SYMMETRIC GROUP TILINGS

The Waldspurger and Mienrenken Decompositions for Type A

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Outline

- Background, Waldspurger and Meinrenken Theorems
- Geometry $\Rightarrow$ Permutations
- Permutations $\Rightarrow$ Alternating Sign Matrices
- Alternating Sign Matrices $\Rightarrow$ Geometry

An element $g \in O(n)$ is a reflection if it sends some nonzero vector $\alpha \in \mathbb{R}^n$ to its negative and fixes the hyperplane orthogonal to α pointwise.

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Let $G \subset O(n)$ be a finite group generated by reflections.

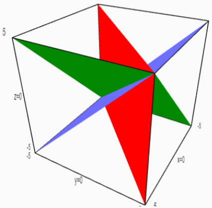


Figure: $\mathfrak{S}_3 \curvearrowright \mathbb{R}^3$

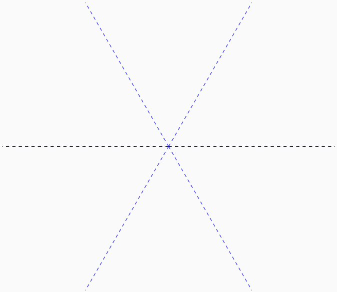
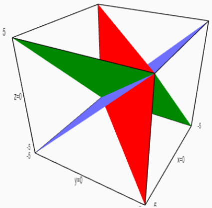


Figure: $\mathfrak{S}_3 \curvearrowright \mathbb{R}^2 = \mathbb{R}^3/(1,1,1)$

The Reflection Representation of $\mathfrak{S}_n$, and the Braid Arrangement



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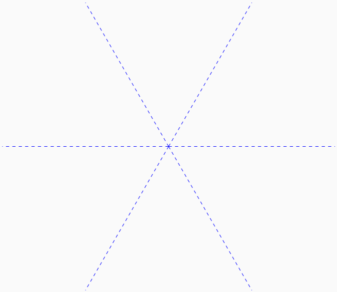
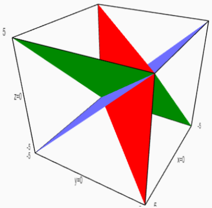


Figure: $\mathfrak{S}_3 \curvearrowright \mathbb{R}^2 = \mathbb{R}^3/(1,1,1)$

Theorem (Coxeter): Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^n$ of reflecting hyperplanes for the reflection group G. Then $G \curvearrowright (\mathbb{R}^n \setminus \cup_{H\in\mathcal{A}} H)$ freely and transitively on the chambers.



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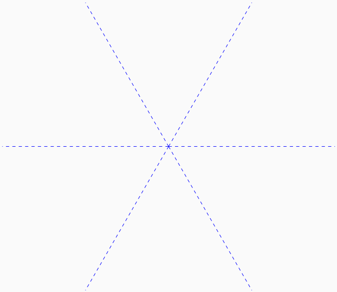

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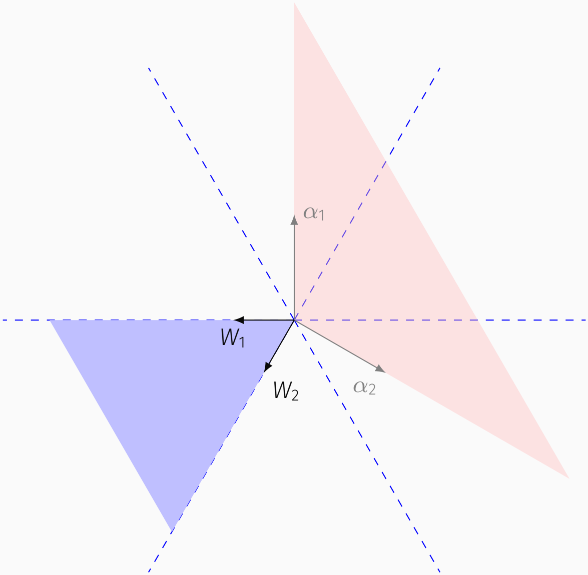
We pick one such chamber and call it the "weight cone" denoted C_W. The cone dual to C_W we call the "root cone" and denote C_R.



