ORDER IDEALS IN WEAK SUBPOSETS OF YOUNG'S LATTICE AND ASSOCIATED UNIMODALITY CONJECTURES

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Abstract. The k-Young lattice Y^k is a weak subposet of the Young lattice containing partitions whose first part is bounded by an integer $k > 0$. The Y^k poset was introduced in connection with generalized Schur functions and later shown to be isomorphic to the weak order on the quotient of the affine symmetric group $\tilde{S}_{k+1}$ by a maximal parabolic subgroup. We prove a number of properties for Y^k including that the covering relation is preserved when elements are translated by rectangular partitions with hook-length k. We highlight the order ideal generated by an $m \times n$ rectangular shape. This order ideal, $L^k(m,n)$, reduces to $L(m,n)$ for large k, and we prove it is isomorphic to the induced subposet of $L(m,n)$ whose vertex set is restricted to elements with no more than $k-m+1$ parts smaller than m. We provide explicit formulas for the number of elements and the rank-generating function of $L^k(m,n)$. We conclude with unimodality conjectures involving q-binomial coefficients and discuss how implications connect to recent work on sieved q-binomial coefficients.

1. Introduction

The Young lattice Y is the poset of integer partitions given by inclusion of diagrams. This poset can be induced from the branching rules of the symmetric group, and certain order ideals of Y are in themselves interesting posets. For example, the induced subposet of partitions whose Ferrers diagrams fit inside an $m \times n$ rectangle satisfies many beautiful properties. These principal order ideals, denoted $L(m,n)$, are graded, self-dual, and strongly sperner lattices [8]. Further, it is known that the number of elements $p_i(m,n)$ of rank i in $L(m,n)$ are coefficients in the generalized Gaussian polynomial, and thus form a unimodal sequence [6, 10]. That is,

$$\sum_{i\geq 0} p_i(m,n)\, q^i = \begin{bmatrix} n+m \\ m \end{bmatrix}_q = \frac{(1-q^{n+1})\cdots(1-q^{m+n})}{(1-q)\cdots(1-q^m)}\,. \tag{1.1}$$

Letting $q \to 1$, the total number of elements in this poset is given by

$$|L(m,n)| = \binom{n+m}{m}\,. \tag{1.2}$$

A weak subposet Y^k of the Young lattice was introduced in connection with functions that generalize the Schur functions [2, 3]. This poset (hereafter called the k-Young lattice) is a lattice defined on the set of partitions whose first part is no larger than fixed integer $k \geq 1$. The order arises from a degree preserving involution on the set of k-bounded partitions $\mathcal{P}^k$ that generalizes partition conjugation. The involution sends one k-bounded partition λ to another, λ^{ω_k}, giving rise to a partial order on $\mathcal{P}^k$ as follows: For λ and μ differing by one box,

Young order on partitions: $\lambda \lessdot \mu$ *when* $\lambda \subseteq \mu$ *(and equivalently* $\lambda' \subseteq \mu'$*).*

k-order on k-bounded partitions: $\lambda \prec\!\cdot\, \mu$ *when* $\lambda \subseteq \mu$ *and* $\lambda^{\omega_k} \subseteq \mu^{\omega_k}$*.*

It happens that $\lambda^{\omega_k} = \lambda'$ for large k implying that the k-order is the Young order in the limit $k \to \infty$.

The k-Young lattice originated from a conjectured formula for multiplying k-Schur functions [2, 3] that is analogous to the Pieri rule. In particular, the conjecture states that the k-Schur functions $s_\mu^{(k)}$

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appearing in the expansion of the product $s_1 s_\lambda^{(k)}$ are exactly those indexed by the successors of λ in the k-Young lattice. That is,

$$s_1 s_\lambda^{(k)} = \sum_{\lambda \prec \mu} s_\mu^{(k)}\,.$$

In [5], it is shown that the k-Young lattice is in fact isomorphic to the weak order on the quotient of the affine symmetric group $\tilde{S}_{k+1}$ by a maximal parabolic subgroup and that the paths in Y^k can be enumerated by certain "k-tableaux", or by reduced words for affine permutations.

Here we investigate general properties of the k-Young lattice. Most notably, we reveal that partitions with rectangular shape and hook-length k, called k-rectangles, play a fundamental role in the structure of this poset. We prove for any k-rectangle $\Box$ and $\lambda, \mu \in \mathcal{P}^k$,

$$\lambda \preceq \mu \quad \text{if and only if} \quad (\lambda \cup \Box) \preceq (\mu \cup \Box)\,,$$

leading to the stronger statement that:

$$\lambda \cup \Box \preceq \mu \iff \mu = \bar{\mu} \cup \Box \quad \textit{and} \quad \lambda \preceq \bar{\mu}\,,$$

for some k-bounded partition $\bar{\mu}$. This is a central property needed to identify the k-Young lattice with a cone in the permutahedron-tiling of the k-dimensional space [11]. The significance of k-rectangles also plays an important role at the symmetric function level in that multiplying a Schur function indexed by a k-rectangle $\Box$ with a k-Schur function is trivial [4]. That is, for any k-bounded partition λ,

$$s_\Box\, s_\lambda^{(k)} = s_{\Box \cup \lambda}^{(k)}\,.$$

Following our study of the k-rectangles and other properties of the k-Young lattice, we discuss a family of induced subposets of $L(m,n)$ whose vertex set consists of the elements that fit inside an $m \times n$ rectangle and have no more than $k-m+1$ parts strictly smaller than m. Surprisingly, we find that these subposets are isomorphic to the principal order ideal of Y^k generated by the shape $m \times n$. As such, we denote these order ideals by $L^k(m,n)$ and note that they are graded, self-dual, distributive lattices of rank mn. We provide explicit formulas for the number of vertices and the rank generating function:

$$\sum_{\lambda \in L^k(m,n)} q^{|\lambda|} = \begin{bmatrix} k+1 \\ m \end{bmatrix}_q + q^{k+1} \frac{1 - q^{m(n-k+m-1)}}{1-q^m} \begin{bmatrix} k \\ m-1 \end{bmatrix}_q\,, \tag{1.3}$$

for $n \geq k-m+1$, which implies that the coefficient of q^i in the right hand side of this expression is the number of partitions in $L^k(m,n)$ with rank i. Further, letting $q \to 1$,

$$\left|L^k(m,n)\right| = \binom{k+1}{m} + (n-k+m-1)\binom{k}{m-1}\,, \tag{1.4}$$

for $n \geq k-m+1$.

Since the vertex set of $L^k(m,n)$ is contained in that of $L^{k+1}(m,n)$, these order ideals provide a natural sequence of subposets of $L(m,n)$. That is, $L(m,n)$ can be constructed from the chain of partitions with no more than one row smaller than m by successively adding sets of partitions with exactly j parts smaller than m for $j \geq 2$. This decomposition aids our investigation of questions pertaining to unimodality. Prompted by the unimodality of $L(m,n)$, we computed examples that suggest $L^k(m,n)$ is unimodal in certain cases. In particular when $k \neq -1 \mod p$ for all prime divisor p of m. When m is prime, we find that the unimodality of $L^k(m,n)$ relies on the conjecture:

If $k \neq -1, 0 \mod m$, then the coefficients of the q powers in

$$\frac{(1-q^{m(n-k+m)})}{(1-q^m)} \begin{bmatrix} k-1 \\ m-2 \end{bmatrix}_q \tag{1.5}$$

form a unimodal sequence for all $n \geq k-m+1$.

We also generalize this conjecture to include the case when m is not prime (see Conjecture 48).

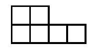
We conclude with a discussion of how our conjectures lead to results coinciding with recent work on sieved binomial polynomials, eg. [1, 9, 12]. Namely, from the unimodality of the coefficients in Eq. (1.5), we recover the identity: the sum of the coefficients of $q^{\ell + *m}$ in $\begin{bmatrix} k-1 \\ m-2 \end{bmatrix}_q$ is equal to $\frac{1}{m}\binom{k-1}{m-2}$ if

$k \neq -1, 0 \mod m$ for a prime $m < k$. Similarly, we use our more general conjecture to suggest a new identity of this type and provide an independent proof (see Proposition 53).

2. Definitions

For definitions and general properties of posets see for example [7]. A partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ is a non-increasing sequence of positive integers. We denote by $|\lambda|$ the degree $\lambda_1 + \cdots + \lambda_m$ of λ, and by $\ell(\lambda)$ its length m. Each partition λ has an associated Ferrers diagram with λ_i lattice squares in the i^{th} row, from the bottom to top. For example,

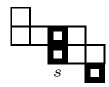
$$\lambda = (4, 2) = \tag{2.1}$$



A partition λ is k-**bounded** if $\lambda_1 \leq k$ and the set of all k-bounded partitions is denoted $\mathcal{P}^k$. For partitions λ and μ, the weakly decreasing rearrangement of their parts is denoted $\lambda \cup \mu$, while $\lambda + \mu$ is the partition obtained by summing their respective parts. Any lattice square in the Ferrers diagram is called a cell, where the cell (i, j) is in the ith row and jth column in the diagram. We say that $\mu \subseteq \lambda$ when $\mu_i \leq \lambda_i$ for all i.

