The Combinatorics of the Waldspurger Decomposition

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March 5, 2016

Reflection Groups

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Example: $G = \mathfrak{S}_n \subset O(n)$ permuting coordinates

Reducibility

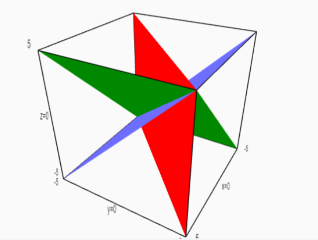
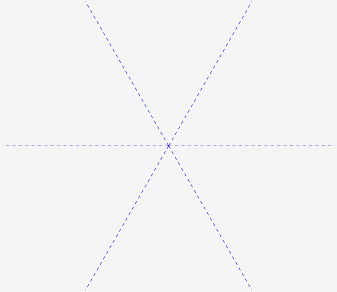


Figure: $\mathfrak{S}_3 \curvearrowright \mathbb{R}^3$



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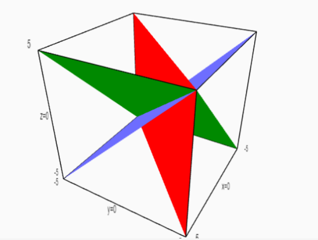

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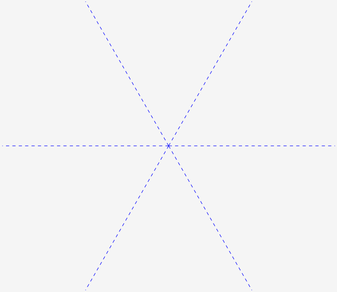

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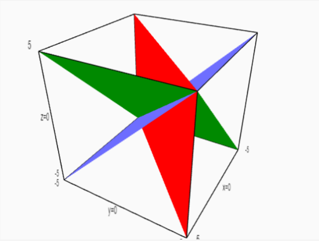

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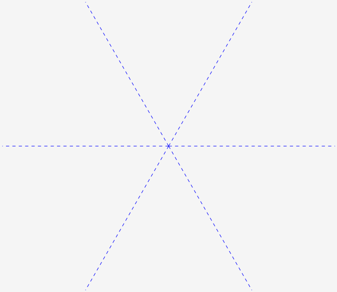

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FACT: Finite reflection groups are completely reducible.

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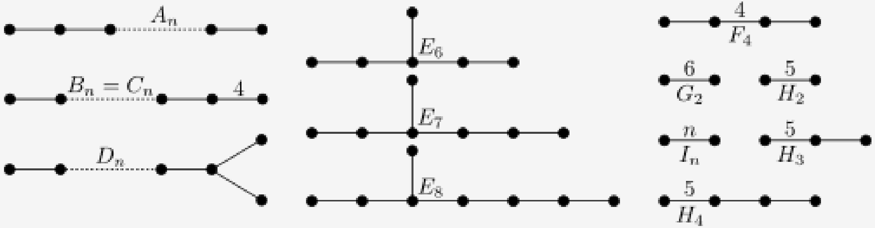
In 1934, building on the work of Möbius, Jordan, Shläfli, Killing, Cartan, and Weyl, HSM Coxeter used diagrams for the classification of finitely generated reflection groups.

Classification of Irreducible Finite Reflection Groups



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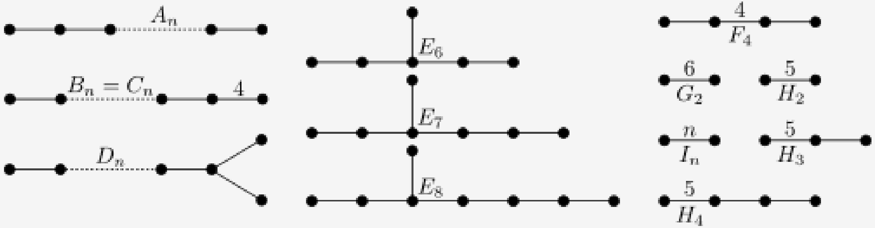

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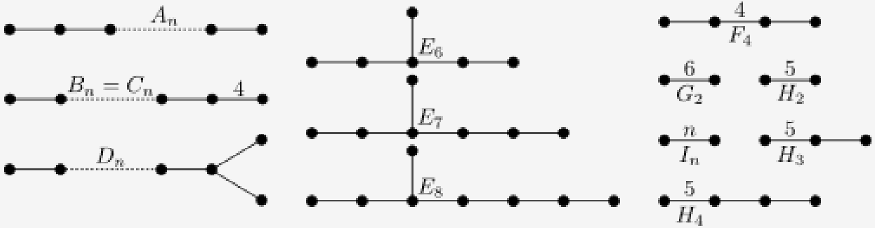

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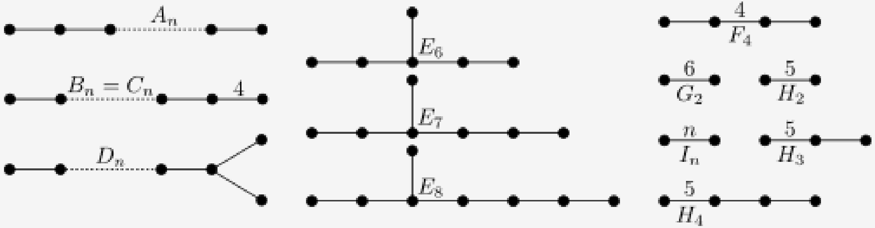

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- Edges labeled k between vertices i and j impose the relation $(S_i S_j)^k = 1$.

Theorem (Coxeter): Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^n$ of reflecting hyperplanes for the reflection group G. Then $G \curvearrowright (\mathbb{R}^n \setminus \cup_{H \in \mathcal{A}} H)$ freely and transitively on the chambers.

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We pick one such chamber and call it the "weight cone" denoted C_W. The cone dual to C_W we call the "root cone" and denote C_R.

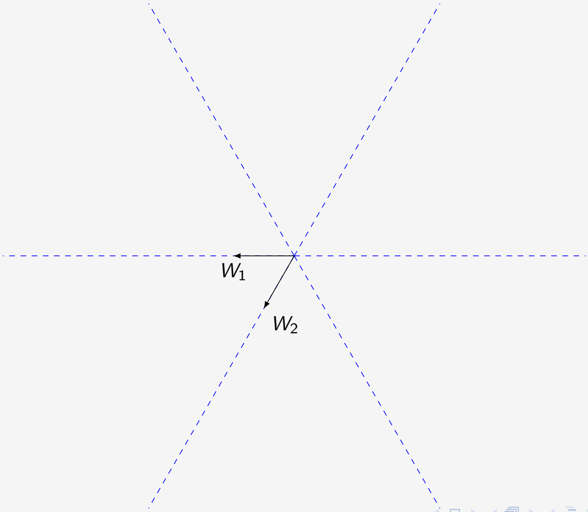
Fact (Coxeter): C_w is a simplicial cone.

