TABLEAUX STATISTICS FOR TWO PART MACDONALD POLYNOMIALS

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ABSTRACT. The Macdonald polynomials expanded in terms of a modified Schur function basis have coefficients called the q,t-Kostka polynomials. We define operators to build standard tableaux and show that they are equivalent to creation operators that recursively build the Macdonald polynomials indexed by two part partitions. We uncover a new basis for these particular Macdonald polynomials and in doing so are able to give an explicit description of their associated q,t-Kostka coefficients by assigning a statistic in q and t to each standard tableau.

1. INTRODUCTION

The Macdonald polynomials, $J_\lambda[X;q,t]$, are a two parameter family of polynomials in N variables, forming a basis for the space of symmetric functions. The polynomials, expanded in a modified Schur function basis $\{S_\lambda[X^t]\}_\lambda$, have coefficients called the q,t-Kostka polynomials, $K_{\lambda\mu}(q,t)$. Macdonald conjectured that $K_{\lambda\mu}(q,t)$ is a polynomial in q and t with positive integer coefficients. This conjecture has been proven [3] based on the representation-theoretic interpretation [2] of the q,t-Kostka polynomials. However, the proof does not reveal a combinatorial interpretation for these coefficients. In this article, we consider the case that $J_\lambda[X;q,t]$ is indexed by partitions with no more than 2 parts. We uncover a new basis for the these particular Macdonald polynomials and in doing so are able to give an explicit description of their associated q,t-Kostka coefficients by assigning a statistic in q and t to each standard tableau.

Our results appeared on the web a few years ago [1]. The recent generalization [6, 7] of the methods introduced here have prompted us to submit this article for publication. Prior to this work, a rigged configuration interpretation for the coefficients associated to Macdonald polynomials indexed by partition with no more than 2 parts was given in [1]. In addition, similar tableaux statistics in the case of partitions whose first part is not larger than 2 were discovered [13] at the same time as ours.

This paper can be seen as a study of the case $k=2$ of the k-Schur functions, $s_\lambda^{(k)}[X;t]$, associated to a filtration of the symmetric function space introduced in [7]. There is strong computational evidence to support conjectures asserting that the k-Schur functions obey k-generalizations of many Schur function positivity properties. The results of this paper prove, for $k=2$, that the k-Schur functions expand positively in terms of Schur functions and that the Macdonald polynomials indexed by partitions whose first part does not exceed 2 can be expanded positively

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in terms of k-Schur functions. Conjectures generalizing this phenomenon are given in [7].

The connection between the work in this paper and that of the k-Schur functions in [7] can be made using the following notational correspondance:

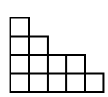
$$\begin{aligned} F_t^{-1}\, H_\lambda[X;q,t] &\longleftrightarrow q^{n(\lambda')} H_{\lambda'}[X;1/q] \\ F_t^{-1} B_2^{(0)}\, F_t &\longleftrightarrow B_2\, t^{-D}\Big|_{t\to 1/q} \\ F_t^{-1} B_2^{(1)}\, F_t &\longleftrightarrow B_{1,1}\, t^{-2D-1}\Big|_{t\to 1/q} \\ F_t^{-1}\, U^{(\epsilon)}_{0^a,1^b} &\longleftrightarrow q^{n(\lambda)} s^{(2)}_\lambda[X;1/q]\,, \end{aligned} \tag{1.1}$$

where on an arbitrary symmetric function $P[X]$, we define $F_t\, P[X] = P[X(1-t)]$ and $F_t^{-1}\, P[X] = P[X/(1-t)]$, and where λ is the partition $\lambda = (2^a, 1^{2b}, \epsilon)$. Note that from Property 5, $F_t^{-1}\, U_v^{(\epsilon)}$ is, for any v, equal up to a constant to a 2-Schur function. Also, letting $t \to 1/q$ has the effect of transforming the 2-Schur functions from a basis of the linear span of Macdonald polynomials indexed by partitions whose first part is not larger than 2, to a basis of the linear span of Macdonald polynomials indexed by partitions whith no more than 2 parts.

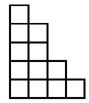
The paper is divided as follows: section two covers basic definitions used in symmetric function theory. We present in the third section, the creation operator $B_2 = tB_2^{(0)} + B_2^{(1)} q^{-D-1}$, showing several properties that include the action of $B_2^{(0)}$ on the Hall-Littlewood polynomials $H_\lambda[X;q,t]$ and an expansion of $J_\lambda[X;q,t]$ in terms of products of $B_2^{(0)}$ and $B_2^{(1)}$ for $\ell(\lambda) \le 2$. Further, a new basis for these Macdonald polynomials having coefficients in the parameters q and t with positive integer coefficients is uncovered. The fourth section begins with basic definitions used in tableaux theory and then introduces operators on tableaux which correspond, under a morphism $\mathcal{F}$, to the operators of section 3. Finally, a statistic on standard tableaux is presented in the fifth section.

2. Definitions

Partitions are sequences of integers $\lambda = (\lambda_1, \lambda_2, \dots)$ such that $\lambda_1 \ge \lambda_2 \ge \dots > 0$. The order of λ is $|\lambda| = \lambda_1 + \lambda_2 + \dots$, the number of non-zero parts in λ is denoted $\ell(\lambda)$ and $n(\lambda)$ refers to $\sum_i (i-1)\lambda_i$. The *dominance order* on partitions is defined for two partitions with $|\lambda| = |\mu|$, by $\lambda \le \mu$ when $\lambda_1 + \dots + \lambda_i \le \mu_1 + \dots + \mu_i$ for all i. A partition λ may be associated to a Young diagram with λ_i lattice squares in the i^{th} row, from the bottom to top. The Young diagram of $\lambda = (5,4,2,1)$ is


. (2.1)

For each square $s = (i,j)$ in the diagram of λ, let $\ell'(s), \ell(s), a(s)$ and $a'(s)$ be respectively the number of squares in the diagram of λ to the south, north, east and west of the square s. The transposition of a Young diagram associated to λ with respect to the main diagonal gives the conjugate partition λ'. For example, the conjugate of (2.1) is


, (2.2)

which gives $\lambda' = (4,3,2,2,1)$.

We shall use λ-rings, needing only the formal ring of symmetric functions $\mathcal{S}ym$ to act on the ring of rational functions in $x_1,\ldots,x_N,q,t$, with coefficients in $\mathbb{R}$. The ring $\mathcal{S}ym$ is generated by power sums Ψ_i, $i=1,2,3\ldots$. The action of Ψ_i on a rational function $\sum_\alpha c_\alpha u_\alpha / \sum_\beta d_\beta v_\beta$ is by definition

$$\Psi_i\left[\frac{\sum_\alpha c_\alpha u_\alpha}{\sum_\beta d_\beta v_\beta}\right] = \frac{\sum_\alpha c_\alpha u_\alpha^i}{\sum_\beta d_\beta v_\beta^i}, \tag{2.3}$$

with $c_\alpha, d_\beta \in \mathbb{R}$ and u_α, v_β monomials in $x_1,\ldots,x_N,q,t$. Since any symmetric function is uniquely expressed in terms of the power sums, (2.3) extends to an action of $\mathcal{S}ym$ on rational functions. In particular, a symmetric function $f(x_1,\ldots,x_N)$ can be denoted $f[x_1+\cdots+x_N]$. We shall use the elements $X := x_1+\cdots+x_N$, $X^{tq} := X(t-1)/(q-1)$ and $X^t := X(t-1)$.

The Macdonald polynomials can now be defined using a scalar product $\langle\ ,\ \rangle_{q,t}$ defined on $\mathcal{S}ym \otimes \mathbb{Q}[q,t]$ by

$$\langle \Psi_\lambda[X], \Psi_\mu[X]\rangle_{q,t} = \delta_{\lambda\mu} z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}, \tag{2.4}$$

where we associate to a parition λ with $m_i(\lambda)$ parts equal to i the number

$$z_\lambda = 1^{m_1} m_1!\, 2^{m_2} m_2! \cdots \tag{2.5}$$

Macdonald polynomials $J_\lambda[X;q,t]$ are thus uniquely specified [11] by

$$\text{(i)}\ \langle J_\lambda, J_\mu\rangle_{q,t} = 0, \qquad \text{if } \lambda \neq \mu, \tag{2.6}$$

$$\text{(ii)}\ J_\lambda[X;q,t] = \sum_{\mu\leq\lambda} v_{\lambda\mu}(q,t) S_\mu[X], \tag{2.7}$$

$$\text{(iii)}\ v_{\lambda\lambda}(q,t) = \prod_{s\in\lambda}(1-q^{a(s)}t^{\ell(s)+1}), \tag{2.8}$$

where $S_\mu[X]$ is the usual Schur function and $v_{\lambda\mu}(q,t) \in \mathbb{Q}[q,t]$.

The expansion coefficients of the Macdonald polynomials when expanded in terms of the basis $\{S_\mu[X^t]\}_\mu$:

$$J_\lambda[X;q,t] = \sum_\mu K_{\mu\lambda}(q,t) S_\mu[X^t]\,, \tag{2.9}$$

are studied here in the case that $\ell(\lambda) \leq 2$.

3. Algebraic Side

A Macdonald polynomial indexed by any partition can be constructed by repeated application of creation operators B_k defined [4, 8] such that

$$B_k\, J_\lambda[X;q,t] = J_{\lambda+1^k}[X;q,t]\,, \qquad \ell(\lambda) \leq k\,, \tag{3.1}$$

or more specifically, when $k=2$,

$$B_2\, J_{m,n}[X;q,t] = J_{m+1,n+1}[X;q,t]\,. \tag{3.2}$$

If $\mathcal{V}$ is the $\mathbb{Q}[q,t]$-linear span of $\{J_\lambda[X;q,t]\}_{\ell(\lambda)\leq 2}$, because it is known [11] that

$$J_\lambda[X;q,t] = \sum_{\mu\geq\lambda} c_{\mu\lambda}(q,t)\, S_\mu[X^{tq}]\,, \tag{3.3}$$

for some $c_{\mu\lambda}(q,t)\in\mathbb{Q}[q,t]$, $\mathcal{V}$ must also be the $\mathbb{Q}[q,t]$-linear span of $\{S_\lambda[X^{tq}]\}_{\ell(\lambda)\leq 2}$. The action of B_2 on $\mathcal{V}$ can thus be defined by its action on $\{S_\lambda[X^{tq}]\}_{\ell(\lambda)\leq 2}$, introduced in [5] as

$$B_2\, S_{m,n}[X^{tq}] = \det \begin{vmatrix} (1-tq^{m+1})S_{m+1}[X^{tq}] & (1-q^{m+2})S_{m+2}[X^{tq}] \\ (1-tq^{n})S_{n}[X^{tq}] & (1-q^{n+1})S_{n+1}[X^{tq}] \end{vmatrix}\,. \tag{3.4}$$

It will be convenient to split the operator B_2 into a sum of two operators,

$$B_2 = tB_2^{(0)} + B_2^{(1)} q^{-D-1}, \tag{3.5}$$

where D is the operator such that $Df[X] = d\,f[X]$ on any homogeneous function of degree d and where $B_2^{(0)}$ and $B_2^{(1)}$ are defined on $\mathcal{V}$ by

$$B_2^{(0)}\, S_{m,n}[X^{tq}] = \det \begin{vmatrix} -q^{m+1}S_{m+1}[X^{tq}] & (1-q^{m+2})S_{m+2}[X^{tq}] \\ -q^{n}S_{n}[X^{tq}] & (1-q^{n+1})S_{n+1}[X^{tq}] \end{vmatrix}\,, \tag{3.6}$$

and

$$B_2^{(1)}\, S_{m,n}[X^{tq}] = q^{m+n+1}\det \begin{vmatrix} S_{m+1}[X^{tq}] & (1-q^{m+2})S_{m+2}[X^{tq}] \\ S_{n}[X^{tq}] & (1-q^{n+1})S_{n+1}[X^{tq}] \end{vmatrix}\,. \tag{3.7}$$

These expressions, obtained by expanding (3.4), provide that (3.5) holds on $\mathcal{V}$.