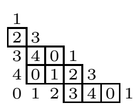
When $\mu \subseteq \lambda$, the skew shape λ/μ is identified with its diagram $\{(i, j) : \mu_i < j \leq \lambda_i\}$. For example,

$$\lambda/\mu = (5, 5, 4, 1)/(4, 2) = \tag{2.2}$$



Lattice squares that do not lie inside a diagram will simply be called squares. We shall say that any $s \in \mu$ lies **below** the diagram of λ/μ. The degree of a skew-shape is the number of cells in its diagram. Associated to λ/μ, the **hook** of any $(i, j) \in \lambda$ is defined by the cells of λ/μ that lie in the L formed with (i, j) as its corner. This definition is well-defined for all squares in λ including those below λ/μ. For example, the framed cells in (2.2) denote the hook of square $s = (1, 3)$. We then let $h_s(\lambda/\mu)$ denote the hook-length of any $s \in \lambda$, i.e. the number of cells in the hook of s. For example, $h_{(1,3)}(5, 5, 4, 1)/(4, 2) = 3$ and $h_{(3,2)}(5, 5, 4, 1)/(4, 2) = 3$ or cell (3,2) has a 3-hook. We also say that the hook of a cell (or a square) is k-bounded if it is not larger than k.

A **removable corner** is a cell $(i, j) \in \lambda/\mu$ such that $(i + 1, j), (i, j + 1) \notin \lambda/\mu$, and an **addable corner** is a square $(i, j) \notin \lambda/\mu$ such that $(i - 1, j), (i, j - 1) \in \lambda/\mu$. We shall include $(1, \lambda_1) \in \lambda/\mu$ as a removable corner, and $(\ell(\lambda) + 1, 1)$ as an addable corner. The k**-residue** of any cell (or square) (i, j) in a skew-shape λ/μ is $j - i \mod k$. That is, the integer in this cell (or square) when λ/μ is periodically labeled with $0, 1, \ldots, k - 1$, where zeros fill the main diagonal. For example, cell $(1, 5)$ has 5-residue 4:



Remark 1. *No two removable corners of any partition fitting inside the shape* $\left(m^{k-m+1}\right)$ *have the same* $k+1$*-residue for any* $1 \leq m \leq k$.

3. Involution on k-bounded partitions

Usual partition conjugation defined by the column reading of diagrams does not send the set of k-bounded partitions to itself. Thus our study of $\mathcal{P}^k$ begins with the need for a degree-preserving involution on this set that extends the notion of conjugation. Such an involution on $\mathcal{P}^k$ was defined in [2] using a certain subset of skew-diagrams. We shall follow the notation of [5], where these skew diagrams are defined by:

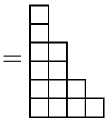
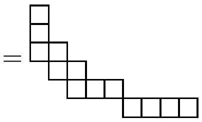
Definition 2. *The* k**-skew diagram** *of a* k*-bounded partition* λ *is the skew diagram, denoted* $\lambda/^k$*, satisfying the conditions:*

(i) row i *of* $\lambda/^k$ *has length* λ_i

(ii) no cell in $\lambda/^k$ *has a hook-length exceeding* k

(iii) every square below the diagram of $\lambda/^k$ *has hook-length exceeding* k

Example 3. *Given $\lambda = (4,3,2,2,1,1)$ and $k=4$,*

$\lambda =$  $\Longrightarrow$ $\lambda/^4 =$ 

It was shown in [5] that $\lambda/^k$ is the unique skew diagram obtained recursively by:

> For any $\lambda = (\lambda_1, \dots, \lambda_\ell) \in \mathcal{P}^k$, $\lambda/^k$ can be obtained by adding to the bottom of $(\lambda_2, \dots, \lambda_\ell)/^k$, a row of λ_1 cells whose first (*i.e.* leftmost) cell s occurs in the leftmost column where $h_s \le k$. That is, row λ_1 lies as far to the left as possible without violating Condition (ii) of $\lambda/^k$, or without creating a non-skew diagram.

As a matter of curiosity, with the skew diagram $\lambda/^k = \gamma/\rho$, it is shown in [5] that a bijection between k-bounded partitions and $k+1$-cores arises by taking $\lambda \to \gamma$.

Since the columns of a k-skew diagram form a partition [5], and the transpose of a k-skew diagram clearly satisfies Conditions (ii) and (iii), an involution on $\mathcal{P}^k$ arises from the column reading of $\lambda/^k$:

Definition 4. *For any k-bounded partition λ, the* k**-conjugate** *of λ is the partition given by the columns of $\lambda/^k$ and denoted λ^{ω_k}. Equivalently, λ^{ω_k} is the unique k-bounded partition such that $(\lambda/^k)' = \lambda^{\omega_k}/^k$.*

Corollary 5. *For a k-bounded partition λ, we have $(\lambda^{\omega_k})^{\omega_k} = \lambda$.*

Example 6. *With λ as in Example 3, the columns of $\lambda/^4$ give $\lambda^{\omega_4} = (3,2,2,1,1,1,1,1,1)$.*

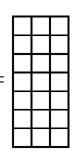
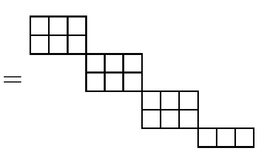
Given a partition λ, the nature of its k-conjugate is not revealed explicitly by Definition 4. However, the k-conjugate can be given explicitly in certain cases. For example,

Remark 7. *If $h_{(1,1)}(\lambda) \le k$, all hooks of λ are k-bounded and thus $\lambda/^k = \lambda$. In this case, $\lambda^{\omega_k} = \lambda'$.*

We can also give a formula for the k-conjugate of a partition with rectangular shape.

Proposition 8. $(m^n)^{\omega_k} = \big((k-m+1)^a, b^m\big)$ *where* $b = n \mod k-m+1$ *and* $a = m\lfloor \frac{n}{k-m+1} \rfloor$.

Proof. Building the k-skew diagram recursively from the partition (m^n) reveals that the top $k-m+1$ rows are stacked in the shape of the rectangle $\square = (m^{k-m+1})$. However, the $k-m+2$-nd row (of size m) cannot lie below any column of $\square$ without creating a $k+1$-hook since all the columns of $\square$ have height $k-m+1$. Thus, it lies strictly to the right of $\square$. By iteration, $(m^n)/^k$ is comprised of a sequence of block diagonal rectangles $\square$ followed by a rectangular block of size $(m^{n \mod k-m+1})$. For example,

$(3^7) =$  $\longrightarrow (3^7)/^4 =$  . (3.1)

The columns of such a skew-shape are thus as indicated. $\square$

The rectangular blocks occurring in the k-skew diagram of (m^n) have the form (m^{k-m+1}) for $1 \le m \le k$. We have found that such rectangles, called k**-rectangles**, play an important role in our study. For starters, we show that k-conjugation can be distributed over the union of any partition and a k-rectangle. To prove this result, we first need to find the k-conjugate of another shape:

Proposition 9. *If $\square = (\ell^{k-\ell+1})$ and $\mu \subseteq (\ell-1)^{k-\ell}$, then*

$$(\square, \mu)/^k = (\square + \mu, \mu)/\mu \quad \textit{implying} \quad (\square, \mu)^{\omega_k} = (\square', \mu')\,. \tag{3.2}$$

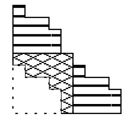


FIGURE 1. $(\square + \mu, \mu)/\mu$, with μ depicted by the partition in horizontal stripes

Proof. Given $\mu \subseteq (\ell - 1)^{k-\ell} \subseteq \square$, $D = (\square + \mu, \mu)/\mu$ is a skew diagram with general shape depicted in Figure 1. If D meets Conditions (ii) and (iii) for a k-skew diagram then $(\square, \mu)/^k = (\square + \mu, \mu)/\mu$ since the rows of D are given by the partition $(\square, \mu)$ and thus $(\square, \mu)^{\omega_k} = (\square', \mu')$ since the columns of D are $(\square', \mu')$. Any square below D has $k - \ell + 1$ cells above it and ℓ to the right implying it has hook $k + 1 > k$. On the other hand, any cell in D has hook-length strictly smaller than this since the columns and rows of D are weakly decreasing. Therefore $D = (\square, \mu)/^k$. □

Theorem 10. $(\lambda \cup \square)^{\omega_k} = \lambda^{\omega_k} \cup \square^{\omega_k}$ *for any k-bounded partition λ and k-rectangle $\square$.*

Proof. Let i be such that $\lambda_i < \ell$ and $\lambda_{i-1} \geq \ell$, and let μ denote the non-skew partition determined by the cells strictly above the bottom row of $(\ell, \lambda_i, \ldots, \lambda_{\ell(\lambda)})/^k$ (Figure 2). The squares in regions (a) and (3) in

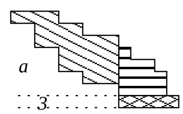


FIGURE 2. $(\ell, \lambda_i, \ldots, \lambda_{\ell(\lambda)})/^k$ with μ depicted by the partition in horizontal stripes

the picture have hooks exceeding k by definition of k-skews. Now consider Figure 3 where the diagram on

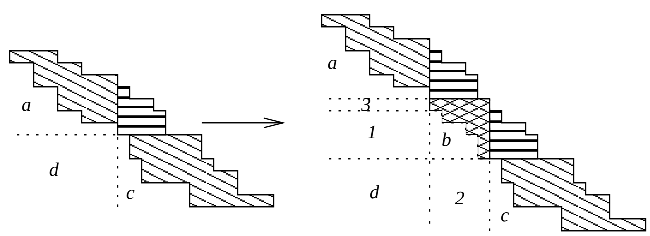


FIGURE 3. Comparison of $\lambda/^k$ and D

the left is $\lambda/^k$ and in the diagram D on the right, region (b) is of shape μ. Since the rows of D are $\lambda \cup \square$, if we prove that D is a k-skew diagram, then $D = (\lambda \cup \square)/^k$. This will then imply $(\lambda \cup \square)^{\omega_k} = \lambda^{\omega_k} \cup \square^{\omega_k}$ since the columns of D are just the columns of $\lambda/^k$ and $\square$.