α_1
W_1
W_2
α_2

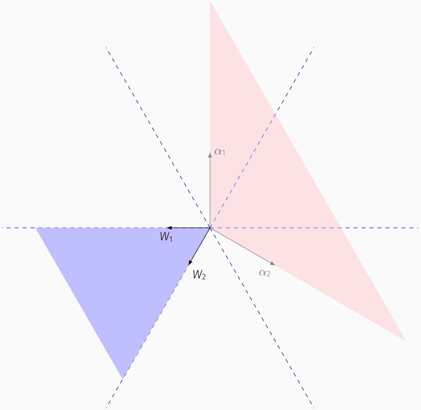
Fact (Coxeter): C_w is a simplicial cone.

Let $w_1, w_2, \ldots, w_n$ be vectors generating the rays of C_W [Jargon: called the "fundamental weights"] Then the dual cone is defined as

$$C_R := \{x \in \mathbb{R}^n : (x, y) \leq 0 \forall y \in C_W\}$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be vectors which generate rays of C_R [Jargon: called the "simple roots"]

The first example A_2 ($\mathfrak{S}_3$)



$$3w_1 = -2\alpha_2 - 1\alpha_1 \qquad -\alpha_1 = 2w_2 - 1w_1$$

$$3w_2 = -1\alpha_2 - 2\alpha_1 \qquad -\alpha_2 = -1w_2 + 2w_1$$

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

In type A, it is conventional to let $\alpha_i = e_i - e_{i+1}$ so $(\alpha_i, \alpha_i) = 2\ \forall i$

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$$-\begin{pmatrix} | & | & | & | \\ | & | & | & | \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \\ | & | & | & | \\ | & | & | & | \end{pmatrix} = \begin{pmatrix} | & | & | & | \\ | & | & | & | \\ w_1 & w_2 & \dots & w_n \\ | & | & | & | \\ | & | & | & | \end{pmatrix}^{-1}$$

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The matrix of α's is called the Cartan Matrix. It gives the coordinates of the simple roots in the basis of fundamental weights.

Consider the reflection representation of the symmetric group

$$\phi : \mathfrak{S}_n \longrightarrow GL_{n-1}(\mathbb{R})$$

Let D be the matrix with columns the fundamental weights in basis of the simple roots (i.e. the $n-1 \times n-1$ inverse of the Cartan matrix).

$$\mathsf{W}(g) := [\phi(1) - \phi(g)]D$$

expressed in the coordinates of simple roots we will call the Waldspurger Matrix of g.

Waldspurger's Theorem (2005):

For G a finite reflection group acting on a Euclidean vector space V,
C_R the (closed) cone over the positive roots, and
$\mathring{C}_W \subset V$ the interior of a fundamental domain for the action of G
(sometimes called the weight cone), one has the following decomposition:

$$C_R = \bigsqcup_{g \in G} (1 - g)\mathring{C}_W$$

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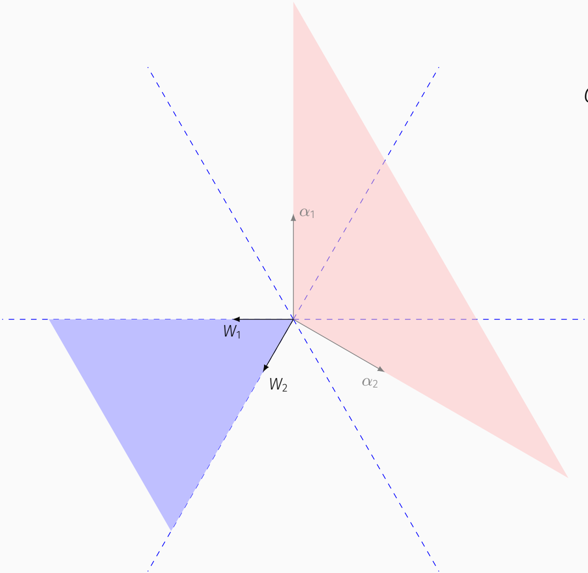
Take away: Convex hulls of columns of Waldspurger matrices (and the zero vector) give a tiling of the $\mathbb{R}^n$!