We now introduce a deformation of the Hall-Littlewood polynomials:

$$H_\lambda[X;q,t] = \sum_{\mu} q^{n(\lambda')} K_{\mu\lambda}(1/q,0) S_{\mu'}[X^t]\,, \tag{3.8}$$

which is a basis for the ring of symmetric functions since the $H_\lambda[X;q,t]$'s are linearly independent; i.e., the $K_{\mu\lambda}(1/q,0)$ matrix is triangular with respect to the partial ordering and $K_{\lambda\lambda}(1/q,0)=1$. In fact, these polynomials are specializations of the Macdonald polynomials. More precisely, let $J_\lambda[X;q,t]^{\{t\}}$ (resp $J_\lambda[X;q,t]^{\{tq\}}$) denote the $\{S_\lambda[X^t]\}$−expansion (resp $\{S_\lambda[X^{tq}]\}$−expansion) of the Macdonald polynomial, $J_\lambda[X;q,t]$. Then

Proposition 1. *$H_\lambda[X;q,t]$ is obtained by taking the coefficient of the maximal t-power in $J_\lambda[X;q,t]^{\{t\}}$ (or $J_\lambda[X;q,t]^{\{tq\}}$). Equivalently,*

$$\begin{aligned} H_\lambda[X;q,t] &= J_\lambda[X;q,t]^{\{t\}}\Big|_{t^{n(\lambda)}} := \sum_{\mu} K_{\mu\lambda}(q,t)\Big|_{t^{n(\lambda)}} S_\mu[X^t] \\ &= J_\lambda[X;q,t]^{\{tq\}}\Big|_{t^{n(\lambda)}} := \sum_{\mu\geq\lambda} c_{\mu\lambda}(q,t)\Big|_{t^{n(\lambda)}} S_\mu[X^{tq}]\,. \end{aligned} \tag{3.9}$$

Proof. Given the first identity, the second follows from (3.3) and the relation:

$$S_\lambda[X^t] = \sum_{\mu} \bar{v}_{\mu\lambda}(q) S_\mu[X^{tq}]\,, \tag{3.10}$$

for some $\bar{v}_{\mu\lambda}(q)\in\mathbb{Q}[q]$. Thus, it suffices to prove the first identity in (3.9). We have $K_{\mu\lambda}(q,t) = q^{n(\lambda')}t^{n(\lambda)}K_{\mu'\lambda}(1/q,1/t)$ [11], which gives

$$K_{\mu\lambda}(q,t)\Big|_{t^{n(\lambda)}} = q^{n(\lambda')} K_{\mu'\lambda}(1/q,1/t)\Big|_{t^0}\,. \tag{3.11}$$

Since $K_{\mu'\lambda}(1/q,1/t)$ is a polynomial in $1/q,1/t$ (eg. [4],[8]), (3.11) can be rewritten

$$K_{\mu\lambda}(q,t)\Big|_{t^{n(\lambda)}} = q^{n(\lambda')}K_{\mu'\lambda}(1/q,0)\,. \tag{3.12}$$

Therefore, the first identity in (3.9) is equivalent to (3.8). □

Substituting the known [11] relation, $J_m[X;q,t]=(q;q)_m S_m[X^{tq}]$, where

$$(q;q)_m = (1-q)(1-q^2)\cdots(1-q^m)\,,\quad m>0\,;\qquad (q;q)_0=1, \tag{3.13}$$

into $H_m[X;q,t]=J_m[X;q,t]^{\{tq\}}\Big|_{t^0}$, that is into Proposition 1 in the case $\lambda=(m)$, yields

Corollary 2.

$$H_m[X;q,t]=J_m[X;q,t]=(q;q)_m S_m[X^{tq}]\,. \tag{3.14}$$

With H_λ now characterized as a specialization of J_λ, we have

Lemma 3. *$\mathcal{V}$ is the $\mathbb{Q}[q,t]$-linear span of $\left\{H_\lambda[X;q,t]\right\}_{\ell(\lambda)\le 2}$.*

Proof. (3.9) gives that the $\mathbb{Q}[q,t]$-linear span of $\left\{H_\lambda[X;q,t]\right\}_{\ell(\lambda)\le 2}$ is included in the $\mathbb{Q}[q,t]$-linear span of $\left\{S_\lambda[X^{tq}]\right\}_{\ell(\lambda)\le 2}$ which is equal to $\mathcal{V}$. The lemma then follows because $\left\{H_\lambda[X;q,t]\right\}_{\ell(\lambda)\le 2}$ is a linearly independent set. □

We now establish several properties of the operators $B_2^{(0)}$ and $B_2^{(1)}$ that will later enable us to associate them to tableaux operators. First, by Lemma 3, we define the action of $B_2^{(0)}$ on $\mathcal{V}$ by finding its action on $H_{m,n}[X;q,t]$.

Property 4. *The action of $B_2^{(0)}$ on $H_{m,n}[X;q,t]$ is given by*

$$B_2^{(0)}H_{m,n}[X;q,t]=H_{m+1,n+1}[X;q,t]\,. \tag{3.15}$$

Proof. Definitions (3.6) and (3.7) show that the action of $B_2^{(0)}$ and $B_2^{(1)}$ on $S_{m,n}[X^{tq}]$ gives coefficients involving only the parameter q when expanded in terms of the $\left\{S_\lambda[X^{qt}]\right\}_{\ell(\lambda)\le 2}$ basis. This implies that the successive action of $B_2^{(0)}$ and $B_2^{(1)}$ on $J_m[X;q,t]=(q;q)_m S_m[X^{tq}]$ produces coefficients involving only q when expanded in the $\left\{S_\lambda[X^{tq}]\right\}_{\ell(\lambda)\le 2}$ basis. Therefore, from

$$J_{m+\ell,\ell}[X;q,t]=\left(B_2\right)^\ell J_m[X;q,t]=\left(tB_2^{(0)}+B_2^{(1)}q^{-D-1}\right)^\ell J_m[X;q,t]\,, \tag{3.16}$$

the coefficient of the maximal t-power in $J_{m+\ell,\ell}[X;q,t]$ is given by

$$J_{m+\ell,\ell}[X;q,t]^{\{tq\}}\Big|_{t^\ell} = \left(B_2^{(0)}\right)^\ell J_m[X;q,t]. \tag{3.17}$$

The assertion follows by replacing left side of this expression with $H_{m+\ell,\ell}[X;q,t]$ by (3.9), and $J_m[X;q,t]=H_m[X;q,t]$. □

Property 5. *On the space $\mathcal{V}$, we have the q-commutation relation*

$$B_2^{(1)}B_2^{(0)}=q\,B_2^{(0)}B_2^{(1)}\,. \tag{3.18}$$

Proof. The action defined in (3.6) and (3.7), and the Pieri rule

$$S_m[X^{tq}]S_n[X^{tq}]=\sum_{\ell=0}^{n}S_{m+n-\ell,\ell}[X^{tq}]\,,\quad \text{where}\quad m\ge n\,, \tag{3.19}$$

yield

$$
\begin{aligned}
B_2^{(0)}B_2^{(1)}S_{m,n}[X^{tq}] &= q^{n+m+2}(q^{m+1}-q^n)\sum_{r=0}^{n}\sum_{\ell=0}^{r}(q^r-q^{m+n+3-r})S_{m+n+4-\ell,\ell}[X^{tq}] \\
&+ q^{2m+2n+5-\ell}(q^{m+1}-q^n)\sum_{\ell=0}^{n}(q^{\ell+1}-1)S_{m+n+3-\ell,\ell+1}[X^{tq}] \\
&+ q^{n+m+1}(1-q^{n+1})(q^{n+1}-q^{m+2})\sum_{\ell=0}^{n+1}S_{m+n+4-\ell,\ell}[X^{tq}] \\
&+ q^{2m+n+3}(1-q^{n+1})(q^{n+2}-1)S_{m+2,n+2}[X^{tq}]
\end{aligned}
\tag{3.20}
$$

and

$$
\begin{aligned}
B_2^{(1)}B_2^{(0)}S_{m,n}[X^{tq}] &= q^{n+m+4}(q^n-q^{m+1})\sum_{r=0}^{n}\sum_{\ell=0}^{r}(q^{m+n+3-r}-q^r)S_{m+n+4-\ell,\ell}[X^{tq}] \\
&+ q^{n+m+3}(q^n-q^{m+1})\sum_{\ell=0}^{n}(1-q^{\ell+1})S_{m+n+3-\ell,\ell+1}[X^{tq}] \\
&+ q^{2m+n+5}(q^{n+1}-1)\sum_{\ell=0}^{n+1}(q^{m+2}-q^{n+1})S_{m+n+4-\ell,\ell}[X^{tq}] \\
&+ q^{2m+n+4}(1-q^{n+1})(q^{n+2}-1)S_{m+2,n+2}[X^{tq}]\,.
\end{aligned}
\tag{3.21}
$$

First exchange the order of summation in the right side of the top line in (3.20) and (3.21), and then use

$$\sum_{r=\ell}^{n}(q^r-q^{m+n+3-r}) = \frac{(1-q^{n-\ell+1})(q^\ell-q^{m+3})}{(1-q)}. \tag{3.22}$$

Next, send $\ell\to\ell-1$ in the right side of the second terms in (3.20) and (3.21). By fixing ℓ, we have that $qB_2^{(0)}B_2^{(1)}S_{m,n}[X^{tq}] = B_2^{(0)}B_2^{(1)}S_{m,n}[X^{tq}]$ if

$$
\begin{aligned}
&\left(\frac{(1-q^{n-\ell+1})(q^\ell-q^{m+3})}{(1-q)} + q^{m+n+4-\ell}(q^\ell-1) + (q^{n+1}-1)\right) \\
&\qquad = \left(q\frac{(1-q^{n-\ell+1})(q^\ell-q^{m+3})}{(1-q)} + (q^\ell-1) + q^{m+3}(q^{n+1}-1)\right),
\end{aligned}
\tag{3.23}
$$

which holds by algebraic manipulation. □

Property 6. *Let $\epsilon\in\{0,1\}$. $B_2^{(0)}$ and $B_2^{(1)}$ are such that*

$$(B_2^{(0)}+B_2^{(1)})^m H_\epsilon[X;q,t] = H_{2m+\epsilon}[X;q,t]\,. \tag{3.24}$$

Proof. Definitions (3.6) and (3.7) give that

$$(B_2^{(0)}+B_2^{(1)})S_k[X^{tq}] = (1-q^{k+1})(1-q^{k+2})S_{k+2}[X^{tq}]\,, \tag{3.25}$$

which, using $H_k[X;q,t]=(q;q)_k S_k[X^{tq}]$, implies that

$$(B_2^{(0)}+B_2^{(1)})H_k[X;q,t] = H_{k+2}[X;q,t]\,. \tag{3.26}$$

We apply this identity m times, starting with $k=\epsilon$, to complete the proof. □

With relation (3.5), this property shows that any Macdonald polynomial indexed by a partition with no more than 2 parts can be built using $B_2^{(0)}$ and $B_2^{(1)}$ since $H_m[X;q,t]=J_m[X;q,t]$.