Let $w_1, w_2, \ldots, w_n$ be vectors generating the rays of C_W [Jargon: called the "fundamental weights"] Then the dual cone is defined as

$$C_R := \{x \in \mathbb{R}^n : (x, y) \leq 0 \forall y \in C_W\}$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be vectors which generate rays of C_R [Jargon: called the "simple roots"]

The first example A_2 ($\mathfrak{S}_3$)



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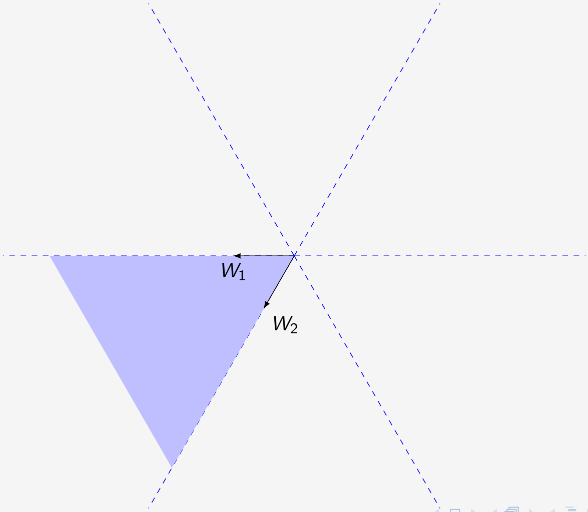

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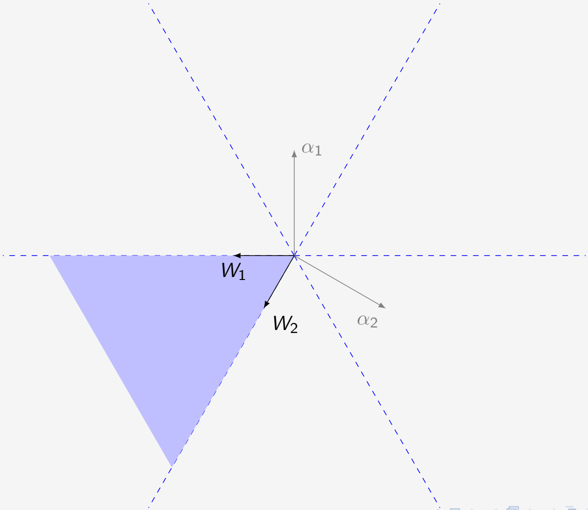

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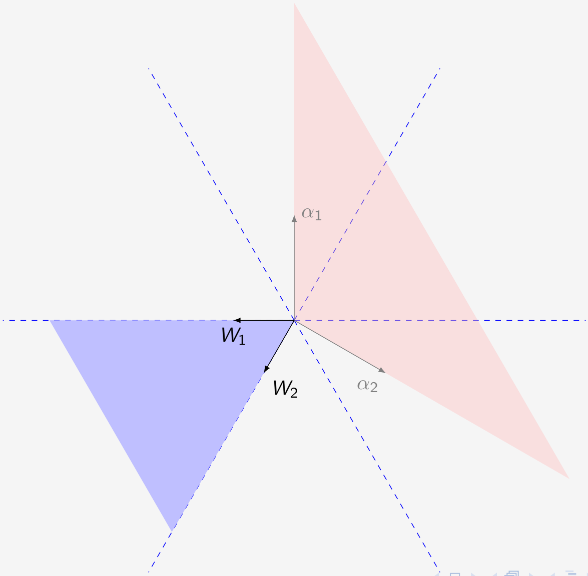

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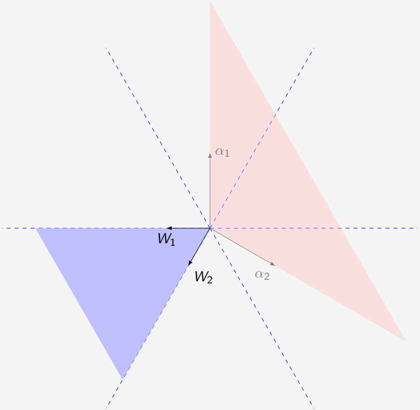

$$3w_1 = -2\alpha_2 - 1\alpha_1$$
$$3w_2 = -1\alpha_2 - 2\alpha_1$$

$$-\alpha_1 = 2w_2 - 1w_1$$
$$-\alpha_2 = -1w_2 + 2w_1$$

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

In type A, it is conventional to let $\alpha_i = e_i - e_{i+1}$ so $(\alpha_i, \alpha_i) = 2 \ \forall i$

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Lengths of weights are then normalized so that

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The matrix of α's is called the Cartan Matrix. It gives the coordinates of the simple roots in the basis of fundamental weights.

Waldspurger's Theorem

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For G a finite reflection group acting on a Euclidean vector space V, C_R the (closed) root cone, and $\mathring{C}_W \subset V$ the interior of the weight cone, one has the following decomposition:

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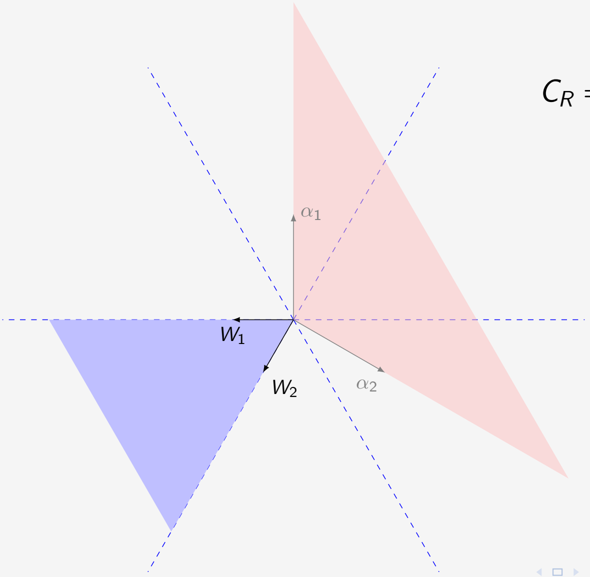
In type A_n what does it tell us about the symmetric group $\mathfrak{S}_{n+1}$?

The Waldspurger Decomposition for A_2 ($\mathfrak{S}_3$)



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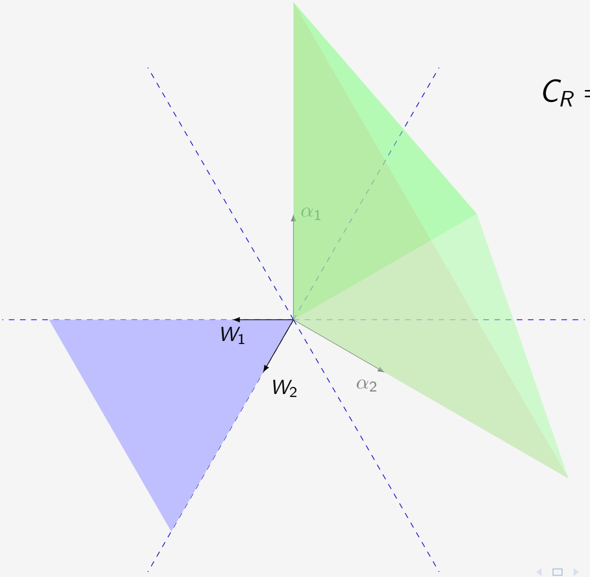

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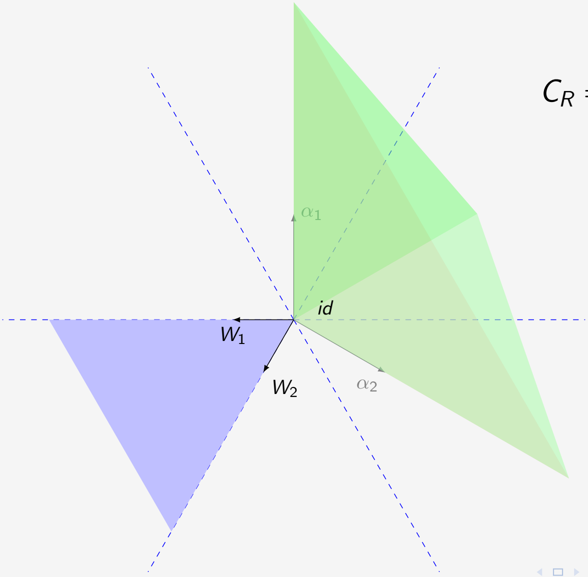