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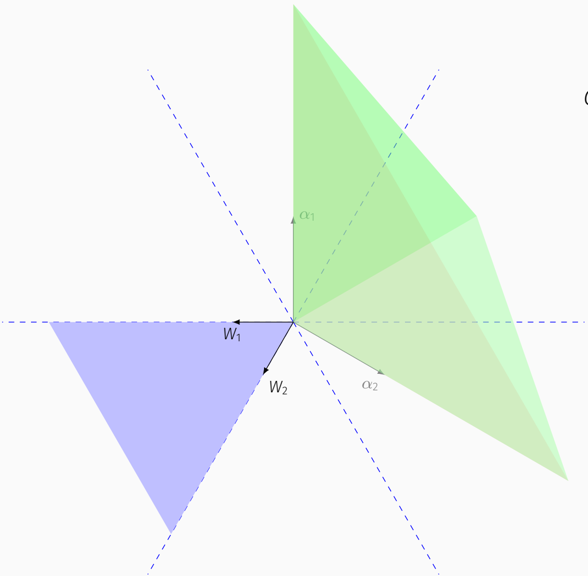
The Waldspurger Decomposition for A_2 ($\mathfrak{S}_3$)



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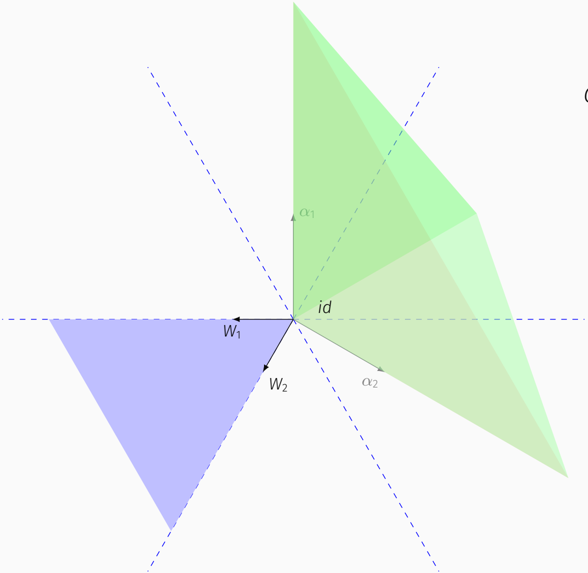

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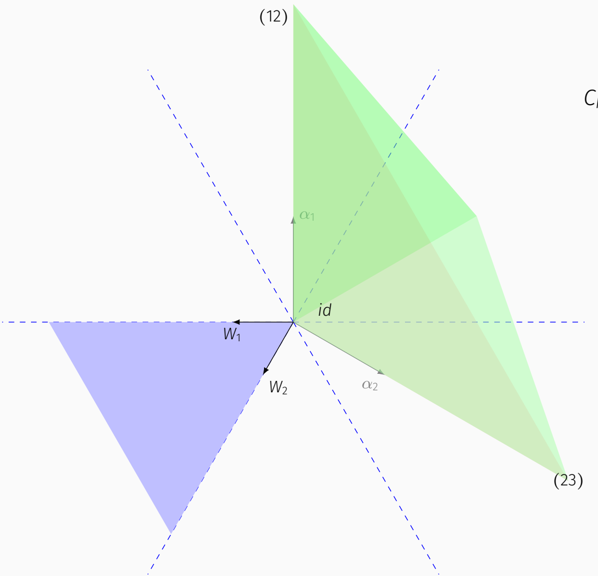

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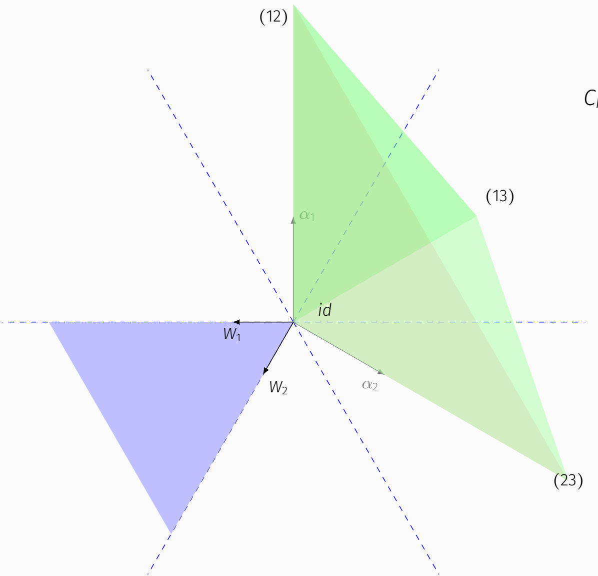