Definition 7. *Let $v=(v_1,\dots,v_k)$ with all $v_i\in\{0,1\}$. For $\epsilon\in\{0,1\}$ we define*

$$U_v^{(\epsilon)}=B_2^{(v_1)}\cdots B_2^{(v_k)}\cdot H_\epsilon[X;q,t]\,. \tag{3.27}$$

Proposition 8. *For $\epsilon, v_i\in\{0,1\}$ we have*

$$J_{2m+\ell+\epsilon,\ell}[X;q,t]=\sum_{v=(v_1,\dots,v_{m+\ell})} q^{(1-d)|v|_\ell+2n(v)_\ell}t^{\ell-|v|_\ell}U_v^{(\epsilon)}, \tag{3.28}$$

where $d=2m+2\ell+\epsilon$, $|v|_\ell=v_1+\cdots+v_\ell$ and $n(v)_\ell=v_2+2v_3+\cdots+(\ell-1)v_\ell$.

Proof. From Property 6, we have that

$$J_{2m+\epsilon}[X;q,t]=\sum_{v=(v_1,\dots,v_m)} U_v^{(\epsilon)}\,, \tag{3.29}$$

where $v_i\in\{0,1\}$, proving (3.28) for $\ell=0$. Proceeding by induction, we assume that (3.28) holds for every ℓ. We thus have, acting with B_2, that

$$(tB_2^{(0)}+B_2^{(1)}q^{-D-1})J_{2m+\ell+\epsilon,\ell}=\sum_{v'=(0,v)} q^{(1-d)|v'|_{\ell+1}+2(n(v')_{\ell+1}-|v'|_{\ell+1})}t^{\ell+1-|v'|_{\ell+1}}U_{v'}^{(\epsilon)}$$
$$+\sum_{v''=(1,v)} q^{(1-d)(|v''|_{\ell+1}-1)+2(n(v'')_{\ell+1}-|v''|_{\ell+1}+1)-d-1}t^{\ell+1-|v''|_{\ell+1}}U_{v''}^{(\epsilon)}\,. \tag{3.30}$$

Combining the two sums, we obtain

$$B_2J_{2m+\ell+\epsilon,\ell}=\sum_{\bar v=(\bar v_1,\dots,\bar v_{m+\ell+1})} q^{\left(1-(d+2)\right)|\bar v|_{\ell+1}+2n(\bar v)_{\ell+1}}t^{\ell+1-|\bar v|_{\ell+1}}U_{\bar v}^{(\epsilon)}, \tag{3.31}$$

which completes the induction argument since $B_2J_{2m+\ell+\epsilon,\ell}=J_{2m+\ell+1+\epsilon,\ell+1}$. □

Example: We have

$$J_{4,2}[X;q,t]=t^2U_{0,0,0}^{(0)}+t^2U_{0,0,1}^{(0)}+tq^{-3}U_{0,1,0}^{(0)}+tq^{-3}U_{0,1,1}^{(0)}$$
$$+tq^{-5}U_{1,0,0}^{(0)}+tq^{-5}U_{1,0,1}^{(0)}+q^{-8}U_{1,1,0}^{(0)}+q^{-8}U_{1,1,1}^{(0)}\,. \tag{3.32}$$

Corollary 9. *The maximal t-power in* (3.28) *is*

$$H_{2m+\ell+\epsilon,\ell}[X;q,t]=\sum_{\bar v} U_{\bar v}^{(\epsilon)}\,, \tag{3.33}$$

summing over all $\bar v=(0^\ell,v)$ where $v=(v_1,\dots,v_m)$ such that $v_i\in\{0,1\}$.

4. Tableaux Side

4.1. **Definition and background.** Let $\mathcal{A}^*$ be the free monoid generated by the alphabet $\mathcal{A}=\{1,2,3,\dots\}$ and $\mathbb{Q}[\mathcal{A}^*]$ be the free algebra of $\mathcal{A}$. The elements of $\mathcal{A}^*$ are called words. The degree of a word w is denoted $|w|$ and its image in the ring of polynomials $\mathbb{Z}[\mathcal{A}]$ is called the evaluation, denoted $ev(w)$. For example, $w=131332$ has degree 6 and evaluation (2,1,3). A word w of degree n is said to be standard iff $ev(w)=(1,1,\dots,1)$.

A tableau T will be the pair (λ, w), where λ is a partition and w is a word, such that $|\lambda| = |w|$. We say that λ is the shape of T. A Young diagram associated to λ filled with the letters of w from left to right and top to bottom is a planar representation of T. For example, $\mathsf{T} = ((4,2,2,1), 114356234)$ corresponds to

$$\mathsf{T} = \begin{array}{|c|c|c|c|} \hline 1 \\ \cline{1-1} 1 & 4 \\ \cline{1-2} 3 & 5 \\ \hline 6 & 2 & 3 & 4 \\ \hline \end{array} \,. \tag{4.1}$$

A semi-standard tableau is a tableau such that the entries in every row are nondecreasing and such that the entries in every column are increasing. In this case, we do not specify λ in the pair (λ, w), since it can be extracted from w. For instance, $T = 67\ 445\ 11123$ has the representation

$$T = \begin{array}{|c|c|c|c|c|} \hline 6 & 7 \\ \cline{1-3} 4 & 4 & 5 \\ \hline 1 & 1 & 1 & 2 & 3 \\ \hline \end{array} \,. \tag{4.2}$$

Notice that a semi-standard tableau is a tableau. Finally, a standard tableau $\mathcal{T}$ is a semi-standard tableau of evaluation $(1,1,\dots,1)$. For example, $\mathcal{T} = 7\ 46\ 1235$ or

$$\mathcal{T} = \begin{array}{|c|c|c|c|} \hline 7 \\ \cline{1-2} 4 & 6 \\ \hline 1 & 2 & 3 & 5 \\ \hline \end{array} \tag{4.3}$$

is a standard tableau.

As with the Ferrers diagrams, we define T^t to be the transpose of a tableau T. For example, with T as given in (4.1), we have

$$\mathsf{T}^t = \begin{array}{|c|c|c|c|} \hline 4 \\ \cline{1-1} 3 \\ \cline{1-3} 2 & 5 & 4 \\ \hline 6 & 3 & 1 & 1 \\ \hline \end{array} \,. \tag{4.4}$$

Notice that the transpositon of a semi-standard tableau may not be a semi-standard tableau while the transposition of a standard tableau must be a standard tableau.

Words can be associated to numbers called the charge and cocharge where

$$\mathrm{charge}(w) = n(ev(w)_P) - \mathrm{cocharge}(w)\,, \tag{4.5}$$

for $ev(w)_P$ the partition obtained by reordering $ev(w)$. The cocharge of a standard word w is defined by the following algorithm;

1. Label the letter 1 in w by $c_1 = 0$
2. If the letter $i+1$ appears at the left of the letter i in w, then $c_{i+1} = c_i + 1$. Otherwise $c_{i+1} = c_i$.
3. $\mathrm{cocharge}(w) = c_1 + \cdots + c_n$.

For instance, $\mathrm{cocharge}(413265) = 0+0+1+2+2+3 = 8$. Recall that semi-standard tableaux and standard tableaux are simply words, and therefore have an associated charge and cocharge.

Lascoux and Schützenberger defined [9] an action of the symmetric group on $\mathbb{Z}[\mathcal{A}^*]$ that sends a word of evaluation $(ev_1, \dots, ev_i, ev_{i+1}, \dots)$ to a word of evaluation $(ev_1, \dots, ev_{i+1}, ev_i, \dots)$ under an elementary transposition, σ_i. Their action induces the usual action of the symmetric group on $\mathbb{Z}[\mathcal{A}]$. For our purposes, we define this action only on words such that $(ev_i, ev_{i+1}) \in \{(1,2),(2,1)\}$:

$$aab \overset{\sigma_i}{\leftrightarrow} abb, \quad aba \overset{\sigma_i}{\leftrightarrow} bba, \quad baa \overset{\sigma_i}{\leftrightarrow} bab\,, \tag{4.6}$$

where a and b stand for i and $i+1$ respectively. This action sends a semi-standard tableau to a semi-standard tableau while preserving its shape. For example, $\sigma_4(215345) = 215344$ and $\sigma_3 \begin{array}{|c|c|c|} \hline 4 \\ \hline 3 & 5 & 5 \\ \hline 1 & 2 & 3 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline 4 \\ \hline 3 & 5 & 5 \\ \hline 1 & 2 & 4 \\ \hline \end{array}$.

We will use several simple linear operations on words. τ_k is a translation of a to $a+k$ for every letter a in an alphabet. For instance, $\tau_2(231567) = 453789$. The restriction of a word w to the alphabet $a, b, c, \dots,$ is denoted $w_{\{a,b,c,\dots\}}$. i.e., $w_{\{3,4\}} = 3343$ for $w = 12334223$. If w is such that $w_{\{a,b\}} = ab$, $r_{(ab\to cd)}$ sends ab to cd. For example, $r_{(23\to 46)}(121543) = 141546$. We further define an operator R_a to remove all letters a in w, and finally, $A_{n+1,n+1}$ is an operator on a tableau T, that adds a horizontal 2-strip of the boxes $n+1$ in all the possible ways to T. For instance

$$A_{5,5}\ \begin{array}{|c|c}\hline 3 \\ \cline{1-1} 2 \\ \hline 1 & \multicolumn{1}{c|}{4} \\ \hline\end{array} = \begin{array}{|c|c|}\hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 2 & 5 \\ \hline 1 & 4 \\ \hline\end{array} + \begin{array}{|c|c|c|}\hline 5 & \multicolumn{2}{c}{} \\ \cline{1-1} 3 & \multicolumn{2}{c}{} \\ \cline{1-1} 2 & \multicolumn{2}{c}{} \\ \hline 1 & 4 & 5 \\ \hline\end{array} + \begin{array}{|c|c|c|}\hline 3 & \multicolumn{2}{c}{} \\ \hline 2 & 5 & \multicolumn{1}{c}{} \\ \hline 1 & 4 & 5 \\ \hline\end{array} + \begin{array}{|c|c|c|c|}\hline 3 & \multicolumn{3}{c}{} \\ \cline{1-1} 2 & \multicolumn{3}{c}{} \\ \hline 1 & 4 & 5 & 5 \\ \hline\end{array}\,. \tag{4.7}$$

There exists a shape and cocharge preserving standardization [9] of semi-standard tableaux, denoted by VS, that we will use only on semi-standard tableaux with evaluation $(ev_1, \dots, ev_k)$, where $ev_i \in \{1,2\}$. In the case of a semi-standard tableau T, it is defined as

1. If $ev_1 = 2$ then $T \to \tau_1 r_{(11\to 01)} T$. Proceed to step 2.
2. If $ev(T) = (1,1,\dots,1)$ then the standardization is complete. Otherwise, $T \to \sigma_1 \cdots \sigma_{i-1} T$ for the smallest i such that $ev_i = 2$. Proceed to step 1.