The hooks in regions (a) and (c) of D are the same as they are in $\lambda/^k$, and thus exceed k. Similarly, the hooks to the right and above (a) and (c) are k-bounded in D. The squares in region (3) lie "below" the k-skew diagram $(\ell, \lambda_i, \ldots,)/^k$ and thus exceed k, as do all squares in regions 1 and d since they can only increase given that the rows of D form a partition. The cells of region (2) also exceed k since there are $k - \ell + 1$ cells above them and at least ℓ to their right.

Finally, the subdiagram of D including region (b) and all cells above and to the right of this region is the diagram of $(\square + \mu, \mu)/\mu$. Since $\mu_1 < \ell$ and $\ell(\mu) \leq k - \ell$ by definition, Proposition 9 implies that this subdiagram is $(\square, \mu)/^k$ and thus meets the conditions of a k-skew. Therefore, D is a k-skew diagram and the theorem follows. □

Ideas to understand the nature of a k-skew diagram containing a k-rectangle that were used in Theorem 10 may be applied to prove the following technical proposition to be used later.

Proposition 11. *For some $1 \leq \ell \leq k$, if ν is a partition containing exactly $k-\ell+1$ rows of length ℓ, where the lowest occurs in some row r, then there are addable corners in row r and $k-\ell+1+r$ of $\nu/^k$ with the same $k+1$-residue.*

Proof. Let $\square = (\ell^{k-\ell+1})$ and $\nu = \lambda \cup \square$ for some partition λ with no parts of size ℓ. We can construct the diagram of $\nu/^k$ as in the previous proof. That is, let i be such that $\lambda_i < \ell$ and $\lambda_{i-1} > \ell$, and let μ denote the non-skew partition determined by the cells strictly above the bottom row of $(\ell, \lambda_i, \ldots, \lambda_{\ell(\lambda)})/^k$. We appeal to Figure 3, where the diagram of $(\lambda \cup \square)/^k$ is on the right.

Note that if r denotes the lowest row in $\nu/^k$ of length ℓ, then row $r-1$ (if it exists) is strictly longer than row r since $\lambda_{i-1} > \ell$. Therefore an addable corner x occurs in row r with some $k+1$-residue j. Furthermore, since row $k-\ell+1+r$ corresponds to row λ_{i-1} (the first row of μ) and $\lambda_{i-1} < \ell$, there is also an addable corner $\bar{x}$ in this row. Thus, it remains to show that $\bar{x}$ has $k+1$-residue j. Let s denote the first cell in row r. If $\bar{x}$ lies in the column of s, then the hook length of cell s is $k-\ell+\ell = k$, and thus $\bar{x}$ also has $k+1$-residue j. We shall now see that $\bar{x}$ does in fact lie in the column containing s. If $\bar{x}$ lies in a column to the right of s, then since $\bar{x}$ is an addable corner, the hook-length of s is larger than k (a contradiction). And if $\bar{x}$ lies in a column to the left of s, then the square in the row of x and the column of $\bar{x}$ has hook-length equal to at most $k-\ell+\ell = k \not> k$. □

4. k-Young lattices

Recall that the Young Lattice Y is the poset of all partitions ordered by inclusion of diagrams, or equivalently $\lambda \leq \mu$ when $\lambda \subseteq \mu$. Since $\lambda \subseteq \mu \iff \lambda' \subseteq \mu'$, it is equivalent to view the Young order as $\lambda \leq \mu$ when $\lambda \subseteq \mu$ *and* $\lambda' \subseteq \mu'$. This interpretation for the Young order refines naturally to an order on k-bounded partitions by using the the k-conjugate on $\mathcal{P}^k$.

Definition 12. *The order $\preceq$ on partitions in $\mathcal{P}^k$ is defined by the transitive closure of the relation*

$$\mu \prec\!\!\cdot\, \lambda \quad \textit{when} \quad \lambda \subseteq \mu \quad \textit{and} \quad \lambda^{\omega_k} \subseteq \mu^{\omega_k} \;\; \textit{and} \;\; |\mu| - |\lambda| = 1\,. \tag{4.1}$$

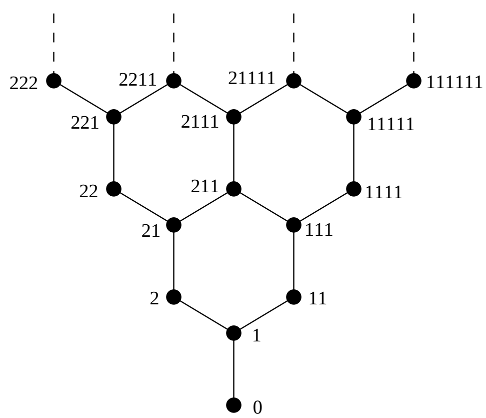


Figure 4. Hasse diagram of the k-Young lattice in the case $k = 2$.

We denote this poset on $\mathcal{P}^k$ by Y^k, and observe that it is a weak subposet of the Young lattice (recall this means that if $\lambda \preceq \mu$ in Y^k, then $\lambda \preceq \mu$ in Y). Furthermore, Y^k reduces to the Young lattice when $k \to \infty$ since [5]:

Property 13. *$\lambda \preceq \mu$ reduces to $\lambda \leq \mu$ when λ and μ are partitions with $h_{(1,1)}(\lambda) \leq k$ and $h_{(1,1)}(\mu) \leq k$.*

While this poset on k-bounded partitions originally arose in connection to a rule for multiplying generalized Schur functions [2], it has been shown in [5] that this poset turns out to be isomorphic to the weak order on the quotient of the affine symmetric group by a maximal parabolic subgroup. Consequently, Y^k is a lattice [11, 13] and we thus call it the *k-Young lattice.*

Although the ordering $\preceq$ is defined by the covering relation $\prec\!\!\cdot$, it follows from the definition that

Property 14. *If $\lambda \preceq \mu$, then $\lambda \subseteq \mu$ and $\lambda^{\omega_k} \subseteq \mu^{\omega_k}$.*

It is important to note that the converse of this statement does not hold. For example, with $k = 3$, $\lambda = (2,2)$, and $\mu = (3,2,1,1,1,1)$: $\lambda^{\omega_k} = \lambda$ and $\mu^{\omega_k} = \mu$ satisfy $\lambda \subseteq \mu$ and $\lambda^{\omega_k} \subseteq \mu^{\omega_k}$, but $\lambda \not\preceq \mu$ (see Theorem 20 and note that λ contains the 3-rectangle $(2,2)$ while μ does not).

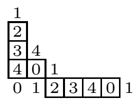
Since the set of μ such that $\mu \subseteq \lambda$ and $|\mu| = |\lambda| - 1$ consists of all partitions obtained by removing a corner box from λ, the set of elements covered by λ with respect to $\preceq$ is a subset of these partitions. The corners that can be removed from λ to give partitions covered by λ are determined as follows:

Theorem 15. [5] *The order $\preceq$ can be characterized by the covering relation*

$$\lambda \prec\!\!\cdot\, \mu \quad\Longleftrightarrow\quad \mu = \lambda + e_r\,, \tag{4.2}$$

where r is any row of $\mu/^k$ with a removable corner whose $k{+}1$-residue does not occur in a higher removable corner, or equivalently for r a row in $\lambda/^k$ with an addable corner whose $k+1$-residue does not occur in a higher addable corner.

Example 16. *With $k = 4$ and $\lambda = (4,2,1,1)$,*

$\lambda/^4 =$  , (4.3)

and thus the partitions that are covered by λ are $(4,1,1,1)$, and $(4,2,1)$, while those that cover it are $(4,2,1,1,1)$ and $(4,2,2,1)$.

Since the conditions of Theorem 15 are always satisfied when choosing the removable corner in the top row of μ we have the corollary:

Corollary 17. *If $\mu = (\mu_1, \ldots, \mu_\ell)$ is a k-bounded partition, then $\lambda \preceq \mu$ for $\lambda = (\mu_1, \ldots, \mu_\ell - 1)$.*

The conditions of Theorem 15 imply that $\mu = \lambda + e_r$. Therefore:

Property 18. *Any row of $\mu/^k$ containing a removable corner whose $k+1$-residue does not occur in a higher removable corner, corresponds to a row of μ with a removable corner.*

As discussed in the introduction, the k-rectangles play a fundamental role in the study of k-Young lattices. Since k-conjugate distributes over the union of k-rectangles with a partition, and the k-Young lattice relies on k-conjugates, we are able to show that the order is preserved under union with a k-rectangle.

Proposition 19. *For any k-rectangle $\square$, $\lambda \preceq \mu$ if and only if $(\lambda \cup \square) \preceq (\mu \cup \square)$.*

Proof. It suffices to consider the case that $\lambda \prec\!\!\cdot\, \mu$. Given $\lambda \subseteq \mu$ and $\lambda^{\omega_k} \subseteq \mu^{\omega_k}$ with $|\mu| - |\lambda| = 1$, clearly $(\lambda \cup \square) \subseteq (\mu \cup \square)$ with $|\mu \cup \square| - |\lambda \cup \square| = 1$. Theorem 10 then implies that $(\lambda \cup \square)^{\omega_k} = (\lambda^{\omega_k} \cup \square^{\omega_k}) \subseteq (\mu^{\omega_k} \cup \square^{\omega_k}) = (\mu \cup \square)^{\omega_k}$. That is, $(\lambda \cup \square) \prec\!\!\cdot\, (\mu \cup \square)$. $\square$

In fact, we have a stronger result that amounts to saying the k-rectangles $\square$ play a trivial role when moving up in the k-Young lattice. That is, the partitions dominating $\lambda \cup \square$ can be obtained by adding the parts of $\square$ to the partitions that dominate λ. The (increasing) covering relations around λ and $\lambda \cup \square$ are isomorphic, and thus these relations are preserved under translation by a k-rectangle.