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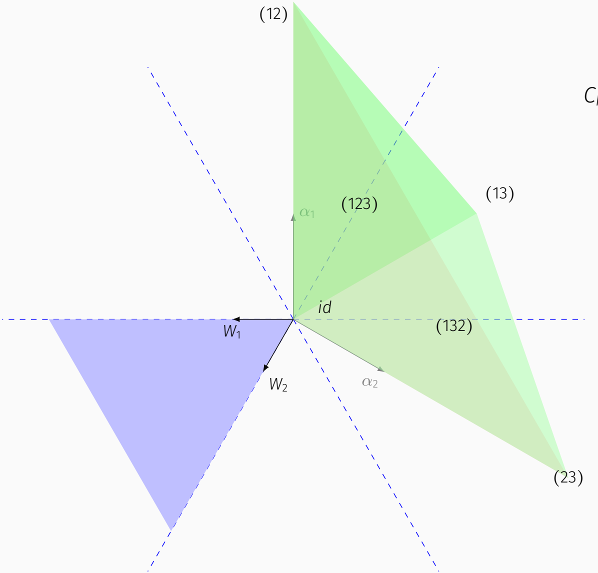

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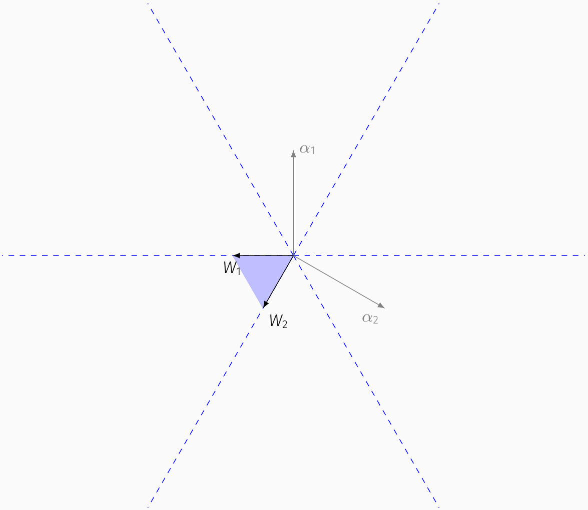
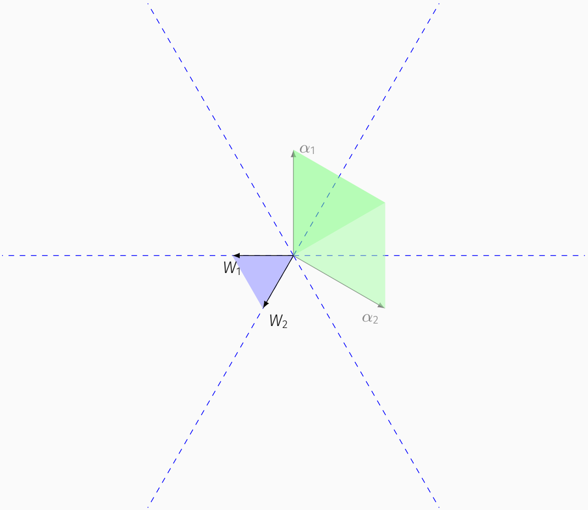