For example, $T = 45\ 235\ 124$ undergoes the following standardization process:

$$\begin{array}{|c|c|c|}\hline 4 & 5 & \multicolumn{1}{c}{} \\ \hline 2 & 3 & 5 \\ \hline 1 & 2 & 4 \\ \hline\end{array} \to \begin{array}{|c|c|c|}\hline 4 & 5 & \multicolumn{1}{c}{} \\ \hline 2 & 3 & 5 \\ \hline 1 & 1 & 4 \\ \hline\end{array} \to \begin{array}{|c|c|c|}\hline 5 & 6 & \multicolumn{1}{c}{} \\ \hline 3 & 4 & 6 \\ \hline 1 & 2 & 5 \\ \hline\end{array} \to \begin{array}{|c|c|c|}\hline 5 & 6 & \multicolumn{1}{c}{} \\ \hline 2 & 3 & 6 \\ \hline 1 & 1 & 4 \\ \hline\end{array} \to \begin{array}{|c|c|c|}\hline 6 & 7 & \multicolumn{1}{c}{} \\ \hline 3 & 4 & 7 \\ \hline 1 & 2 & 5 \\ \hline\end{array} \to \begin{array}{|c|c|c|}\hline 5 & 6 & \multicolumn{1}{c}{} \\ \hline 2 & 3 & 7 \\ \hline 1 & 1 & 4 \\ \hline\end{array} \to \begin{array}{|c|c|c|}\hline 6 & 7 & \multicolumn{1}{c}{} \\ \hline 3 & 4 & 8 \\ \hline 1 & 2 & 5 \\ \hline\end{array}\,. \tag{4.8}$$

Note that $VS^{(n)}$ will denote the standardization of only the first n letters of a semi-standard tableau of degree $N \geq n$ which sends such a tableau to a semi-standard tableau of evaluation $(1^n, ev_{n+1}, \dots, ev_N)$.

4.2. **Tableaux operators.** The primary goal of this section is to present tableaux operators that can be related to $B_2^{(0)}$ and $B_2^{(1)}$ using the linear morphism on semi-standard tableaux,

$$F : T \to q^{\text{cocharge}(T)} S_{\text{shape}(T)}[X^t]\,. \tag{4.9}$$

It is known [10] that the Hall-Littlewood polynomials

$$Q_\lambda[X;t] = \sum_\mu K_{\mu\lambda}(0,t) S_\mu[X^t], \tag{4.10}$$

are equivalently expressed as a sum over semi-standard tableaux T such that

$$Q_\lambda[X;t] = \sum_{T;ev(T)=\lambda} t^{\text{charge}(T)} S_{\text{shape}(T)}[X^t]. \tag{4.11}$$

The substitution of $K_{\mu\lambda}(1/q,0) = K_{\mu'\lambda'}(0,1/q)$ [11] in expression (3.8) thus yields

$$\begin{aligned} H_\lambda[X;q,t] &= \sum_\mu q^{n(\lambda')} K_{\mu'\lambda'}(0,1/q) S_{\mu'}[X^t] \\ &= \sum_{T;ev(T)=\lambda'} q^{\text{cocharge}(T)} S_{\text{shape}(T)}[X^t]\,, \end{aligned} \tag{4.12}$$

using $\text{cocharge}(T) = n(ev(T)) - \text{charge}(T)$. Since standardization VS preserves cocharge and shape, we have

$$H_\lambda[X;q,t] = \sum_{T;ev(T)=\lambda'} q^{\text{cocharge}(VS(T))} S_{\text{shape}\left(VS(T)\right)}[X^t]\,. \tag{4.13}$$

This given, we can equivalently express $H_\lambda[X;q,t]$ as a sum of semi-standard tableaux under the action of $\mathcal{F}$. More precisely,

Definition 10. *For all partitions λ where $\ell(\lambda) \le 2$, let*

$$\mathbb{H}_\lambda = \sum_{T;ev(T)=\lambda'} VS(T)\,. \tag{4.14}$$

Therefore, (4.13) gives that

$$\mathcal{F}(\mathbb{H}_\lambda) = H_\lambda[X;q,t]\,. \tag{4.15}$$

In the spirit of section 3, we define two linear operators on standard tableaux.

Definition 11. *On any standard tableau $\mathcal{T}$ such that $|\mathcal{T}| = n$, let*

$$\mathbb{B}_2^{(0)} : \mathcal{T} \to VS(A_{n+1,n+1}\mathcal{T}) = \tau_1 r_{(11\to 01)}\sigma_1\cdots\sigma_n A_{n+1,n+1}\mathcal{T} \tag{4.16}$$

and

$$\mathbb{B}_2^{(1)} : \mathcal{T} \to \left(\mathbb{B}_2^{(0)}\mathcal{T}^t\right)^t\,. \tag{4.17}$$

Example: Given a standard tableau, $\mathcal{T} = \begin{smallmatrix} 3 & & \\ 1 & 2 & 4 \end{smallmatrix}$, we add all possible horizontal 2-strips containing the letter 5 and then standardize:

$$\begin{aligned} \mathbb{B}_2^{(0)}\left(\begin{smallmatrix} 3 & & \\ 1 & 2 & 4 \end{smallmatrix}\right) &= VS\left(\begin{smallmatrix} 5 & & \\ 3 & 5 & \\ 1 & 2 & 4 \end{smallmatrix} + \begin{smallmatrix} 5 & & & \\ 3 & & & \\ 1 & 2 & 4 & 5 \end{smallmatrix} + \begin{smallmatrix} 3 & 5 & 5 \\ 1 & 2 & 4 \end{smallmatrix} + \begin{smallmatrix} 3 & 5 & & \\ 1 & 2 & 4 & 5 \end{smallmatrix} + \begin{smallmatrix} 3 & & & & \\ 1 & 2 & 4 & 5 & 5 \end{smallmatrix}\right) \\ &= \begin{smallmatrix} 5 & & \\ 4 & 6 & \\ 1 & 2 & 3 \end{smallmatrix} + \begin{smallmatrix} 6 & & & \\ 4 & & & \\ 1 & 2 & 3 & 5 \end{smallmatrix} + \begin{smallmatrix} 3 & 4 & 6 \\ 1 & 2 & 5 \end{smallmatrix} + \begin{smallmatrix} 4 & 6 & & \\ 1 & 2 & 3 & 5 \end{smallmatrix} + \begin{smallmatrix} 4 & & & & \\ 1 & 2 & 3 & 5 & 6 \end{smallmatrix}\,. \end{aligned} \tag{4.18}$$

Note that $\mathbb{B}_2^{(0)}$ and $\mathbb{B}_2^{(1)}$ send $\mathcal{T}$ to a sum of standard tableaux.

We now undertake the task of proving that these operators satisfy properties analogous to Properties 4, 5 and 6 as this will imply that sequences of $\mathbb{B}_2^{(0)}$ and $\mathbb{B}_2^{(1)}$ correspond, under $\mathcal{F}$, to similar sequences of $B_2^{(0)}$ and $B_2^{(1)}$ (see Theorem 26). First we prove the analog of Property 4.

Property 12. *With λ a partition such that $\ell(\lambda) \le 2$,*

$$\mathbb{B}_2^{(0)}\mathbb{H}_\lambda = \mathbb{H}_{\lambda+1^2}\,. \tag{4.19}$$

Proof. Since $A_{n+1,n+1}$ commutes with $VS^{(n)}$, using the action of $A_{n+1,n+1}$ on a semi-standard tableau of degree $n = |\lambda|$ and (4.14),

$$A_{n+1,n+1}\sum_{T;ev(T)=\lambda'} VS^{(n)}(T) = \sum_{T;ev(T)=(\lambda',2)} VS^{(n)}(T)\,, \tag{4.20}$$

where λ' is a vector of length n. Thus by the definition of $\mathbb{B}_2^{(0)}$,

$$\mathbb{B}_2^{(0)}\sum_{T;ev(T)=\lambda'} VS^{(n)}(T) = \sum_{T;ev(T)=(\lambda',2)} \tau_1 r_{(11\to 01)}\sigma_1\cdots\sigma_n VS^{(n)}(T)\,. \tag{4.21}$$

and further by the recursion $VS^{(n+2)} = \tau_1 r_{(11\to 01)}\sigma_1\cdots\sigma_n VS^{(n)}$, we have

$$\mathbb{B}_2^{(0)}\sum_{T;ev(T)=\lambda'} VS^{(n)}(T) = \sum_{T;ev(T)=(\lambda',2)} VS^{(n+2)}(T)\,. \tag{4.22}$$

Because the symmetric group action (4.6) preserves charge and shape, for $\beta(\mu)$ any permutation of the vector μ, we have

$$\sum_{T;ev(T)=\mu} VS(T) = \sum_{\bar{T};ev(\bar{T})=\beta(\mu)} VS(\bar{T}), \tag{4.23}$$

which then implies

$$\mathbb{B}_2^{(0)} \sum_{T;ev(T)=\lambda'} VS^{(n)}(T) = \sum_{\bar{T};ev(\bar{T})=(\lambda+1^2)'} VS^{(n+2)}(\bar{T}). \tag{4.24}$$

Property 12 follows now from definition (4.14). □

The analog of Property 6 in the tableaux world is stated:

Property 13. *For $\epsilon \in \{0,1\}$, we have that*

$$\mathbb{H}_{2m+\epsilon} = (\mathbb{B}_2^{(0)} + \mathbb{B}_2^{(1)})^m \, \mathbb{H}_\epsilon . \tag{4.25}$$

Proof. ¿From Definition 10,

$$\mathbb{H}_n = \sum_{T;ev(T)=(1^n)} VS(T) = \sum_{\mathcal{T}} \mathcal{T}, \tag{4.26}$$

and

$$\mathbb{H}_{n+1,1} = \sum_{T;ev(T)=(2,1^n)} VS(T) = \sum_{\mathcal{T}';\mathcal{T}'_{\{1,2\}}=12} \mathcal{T}' \tag{4.27}$$

where $\mathcal{T}$ and $\mathcal{T}'$ are standard tableaux of degree n and $n+2$ respectively. Property 12 gives that $\mathbb{B}_2^{(0)} \mathbb{H}_n = \mathbb{H}_{n+1,1}$, implying

$$\mathbb{B}_2^{(0)} \mathbb{H}_n = \sum_{\mathcal{T}';\mathcal{T}'_{\{1,2\}}=12} \mathcal{T}'. \tag{4.28}$$

On the other hand, since $\mathbb{B}_2^{(1)} \mathcal{T} = (\mathbb{B}_2^{(0)} \mathcal{T}^t)^t$ and $\mathbb{H}_n = \mathbb{H}_n^t$, we have

$$\mathbb{B}_2^{(1)} \mathbb{H}_n = \left(\mathbb{B}_2^{(0)} \mathbb{H}_n\right)^t = \sum_{\mathcal{T}';\mathcal{T}'_{\{1,2\}}=21} \mathcal{T}'. \tag{4.29}$$

$\mathbb{H}_{n+2}$ is the sum of all standard tableaux of order $n+2$, of which each tableaux contain the subword 12 or 21. Therefore we have

$$\mathbb{H}_{n+2} = (\mathbb{B}_2^{(0)} + \mathbb{B}_2^{(1)}) \, \mathbb{H}_n , \tag{4.30}$$

from which (4.25) follows. □

It remains to show that $\mathbb{B}_2^{(0)}$ and $\mathbb{B}_2^{(1)}$ satisfy a relation similar to the q-commutation relation of $B_2^{(0)}$ and $B_2^{(1)}$. The proof of this last important property (stated in Property 25) requires a lengthy development of lemmas and identities.