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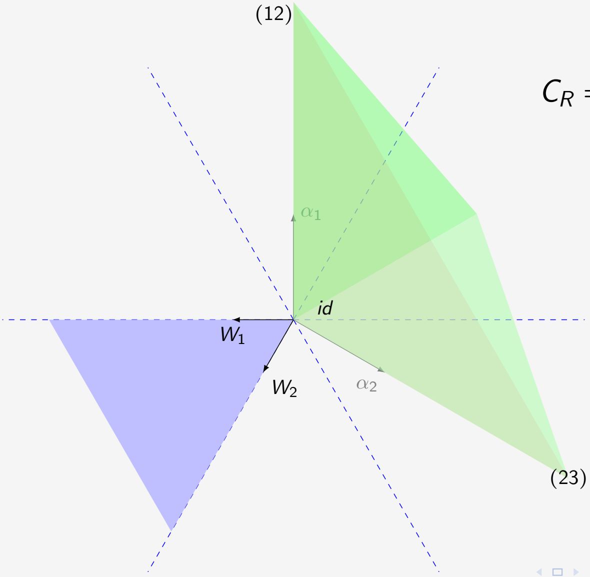

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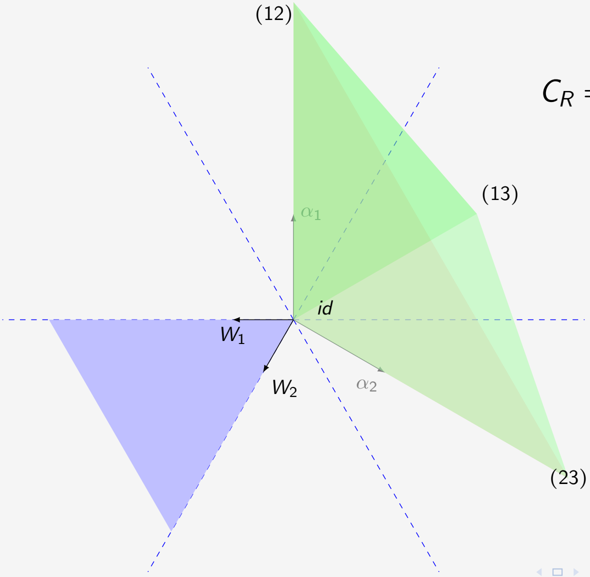

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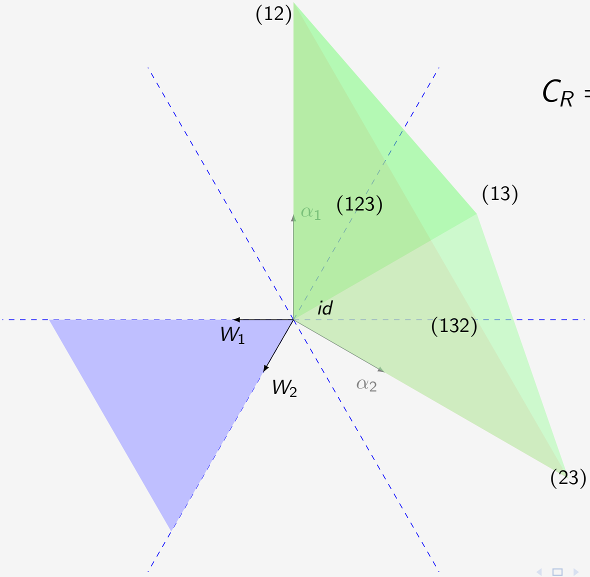

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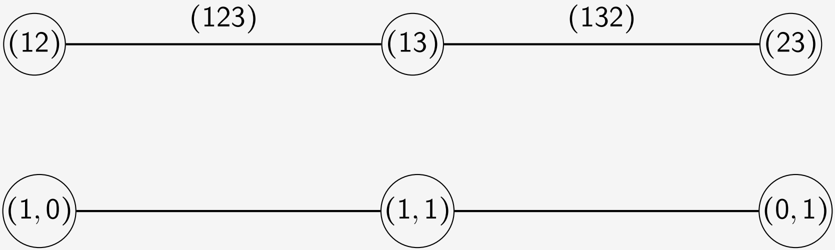
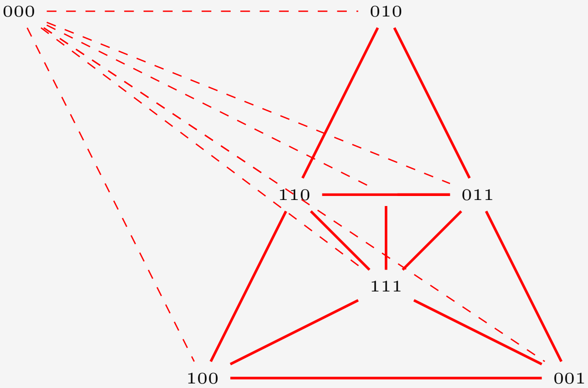
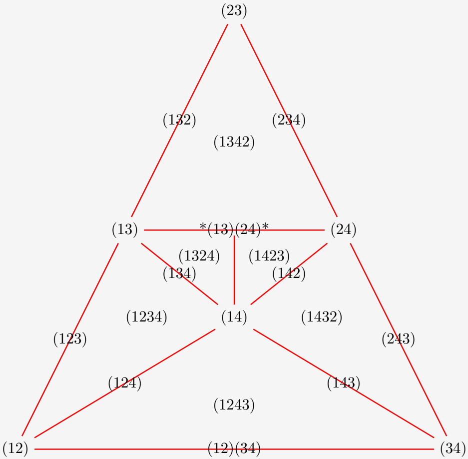
Slice it, put it in root coordinates



The Waldspurger Decomposition for A_3 ($\mathfrak{S}_4$)



Classical Permutation Statistics, something weird...



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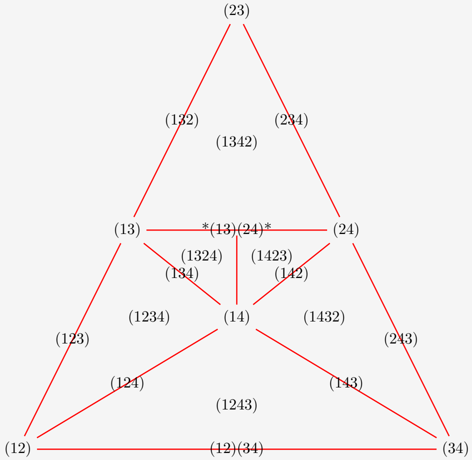

- four copies of $\mathfrak{S}_3$ picture

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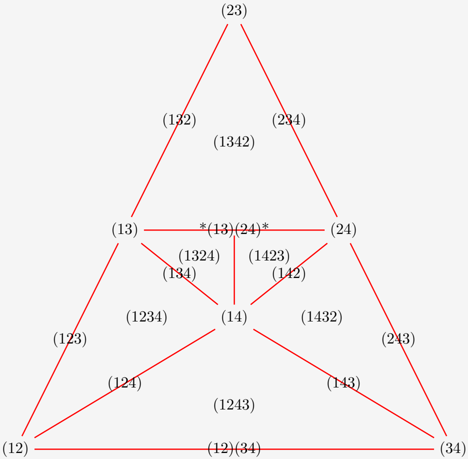