Theorem 20. *For $\lambda, \mu \in \mathcal{P}^k$ and a k-rectangle $\square$,*

$$\lambda \cup \square \preceq \mu \iff \mu = \bar\mu \cup \square \quad \textit{and} \quad \lambda \preceq \bar\mu\,,$$

for some k-bounded partition $\bar\mu$.

Proof. Let the k-rectangle be $\square = (\ell^{k-\ell+1})$. ($\Longleftarrow$) follows from Proposition 19. For ($\Longrightarrow$), it suffices to consider $\lambda \cup \square \prec\!\!\cdot\, \mu$. Theorem 15 implies that $\mu = (\lambda \cup \square) + e_r$ for some row r with an addable corner o whose $k+1$-residue does not occur in any higher addable corner of $(\lambda \cup \square)/^k$. Assume by contradiction that $(\lambda \cup \square) + e_r \neq \bar\mu \cup \square$ for any k-bounded partition $\bar\mu$. The only scenario where the number of rows of length ℓ is reduced by adding a box is if $\lambda \cup \square$ has exactly $k - \ell + 1$ rows of length ℓ and row r is the lowest row of length ℓ. Thus, by Proposition 11, there is an addable corner in row $k-\ell+1+r$ of $(\lambda\cup\Box)/^k$ with the same $k+1$-residue as o. However, row $k-\ell+1+r$ is higher than row r and by contradiction, $\mu=\bar\mu\cup\Box$ for some $\bar\mu$. Finally, given $\lambda\cup\Box\preceq\bar\mu\cup\Box$, the previous proposition implies $\lambda\preceq\bar\mu$. □

Remark 21. *Consider $\lambda,\mu\in\mathcal{P}^k$ and a k-rectangle $\Box$. Notice that in general,*

$$\lambda \prec\!\!\cdot\, \mu\cup\Box \not\Longleftrightarrow \lambda=\bar\lambda\cup\Box \text{ and } \bar\lambda\prec\!\!\cdot\,\mu\,.$$

For example, with $k=3$: the 3-rectangle $(2,1)\prec\!\!\cdot\,(2,2)$ while $(2,1)\not\supseteq(2,2)$. However,

$$\Box\subseteq\mu \text{ and } \lambda\prec\!\!\cdot\,\mu\cup\Box \iff \lambda=\bar\lambda\cup\Box \text{ and } \bar\lambda\prec\!\!\cdot\,\mu$$

follows from Proposition 19. In this case, what occurs above and below μ is replicated at $\mu\cup\Box$. Interpreting the poset as a cone in a tiling of k-space by permutahedrons [11], this implies that a vertex μ lying at least a distance $|\Box|$ from the boundary of the cone can not be distinguished from $\mu\cup\Box$, and thus the k-rectangles are the vectors of translation invariance in the tiling.

5. Principal order ideal

Let Y_λ denote the principal order ideal generated by λ in the Young lattice. When λ is a rectangle, the order ideal is denoted $L(m,n)$ and is the induced poset of partitions with at most n parts and largest part at most m. This order ideal is a graded, self-dual, and distributive lattice. Further, the rank-generating function of $L(m,n)$, the Gausssian polynomial, is known to be unimodal. The next several sections concern the study of properties for order ideals in the k-Young lattice that are analogous to those held by $L(m,n)$.

Let Y^k_λ denote the principal order ideal generated by λ in the poset Y^k. That is,

$$Y^k_\lambda=\{\mu:\mu\preceq\lambda\}\ .$$

As $k\to\infty$, the poset Y^k_λ reduces to Y_λ. More precisely,

Property 22. *If λ is a partition with $h_{(1,1)}(\lambda)\le k$, then $Y^k_\lambda=Y_\lambda$.*

Proof. Since $\mu\preceq\lambda$ implies that $\mu\subseteq\lambda$, we have that $h_{(1,1)}(\mu)\le h_{(1,1)}(\lambda)\le k$. Thus, for all $\mu,\nu\in Y^k_\lambda$ we have $\mu\preceq\nu \iff \mu\le\nu$ by Property 13, that is, $Y^k_\lambda=Y_\lambda$. □

Proposition 23. *Y^k_λ is graded of rank $|\lambda|$.*

Proof. If $\lambda^{(1)},\lambda^{(2)},\ldots\lambda^{(n)}$ is a saturated chain in Y^k_λ then $|\lambda^{(i)}|=|\lambda^{(i+1)}|-1$ from the definition of the order $\preceq$. Corollaries 17 implies that a maximal chain in Y^k_λ must begin with the empty partition. Therefore, since by definition of Y^k_λ all maximal chains start with λ, we have our claim. □

We are interested in proving properties of Y^k_λ when λ is a rectangular partition, denoted by:

Definition 24. *The principal order ideal of Y^k generated by the partition m^n will be denoted*

$$L^k(m,n)=\{\mu:\mu\preceq(m^n)\}\ . \tag{5.1}$$

Note that $m\le k$ since Y^k contains only elements of $\mathcal{P}^k$. Further,

Remark 25. *From Property 22, $L^k(m,n)=L(m,n)$, for $n\le k-m+1$. Therefore, all cases are covered when considering $n\ge k-m+1$. That is, all $L^k(m,n)$ distinct from $L(m,n)$, plus the non distinct case $L^{n+m-1}(m,n)=L(m,n)$.*

We shall prove that $L^k(m,n)$ is a graded, self-dual, and distributive lattice. Further, we shall conjecture that its rank-generating function is unimodal in certain cases. To start, following from Proposition 23,

Corollary 26. *$L^k(m,n)$ is graded of rank mn.*

It will develop that for each k, the principal order ideal generated by (m^n) in Y^k is isomorphic to an induced subposet of the principal order ideal generated by (m^n) in the Young Lattice. From this, we can then derive a number of properties for the posets $L^k(m,n)$. To this end, we first explicitly determine the vertices of the order ideal.

Theorem 27. *The set of partitions in $L^k(m,n)$ are those that fit inside an $m\times n$ rectangle and have no more than $k-m+1$ rows shorter than m.*

Proof. We first show that any $\lambda\in L^k(m,n)$ can have at most $k-m+1$ rows of length shorter than m. Suppose the contrary, and note that the top $k-m+2$ rows of $\lambda/^k$ form a partition since the rows are all shorter than m and thus no hook exceeds k. Therefore, the first column of $\lambda/^k$ has height at least $k-m+2$ and λ^{ω_k} has a row of length at least $k-m+2$ by definition of k-conjugate. However, the rows of $(m^n)^{\omega_k}$ do not exceed $k-m+1$ by Proposition 8 and thus $\lambda^{\omega_k}\not\subseteq(m^n)^{\omega_k}$. Therefore $\lambda\notin L^k(m,n)$ by Property 14.

On the other hand, to prove that any $\lambda=(m^a,\mu)\subseteq m^n$ with $\mu\subseteq(m^{k-m+1})$ lies in $L^k(m,n)$, it suffices to prove $\lambda\in L^k(m,\ell(\lambda))$ since $m^{\ell(\lambda)}\preceq m^n$ – i.e. using Corollary 17 m times, we obtain $m^{n-1}\preceq m^n$, and by iteration $m^{\ell(\lambda)}\preceq m^n$. To this end, note that the top $\ell(\mu)\le k-m+1$ rows of $(m^{\ell(\lambda)})/^k$ fit inside the shape (m^{k-m+1}) implying every diagonal has a distinct $k+1$-residue by Remark 1. Therefore, Theorem 15 implies removable corners can be successively removed from the top $\ell(\mu)$ rows in $(m^{\ell(\lambda)})$ to obtain partitions $\lambda^{(1)},\dots,\lambda^{(i)}$ where $\lambda\prec\lambda^{(1)}\prec\cdots\prec\lambda^{(i)}\prec(m^{\ell(\lambda)})$. □

The theorem reveals that when $n\ge k-m+1$, the elements of $L^k(m,n)$ are of the form (m^a,μ) for $\mu\subseteq(m^{k-m+1})$ and $a\le n-(k-m+1)$. By Remark 25, we have thus identified all the vertices.

Corollary 28. *For $n\ge k-m+1$ and $m\le k$, the set of partitions in $L^k(m,n)$ is the disjoint union*

$$\left\{\mu\subseteq(m^{k-m+1})\right\}\quad\bigcup_{i=1}^{n-(k-m+1)}\quad\left\{\left(m^i,\mu_1+1,\dots,\mu_{k-m+1}+1\right):\mu\subseteq(m-1)^{k-m+1}\right\}.$$

It also follows from Theorem 27 that the vertices of $L^m(m,n)$ are simply the partitions with at most one row smaller than m.

Corollary 29. *The set of partitions in $L^m(m,n)$ is*

$$L^m(m,n)=\left\{(m^j,i)\,:\,0\le i\le m\text{ and }0\le j\le n-1\right\}. \tag{5.2}$$

6. Further properties of k-Young lattice ideals

Equipped with a simple characterization of the vertex set of $L^k(m,n)$, we can now investigate a connection between the k-Young lattice and the Young lattice. As it turns out, the principal order ideal generated by (m^n) in Y^k is isomorphic to the induced subposet of $L(m,n)$ containing only the subset of partitions with no more than $k-m+1$ rows smaller than m. Consequently, it is easy to grasp which elements are covered by λ in $L^k(m,n)$ and to deduce that the posets are self-dual and distributive.