We need two linear operators on standard tableaux, defined by:

Definition 14. *On any standard tableau $\mathcal{T}$ with $\mathcal{T}_{\{1,2\}} = 12$ and $|\mathcal{T}| = n$,*

$$\overset{*}{\mathbb{B}}{}_2^{(0)} : \mathcal{T} \to R_{n-1}\sigma_{n-2} \cdots \sigma_1 r_{(01\to 11)} \tau_{-1} \mathcal{T}, \tag{4.31}$$

and on any standard tableau $\mathcal{T}$ such that $\mathcal{T}_{\{1,2\}} = 21$ and $|\mathcal{T}| = n$,

$$\overset{*}{\mathbb{B}}{}_2^{(1)} : \mathcal{T} \to (\overset{*}{\mathbb{B}}{}_2^{(0)} \mathcal{T}^t)^t . \tag{4.32}$$

Example: Acting with $\overset{*}{\mathbb{B}}{}_2^{(1)}$ on $\begin{array}{l} 5 \\ 2\,6 \\ 1\,3\,4 \end{array}$, we go through the following steps:

$$\begin{array}{l} 5 \\ 2\,6 \\ 1\,3\,4 \end{array} \overset{t}{\to} \begin{array}{l} 4 \\ 3\,6 \\ 1\,2\,5 \end{array} \overset{(1)}{\to} \begin{array}{l} 3 \\ 2\,5 \\ 1\,1\,4 \end{array} \overset{(2)}{\to} \begin{array}{l} 3 \\ 2\,5 \\ 1\,2\,4 \end{array} \overset{(3)}{\to} \begin{array}{l} 3 \\ 2\,5 \\ 1\,3\,4 \end{array} \overset{(4)}{\to} \begin{array}{l} 3 \\ 2\,5 \\ 1\,4\,4 \end{array} \overset{(5)}{\to} \begin{array}{l} 3 \\ 2\,5 \\ 1\,4\,5 \end{array} \overset{(6)}{\to} \begin{array}{l} 3 \\ 2 \\ 1\,4 \end{array} \overset{t}{\to} \begin{array}{l} 4 \\ 1\,2\,3 \end{array}. \tag{4.33}$$

For any sum $S = \sum_k \mathsf{T}^{(k)}$ with $\mathsf{T}^{(k)} \neq \mathsf{T}^{(k')}$ for all $k \neq k'$, we shall say that $\mathsf{T} \in S$ if and only if $\mathsf{T} = \mathsf{T}^{(k)}$ for some k. Then, the operators $\overset{*}{\mathbb{B}}{}_2^{(0)}$ and $\overset{*}{\mathbb{B}}{}_2^{(1)}$ can be seen as the inverse of operations $\mathbb{B}_2^{(0)}$ and $\mathbb{B}_2^{(1)}$. That is,

Property 15. *For $\mathcal{T}_1' \in B_2^{(0)}\mathcal{T}_1$ and $\mathcal{T}_2' \in B_2^{(1)}\mathcal{T}_2$, we have*

$$\overset{*}{\mathbb{B}}{}_2^{(0)} \mathcal{T}_1' = \mathcal{T}_1 \quad \textit{and} \quad \overset{*}{\mathbb{B}}{}_2^{(1)} \mathcal{T}_2' = \mathcal{T}_2 \,. \tag{4.34}$$

Proof. These identities follow directly from the definition of the elementary operations comprising the operators and the fact that $R_a\mathsf{T}' = \mathsf{T}$ for any tableau $\mathsf{T}' \in A_{a,a}\mathsf{T}$. □

It is useful to alternatively define the action of $\overset{*}{\mathbb{B}}{}_2^{(1)}$. This is accomplished by introducing a new permutation $\bar{\sigma}_a$, defined by its action on a and $b = a+1$;

$$aab \overset{\bar{\sigma}_a}{\leftrightarrow} bab, \quad aba \overset{\bar{\sigma}_a}{\leftrightarrow} abb, \quad baa \overset{\bar{\sigma}_a}{\leftrightarrow} bba \tag{4.35}$$

Note that $\bar{\sigma}_a$ is such that on a word w with $ev(w_{\{a,b\}}) \in \{(1,2),(2,1)\}$, we have $\sigma_a w^R = (\bar{\sigma}_a w)^R$, where the superscript R is the operation sending a word $w = w_1 \cdots w_n$ to the word $w^R = w_n \cdots w_1$.

Property 16. *The action of $\overset{*}{\mathbb{B}}{}_2^{(1)}$ on any standard tableau $\mathcal{T}$ where $\mathcal{T}_{\{1,2\}} = 21$ and $|\mathcal{T}| = n$, can equivalently be expressed as*

$$\overset{*}{\mathbb{B}}{}_2^{(1)} : \mathcal{T} \to R_{n-1}\bar{\sigma}_{n-2}\cdots\bar{\sigma}_1 r_{(10\to 11)}\tau_{-1}\mathcal{T} \,. \tag{4.36}$$

Example: This action of $\overset{*}{\mathbb{B}}{}_2^{(1)}$ on $\begin{array}{l} 5 \\ 2\,6 \\ 1\,3\,4 \end{array}$ consists of the following steps:

$$\begin{array}{l} 5 \\ 2\,6 \\ 1\,3\,4 \end{array} \overset{(1)}{\to} \begin{array}{l} 4 \\ 1\,5 \\ 1\,2\,3 \end{array} \overset{(2)}{\to} \begin{array}{l} 4 \\ 2\,5 \\ 1\,2\,3 \end{array} \overset{(3)}{\to} \begin{array}{l} 4 \\ 3\,5 \\ 1\,2\,3 \end{array} \overset{(4)}{\to} \begin{array}{l} 4 \\ 4\,5 \\ 1\,2\,3 \end{array} \overset{(5)}{\to} \begin{array}{l} 5 \\ 4\,5 \\ 1\,2\,3 \end{array} \overset{(6)}{\to} \begin{array}{l} 4 \\ 1\,2\,3 \end{array}. \tag{4.37}$$

Note that we avoid the transposition steps shown in Example (4.33), and that each tableau here is the transposition of the corresponding tableau from Example (4.33).

Proof. Definition 14 gives that for any standard tableau $\mathcal{T}$ such that $\mathcal{T}_{\{1,2\}} = 21$ and $|\mathcal{T}| = n$, we must show

$$\left(\sigma_{n-2}\cdots\sigma_1 r_{(01\to 11)}\tau_{-1}\mathcal{T}^t\right)^t = \bar{\sigma}_{n-2}\cdots\bar{\sigma}_1 r_{(10\to 11)}\tau_{-1}\mathcal{T} \,, \tag{4.38}$$

where we have used that R_{n-1} commutes with transposition. If we let $\bar{T}$ be the semi-standard tableau $\bar{T} = r_{(01\to 11)}\tau_{-1}\mathcal{T}^t$, we get that $\bar{T}^t = r_{(10\to 11)}\tau_{-1}\mathcal{T}$. Expression (4.38) is then verified since we can obtain $(\sigma_{n-2}\cdots\sigma_1\bar{T})^t = \bar{\sigma}_{n-2}(\sigma_{n-3}\cdots\sigma_1\bar{T})^t = \cdots = \bar{\sigma}_{n-2}\cdots\bar{\sigma}_1\bar{T}^t$ by repeatedly using the identity,

$$(\sigma_a T)^t = \bar{\sigma}_a T^t \text{ for any semi-standard } T \text{ with } ev(T_{\{a,b=a+1\}}) \in \{(1,2),(2,1)\} \,. \tag{4.39}$$

This identity can be proven by observing that under such conditions, we have $(T_{\{a,b\}})^R = T^t_{\{a,b\}}$. For example, $T = \begin{array}{l} 4 \\ 1\,2\,3\,3 \end{array}$ is such that $T_{\{3,4\}} = 433$ and $T^t_{\{3,4\}} = 334$. The only possible planar distribution of a and b that could cause this to fail are

$\begin{smallmatrix} & a \\ b & \end{smallmatrix}$ or $\begin{smallmatrix} & b \\ a & \end{smallmatrix}$, since $T_{\{a,b\}}$ would be the same as $T^t_{\{a,b\}}$. The first case never holds and the second occurs only if we have both another a and another b, which we do not. Thus, using $T^R_{\{a,b\}} = (T^R)_{\{a,b\}} = (T_{\{a,b\}})^R = T^t_{\{a,b\}}$, we have $\bar{\sigma}_a T^t_{\{a,b\}} = \bar{\sigma}_a T^R_{\{a,b\}} = (\sigma_a T_{\{a,b\}})^R = (\sigma_a T)^R_{\{a,b\}} = (\sigma_a T)^t_{\{a,b\}}$, where the second equality follows from the definition of $\bar{\sigma}_a$. This gives that $(\sigma_a T)^t = \bar{\sigma}_a T^t$ on a semi-standard tableau. □

Let us now define an operation Σ_i to act on pairs of words as follows;

$$\Sigma_i : (w_1, w_2) \to (\bar{\sigma}_i \sigma_{i+1} w_1, \sigma_i \bar{\sigma}_{i+1} w_2)\,, \tag{4.40}$$

and consider 6 pairs of words with a and $b = a+1$;

$$\begin{aligned} &C_1(a) = (aabb, baab), \qquad C_2(a) = (abab, abab), \qquad C_3(a) = (abba, aabb) \\ &C_4(a) = (baab, bbaa), \qquad C_5(a) = (baba, baba), \qquad C_6(a) = (bbaa, abba)\,. \end{aligned} \tag{4.41}$$

We insert letters into such pairs by defining an insertion operator on words,

$$I^{(a)}_k w_1 \cdots w_n = w_1 \cdots w_{k-1} a w_k \cdots w_n \tag{4.42}$$

with the understanding that $I^{(a)}_1 w = aw$ and $I^{(a)}_{n+1} w = wa$. $I^{(a)}_k$ acts on pairs of words by $I^{(a)}_k (w_1, w_2) = (I^{(a)}_k w_1, I^{(a)}_k w_2)$.

Computer experimentation using *ACE* revealed that applying Σ_i to any pair $C_j(i)$ with the letter $(i+2)$ inserted recovered a pair of the same type where $i \to i+1$ with the extra letter i occuring in the same position of both elements of the pair. More exactly,

Lemma 17. *Let $1 \le k, k' \le 5$, $1 \le j, j' \le 6$ and $i > 1$. For any k, j and i,*

$$\Sigma_i \left(I^{(i+2)}_k C_j(i) \right) = I^{(i)}_{k'} C_{j'}(i+1), \tag{4.43}$$

for some k' and j'.

Example: We have

$$\begin{aligned} \Sigma_4 \left(I^{(6)}_2 C_3(4) \right) &= \Sigma_4(46554, 46455) \\ &= (\bar{\sigma}_4 \sigma_5 46554, \sigma_4 \bar{\sigma}_5 46455) \\ &= (46565, 46565) = I^{(4)}_1 C_5(5)\,. \end{aligned} \tag{4.44}$$

Proof. This is a lemma having 30 possible configurations which are easily verified with a computer. □

Proposition 18. *Let $\mathcal{T}_1$ be any standard tableau of degree n such that $\mathcal{T}_{1\{1,2,3,4\}}$ is* 4312 *or* 3124 *and let $\mathcal{T}_2 = \mathcal{T}_1^{2\leftrightarrow 3}$, i.e. $\mathcal{T}_2$ is obtained by permuting* 2 *and* 3 *in $\mathcal{T}_1$. If*

$$\mathbb{B} : (\mathcal{T}_1, \mathcal{T}_2) \to \left(\overset{*}{\mathbb{B}}{}^{(1)}_2 \overset{*}{\mathbb{B}}{}^{(0)}_2 \mathcal{T}_1, \overset{*}{\mathbb{B}}{}^{(0)}_2 \overset{*}{\mathbb{B}}{}^{(1)}_2 \mathcal{T}_2 \right), \tag{4.45}$$

then

$$\mathbb{B}(\mathcal{T}_1, \mathcal{T}_2) = (\bar{\mathcal{T}}, \bar{\mathcal{T}})\,, \tag{4.46}$$

for some standard tableau $\bar{\mathcal{T}}$.