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  - Dimension

$$\updownarrow$$

$\#$ cycles $(c(n,k))$

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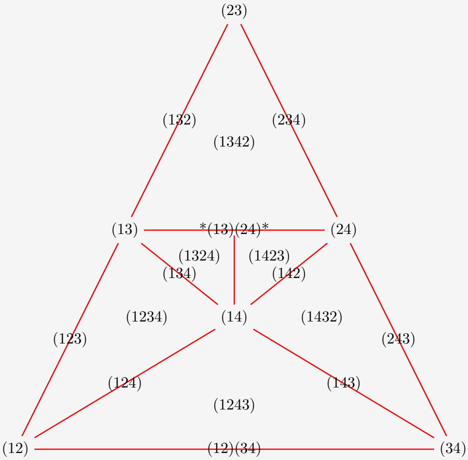

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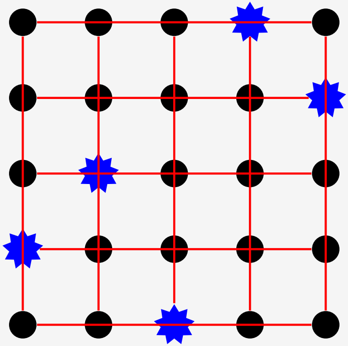
    $\#$ cycles $(c(n,k))$
- NOT A CW-complex!

Theorem (Armstrong, M. 2015): The following algorithm turns linear algebra into combinatorics:

A Cute Algorithm

Consider $43512 \in \mathfrak{S}_5$



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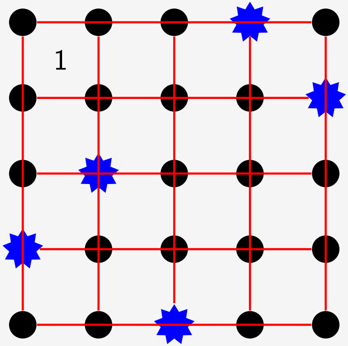

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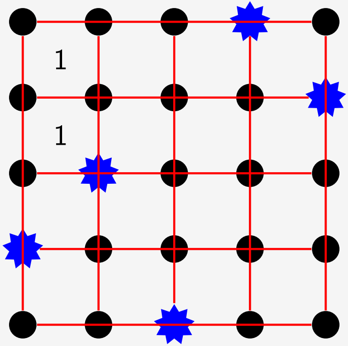

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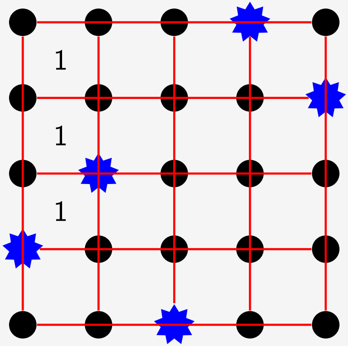

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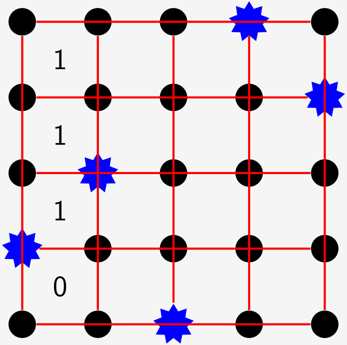

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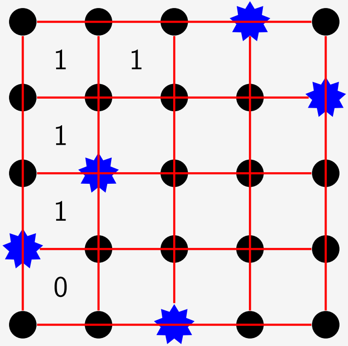

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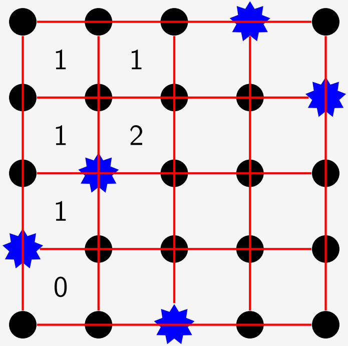

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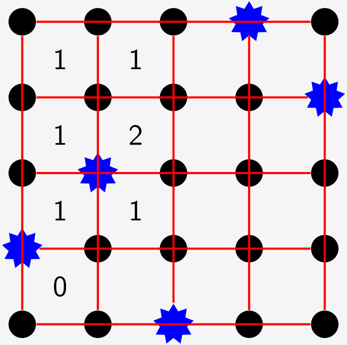

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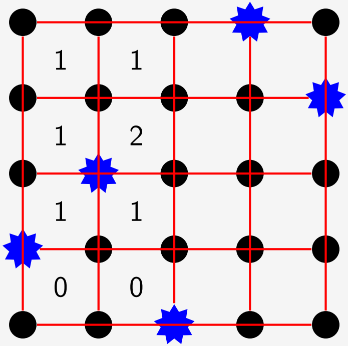

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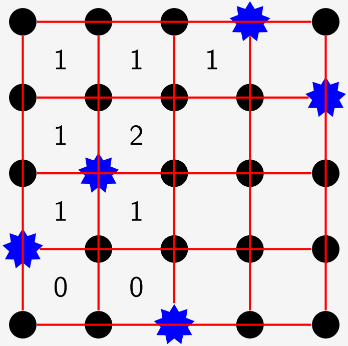

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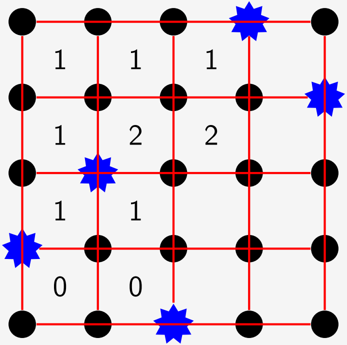

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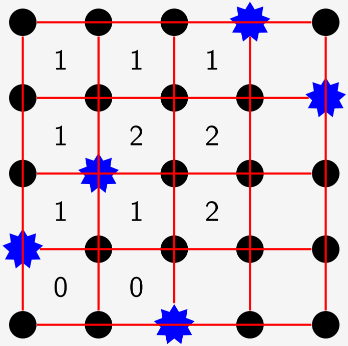

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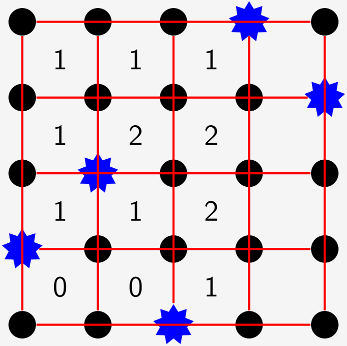

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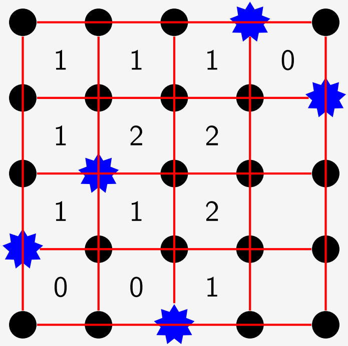