$$V = \bigsqcup_{g \in G} (1-g)\mathring{A}_W$$



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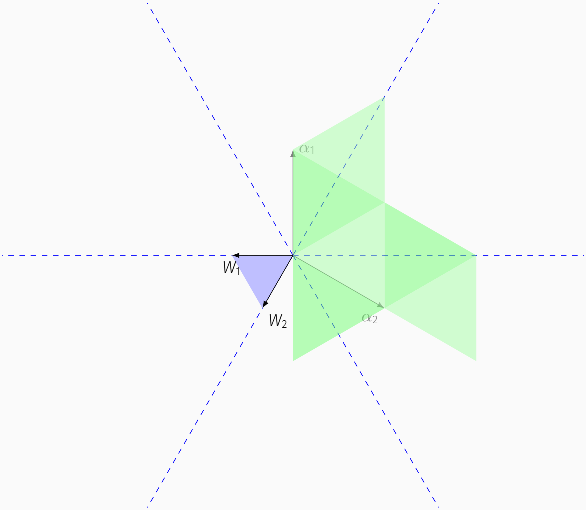

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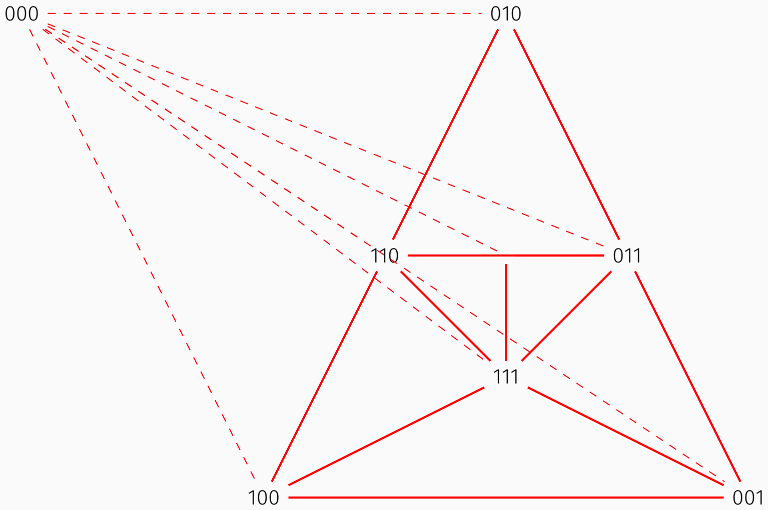
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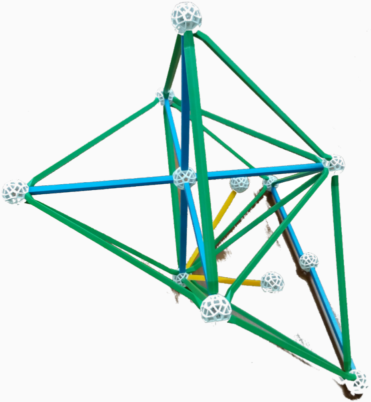
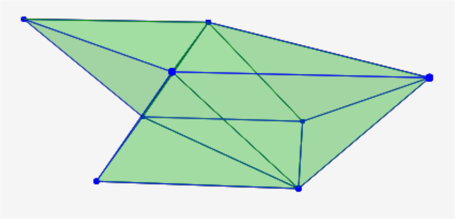
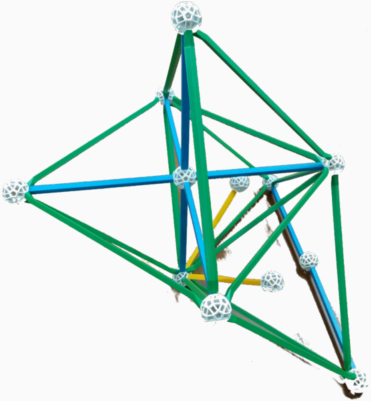
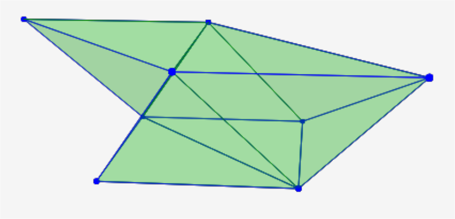
Meinrenken Tile for $A_3 = \mathfrak{S}_4$