Example: Given $\mathcal{T}_1 = \begin{array}{l} 7 \\ 3\,5 \\ 1\,2\,4\,6\,8 \end{array}$, we have the pair,

$$\mathbb{B}(\mathcal{T}_1, \mathcal{T}_2) = \mathbb{B}\left(\begin{array}{l} 7 \\ 3\,5 \\ 1\,2\,4\,6\,8 \end{array}, \begin{array}{l} 7 \\ 2\,5 \\ 1\,3\,4\,6\,8 \end{array} \right) = \left(\begin{array}{l} 2 \\ 1\,3\,4 \end{array}, \begin{array}{l} 2 \\ 1\,3\,4 \end{array} \right)\,. \tag{4.47}$$

Proof. $\overset{*}{\mathbb{B}}_2^{(0)}$ reduces the degree of $\mathcal{T}_1$ to $n-2$, giving by Definition 14 and Property 16,

$$\begin{aligned}\overset{*}{\mathbb{B}}_2^{(1)}\overset{*}{\mathbb{B}}_2^{(0)}\mathcal{T}_1 &= R_{n-3}\bar{\sigma}_{n-4}\cdots\bar{\sigma}_1 r_{(10\to 11)}\tau_{-1}R_{n-1}\sigma_{n-2}\cdots\sigma_1 r_{(01\to 11)}\tau_{-1}\mathcal{T}_1\\ &= R_{n-3}R_{n-2}\big(\bar{\sigma}_{n-4}\sigma_{n-3}\big)\cdots\big(\bar{\sigma}_1\sigma_2\big)r_{(10\to 11)}\tau_{-1}\sigma_2\sigma_1 r_{(01\to 11)}\tau_{-1}\mathcal{T}_1\,, \quad (4.48)\end{aligned}$$

where we have considered the relations $\tau_{-1}R_{n-1} = R_{n-2}\tau_{-1}$ and $\tau_{-1}\sigma_i = \sigma_{i-1}\tau_{-1}$. Similarly, acting first with $\overset{*}{\mathbb{B}}_2^{(1)}$,

$$\begin{aligned}\overset{*}{\mathbb{B}}_2^{(0)}\overset{*}{\mathbb{B}}_2^{(1)}\mathcal{T}_2 &= R_{n-3}\sigma_{n-4}\cdots\sigma_1 r_{(01\to 11)}\tau_{-1}R_{n-1}\bar{\sigma}_{n-2}\cdots\bar{\sigma}_1 r_{(10\to 11)}\tau_{-1}\mathcal{T}_2\\ &= R_{n-3}R_{n-2}\big(\sigma_{n-4}\bar{\sigma}_{n-3}\big)\cdots\big(\sigma_1\bar{\sigma}_2\big)r_{(01\to 11)}\tau_{-1}\bar{\sigma}_2\bar{\sigma}_1 r_{(10\to 11)}\tau_{-1}\mathcal{T}_2\,. \quad (4.49)\end{aligned}$$

We act first with $r_{(10\to 11)}\tau_{-1}\sigma_2\sigma_1 r_{(01\to 11)}\tau_{-1}$ on $\mathcal{T}_1$ in the case that $\mathcal{T}_{1\{1,2,3,4\}}$ is 3124, obtaining a word with subword 1122 and letters $3,\ldots,n-2$ occurring exactly once. $\mathcal{T}_2$, defined by permuting 2 and 3 in $\mathcal{T}_1$, thus contains the subword 2134 which is sent to 2112 under $r_{(01\to 11)}\tau_{-1}\bar{\sigma}_2\bar{\sigma}_1 r_{(10\to 11)}\tau_{-1}$, while the remaining letters occur exactly as they do in $r_{(10\to 11)}\tau_{-1}\sigma_2\sigma_1 r_{(01\to 11)}\tau_{-1}\mathcal{T}_1$. Consequently,

$$\mathbb{B}(\mathcal{T}_1,\mathcal{T}_2) = R_{n-3}R_{n-2}\Sigma_{n-4}\cdots\Sigma_1\left(I_{k_1}^{(n-2)}\cdots I_{k_{n-5}}^{(4)}I_{k_{n-4}}^{(3)}(1122,2112)\right)\,, \quad (4.50)$$

for some $k_1,\ldots,k_{n-4}$. Since Σ_1 acts only on the letters 1,2 and 3, we may now use Lemma 17, where $i=j=1$, to determine that the action of Σ_1 results in

$$\mathbb{B}(\mathcal{T}_1,\mathcal{T}_2) = R_{n-3}R_{n-2}\Sigma_{n-4}\cdots\Sigma_2\left(I_{k_1}^{(n-2)}\cdots I_{k_{n-5}}^{(4)}I_r^{(1)}C_j(2)\right)\,, \quad (4.51)$$

for some j and r. Lemma 17, applied repeatedly in this manner, gives

$$\mathbb{B}(\mathcal{T}_1,\mathcal{T}_2) = R_{n-3}R_{n-2}\left(I_{r_1}^{(1)}I_{r_2}^{(2)}\cdots I_{r_{n-4}}^{(n-4)}C'_j(n-3)\right)\,, \quad (4.52)$$

for some $r_1,\ldots,r_{n-4}$ and some j'. This is to say that the action of $\mathbb{B}$ on such a pair is equivalent to acting with $R_{n-3}R_{n-2}$ on a pair of tableaux that are identical in all letters except $n-3$ and $n-2$. Further, since $R_{n-3}R_{n-2}$ removes these letters, we have proved the identity in the case 3124. A sequence of similar arguments may be used in the case that $\mathcal{T}_{1\{1,2,3,4\}}$ is 4312, and we get

$$\mathbb{B}(\mathcal{T}_1,\mathcal{T}_2) = R_{n-3}R_{n-2}\Sigma_{n-4}\cdots\Sigma_1\left(I_{k_1}^{(n-2)}\cdots I_{k_{n-5}}^{(4)}I_{k_{n-4}}^{(3)}(2112,2211)\right)\,. \quad (4.53)$$

Again, successive applications of Lemma 17, beginning with the case $i=1$ and $j=4$, prove the identity. □

Lemma 19. *For $\mathcal{T}'$ a standard tableau of degree $n\geq 4$, we have that*

$$\begin{aligned}\mathcal{T}' \in \mathbb{B}_2^{(0)}\mathbb{B}_2^{(0)}\sum_{\mathcal{T};|\mathcal{T}|=n-4}\mathcal{T} &\iff \mathcal{T}'_{\{1,2,3,4\}}\in\{1234,4123,3412\}\,,\\ \mathcal{T}' \in \mathbb{B}_2^{(0)}\mathbb{B}_2^{(1)}\sum_{\mathcal{T};|\mathcal{T}|=n-4}\mathcal{T} &\iff \mathcal{T}'_{\{1,2,3,4\}}\in\{4312,3124\}\,,\\ \mathcal{T}' \in \mathbb{B}_2^{(1)}\mathbb{B}_2^{(0)}\sum_{\mathcal{T};|\mathcal{T}|=n-4}\mathcal{T} &\iff \mathcal{T}'_{\{1,2,3,4\}}\in\{4213,2134\}\,,\\ \mathcal{T}' \in \mathbb{B}_2^{(1)}\mathbb{B}_2^{(1)}\sum_{\mathcal{T};|\mathcal{T}|=n-4}\mathcal{T} &\iff \mathcal{T}'_{\{1,2,3,4\}}\in\{4321,3214,2413\}\,.\end{aligned} \quad (4.54)$$

Proof. We begin by simultaneously proving the first two cases of ($\Rightarrow$) and the others of ($\Rightarrow$) follow by transposition. Notice first that $B_2^{(0)}\mathcal{T} = \sum_{\mathcal{T}''}\mathcal{T}''$, where $\mathcal{T}''_{\{1,2\}}$ is 12 and $B_2^{(1)}\mathcal{T} = \sum_{\mathcal{T}''}\mathcal{T}''$, where $\mathcal{T}''_{\{1,2\}}$ is 21 (see the proof of Property 13). The following action of $B_2^{(0)}$ begins with $A_{n+1,n+1}$ adding a horizontal 2 strip to $\mathcal{T}''$ resulting in tableaux that are all semi-standard and containing the subword 12 (or 21). We act next with the succession of $\sigma_n, \sigma_{n-1}, \ldots, \sigma_3$ implying that the tableaux remain semi-standard and thus must each contain, in the first case, the subword 1233, 3123 or 3312, and in the second, 3213 or 2133. The remaining operations, aside from τ_1, act exclusively on these subwords as follows;

$$\begin{aligned}
&\tau_1 r_{(11\to01)}\sigma_1\sigma_2 1233 = 1234\,,\\
&\tau_1 r_{(11\to01)}\sigma_1\sigma_2 3123 = 4123\,,\\
&\tau_1 r_{(11\to01)}\sigma_1\sigma_2 3312 = 3412\,,
\end{aligned} \tag{4.55}$$

for the first case and

$$\begin{aligned}
&\tau_1 r_{(11\to01)}\sigma_1\sigma_2 3213 = 4312\,,\\
&\tau_1 r_{(11\to01)}\sigma_1\sigma_2 2133 = 3124\,,
\end{aligned} \tag{4.56}$$

for the second, thus proving ($\Rightarrow$) for the two first cases. To prove ($\Leftarrow$), we are given a standard tableau $\mathcal{T}'$ with subword $\mathcal{T}'_{\{1,2,3,4\}}$ in one of the four defined disjoint sets; call this set $S_{\epsilon_1,\epsilon_2}$. Property 13 gives that $\mathcal{T}'$, which is $\in \mathbb{H}_n$, for $n = |\mathcal{T}'|$, is such that $\mathcal{T}' \in \mathbb{B}_2^{(\bar\epsilon_1)}\mathbb{B}_2^{(\bar\epsilon_2)}\mathbb{H}_{n-4} = \mathbb{B}_2^{(\bar\epsilon_1)}\mathbb{B}_2^{(\bar\epsilon_2)}\sum_{\{|\mathcal{T}|=n-4\}}\mathcal{T}$, for some $\bar\epsilon_i \in \{0,1\}$. But since we have just proven that for such $\mathcal{T}'$, $\mathcal{T}'_{\{1,2,3,4\}}$ is contained in the set $S_{\bar\epsilon_1,\bar\epsilon_2}$, we see that $\bar\epsilon_i = \epsilon_i$ and the lemma is proven. □

Proposition 20. *On any standard tableau $\mathcal{T}$, we have*

$$\mathbb{B}_2^{(0)}\mathbb{B}_2^{(1)}\mathcal{T} = \left(\mathbb{B}_2^{(1)}\mathbb{B}_2^{(0)}\mathcal{T}\right)^{2\leftrightarrow3}\,, \tag{4.57}$$

where $2\leftrightarrow3$ denotes a permutation of the letters 2 and 3 in each tableaux.