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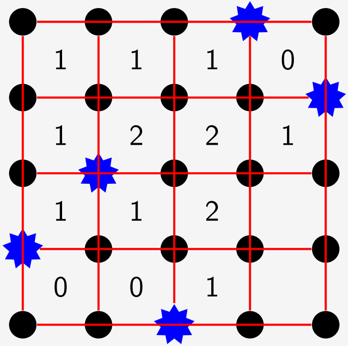

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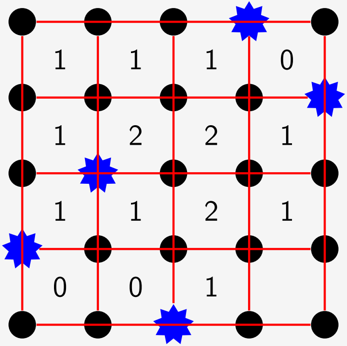

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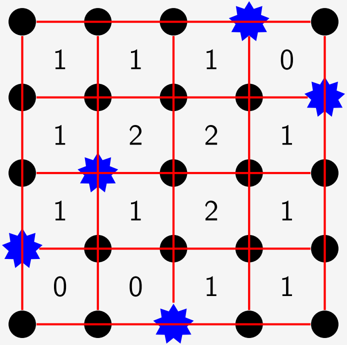

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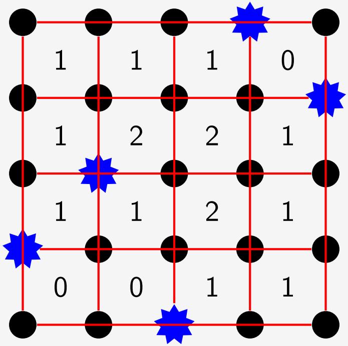

that is,

$$43512 \mapsto \left\{ a\begin{pmatrix}1\\1\\1\\0\end{pmatrix} + b\begin{pmatrix}1\\2\\1\\0\end{pmatrix} + c\begin{pmatrix}1\\2\\2\\1\end{pmatrix} + d\begin{pmatrix}0\\1\\1\\1\end{pmatrix} \middle| a, b, c, d \in \mathbb{R}_{\geq 0} \right\}$$

proof

Proof: Let P be the $(n-1)\times(n-1)$ matrix for the permutation $\pi \in S_n$ expressed in root coordinates. Let C be the $(n-1)\times(n-1)$ Cartan matrix and let D be the $(n-1)\times(n-1)$ matrix

$$D_{i,j} = \begin{cases} \sum\limits_{\substack{a\leq i\\ b>j}} \pi_{a,b} & i \leq j \\ \sum\limits_{\substack{a>i\\ b\leq j}} \pi_{a,b} & i \geq j \end{cases}.$$

We will show $(\mathbf{I}-\mathbf{P}) = \mathbf{DC}$.

proof

We use the fact that $C = A^T A$ where A is the $n \times (n-1)$ matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & -1 \end{pmatrix}$$

to rewrite the conjecture :

$$P = I - DA^T A$$

We multiply both sides on the left by A:

$$AP = A - ADA^T A$$

Substitute $AP = \pi A$ and cancel the A's on the right:

$$\pi = I - ADA^T$$

This we will verify.

Simply multiplying A and D we see that $(AD)_{i,j} = D_{i,j} - D_{i-1,j}$ with the understanding $D_{0,k} := 0$ for all k. One more multiplication gives us that

$$(ADA^T)_{i,j} = D_{i,j} - D_{i-1,j} - D_{i,j-1} + D_{i-1,j-1}$$

once again, with the understanding that if either $i = 0$ or $j = 0$ then $D_{i,j} := 0$

Case 1

If $i = j$ then

$$
\begin{aligned}
(ADA^T)_{i,j} &= D_{i,j} - D_{i-1,j} - D_{i,j-1} + D_{i-1,j-1} \\
&= \sum_{\substack{a \le i \\ b > j}} \pi_{a,b} - \sum_{\substack{a \le i-1 \\ b > j}} \pi_{a,b} - \sum_{\substack{a > i \\ b \le j-1}} \pi_{a,b} + \sum_{\substack{a > i-1 \\ b \le j-1}} \pi_{a,b} \\
&= \sum_{k \ne j} \pi_{i,k} \\
&= \begin{cases} 0 & \pi_{i,j} = 1 \\ 1 & \pi_{i,j} = 0 \end{cases}
\end{aligned}
$$

If the second to last equality seems like a bit of a jump consider that we are summing over the following terms of permutation matrices:

$$
\begin{pmatrix}
\ddots & \vdots & \vdots & \vdots & \iddots \\
\cdots & \pi_{i-1,j-1} & \pi_{i-1,j} & \pi_{i-1,j+1} & \cdots \\
\cdots & \pi_{i,j-1} & \pi_{i,j} & \pi_{i,j+1} & \cdots \\
\cdots & \pi_{i+1,j-1} & \pi_{i+1,j} & \pi_{i+1,j+1} & \cdots \\
\iddots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
-
\begin{pmatrix}
\ddots & \vdots & \vdots & \vdots & \iddots \\
\cdots & \pi_{i-1,j-1} & \pi_{i-1,j} & \pi_{i-1,j+1} & \cdots \\
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\end{pmatrix}
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\cdots & \pi_{i+1,j-1} & \pi_{i+1,j} & \pi_{i+1,j+1} & \cdots \\
\iddots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
+
\begin{pmatrix}
\ddots & \vdots & \vdots & \vdots & \iddots \\
\cdots & \pi_{i-1,j-1} & \pi_{i-1,j} & \pi_{i-1,j+1} & \cdots \\
\cdots & \pi_{i,j-1} & \pi_{i,j} & \pi_{i,j+1} & \cdots \\
\cdots & \pi_{i+1,j-1} & \pi_{i+1,j} & \pi_{i+1,j+1} & \cdots \\
\iddots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

Thus, $(I - ADA^T)_{i,j} = \pi_{i,j}$ for this case.

Case 2

If $i < j$ then

$$\begin{aligned}(ADA^T)_{i,j} &= D_{i,j} - D_{i-1,j} - D_{i,j-1} + D_{i-1,j-1} \\ &= \sum_{\substack{a\leq i\\ b>j}} \pi_{a,b} - \sum_{\substack{a\leq i-1\\ b>j}} \pi_{a,b} - \sum_{\substack{a\leq i\\ b>j-1}} \pi_{a,b} + \sum_{\substack{a\leq i-1\\ b>j-1}} \pi_{a,b} \\ &= -\pi_{i,j}\end{aligned}$$

This last equality is, again, perhaps more easily understood visually:

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & ⋰ \\ \cdots & \pi_{i-1,j-1} & \pi_{i-1,j} & \pi_{i-1,j+1} & \cdots \\ \cdots & \pi_{i,j-1} & \pi_{i,j} & \pi_{i,j+1} & \cdots \\ \cdots & \pi_{i+1,j-1} & \pi_{i+1,j} & \pi_{i+1,j+1} & \cdots \\ ⋰ & \vdots & \vdots & \vdots & \ddots \end{pmatrix} - \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & ⋰ \\ \cdots & \pi_{i-1,j-1} & \pi_{i-1,j} & \pi_{i-1,j+1} & \cdots \\ \cdots & \pi_{i,j-1} & \pi_{i,j} & \pi_{i,j+1} & \cdots \\ \cdots & \pi_{i+1,j-1} & \pi_{i+1,j} & \pi_{i+1,j+1} & \cdots \\ ⋰ & \vdots & \vdots & \vdots & \ddots \end{pmatrix} - \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & ⋰ \\ \cdots & \pi_{i-1,j-1} & \pi_{i-1,j} & \pi_{i-1,j+1} & \cdots \\ \cdots & \pi_{i,j-1} & \pi_{i,j} & \pi_{i,j+1} & \cdots \\ \cdots & \pi_{i+1,j-1} & \pi_{i+1,j} & \pi_{i+1,j+1} & \cdots \\ ⋰ & \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & ⋰ \\ \cdots & \pi_{i-1,j-1} & \pi_{i-1,j} & \pi_{i-1,j+1} & \cdots \\ \cdots & \pi_{i,j-1} & \pi_{i,j} & \pi_{i,j+1} & \cdots \\ \cdots & \pi_{i+1,j-1} & \pi_{i+1,j} & \pi_{i+1,j+1} & \cdots \\ ⋰ & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Thus, $(I - ADA^T)_{i,j} = \pi_{i,j}$ for this case as well.