Proposition 30. *Let $\lambda,\mu\in L^k(m,n)$. Then $\mu\prec\!\!\cdot\,\lambda$ if and only if $\mu<\!\!\cdot\,\lambda$.*

Proof. First consider $\lambda\in L^k(m,n)$ where $\lambda\subseteq(m^{k-m+1})$. Since $h_{(1,1)}(\lambda)\le k$, $\mu\preceq\lambda$ reduces to $\mu\le\lambda$ by Property 13. Any other element of $L^k(m,n)$ has the form $\lambda=(m^b,\nu)$ for some $\nu\in\mathcal{P}^m$ with $\ell(\nu)=k-m+1$ by Corollary 28. Given λ of this form, since $\mu\prec\!\!\cdot\,\lambda$ and $\mu<\!\!\cdot\,\lambda$ both require that $\mu=\lambda-e_r$ where r is a row of λ with a removable corner, we need only consider $b\le r\le b+\ell(\nu)$. Further, $(\lambda-e_r)\notin L^k(m,n)$ if $r=b$ (the partition would have more than $k-m+1$ rows shorter than m). Thus it suffices to show that for $b<r\le b+\ell(\nu)$, there is a removable corner in row r of λ if and only if $(\lambda-e_r)\prec\!\!\cdot\,\lambda$ – equivalently by Theorem 15 – if and only if there is a removable corner in row r of $\lambda/^k$ whose $k+1$-residue does not occur in any higher removable corner.

When row r of $\lambda/^k$ has a removable corner that is the highest of a given $k+1$-residue, there is a removable corner in row r of λ by Property 18. On the other hand, if there is a removable corner in row $b<r\le b+\ell(\nu)$ of λ then there is also a removable corner in this row of $\lambda/^k$ since the top $\ell(\nu)$ rows of $(m^b,\nu)/^k$ coincide with the diagram of ν given that $\ell(\nu)=k-m+1$. Furthermore, $\nu\subseteq(m^{k-m+1})$ also implies that there cannot be another removable corner above the one in row r of the same $k+1$-residue because the diagonals in (m^{k-m+1}) all have distinct $k+1$-residue. □

Theorem 31. *$L^k(m,n)$ is isomorphic to the induced subposet of $L(m,n)$ with vertices restricted to the partitions in $L^k(m,n)$. Equivalently, for $\lambda,\mu \in L^k(m,n)$,*

$$\mu \preceq \lambda \quad \iff \quad \mu \subseteq \lambda\,. \tag{6.1}$$

Proof. Let $\lambda,\mu \in L^k(m,n)$. From Property 14, $\mu \preceq \lambda \implies \mu \subseteq \lambda$. It thus remains to show that there exists a chain from μ to λ in $L^k(m,n)$ when $\mu \subseteq \lambda$. Equivalently by the previous proposition, it suffices to show that we can reach μ by successively adding boxes to λ in such a way that no intermediate step gives a partition with more than $k-m+1$ rows of length less than m. This is achieved as follows: given $\mu \subseteq \lambda$ in $L^k(m,n)$, consider the chain of partitions $\mu=\mu^0 \subseteq \mu^1 \subseteq \cdots \subseteq \mu^j = \lambda$ where μ^{i+1} is obtained by adding one box to the first row in μ^i that is strictly less than the corresponding row in λ. Since the chain starts from $\mu=(m^a,\nu)$ where $\ell(\nu)\le k-m+1$, the number of rows of length less than m does not exceed $k-m+1$ in any μ^i by construction. □

We can now derive a number of properties for the order ideals $L^k(m,n)$ based on the identification with induced subposets of $L(m,n)$ under inclusion of diagrams. First, given the explicit description Eq. (5.2) for the vertices in $L^m(m,n)$ we have

Proposition 32. *The order ideal $L^m(m,n)$ is isomorphic to the saturated chain of partitions:*

$$\emptyset \subseteq (1) \subseteq \cdots \subseteq (m) \subseteq (m,1) \subseteq \cdots \subseteq (m^j,i) \subseteq \cdots \subseteq (m^{n-1},m-1) \subseteq (m^n)\,.$$

Proposition 33. *For $k \ge m$, $L^k(m,n)$ is an induced subposet of $L^{k+1}(m,n)$.*

Proof. From Theorem 27, the elements of $L^k(m,n)$ (or $L^{k+1}(m,n)$) are partitions contained in (m^n) with at most $k-m+1$ (resp. $k-m+2$) rows that are smaller than m. Therefore, by Theorem 31, it suffices to note that under inclusion of diagrams, the poset of partitions that fit inside an $m\times n$ rectangle with no more than $k-m+1$ parts smaller than m is an induced subposet of the poset of partitions that fit inside an $m\times n$ rectangle with no more than $k-m+2$ parts smaller than m. □

Proposition 34. *Let $\lambda,\mu \in L^k(m,n)$.*

(i) $\lambda\wedge\mu$ is the partition determined by the intersection of the cells in the diagrams in λ and μ.

(ii) $\lambda\vee\mu$ is the partition whose diagram is determined by the union of the cells in λ and μ.

Proof. Since the meet and join of elements in $L(m,n)$ is given by the intersection and union of diagrams respectively, and $L^k(m,n)$ is isomorphic to an induced subposet of $L(m,n)$ by Theorem 31, it suffices to show that $L^k(m,n)$ is closed under the intersection and the union of diagrams. Equivalently, by Theorem 27, we must prove that the intersection and union of such partitions do not have more than $k-m+1$ rows with length less than m. Let $\lambda=(m^a,\bar\lambda)$ with $\ell(\bar\lambda)\le k-m+1$ and $\mu=(m^b,\bar\mu)$ with $\ell(\bar\mu)\le k-m+1$ and assume $a\ge b$ without loss of generality. As such, the diagram of λ intersected with μ has at least b rows of length m, and at most $\ell(\bar\mu)$ rows with length less than m. Therefore, there are no more than $k-m+1$ rows of length smaller than m. Similarly, the union has at least a rows of length m, and at most $\max\big\{\ell(\bar\lambda),\ell(\bar\mu)-(a-b)\big\}$ rows less than m. Again, no more than $k-m+1$ rows of length smaller than m. □

Now given that the meet and join of $L^k(m,n)$ coincide with those of $L(m,n)$. Therefore, since $L^k(m,n)$ is an induced subposet of the *lattice* $L(m,n)$, we have[1]

Corollary 35. *For each $k\ge m$, $L^k(m,n)$ is a lattice.*

Furthermore, since $L(m,n)$ is a distributive lattice, the relations

$$\lambda\vee(\mu\wedge\nu)=(\lambda\vee\mu)\wedge(\lambda\vee\nu)\,,\qquad \lambda\wedge(\mu\vee\nu)=(\lambda\wedge\mu)\vee(\lambda\wedge\nu)\,, \tag{6.2}$$

must hold. Therefore, these relations hold in the induced subposets $L^k(m,n)$ and we find

Corollary 36. *For each $k\ge m$, $L^k(m,n)$ is distributive.*

[1]Note that the corollary also follows immediately from the fact that Y^k is a lattice.

In addition to having that each of the $L^k(m,n)$ are distributive lattices, we can also prove that they are symmetric.

Theorem 37. *For any $k \geq m$, $L^k(m,n)$ is self-dual.*

Proof. Let $\bar{L}^k(m,n)$ denote the dual of $L^k(m,n)$ and consider the mapping $\phi(\lambda) = \lambda^c$ where λ^c is the partition determined by rotating the complement of λ in (m^n) by 180^o. Since $\lambda \in L^k(m,n)$ implies that $\lambda = (m^a, \mu)$ for some $a \leq n-(k-m+1)$ and $\mu \subseteq (m^{k-m+1})$ by Corollary 28, $\lambda^c = (m^{n-(k-m+1)-a}, \mu^c)$ for some $\mu^c \subseteq (m^{k-m+1})$ satisfying $|\lambda| + |\lambda^c| = mn$. Therefore $\phi : L^k(m,n) \to \bar{L}^k(m,n)$ and it suffices to show that ϕ is an order-preserving bijection. Equivalently, that $\mu \preceq \lambda$ in $L^k(m,n) \iff \phi(\lambda) \preceq \phi(\mu)$ in $\bar{L}^k(m,n)$. By Theorem 31 and the definition of $\bar{L}^k(m,n)$, this is equivalent to $\mu \subseteq \lambda \iff \lambda^c \subseteq \mu^c$ for elements $\lambda, \mu \in L^k(m,n)$, which is true. □

Corollary 38. *$L^k(m,n)$ is rank-symmetric.*

7. Rank-generating function

The explicit description of the partitions in the posets $L^k(m,n)$ can also be used to determine the rank-generating functions. Recall that the number of elements of rank i in $L(m,n)$ is the coefficient of q^i in the Gaussian polynomial:

$$\sum_{i=0}^{mn} p_i(m,n)\, q^i = \begin{bmatrix} m+n \\ m \end{bmatrix}_q . \tag{7.1}$$

Similarly, we can determine the rank-generating functions for $L^k(m,n)$.