Proof. Suppose there exists $\mathcal{T}' \in \mathbb{B}_2^{(0)}\mathbb{B}_2^{(1)}\mathcal{T}$ such that $\mathcal{T}'^{2\leftrightarrow3} \notin \mathbb{B}_2^{(1)}\mathbb{B}_2^{(0)}\mathcal{T}$. Lemma 19 gives that every element in $\mathbb{B}_2^{(0)}\mathbb{B}_2^{(1)}\mathcal{T}$, in particular $\mathcal{T}'$, must contain either the subword 4312 or 3124. This implies that $\mathcal{T}'^{2\leftrightarrow3}$ must contain either 4213 or 2134 and thus by the same lemma we have that

$$\mathcal{T}'^{2\leftrightarrow3} \in \mathbb{B}_2^{(1)}\mathbb{B}_2^{(0)}\mathcal{T}'' \tag{4.58}$$

for some $\mathcal{T}'' \neq \mathcal{T}$. Observe now that $\overset{*}{\mathbb{B}}{}_2^{(1)}\overset{*}{\mathbb{B}}{}_2^{(0)}\mathcal{T}' = \mathcal{T}$, implies by Proposition 18 that $\overset{*}{\mathbb{B}}{}_2^{(0)}\overset{*}{\mathbb{B}}{}_2^{(1)}\mathcal{T}'^{2\leftrightarrow3}$ must also be $\mathcal{T}$. Expression (4.58) then yields

$$\overset{*}{\mathbb{B}}{}_2^{(0)}\overset{*}{\mathbb{B}}{}_2^{(1)}\left(\mathbb{B}_2^{(1)}\mathbb{B}_2^{(0)}\mathcal{T}''\right) = \mathcal{T} + \text{other terms} \tag{4.59}$$

which by Property 15 gives that $\mathcal{T}'' = \mathcal{T}$ and we reach a contradiction. We thus have that $\mathbb{B}_2^{(0)}\mathbb{B}_2^{(1)}\mathcal{T} \subseteq \left(\mathbb{B}_2^{(1)}\mathbb{B}_2^{(0)}\mathcal{T}\right)^{2\leftrightarrow3}$. We can also show in the same manner that $\mathbb{B}_2^{(0)}\mathbb{B}_2^{(1)}\mathcal{T} \supseteq \left(\mathbb{B}_2^{(1)}\mathbb{B}_2^{(0)}\mathcal{T}\right)^{2\leftrightarrow3}$, which proves the proposition. □

We now define four pairs of words on consecutive numbers; $a, b = a+1, c = b+1$ and $d = c+1$.

$$\begin{aligned} D_1(a) &= (bacd, cabd), & D_2(a) &= (dbac, dcab), \\ D_3(a) &= (acdb, abdc), & D_4(a) &= (cdba, bdca)\,. \end{aligned} \tag{4.60}$$

These pairs appear as the only distinct subwords in certain pairs of semi-standard tableaux. More precisely, such a pair of semi-standard tableaux, called $(T_1, T_2)_{D_j(a)}$, satisfies $T_1 = T_2^{b \leftrightarrow c}$ and $(T_{1_{\{a,b,c,d\}}}, T_{2_{\{a,b,c,d\}}}) = D_j(a)$. For example,

$$\left(\begin{array}{|c|c|c|c|} \hline 5 \\ \cline{1-1} 4 & 7 & 8 \\ \cline{1-3} 1 & 2 & 3 & 6 \\ \hline \end{array} \, , \, \begin{array}{|c|c|c|c|} \hline 5 \\ \cline{1-1} 4 & 6 & 8 \\ \cline{1-3} 1 & 2 & 3 & 7 \\ \hline \end{array} \right)_{D_3(5)} \tag{4.61}$$

is such a pair. One should note that $(T_1, T_2)_{D_j(a)}$ is a pair of tableaux of the same shape since in any such semi-standard tableaux, b and c never occur in the same row or column. With Ω_i defined such that on pairs of words

$$\Omega_i : (w_1, w_2) \to (\sigma_i \sigma_{i+1} \sigma_{i+2} \sigma_{i+3} w_1, \sigma_i \sigma_{i+1} \sigma_{i+2} \sigma_{i+3} w_2)\,; \tag{4.62}$$

Lemma 21. *Let* $1 \le k_2, k_2', k_2'' \le 5$, $1 \le k_1, k_1', k_1'' \le 6$, $1 \le j, j', j'' \le 4$ *and* $i > 0$. *For any such* k_1, k_2, j *and* i, *we have*

$$r_{(ii \to ii-1)} \Omega_i I_{k_1}^{(i+4)} I_{k_2}^{(i+4)} D_j(i) \in \left\{ I_{k_1'}^{(i-1)} I_{k_2'}^{(i)} D_{j'}(i+1)\,,\; I_{k_1'}^{(i-1)} I_{k_2'}^{(i+4)} D_{j'}(i) \right\} \tag{4.63}$$

and

$$r_{(ii \to i-1i)} \Omega_i I_{k_1}^{(i+4)} I_{k_2}^{(i+4)} D_j(i) \in \left\{ I_{k_1''}^{(i-1)} I_{k_2''}^{(i)} D_{j''}(i+1)\,,\; I_{k_1''}^{(i-1)} I_{k_2''}^{(i+4)} D_{j''}(i) \right\}\,, \tag{4.64}$$

for some $k_1', k_1'', k_2', k_2'', j'$ *and* j''.

Example: Starting with $I_4^{(9)} I_2^{(9)} D_4(5) = (798965, 698975)$, we get

$$\Omega_4(798965, 698975) = (597865, 596875)\,. \tag{4.65}$$

Under $r_{(55 \to 45)}$, we recover $I_1^{(4)} I_1^{(9)} D_4(5)$ and under $r_{(55 \to 54)}$, $I_6^{(4)} I_2^{(9)} D_3(5)$.

Proof. For each i there are 60 cases that have been verified using a computer. □

Lemma 22. *Let* $\mathcal{T}$ *and* $\mathcal{T}'$ *be standard tableaux of type* $(\mathcal{T}, \mathcal{T}')_{D_j(i)}$ *for some* i, j. *If standard tableaux* $\bar{\mathcal{T}} \in \mathbb{B}_2^{(\epsilon)} \mathcal{T}$ *and* $\bar{\mathcal{T}}' \in \mathbb{B}_2^{(\epsilon)} \mathcal{T}'$ *are standard tableaux of the same shape for* $\epsilon \in \{0,1\}$, *then* $\bar{\mathcal{T}}$ *and* $\bar{\mathcal{T}}'$ *is a pair of type* $(\bar{\mathcal{T}}, \bar{\mathcal{T}}')_{D_{j'}(i+1)}$ *or* $(\bar{\mathcal{T}}, \bar{\mathcal{T}}')_{D_{j'}(i+2)}$, *for some* j'.

Proof. We start with the case $\epsilon = 0$ and split the action of $B_2^{(0)}$ into a sequence of operations beginning with $A_{n+1,n+1}$. As such, we consider a semi-standard tableau T obtained by adding an arbitrary horizontal 2-strip to $\mathcal{T}$. We denote by T', the semi-standard tableau of the same shape that is obtained by adding this horizontal 2-strip to $\mathcal{T}'$ and thus T and T' are a pair of type $(T, T')_{D_j(i)}$. Next in the sequence of operations defining $B_2^{(0)}$ is $\sigma_{i+4} \cdots \sigma_n$ which, acting only on the letters $i+4, \ldots, n$, must preserve the similarity in T and T'. If $i \neq 1$, since acting with σ_{i-1} amounts to applying either $r_{(ii \to ii-1)}$ or $r_{(ii \to i-1i)}$, acting on both elements with $\sigma_{i-1} \sigma_i \sigma_{i+1} \sigma_{i+2} \sigma_{i+3}$ using Lemma 21, gives a pair of semi-standard tableaux of type $(\bar{T}, \bar{T}')_{D_{j'}(i+1)}$ or $(\bar{T}, \bar{T}')_{D_{j'}(i)}$. There remains to act with $\tau_1 r_{11 \to 01} \sigma_1 \ldots \sigma_{i-2}$, which leads to pairs of standard tableaux of type $(\bar{\mathcal{T}}, \bar{\mathcal{T}}')_{D_{j'}(i+2)}$ or $(\bar{\mathcal{T}}, \bar{\mathcal{T}}')_{D_{j'}(i+1)}$. In the case where $i = 1$, acting on both elements with $\tau_1 r_{(11 \to 01)} \sigma_1 \sigma_2 \sigma_3 \sigma_4$, gives, from Lemma 21, pairs of standard tableaux of type $(\tilde{\mathcal{T}}, \tilde{\mathcal{T}}')_{D_{j'}(3)}$ or $(\tilde{\mathcal{T}}, \tilde{\mathcal{T}}')_{D_{j'}(2)}$, finally proving the lemma for $\epsilon = 0$. To prove the lemma in the case $\mathbb{B}_2^{(1)}$, we observe that, for $\mathcal{T}_1$ and $\mathcal{T}_2$ standard tableaux,

$$\begin{aligned} (\mathcal{T}_1,\mathcal{T}_2)_{D_1(a)} &\implies (\mathcal{T}_2^t,\mathcal{T}_1^t)_{D_2(a)} \qquad (\mathcal{T}_1,\mathcal{T}_2)_{D_2(a)} \implies (\mathcal{T}_2^t,\mathcal{T}_1^t)_{D_1(a)}\,, \\ (\mathcal{T}_1,\mathcal{T}_2)_{D_3(a)} &\implies (\mathcal{T}_2^t,\mathcal{T}_1^t)_{D_4(a)} \qquad (\mathcal{T}_1,\mathcal{T}_2)_{D_4(a)} \implies (\mathcal{T}_2^t,\mathcal{T}_1^t)_{D_3(a)}\,. \end{aligned} \tag{4.66}$$

$\mathbb{B}_2^{(1)}\mathcal{T} = (\mathbb{B}_2^{(0)}\mathcal{T}^t)^t$ thus implies that the proof in this case is exactly the proof for $\mathbb{B}_2^{(0)}$ with every pair reversed plus an additional reversal of the pairs at the end, accounting for the last transposition in $\mathbb{B}_2^{(1)}$. □

Lemma 23. *If $\mathcal{T}$ and $\mathcal{T}'$ are a pair of standard tableaux of type $(\mathcal{T},\mathcal{T}')_{D_j(i)}$ then*

$$F(\mathcal{T}) = qF(\mathcal{T}')\,. \tag{4.67}$$

Proof. We have already noted that $(\mathcal{T},\mathcal{T}')_{D_j(i)}$ is a pair of tableaux with the same shape. Further, the definition of cocharge gives $\mathrm{cocharge}(\mathcal{T}) = \mathrm{cocharge}(\mathcal{T}') + 1$; for example, for $D_1(a) = (bacd, cabd)$, we have

$$\begin{aligned} \mathrm{cocharge}(\mathcal{T}) &= c_1 + \cdots + c_a + (c_a+1) + (c_a+1) + (c_a+1) + c_{a+4} + \cdots + c_n \\ \mathrm{cocharge}(\mathcal{T}') &= c_1 + \cdots + c_a + (c_a) + (c_a+1) + (c_a+1) + c_{a+4} + \cdots + c_n\,, \end{aligned} \tag{4.68}$$

giving $\mathrm{cocharge}(\mathcal{T}) = \mathrm{cocharge}(\mathcal{T}') + 1$ as claimed. □

We now finally have all the ingredients needed to prove the commutation relation.