Case 3

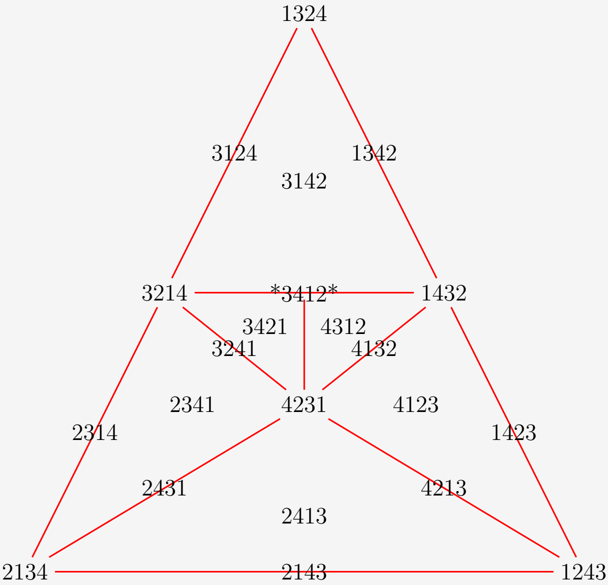
If $i > j$ then

$$\begin{aligned}(ADA^T)_{i,j} &= D_{i,j} - D_{i-1,j} - D_{i,j-1} + D_{i-1,j-1} \\ &= \sum_{\substack{a>i \\ b\leq j}} \pi_{a,b} - \sum_{\substack{a>i-1 \\ b\leq j}} \pi_{a,b} - \sum_{\substack{a>i \\ b\leq j-1}} \pi_{a,b} + \sum_{\substack{a>i-1 \\ b\leq j-1}} \pi_{a,b} \\ &= -\pi_{i,j}\end{aligned}$$

Here once more, the visual aid comes to the rescue and makes the last equality apparent.

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \pi_{i-1,j-1} & \pi_{i-1,j} & \pi_{i-1,j+1} & \cdots \\ \cdots & \pi_{i,j-1} & \pi_{i,j} & \pi_{i,j+1} & \cdots \\ \cdots & \pi_{i+1,j-1} & \pi_{i+1,j} & \pi_{i+1,j+1} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} - \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \pi_{i-1,j-1} & \pi_{i-1,j} & \pi_{i-1,j+1} & \cdots \\ \cdots & \pi_{i,j-1} & \pi_{i,j} & \pi_{i,j+1} & \cdots \\ \cdots & \pi_{i+1,j-1} & \pi_{i+1,j} & \pi_{i+1,j+1} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} - \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \pi_{i-1,j-1} & \pi_{i-1,j} & \pi_{i-1,j+1} & \cdots \\ \cdots & \pi_{i,j-1} & \pi_{i,j} & \pi_{i,j+1} & \cdots \\ \cdots & \pi_{i+1,j-1} & \pi_{i+1,j} & \pi_{i+1,j+1} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \pi_{i-1,j-1} & \pi_{i-1,j} & \pi_{i-1,j+1} & \cdots \\ \cdots & \pi_{i,j-1} & \pi_{i,j} & \pi_{i,j+1} & \cdots \\ \cdots & \pi_{i+1,j-1} & \pi_{i+1,j} & \pi_{i+1,j+1} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Thus, $(I - ADA^T)_{i,j} = \pi_{i,j}$ in this final case.


1324
3124
1342
3142
3214
3412
1432
3421
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4231
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1423
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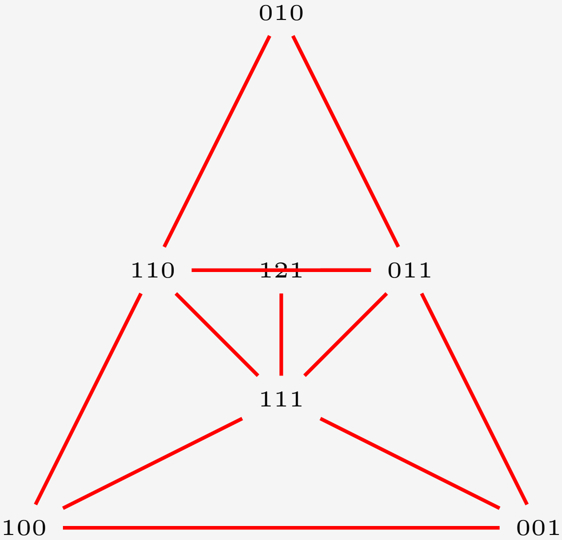
4213
2413
2134
2143
1243


010
110
121
011
111
100
001

Consequences of the algorithm

Theorem:
2^n vertices arise as columns of Waldspurger matrices for type A_n.

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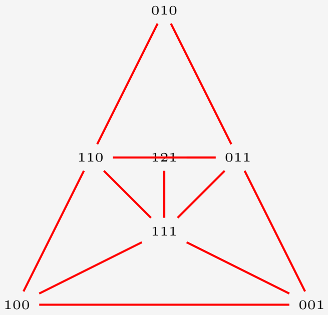
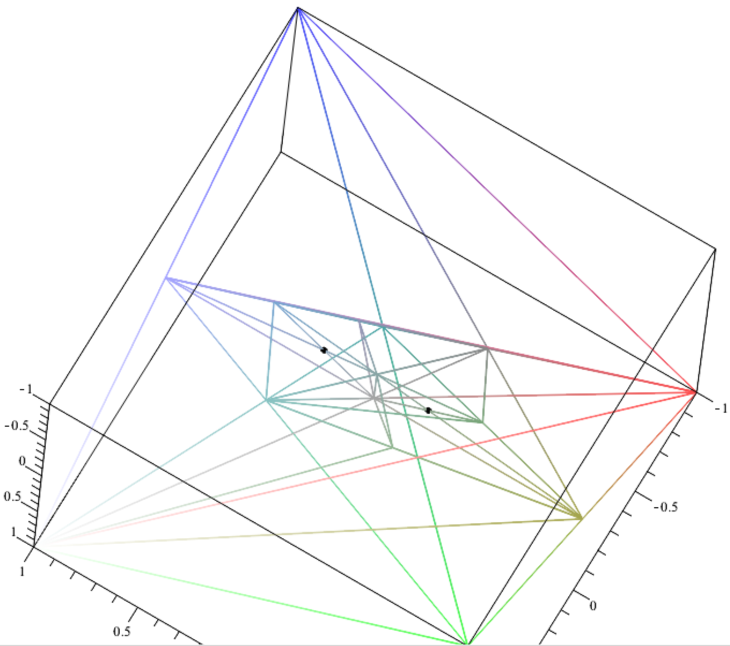
- Entries in the columns can only increase or decrease by one.
- There are 2^{n-1} Unimodal Motzkin Paths of length n.
- Given any column with these properties, one has enough freedom to complete it to a Waldspurger matrix.