Theorem 39. *For $n \geq k-m+1$ and $m \leq k$, the number of elements of degree i in $L^k(m,n)$ is the coefficient of q^i in*

$$\sum_{i=0}^{mn} p_i^k(m,n)\, q^i := \begin{bmatrix} k+1 \\ m \end{bmatrix}_q + q^{k+1} \frac{1-q^{m(n-k+m-1)}}{1-q^m} \begin{bmatrix} k \\ m-1 \end{bmatrix}_q . \tag{7.2}$$

Proof. Recall that Corollary 28 provides an interpretation for the vertices of $L^k(m,n)$ as a disjoint union of sets whose elements can be understood in terms of certain $L(a,b)$:

$$\{\mu \subseteq (m^{k-m+1})\} \bigcup_{i=1}^{n-(k-m+1)} \left\{ \left(m^i, \mu_1+1, \ldots, \mu_{k-m+1}+1\right) : \mu \subseteq (m-1)^{k-m+1} \right\} .$$

Identity (7.1) then gives that the number of elements of rank i in $L^k(m,n)$ is the coefficient of q^i in

$$\begin{bmatrix} k+1 \\ m \end{bmatrix}_q + \sum_{i=1}^{n-(k-m+1)} q^{m(i)+k-m+1} \begin{bmatrix} k \\ m-1 \end{bmatrix}_q . \tag{7.3}$$

□

Letting $q \to 1$ in Eq. (7.3) then gives the number of vertices:

Corollary 40. *For $n \geq k-m+1$ and $m \leq k$, the number of vertices in $L^k(m,n)$ is*

$$\sum_{i=0}^{mn} p_i^k(m,n) = \binom{k+1}{m} + (n-k+m-1)\binom{k}{m-1} . \tag{7.4}$$

In the next section, we will see that the study of the order ideals $L^k(m,n)$ is simplified by a close examination of certain subsets of the vertices that are determined by k.

Definition 41. *For $n \geq k-m+1$ and $k > m$, let $\Gamma^k(m,n)$ denote the set of all partitions in $L^k(m,n)$ that are not in $L^{k-1}(m,n)$.*

We conclude the section with a discussion of the sets $\Gamma^k(m,n)$, starting with an explicit description of the elements.

Proposition 42. *The elements of $\Gamma^k(m,n)$ are partitions that fit inside an $m\times n$ rectangle and have exactly $k-m+1$ rows with length smaller than m. Equivalently,*

$$\Gamma^k(m,n)=\Big\{(m^a,\mu_1+1,\ldots,\mu_{k-m+1}+1)\,:\,0\le a\le n-(k-m+1)\,,\,\mu\subseteq(m-2)^{k-m+1}\Big\} \tag{7.5}$$

Proof. Theorem 28 indicates that the vertices of $L^k(m,n)$ are partitions with no more than $k-m+1$ rows of length smaller than m while those of $L^{k-1}(m,n)$ are partitions with no more than $k-m$ rows of length smaller than m. The result thus follows. □

Since the vertices of $L^m(m,n)$ are given in Eq. (5.2) to be the partitions with no more than one row of length smaller than m, the set of vertices for $L^k(m,n)$ can be obtained by adding the elements of $\Gamma^i(m,n)$ for $i=m+1,\ldots,k$ to this set of partitions. That is,

Proposition 43. *For $n\ge k-m+1$ and $k>m$, the vertices of $L^k(m,n)$ are the disjoint unions:*

$$L^k(m,n)=L^m(m,n)\ \cup_{j=m+1}^{k}\Gamma^j(m,n)\,. \tag{7.6}$$

Given this decomposition for the set of vertices in $L^k(m,n)$, we note that there are no edges between partitions in $\Gamma^j(m,n)$ and partitions in $\Gamma^{j-i}(m,n)$ for any $i>1$ since adding or deleting a box to a partition with exactly a rows smaller than length m never gives rise to a partition with more than $a+1$ rows smaller than m. Now, $L^{m+1}(m,n)$ can be constructed by connecting elements of $\Gamma^{m+1}(m,n)$ to the appropriate vertices in the saturated chain $L^m(m,n)$ of Proposition 32. $L^{m+2}(m,n)$ can then be constructed by connecting *only* elements of $\Gamma^{m+1}(m,n)$ to the appropriate elements of $\Gamma^{m+2}(m,n)$. In this manner, the poset $L^k(m,n)$ can be constructed from the saturated chain $L^m(m,n)$. In particular, $L(m,n)=L^{m+n-1}(m,n)$ can be obtained using this process (see Figure 5 for an example). In summary,

Remark 44. *The poset $L^k(m,n)$ can be obtained from $L^{k-1}(m,n)$ by adding edges from $\lambda\in\Gamma^{k-1}(m,n)$ to all partitions in $\Gamma^k(m,n)$ that contain or are contained in λ.*

We now obtain the rank-generating function for $\Gamma^k(m,n)$ from Proposition 42 by using the Gaussian polynomial for partitions inside an $(k-m+1)\times(m-2)$ rectangle.

Proposition 45. *For $n\ge k-m+1$ and $k>m$, the rank-generating function for $\Gamma^k(m,n)$ is*

$$\sum_{i\ge0}u_i^k(m,n)\,q^i\ :=\sum_{\lambda\in\Gamma^k(m,n)}q^{|\lambda|}\ =\ q^{k-m+1}\frac{(1-q^{m(n-k+m)})}{(1-q^m)}\begin{bmatrix}k-1\\m-2\end{bmatrix}_q. \tag{7.7}$$

In this equation, $u_i^k(m,n)$ is defined to be the number of elements of degree i in $\Gamma^k(m,n)$.

It turns out that the sequence of coefficients in this expression is rank-symmetric.

Proposition 46. *$u_i^k(m,n)=u_{mn-i}^k(m,n)$ for all $i=0,\ldots,\lfloor\frac{mn}{2}\rfloor$. That is, the vector of coefficients $\vec u^k=(u_0^k(m,n),\ldots,u_{mn}^k(m,n))$ is symmetric about the middle (i.e. $mn/2$).*

Proof. By Eq. (7.7), $u_i^k(m,n)$ is the coefficient of q^i in a product of three rank-symmetric polynomials: q^{k-m+1}, $\frac{(1-q^{m(n-k+m)})}{(1-q^m)}$, and $\begin{bmatrix}k-1\\m-2\end{bmatrix}_q$. Therefore, since the term of lowest degree in the expansion of these three polynomials has degree $k-m+1$ and the highest degree term has degree $k-m+1+m(n-k+m-1)+(m-2)(k-m+1)=mn-k+m-1$, the polynomial must be rank-symmetric about $\big(k-m+1+(mn-k+m-1)\big)/2=mn/2$. □

8. Unimodality and sieved sums

Recall that a poset P of rank d is **unimodal** if

$$p_0\le p_1\le\cdots\le p_i\ge p_{i+1}\ge\cdots\ge p_d \text{ for some } 0\le i\le d\,,$$

where p_i is the number of elements with rank i in P. We shall say the rank-generating function for P is unimodal when the vector $(p_0,\ldots,p_d)$, with p_i the coefficient of q^i, forms a unimodal sequence.

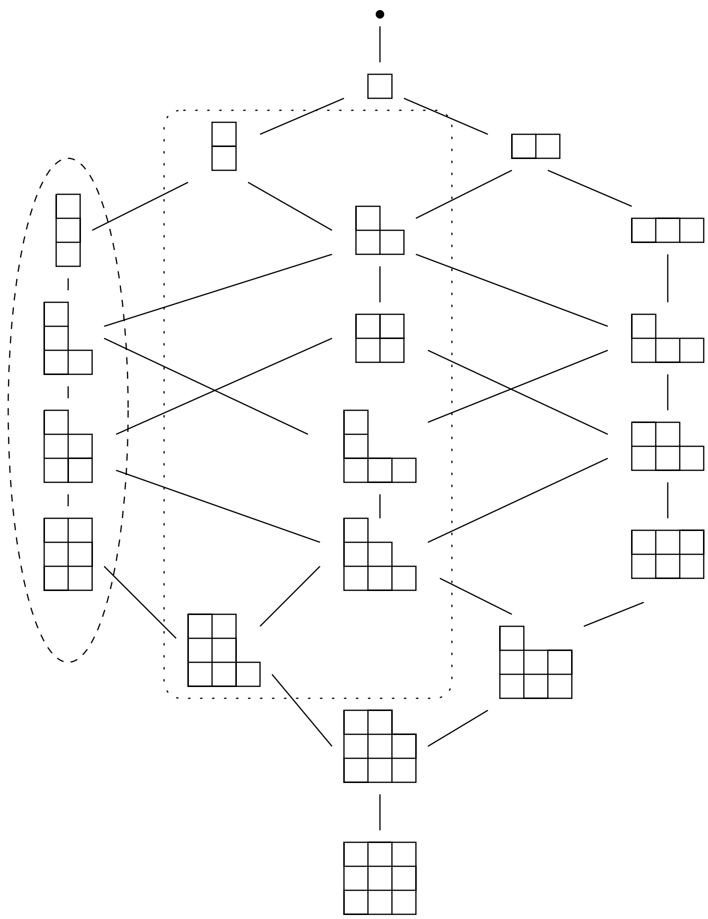


FIGURE 5. Decomposing $L^5(3,3)$ into $L^3(3,3) \cup \Gamma^4(3,3) \cup \Gamma^5(3,3)$

For example, the Gaussian polynomial (7.1) is unimodal because it is known [6, 10] that the coefficients $(p_0, \ldots, p_{mn})$ form a unimodal sequence. Equivalently, $L(m,n)$ is a unimodal poset – one of the deeper properties of this order ideal.

In this section, we address the question of unimodality for the posets $L^k(m,n)$, illustrating the role of k in our study. As such, by Eq. (7.2), the order ideal $L^k(m,n)$ will be unimodal when

$$\sum_{i=0}^{mn} p_i^k(m,n)\, q^i \;=\; \begin{bmatrix} k+1 \\ m \end{bmatrix}_q + q^{k+1} \frac{1-q^{m(n+m-k-1)}}{1-q^m} \begin{bmatrix} k \\ m-1 \end{bmatrix}_q \tag{8.1}$$

is unimodal. Our work in the preceding section pays off here by enabling us to rewrite this expression as a sum of rank-symmetric components.