Definition 24. *We define, for $v = (v_1,\ldots,v_k)$ with $v_i \in \{0,1\}$,*

$$\mathbb{U}_v^{(\epsilon)} = \mathbb{B}_2^{(v_1)}\cdots\mathbb{B}_2^{(v_k)}\mathbb{H}_\epsilon\,, \qquad \epsilon \in \{0,1\}\,. \tag{4.69}$$

Property 25. *For any $v = (v_1,\ldots,v_k)$ and $\bar v = (\bar v_1,\ldots,\bar v_{k'})$, with $v_i, \bar v_i \in \{0,1\}$ and $k,k' \geq 0$, we have*

$$F(\mathbb{U}^{(\epsilon)}_{v,1,0,\bar v}) = qF(\mathbb{U}^{(\epsilon)}_{v,0,1,\bar v}) \qquad \epsilon \in \{0,1\}\,. \tag{4.70}$$

Proof. We begin by showing that this identity holds in the case that v is empty. Since $\mathbb{U}^{(\epsilon)}_{1,0,\bar v} = \mathbb{B}_2^{(1)}\mathbb{B}_2^{(0)}\sum_{\mathcal{T}}\mathcal{T}$ and $\mathbb{U}^{(\epsilon)}_{0,1,\bar v} = \mathbb{B}_2^{(0)}\mathbb{B}_2^{(1)}\sum_{\mathcal{T}}\mathcal{T}$, each tableau $\mathcal{T}_1 \in \mathbb{U}^{(\epsilon)}_{1,0,\bar v}$ can be paired with some $\mathcal{T}_1' \in \mathbb{U}^{(\epsilon)}_{0,1,\bar v}$ such that $\mathcal{T}_1' = \mathcal{T}_1^{2\leftrightarrow 3}$ by Proposition 20 and such that $\mathcal{T}_{1\{1,2,3,4\}} \in \{4213, 2134\}$ by Lemma 19. This implies that $\mathcal{T}_1$ and $\mathcal{T}_1'$ are of type $(\mathcal{T}_1,\mathcal{T}_1')_{D_j(1)}$ where $j=1$ or 2 which, using Lemma 23, proves the identity for $v = ()$. We now proceed to the case when $v = (0)$ or (1) by acting with $\mathbb{B}_2^{(\epsilon)}$ on the pairs obtained when $v = ()$. These pairs $(\mathcal{T}_1,\mathcal{T}_1')_{D_j(1)}$ are thus sent to a pair of sums of standard tableaux that, by Lemma 22, can be paired by types $(\mathcal{T}_2,\mathcal{T}_2')_{D_j(i)}$ for some $j = 1,2,3$ or 4 and some $i = 3$ or 4. For any v, we repeat this process and obtain that each $\bar{\mathcal{T}} \in \mathbb{U}^{(\epsilon)}_{v,1,0,\bar v}$ can be paired with some standard tableau $\bar{\mathcal{T}}' \in \mathbb{U}^{(\epsilon)}_{v,0,1,\bar v}$ where this pair is of type $(\bar{\mathcal{T}},\bar{\mathcal{T}}')_{D_j(i)}$, for $1 \leq j \leq 4$ and $1 \leq i \leq n-3$. Lemma 23 then proves the property. □

Given Properties 12, 13 and 25, analogous to those proven in section 3, we can finally prove the main result of this section.

Theorem 26. *Let $\epsilon, v_i \in \{0,1\}$. For any $v = (v_1,\ldots,v_k)$ we have*

$$F(\mathbb{U}_v^{(\epsilon)}) = U_v^{(\epsilon)}\,. \tag{4.71}$$

Proof. Recall that the action of $B_2^{(0)}$ and $B_2^{(0)}+B_2^{(1)}$, determined in Properties 4 and 6, led to Corollary 9. That is, to

$$H_{2m+\ell+\epsilon,\ell}[X;q,t]=\sum_{\bar v} U_{\bar v}^{(\epsilon)}\,, \tag{4.72}$$

where $\bar v=(0^\ell,v)$ for some $v=(v_1,\dots,v_m)$. Observe that we have proved equivalent actions for the operators $\mathbb{B}_2^{(0)}$ and $\mathbb{B}_2^{(0)}+\mathbb{B}_2^{(1)}$ in Properties 12 and 13, giving

$$\mathbb{H}_{2m+\ell+\epsilon,\ell}=\sum_{\bar v}\mathbb{U}_{\bar v}^{(\epsilon)}\,, \tag{4.73}$$

where $\bar v$ is as before. We thus have, since $\mathcal{F}(\mathbb{H}_{m,\ell})=H_{m,\ell}[X;q,t]$,

$$H_{2m+\ell+\epsilon,\ell}[X;q,t]=\sum_{\bar v}\mathcal{F}(\mathbb{U}_{\bar v}^{(\epsilon)})=\sum_{\bar v}U_{\bar v}^{(\epsilon)}\,. \tag{4.74}$$

We convert the expression such that we are summing only over dominant vectors $v_d=(0^{m+\ell-k},1^k)$ for some k by using the following implication of the q-commutation relations proven in Properties 5 and 25: $U_{\beta(v)}^{(\epsilon)}=q^{\ell(\beta)}U_{v_d}^{(\epsilon)}$ and $\mathcal{F}(\mathbb{U}_{\beta(v)}^{(\epsilon)})=q^{\ell(\beta)}\mathcal{F}(\mathbb{U}_{v_d}^{(\epsilon)})$, where $\ell(\beta)$ is the length of the permutation β such that $\beta(v)=v_d$. This gives

$$H_{2m+\ell+\epsilon,\ell}[X;q,t]=\sum_{v_d}d_{v_d}^{m,\ell}(q)\mathcal{F}(\mathbb{U}_{v_d}^{(\epsilon)})=\sum_{v_d}d_{v_d}^{m,\ell}(q)U_{v_d}^{(\epsilon)}\,, \tag{4.75}$$

where $d_{v_d}^{m,\ell}(q)=\sum_{\beta(v)=v_d}q^{\ell(\beta)}$. For $2m+2\ell+\epsilon=n$, the number of possible v_d is $\lfloor n/2\rfloor+1$, exactly the number of partitions of n of length ≤ 2. We thus have, from (4.75), that $U_{v_d}^{(\epsilon)}$ and $\mathcal{F}(\mathbb{U}_{v_d}^{(\epsilon)})$ are both bases for $\mathcal{V}$, the $\mathbb{Q}[q,t]$-linear span of $\{H_\lambda[X;q,t]\}_{\ell(\lambda)\le 2}$. We see from expression (4.75) again, that the transition matrices from $\{H_\lambda[X;q,t]\}_{\ell(\lambda)\le 2}$ to $\{U_{v_d}^{(\epsilon)}\}_{v_d}$ and from $\{H_\lambda[X;q,t]\}_{\ell(\lambda)\le 2}$ to $\{\mathcal{F}(\mathbb{U}_{v_d}^{(\epsilon)})\}_{v_d}$ are identical. Since these are invertible matrices, we have that $U_{v_d}^{(\epsilon)}=\mathcal{F}(\mathbb{U}_{v_d}^{(\epsilon)})$, which can be extended to $U_v^{(\epsilon)}=\mathcal{F}(\mathbb{U}_v^{(\epsilon)})$ using Properties 5 and 25. □

5. A statistic for Macdonald polynomials in 2 parts

It is now clear from the previous theorem and (3.28) that for $d=2m+2\ell+\epsilon$, we have

$$J_{2m+\ell+\epsilon,\ell}[X;q,t]=\sum_{v=(v_1,\dots,v_{m+\ell})}q^{(1-d)|v|_\ell+2n(v)_\ell}t^{\ell-|v|_\ell}\mathcal{F}(\mathbb{U}_v^{(\epsilon)}), \tag{5.1}$$

where $|v|_\ell$ and $n(v)_\ell$ are as defined in Proposition 8. To provide an expression for $J_\lambda[X;q,t]$ with coefficients that are determined by statistics, we associate to any standard tableau a vector $\in\{0,1\}^k$ called a "domino" vector. The domino vector is determined by the succession of of operators $\mathbb{B}_2^{(0)}$ and $\mathbb{B}_2^{(1)}$ that build the associated standard tableau. Since any standard tableau $\mathcal{T}$ such that $\mathcal{T}_{\{1,2\}}=12$ (or 21) can be obtained by acting with $\mathbb{B}_2^{(0)}$ (or $\mathbb{B}_2^{(1)}$) on a predecessor $\mathcal{T}'$, such a succession is determined recursively using Property 15.

Theorem 27. *The Macdonald polynomials indexed by partitions with no more than 2 parts are given by*

$$J_{2m+\ell+\epsilon,\ell}=\sum_{|\mathcal{T}|=d}\mathrm{Stat}(\mathcal{T})S_{\mathrm{shape}(\mathcal{T})}[X^t], \tag{5.2}$$

where $d = 2m + 2\ell + \epsilon$ *and*

$$\mathrm{Stat}(\mathcal{T}) = q^{\mathrm{cocharge}(\mathcal{T})} q^{(1-d)|\mathrm{dv}(\mathcal{T})|_\ell + 2n(\mathrm{dv}(\mathcal{T}))_\ell} t^{\ell - |\mathrm{dv}(\mathcal{T})|_\ell}, \tag{5.3}$$

with the domino vector, $\mathrm{dv}(\mathcal{T}) = (\mathrm{dv}_1, \ldots, \mathrm{dv}_{m+\ell})$*, obtained recursively by*

$$\mathrm{dv}(\mathcal{T}) = \begin{cases} \left(0, \mathrm{dv}\left(\overset{*}{\mathbb{B}}{}_2^{(0)} \mathcal{T}\right)\right) & \text{if } \mathcal{T}_{\{1,2\}} = 12 \\ \left(1, \mathrm{dv}\left(\overset{*}{\mathbb{B}}{}_2^{(1)} \mathcal{T}\right)\right) & \text{if } \mathcal{T}_{\{1,2\}} = 21 \\ \qquad \emptyset & \text{if } \mathcal{T} \text{ has degree} \leq 1 \end{cases} . \tag{5.4}$$

Proof. The theorem follows directly from (5.1) and Property 15. □

Example: The statistic associated to a standard tableau $\mathcal{T} = \begin{array}{|c|c|c} \hline 4 & 8 & \\ \hline 3 & 5 & 7 \\ \hline 1 & 2 & 6 \\ \hline \end{array}$ in the Macdonald polynomial $J_{6,2}[X; q, t]$ is determined by finding the domino vector of $\mathcal{T}$.

$$\begin{aligned} \left(0, \mathrm{dv}\left(\overset{*}{\mathbb{B}}{}_2^{(0)} \begin{array}{ccc} 4 & 8 & \\ 3 & 5 & 7 \\ 1 & 2 & 6 \end{array} = \begin{array}{ccc} 4 & & \\ 2 & 6 & \\ 1 & 3 & 5 \end{array} \right)\right) &= \left(0, 1, \mathrm{dv}\left(\overset{*}{\mathbb{B}}{}_2^{(1)} \begin{array}{ccc} 4 & & \\ 2 & 6 & \\ 1 & 3 & 5 \end{array} = \begin{array}{cc} 4 & \\ 3 & \\ 1 & 2 \end{array} \right)\right) \\ &= \left(0, 1, 0, \mathrm{dv}\left(\overset{*}{\mathbb{B}}{}_2^{(0)} \begin{array}{cc} 4 & \\ 3 & \\ 1 & 2 \end{array} = \begin{array}{c} 2 \\ 1 \end{array} \right)\right) \\ &= (0, 1, 0, 1) . \end{aligned} \tag{5.5}$$

This gives that $|\mathrm{dv}(\mathcal{T})|_2 = 1$ and $n(\mathrm{dv}(\mathcal{T}))_2 = 1$. The cocharge of $\mathcal{T} = 48\ 357\ 126$ is $0 + 0 + 1 + 2 + 2 + 2 + 3 + 4 = 14$, and we have

$$\mathrm{Stat}(\mathcal{T}) = q^{14} q^{-7+2} t^{2-1} = q^9 t . \tag{5.6}$$

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