Not all [virtual] vertices come from Waldspurger Matrices!

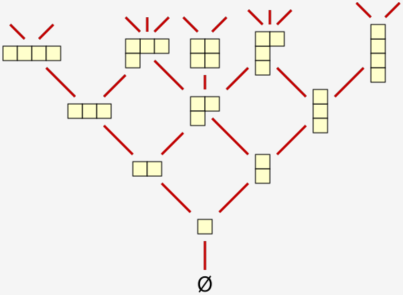


Our plan was to complete the Waldspurger decomposition to a polytopal complex intersecting facets to get faces of codimension one, intersecting those to get faces of codimension two, etc. This is problematic.

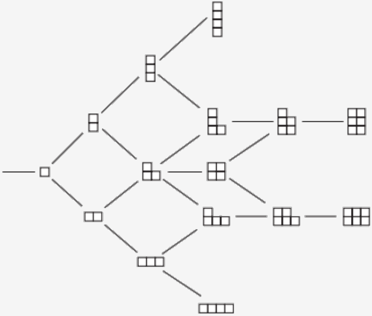

-1
-0.5
0
0.5
1
1
0.5
-1
-0.5
0

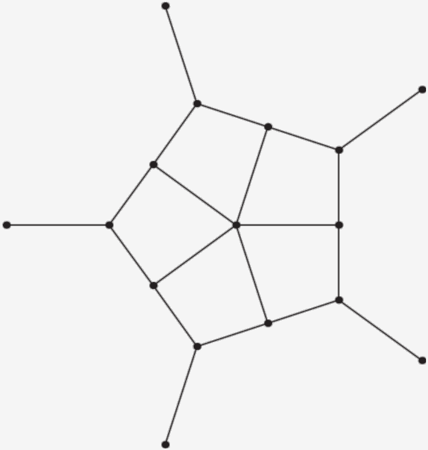
Connection to Young's lattice

In 2002 Ruedi Suter exhibited a subposet of Young's lattice with dihedral symmetry.

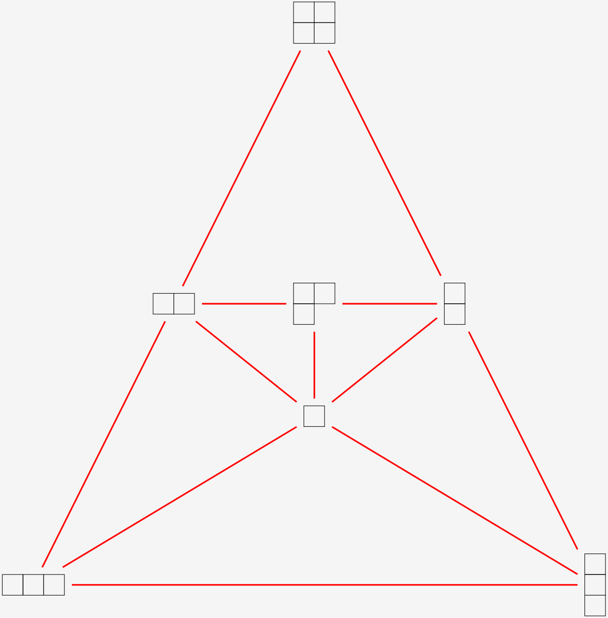


For $n \geq 3$ define Y_n to be the induced subgraph of partitions with hooklength less than or equal to n. Y_n has they same dihedral symmetry as a regular n-gon.





Y_n has 2^{n-1} elements! (counting the empty partition)



Why this bijection?... Abelian Ideals!

- Ruedi Suter showed that elements in Y_n represent abelian ideals of the Borel subalgebra of $\mathfrak{sl}_n(\mathbb{C})$

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Why this bijection?... Abelian Ideals!

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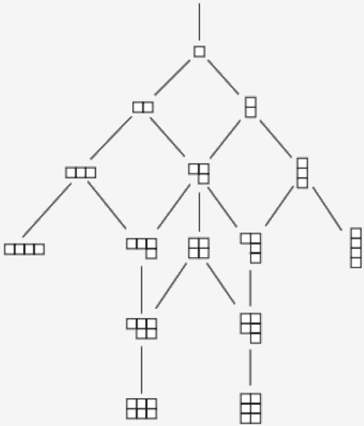
- An ideal of a Lie algebra is a set with the absorbing property with respect to the bracket.
- An ideal of a Lie algebra is called abelian if the Lie bracket vanishes on it.

The Borel subalgebra of $\mathfrak{sl}_5(\mathbb{C})$ consists of all upper triangular matrices with trace$= 0$. These partitions represent each of its abelian ideals.



Open Questions:

$$\text{Lie Algebra} \\ \updownarrow \\ \text{The Waldspurger Decomposition}$$

Does this connection with abelian ideals hold in other types?

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Does the dihedral symmetry say anything about the Waldspurger picture?

Open Questions:

Lie Algebra

$\updownarrow$

The Waldspurger Decomposition

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Is there more going on here?

Original Goal, backtracking

- Complete the Waldspurger decomposition to a CW-complex and compute its f-vector.

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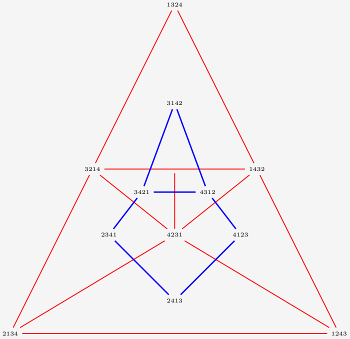
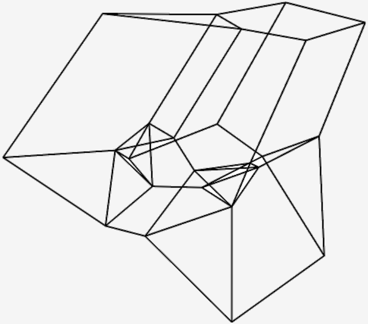
- This gives even more "virtual vertices" than those from Waldspurger matrices.
- New approach: Use the recursive structure and consider facets.

Theorem (Bibikov, Zhgoon): Two facets c_1 and c_2 share a codimension one boundary iff $c_1 s_i = c_2 s_j$ for s_i and s_j adjacent transpositions.

This defines a graph on $n - cycles$.





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3	2																				
4		4	2																		
5			8	10	4	2															
6				16	34	30	24	8	6	2											
7					32	98	138	158	106	80	58	28	16	4	2						
8						64	258	504	764	774	692	632	492	380	220	122	74	46	10	6	2

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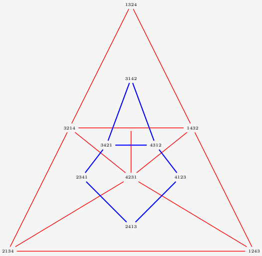

"Adjacent Adjacent Transpositions" and Bounds on degrees

Lemma: Let c be an n-cycle and s_i be the transposition switching i and $i+1$. Either cs_is_{i+1} is an n-cycle, or $cs_{i+1}s_i$ is an n-cycle. Not both.