Proposition 47. *The number of elements in $L^k(m,n)$ is given by the coefficient of q^i in*

$$\sum_{i=0}^{mn} p_i^k(m,n)\, q^i = \sum_{i=0}^{mn} \left(1 \;+ \sum_{r=m+1}^{k} u_i^r(m,n) \right) q^i\,, \tag{8.2}$$

where $u_i^r(m,n)$ is defined in Eq. (7.7).

Proof. Proposition 43 decomposes the set of vertices of $L^k(m,n)$ into $L^m(m,n) \cup_{i=m+1}^k \Gamma^i(m,n)$, implying that the coefficients $p_i^k(m,n)$ occur in the expression:

$$\sum_{i=0}^{mn} p_i^k(m,n)\, q^i = \sum_{\lambda \in L^m(m,n)} q^{|\lambda|} + \sum_{r=m+1}^{k} \sum_{\lambda \in \Gamma^r(m,n)} q^{|\lambda|}\,. \tag{8.3}$$

The result then follows since $L^m(m,n)$ is just a saturated chain by Proposition 32, and the elements in $\Gamma^r(m,n)$ are given in Proposition 45 by $u_i^r(m,n)$. □

Since $u_i^r(m,n) = u_{mn-i}^r(m,n)$ by Proposition 46, (8.2) reveals $\vec{p}^{\,k}(m,n) = (p_0^k(m,n),\ldots,p_{mn}^k(m,n))$ is a sum of sequences symmetric about the middle. Thus, the question of unimodality of $L^k(m,n)$ reduces to a study of the sequences $\vec{u}^{\,r}(m,n)$. We have experimentally discovered that certain sums of these sequences are unimodal under conditions on m expressed in Conjecture 48, and more generally in Conjecture 51. This given, we can deduce that $\vec{p}^{\,k}(m,n)$ is unimodal in these cases and can be written as a sum of rank-unimodal sequences (see Remark 52). To this end, we start with the special case when m is prime.

Conjecture 48. *If $n \geq k-m+1$ and $k \neq -1, 0 \mod m$ for a prime number $m < k$, then*

$$\sum_{i=0}^{mn} u_i^k(m,n)\, q^i \;=\; q^{k-m+1}\, \frac{(1-q^{m(n-k+m)})}{(1-q^m)} \begin{bmatrix} k-1 \\ m-2 \end{bmatrix}_q \tag{8.4}$$

is unimodal. Further, when $n > k-m+1$ and $k = -1 \mod m$,

$$\sum_{i=0}^{mn} \Big(u_i^k(m,n) + u_i^{k+1}(m,n) \Big) q^i$$
$$= q^{k-m+1}\, \frac{(1-q^{m(n-k+m)})}{(1-q^m)} \begin{bmatrix} k-1 \\ m-2 \end{bmatrix}_q + q^{k-m+2}\, \frac{(1-q^{m(n-k+m-1)})}{(1-q^m)} \begin{bmatrix} k \\ m-2 \end{bmatrix}_q \tag{8.5}$$

is unimodal.

There are two consequences of this conjecture; the first relating to the unimodality of $L^k(m,n)$.

Consequence of Conjecture 48. *If $n > k-m+1$ and $k \neq -1 \mod m$ for a prime $m < k$, then the order ideal $L^k(m,n)$ is unimodal. Equivalently,*

$$\sum_{i=0}^{mn} p_i^k(m,n)\, q^i \;=\; \begin{bmatrix} k+1 \\ m \end{bmatrix}_q + q^{k+1} \frac{1-q^{m(n+m-k-1)}}{1-q^m} \begin{bmatrix} k \\ m-1 \end{bmatrix}_q \tag{8.6}$$

is unimodal under these conditions.

Proof following from Conjecture 48. By Eq. (8.2), it suffices to show that $\sum_{j=m+1}^{k} \vec{u}^{\,j}(m,n)$ is unimodal. For all $m < j < n+m-1$, the sequences $\vec{u}^{\,j}(m,n)$ are rank-symmetric about $mn/2$ by Proposition 46. Therefore, if $\vec{u}^{\,j}(m,n)$ is unimodal for $j \neq -1, 0 \mod m$ and $\vec{u}^{\,j}(m,n) + \vec{u}^{\,j+1}(m,n)$ is unimodal when $j = -1 \mod m$, then $\sum_{j=m+1}^{k} \vec{u}^{\,j}(m,n)$ is unimodal unless $k = -1 \mod m$. That is, except in the case that $\vec{u}^{\,j+1}(m,n)$ cannot be added to restore unimodality. □

Interestingly, a second consequence of Conjecture 48 relates to sieved q-binomial coefficients (see for example, [1, 9, 12]). In this result, "the sum of the coefficients of $q^{\ell+*m}$" in an expression refers to the sum of coefficients of q^i for every $i = \ell \mod m$.

Proposition 49. *If $k \neq -1, 0 \mod m$ for a prime $m < k$, then for any $0 \leq \ell \leq m-1$, the sum of the coefficients of $q^{\ell+*m}$ in $\begin{bmatrix} k-1 \\ m-2 \end{bmatrix}_q$ is $\binom{k-1}{m-2}/m$. Equivalently,*

$$\sum_{j\geq 0} p_{\ell+jm}(m-2, k-m+1) = \frac{1}{m}\binom{k-1}{m-2}\,, \quad \textit{for fixed } \ell = 0,\ldots,m-1\,. \tag{8.7}$$

We shall demonstrate how this proposition follows from Conjecture 48 to give evidence supporting the conditions under which we believe unimodality to hold. However, the result is implied from the main result in [12] (and more recently from [9]) as follows:

Proof. Condition 3 of the main result in [12] says that the sum of the coefficients of $q^{\ell+*m}$ in $\begin{bmatrix} n \\ t \end{bmatrix}_q$ is $\binom{n}{t}/m$ for all ℓ iff for all $d > 1$ that divide m, we have $t \mod d > n \mod d$. Since only $d = m$ divides

m when m is prime, letting $n = k-1$ and $t = m-2$, we note that $m-2 \mod m > k-1 \mod m$ is equivalent to the condition $k \neq -1, 0 \mod m$. ☐

Proof following from Conjecture 48. For $k \neq -1, 0 \mod m$ for m prime, Conjecture 48 implies

$$\frac{(1-q^{m(n-k+m)})}{(1-q^m)} \begin{bmatrix} k-1 \\ m-2 \end{bmatrix}_q$$

is unimodal. Letting $n \to \infty$ in this expression then produces an increasing (unimodal and symmetric about infinity) sequence $(v_0^k(m), v_1^k(m), \ldots)$, where

$$\sum_i v_i^k(m)\, q^i = (1 + q^m + q^{2m} + \cdots) \begin{bmatrix} k-1 \\ m-2 \end{bmatrix}_q . \tag{8.8}$$

Using the definition of $p_i(m,n)$ in Eq. (7.1), the coefficient of q^i in this expression satisfy

$$v_i^k(m) = p_i(m-2, k-m+1) + p_{i-m}(m-2, k-m+1) + p_{i-2m}(m-2, k-m+1) + \cdots .$$

This sequence thus meets the conditions in the following proposition with $\vec{a} = \vec{p}(m-2, k-m+1)$, where the claim follows from Eq. (8.9) since $\sum_i p_i(m-2, k-m+1) = \binom{k-1}{m-2}$. ☐

Proposition 50. *Let $v_i = \sum_{t\geq 0} a_{i-tm}$ for some $\vec{a} = (a_0, \ldots, a_D)$ and $a_j = 0$ when $j < 0$ or $j > D$. If the infinite sequence $\vec{v} = (v_0, v_1, \ldots,)$ is weakly increasing, then*

(i) $v_i = v_{D-m+1}$ for $i \geq D-m+1$.

(ii) For any $\ell = 0, \ldots, m-1$,

$$\sum_{j\geq 0} a_{\ell+jm} = \frac{1}{m}\sum_{j=0}^{D} a_j . \tag{8.9}$$

Proof. (i) Since $i + m > D$ when $i \geq D-m+1$, we have $a_{i+m} = 0$. The definition of v_{i+m} implies $v_{i+m} = a_{i+m} + v_i$ and thus $v_{i+m} = v_i$ when $i \geq D-m+1$. To show $v_i = v_{D-m+1}$ for all $i \geq D-m+1$, note that $v_{D-m+1} \leq v_{D-m+2} \leq \cdots \leq v_{D+1}$ since $\vec{v}$ is increasing. However, $v_{D-m+1} = v_{D+1}$ by the preceding argument, implying $v_{D-m+1} = v_{D-m+2} \cdots = v_{D+1}$. By iteration, $v_i = v_{D-m+1}$ for all $i \geq D-m+1$.

