspherical polynomials will be relevant in our efforts to provide their correlation with the bivariate Knop-Sahi polynomials. It seems that this result could be a known result, but for lack of a reference, we will include a sentence for verification.

Theorem 2. *Defining v_1 and v_2 to be*

$$v_1 = xy\,, \qquad\qquad v_2 = \frac{x+y}{2\sqrt{xy}}\,, \tag{15}$$

we have

$$P_n(x,y;q,t) \;=\; \frac{(q;q)_n}{(t;q)_n}\,(v_1)^{n/2}\,C_n(v_2;t\,|\,q)\,. \tag{16}$$

Proof The decomposition of these Macdonald polynomials in terms of the q-ultraspherical polynomials is facilitated by a change of variables made in the q-ultraspherical polynomials, as defined in (13);

$$C_n\left(\frac{1+\tan\theta}{2\sqrt{\tan\theta}};\beta|q\right) \;=\; \sum_{k=0}^{n}\frac{(\beta;q)_k\,(\beta;q)_{n-k}}{(q;q)_k\,(q;q)_{n-k}}\,(\tan\theta)^{n/2-k} \tag{17}$$

(this expression can be verified by simple manipulation and the q-binomial formula). The substitution of $\tan\theta = x/y$ and $\beta = t$ in equation (17) then gives that the right hand side of (16) is exactly the Macdonald polynomial as given in (14). □

The Knop-Sahi polynomials in two variables may be expressed as a difference of symmetric Macdonald polynomials indexed by one part partitions. The basic identity is stated as follows:

Theorem 3. *If $c_{n,r}$ denotes the coefficient,*

$$c_{n,r} \;=\; \frac{(-t)^r\,q^{\binom{r}{2}}\,(t^2;q)_{n+1}}{(t^2;q)_{r+1}\,(q;q)_{n-r}}\,, \tag{18}$$

we have

$$G_{(n,0)}(x,y;q,t) \;=\; \sum_{r=0}^{n}\frac{(t;q)_r}{(q;q)_r}\,c_{n,r}\,\big(P_r(x,qy;q,t) \;-\; t\,q^r\,P_r(x,y;q,t)\big)\;. \tag{19}$$

Proof The proof of this relation requires that we use the monomial expansion for the bivariate Knop-Sahi polynomials given by Theorem 1. Such an expansion allows us to express $G_{(n,0)}(x,y;q,t)$ as a sum of homogeneous components in the following manner;

$$G_{(n,0)}(x,y;q,t) \;=\; \sum_{r=0}^{n}\left(\sum_{l=0}^{r}\frac{(-t)^r\,q^{\binom{r}{2}+l}\,(t^2;q)_{n+1}(t;q)_{r+1-l}\,(t;q)_l}{(t^2;q)_{r+1}\,(q;q)_{n-r}(q;q)_{r-l}\,(q;q)_l}\,x^{r-l}\,y^l\right)\;. \tag{20}$$

We then obtain the following identity as a result of factoring out the $(r+1-l)$th term from $(t;q)_{r+1-l}$:

$$G_{(n,0)}(x,y;q,t) = \sum_{r=0}^{n} \left(\sum_{l=0}^{r} \frac{(-t)^r\, q^{\binom{r}{2}+l}\,(t^2;q)_{n+1}\,(t;q)_{r-l}\,(t;q)_l}{(t^2;q)_{r+1}\,(q;q)_{n-r}(q;q)_{r-l}\,(q;q)_l}\,(1-t\,q^{r-l})\,x^{r-l}\,y^l \right)$$

$$\begin{aligned} &= \sum_{r=0}^{n} \frac{(-t)^r\, q^{\binom{r}{2}}\,(t^2;q)_{n+1}}{(t^2;q)_{r+1}\,(q;q)_{n-r}} \qquad (21)\\ &\quad \times \left(\sum_{l=0}^{r} \frac{(t;q)_{r-l}(t;q)_l}{(q;q)_{r-l}(q;q)_l} x^{r-l}(qy)^l - tq^{r-l} \sum_{l=0}^{r} \frac{(t;q)_{r-l}(t;q)_l}{(q;q)_{r-l}(q;q)_l} x^{r-l} y^l \right). \end{aligned}$$

A difference of the Macdonald polynomials as given in (14) appears in the right hand side of this expression. I.e., we now have that the two variable Knop-Sahi polynomials can be expressed in terms of symmetric Macdonald polynomials as

$$G_{(n,0)}(x,y;q,t) = \sum_{r=0}^{n} \frac{(-t)^r\, q^{\binom{r}{2}}\,(t^2;q)_{n+1}\,(t;q)_r}{(t^2;q)_{r+1}\,(q;q)_{n-r}\,(q;q)_r} \left(P_r(x,qy;q,t) - t\,q^{r-l} P_r(x,y;q,t)\right), \qquad (22)$$

precisely as stated in Theorem 3. $\square$

We may now provide the final relations among the the bivariate Knop-Sahi polynomials, the non-symmetric Macdonald polynomials, and the q-ultraspherical polynomials.

Theorem 4. *With $c_{n,r}$ as defined in (18), we have*

$$G_{(n,0)}(x,y;q,t) \;=\; \sum_{r=0}^{n} c_{n,r} \left((u_1)^{r/2}\, C_r\,(u_2;t|q) - t\,q^r\,(v_1)^{r/2}\, C_r\,(v_2;t|q) \right) \qquad (23)$$

$$\textit{where} \qquad u_1 = qxy\,, \qquad u_2 = \frac{x+qy}{2\sqrt{qxy}}\,. \qquad (24)$$

$$\textit{and} \qquad v_1 = xy\,, \qquad v_2 = \frac{x+y}{2\sqrt{xy}}\,. \qquad (25)$$

A similar expansion for the non-symmetric Macdonald polynomials is revealed simultaneously since these polynomials are merely the top homogeneous components of the Knop-Sahi polynomials; i.e., the formula for the non-symmetric Macdonald polynomials in terms of q-ultraspherical polynomials is obtained simply by letting $r=n$ in Theorem 4.

Corollary 1. *With u_1, u_2 as defined in (24) and v_1, v_2 as in (25), we have*

$$E_{(n,0)}(x,y;q,t) = (-t)^n\, q^{\binom{n}{2}} \left((u_1)^{n/2}\, C_n\,(u_2;t|q) \;-\; t\,q^n\,(v_1)^{n/2}\, C_n\,(v_2;t|q) \right). \qquad (26)$$

Proof of Theorem 4 The determination of the relationship between the bivariate Knop-Sahi polynomials and the q-ultraspherical polynomials can be achieved by using Theorem 3;

$$G_{(n,0)}(x,y;q,t) \;=\; \sum_{r=0}^{n} \frac{(t;q)_r}{(q;q)_r}\, c_{n,r}\left(P_r(x,qy;q,t) \;-\; t\,q^r P_r(x,y;q,t)\right)\;, \tag{27}$$

and substituting the Macdonald polynomials as defined by Theorem 2 into the right hand side of this expression. □

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