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"Non-adjacent Adjacent transpositions"

Lemma 2: If $j > i + 1$, then there are two classes of n-cycles related by $(i, i+1)(j, j+1)$

$$(\mathbf{i}, a_1, \dots, a_k, \mathbf{j}, b_1, \dots, b_l, \mathbf{i+1}, c_1, \dots, c_m, \mathbf{j+1}, d_1, \dots, d_n) \quad (1)$$

and

$$(\mathbf{i}, a_1, \dots, a_k, \mathbf{j+1}, b_1, \dots, b_l, \mathbf{i+1}, c_1, \dots, c_m, \mathbf{j}, d_1, \dots, d_n). \quad (2)$$

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Corollary 1: If cs_is_j is an n-cycle, then so is $c^{-1}s_is_j$. In particular, c and c^{-1} have the same degree in $\mathfrak{S}_n$ dual graph.

When does $s_i s_j$ take you to another n-cycle?

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and

$$(\mathbf{i}, a_1, \ldots, a_k, \mathbf{j{+}1}, b_1, \ldots, b_l, \mathbf{i{+}1}, c_1, \ldots, c_m, \mathbf{j}, d_1, \ldots, d_n). \tag{4}$$

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Corollary 2: Written in cycle notation beginning with a 1, the n-cycles with vertex degree $n-2$ in $\mathfrak{S}_n$ correspond to unimodal sequences.

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Example: $(1,2,3,4)$ $(1,3,4,2)$ $(1,4,3,2)$ $(1,2,4,3)$ are the four vertices of degree two in the $\mathfrak{S}_4$ dual graph.

Lemma 2: If $j > i+1$, then there are two classes of n-cycles related by $(i, i+1)(j, j+1)$

$$(\mathbf{i}, a_1, \ldots, a_k, \mathbf{j}, b_1, \ldots, b_l, \mathbf{i{+}1}, c_1, \ldots, c_m, \mathbf{j{+}1}, d_1, \ldots, d_n)$$

and

$$(\mathbf{i}, a_1, \ldots, a_k, \mathbf{j{+}1}, b_1, \ldots, b_l, \mathbf{i{+}1}, c_1, \ldots, c_m, \mathbf{j}, d_1, \ldots, d_n).$$

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4		4	2																		
5			8	10	4	2															
6				16	34	30	24	8	6	2											
7					32	98	138	158	106	80	58	28	16	4	2						
8						64	258	504	764	774	692	632	492	380	220	122	74	46	10	6	2

Lemma 2: If $j > i+1$, then there are two classes of n-cycles related by $(i, i+1)(j, j+1)$

$$(\mathbf{i}, a_1, \ldots, a_k, \mathbf{j}, b_1, \ldots, b_l, \mathbf{i{+}1}, c_1, \ldots, c_m, \mathbf{j{+}1}, d_1, \ldots, d_n)$$

and

$$(\mathbf{i}, a_1, \ldots, a_k, \mathbf{j{+}1}, b_1, \ldots, b_l, \mathbf{i{+}1}, c_1, \ldots, c_m, \mathbf{j}, d_1, \ldots, d_n).$$

$n \backslash deg$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
3	2																				
4		4	2																		
5			8	10	4	2															
6				16	34	30	24	8	6	2											
7					32	98	138	158	106	80	58	28	16	4	2						
8						64	258	504	764	774	692	632	492	380	220	122	74	46	10	6	2

Corollary 3: $(1, 3, 5, \ldots, 2, 4, 6, \ldots)$ and its inverse are the two vertices of degree $\binom{n-1}{2}$.

"Adjacent Adjacent" vs "Non-adjacent Adjacent"

Our Dual graph $G = G_A + G_B$ where G_A has edges from $\{s_i s_{i+1}\}$ and G_B has edges from the other $\{s_i s_j\}$.

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G_B has the same degree histogram as G without the leading zeros

$n \backslash deg$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	2															
4	4	2														
5	8	10	4	2												
6	16	34	30	24	8	6	2									
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8	64	258	504	764	774	692	632	492	380	220	122	74	46	10	6	2

The trailing numbers appear to stabilize for even and odd n.

More Observations, Conjectures and Questions about the dual graph

$n \backslash deg$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
3	1																				
4		2	1																		
5			4	5	2	1															
6				8	17	15	12	4	3	1											
7					16	49	69	79	53	40	29	14	8	2	1						
8						32	129	252	382	287	346	316	246	190	110	61	37	23	5	3	1

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7					16	49	69	79	53	40	29	14	8	2	1						
8						32	129	252	382	287	346	316	246	190	110	61	37	23	5	3	1

- $1, 5, 17, 49, 129, \ldots$ counts the number of Dyck paths with exactly one peak at or above height three.

More Observations, Conjectures and Questions about the dual graph

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3	1																				
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- $1, 5, 17, 49, 129, \ldots$ counts the number of Dyck paths with exactly one peak at or above height three.
- $2, 15, 69, 252, \ldots$ counts the number of Dyck paths with exactly two "long ascents."

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- $1, 5, 17, 49, 129, \ldots$ counts the number of Dyck paths with exactly one peak at or above height three.
- $2, 15, 69, 252, \ldots$ counts the number of Dyck paths with exactly two "long ascents."
- The OEIS doesn't know any more diagonals.

More Observations, Conjectures and Questions about the dual graph

n	$\#edges$
3	1
4	7
5	48
6	360
7	3000
8	27720
9	282240
10	3144960
$\vdots$	$\vdots$
n	$\frac{(n+3)(n-2)(n-1)!}{12}$

These numbers appear in the OEIS in relation to two things:

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These numbers appear in the OEIS in relation to two things:

- # "series parallel networks" with n vertices and $2n-1$ edges
- Labeling tableaux of shape $(3, 1^{n-2})$ with [n-1] such that the first row is decreasing and the first column has $n-1$ labels (two blank spots):
- There are 7 ways of labeling the tableau of shape (3,1,1) with 1, 2 and 3 (with each label being used once) such that the first row is decreasing and the first column has 1 label.

What next?

- Further interpret the degree sequence of the Waldspurger dual graph of $\mathfrak{S}_n$.

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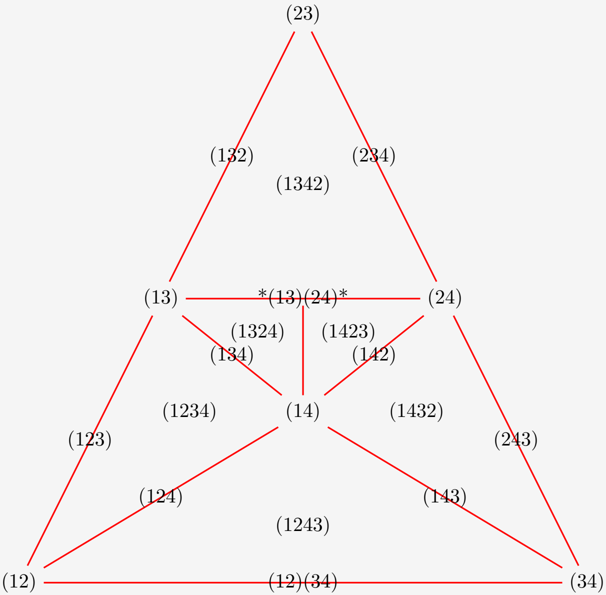
What next?

- Further interpret the degree sequence of the Waldspurger dual graph of $\mathfrak{S}_n$.
- The "adjacent adjacent" transpositions $\{s_i s_{i+1} : 1 \leq i \leq n-2\}$, form a generating set for the alternating group A_n. In the Waldspurger decomposition, they act like hyperplanes and give the alternating group a "Coxeter like structure." Make this precise and understand it better.
- Is there a better dual object, say a dual complex that would help better understand the Waldspurger decomposition?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Thanks!