(ii) Since $a_i = 0$ for $i < 0$, we have for all $1 \leq \ell \leq m$ that

$$v_{D-\ell+1} = a_{D-\ell+1} + a_{D-\ell+1-m} + \cdots + a_{D-\ell+1-\lfloor (D-\ell+1)/m\rfloor m} . \tag{8.10}$$

That is, $v_{D-\ell+1} = \sum_n a_n$ for $n = D-\ell+1 \mod m$. The sum $v_{D-m+1} + \cdots + v_{D-1} + v_D$ is thus the sum of all entries of $\vec{a}$. However, (i) implies that $v_{D-m+1} = \cdots = v_{D-1} = v_D$ and therefore, $m \cdot v_{D-\ell+1} = \sum_{j=0}^{D} a_j$, for $\ell = 1, \ldots, m$. From Eq. (8.10), $v_{D-\ell+1}$ is the sum of entries in $\vec{a}$ indexed by $D-\ell+1$ modulo m, thus implying the claim since as ℓ runs over 1 to m, $D-\ell+1$ runs over all possible values modulo m. ☐

In summary, when m is prime, proving unimodality of the posets $L^k(m,n)$ reduces to proving the unimodality of $\vec{u}^{\,r}(m,n) + \vec{u}^{\,r+1}(m,n)$. As such, to extend this idea for m not prime, we study the unimodality of more general sums of $\vec{u}^{\,r}(m,n)$.

Conjecture 51. *Consider integers a and b such that $m \leq a < b \leq n+m-1$. If $a, b \neq -1 \mod p$ for every prime divisor p of m, then*

$$\sum_{i=0}^{mn}\left(\sum_{j=a+1}^{b} u_i^j(m,n)\right) q^i \;=\; \sum_{j=a+1}^{b} q^{j-m+1}\frac{(1-q^{m(n-j+m)})}{(1-q^m)}\begin{bmatrix} j-1 \\ m-2 \end{bmatrix}_q \tag{8.11}$$

is unimodal.

We recover Conjecture 48 by taking m prime and letting $a = k-1$ and $b = k$ to get the unimodality of $\vec{u}^k(m,n)$. Further, $a = k-1$ and $b = k+1$ implies the unimodality of $\vec{u}^k(m,n) + \vec{u}^{k+1}(m,n)$. As with the case of m prime, we can extract two consequences of Conjecture 51.

Consequence of Conjecture 51. *Let $k > m$. If $k \neq -1 \mod p$ for every prime divisor p of m, then $L^k(m,n)$ is unimodal for all $n > k-m+1$. Equivalently,*

$$\begin{bmatrix} k+1 \\ m \end{bmatrix}_q + q^{k+1}\frac{1-q^{m(n+m-k-1)}}{1-q^m}\begin{bmatrix} k \\ m-1 \end{bmatrix}_q \tag{8.12}$$

is unimodal under these conditions.

Proof following from Conjecture 51. Given $k \neq -1 \mod p$ for all prime divisors p of m and noting that $m \neq -1 \mod p$, we meet the conditions of Conjecture 51 with $a = m$ and $b = k$. Therefore, $\vec{p}^{\,k}(m,n)$ is unimodal using the decomposition given in Proposition 47. □

Remark 52. *Following trivially from Proposition 47, we can obtain an alternative decomposition for $\vec{p}^{\,k}(m,n)$ in terms of sums of $\vec{u}^{\,r}(m,n)$: for nonnegative integers $k_1 < k_2 < \cdots < k_\ell = k$, when each $k_j \neq -1 \mod p$ for all prime divisors p of $m \leq k$,*

$$\sum_{i=0}^{mn} p_i^k(m,n)\,q^i = \sum_{i=0}^{mn}\left(1 + \sum_{r=m+1}^{k_1} u_i^r(m,n) + \sum_{r=k_1+1}^{k_2} u_i^r(m,n) + \cdots + \sum_{r=k_{\ell-1}+1}^{k_\ell} u_i^r(m,n)\right) q^i \,. \tag{8.13}$$

Thus, by Conjecture 51, the order ideals $L^k(m,n)$ can be decomposed in terms of $\ell+1$ unimodal pieces rank-symmetric about the same point $mn/2$.

Our second consequence is a more general result on sieved q-binomial coefficients. We state this consequence as a proposition since we will provide a proof that is independent of our conjecture.

Proposition 53. *If $m \leq a < b$ are non-negative integers where $a, b \neq -1 \mod p$ for every prime divisor p of m, then the sum of the coefficients of $q^{\ell+*m}$ in $\sum_{j=a+1}^{b} q^{j-(a+1)}\begin{bmatrix} j-1 \\ m-2 \end{bmatrix}_q$ is $\frac{1}{m}\sum_{j=a+1}^{b}\binom{j-1}{m-2}$, for any $0 \leq \ell \leq m-1$.*

Note that even when m is prime, this is more general than Proposition 49. For example, the case that $k = -1 \mod m$ is included when $a = k-1$ and $b = k+1$. Before proving this result, we shall show how it follows from Conjecture 51 to support the validity of our conjecture.

Proof following from Conjecture 51. Letting $n \to \infty$ in Eq. (8.11), (and multiplying by q^{m-a-2} for convenience), the coefficients $(v_0^{a,b}(m), v_1^{a,b}(m), \dots)$ defined by

$$\sum_{i\geq 0} v_i^{a,b}(m)\,q^i \;=\; (1+q^m+q^{2m}+\cdots)\sum_{j=a+1}^{b} q^{j-(a+1)}\begin{bmatrix} j-1 \\ m-2 \end{bmatrix}_q \tag{8.14}$$

form an increasing sequence given Conjecture 51. Thus, defining

$$\sum_{i\geq 0} p_i^{a,b}(m)\,q^i = \sum_{j=a+1}^{b} q^{j-(a+1)}\begin{bmatrix} j-1 \\ m-2 \end{bmatrix}_q \,, \tag{8.15}$$

we have

$$v_i^{a,b}(m) = p_i^{a,b}(m) + p_{i-m}^{a,b}(m) + p_{i-2m}^{a,b}(m) + \cdots \,. \tag{8.16}$$

The proposition thus follows from Proposition 50 with $\vec{a} = (p_0^{a,b}(m), p_1^{a,b}(m), \ldots)$, since $\sum_i p_i^{a,b}(m) = \sum_{j=a+1}^{b}\binom{j-1}{m-2}$. □

Proof. From

$$\sum_{\omega:\omega^m=1}\omega^r = \begin{cases} m & \text{if } m|r \\ 0 & \text{otherwise} \end{cases}\,, \tag{8.17}$$

we have that the sum of coefficients of $q^{\ell+*m}$ in any polynomial $P(q)$ is

$$\frac{1}{m}\sum_{\omega:\omega^m=1}\omega^{-\ell}P(\omega)\,. \tag{8.18}$$

Thus, to prove the sum of the coefficients of $q^{\ell+*m}$ in $\sum_{j=a+1}^{b} q^{j-(a+1)} \begin{bmatrix} j-1 \\ m-2 \end{bmatrix}_q$ is $\sum_{j=a+1}^{b} \binom{j-1}{m-2}/m$, it suffices to prove

$$\frac{1}{m}\sum_{j=a+1}^{b}\sum_{\omega:\omega^m=1}\omega^{-(\ell-j+a+1)}\begin{bmatrix} j-1 \\ m-2 \end{bmatrix}_\omega = \frac{1}{m}\sum_{j=a+1}^{b}\binom{j-1}{m-2}. \tag{8.19}$$

Or equivalently, since the right hand side equals the $\omega = 1$ term in the left hand side,

$$\frac{1}{m}\sum_{j=a+1}^{b}\sum_{\substack{\omega^m=1\\ \omega\neq 1}}\omega^{-(\ell-j+a+1)}\begin{bmatrix} j-1 \\ m-2 \end{bmatrix}_\omega = 0\,. \tag{8.20}$$

To this end, we shall demonstrate that for all ω such that $\omega^m = 1$ and $\omega \neq 1$,

$$\sum_{j=a+1}^{b}\omega^j\begin{bmatrix} j-1 \\ m-2 \end{bmatrix}_\omega = 0 \tag{8.21}$$

by proving this identity holds when ω is a primitive d^{th} root of unity, for all $d|m$ not equal to 1.

If $j \neq -1, 0 \mod d$, then $\begin{bmatrix} j-1 \\ m-2 \end{bmatrix}_\omega = 0$ since the numerator of $\begin{bmatrix} j-1 \\ m-2 \end{bmatrix}_\omega$ has one more zero than its denominator, given that $i \mod d = 0$ for some $j-d+2 \leq i \leq j-1$. On the other hand, when $j = -1 \mod d$ (see Lemma 1(3) of [9])

$$\begin{bmatrix} j-1 \\ m-2 \end{bmatrix}_\omega = \binom{n-1}{m/d-1}, \tag{8.22}$$

for $n = (j+1)/d$. Thus, using

$$\begin{bmatrix} j \\ m-2 \end{bmatrix}_\omega = \frac{1-\omega^j}{1-\omega^{j-m+2}}\begin{bmatrix} j-1 \\ m-2 \end{bmatrix}_\omega,$$

we also have

$$\begin{bmatrix} j \\ m-2 \end{bmatrix}_\omega = \frac{1-\omega^{-1}}{1-\omega}\binom{n-1}{m/d-1}. \tag{8.23}$$

However, the conditions $a, b \neq -1 \mod p$ for every prime divisor p of m imply that $a, b \neq -1 \mod d$ for any $d|m$. Therefore, the cases $a+1 = 0 \mod d$, and $b = -1 \mod d$ cannot be limits in the sum (8.21) implying that if a term of the form $j = -1 \mod d$ occurs in the sum then so does $j+1 = 0 \mod d$. Therefore, since these pairs of terms satisfy

$$\omega^j\begin{bmatrix} j-1 \\ m-2 \end{bmatrix}_\omega + \omega^{j+1}\begin{bmatrix} j \\ m-2 \end{bmatrix}_\omega = \omega^{-1}\binom{n-1}{m/d-1} + \frac{1-\omega^{-1}}{1-\omega}\binom{n-1}{m/d-1} = 0\,,$$

the proposition holds. □

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