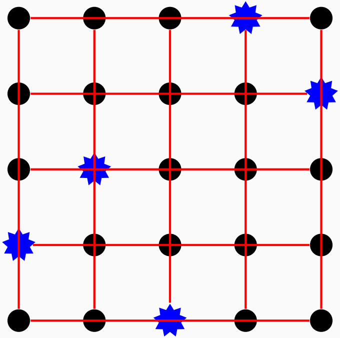






Part 2: Geometry $\Rightarrow$ Permutations

Consider $43512 \in \mathfrak{S}_5$



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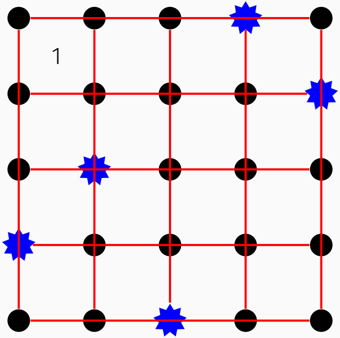

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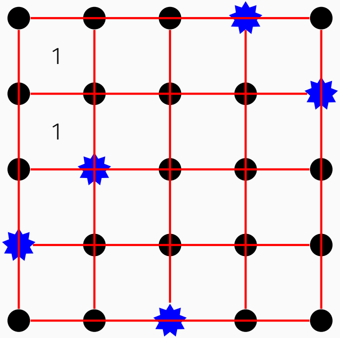

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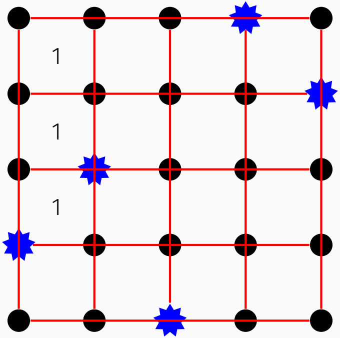

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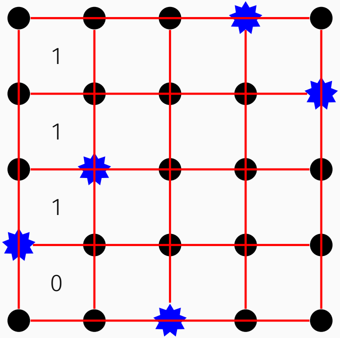

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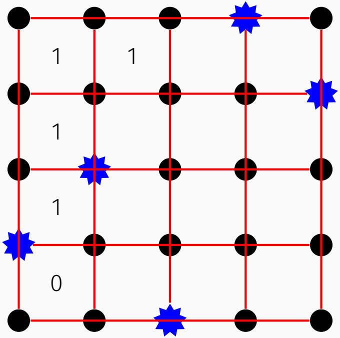

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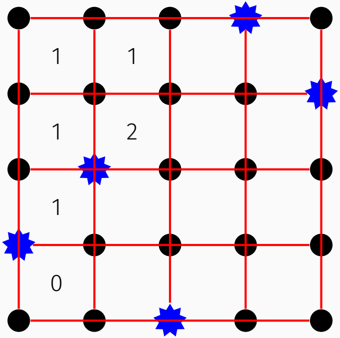

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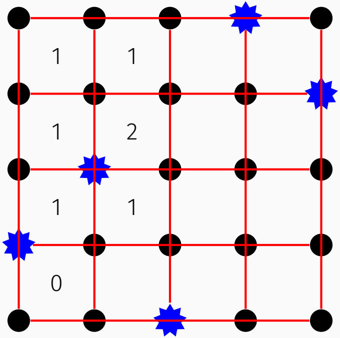

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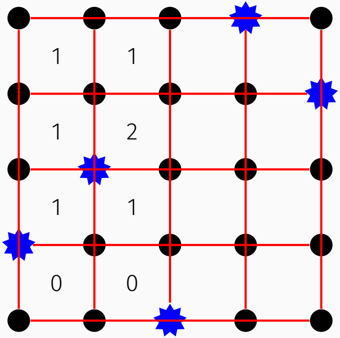

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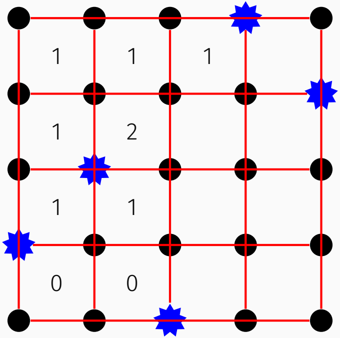

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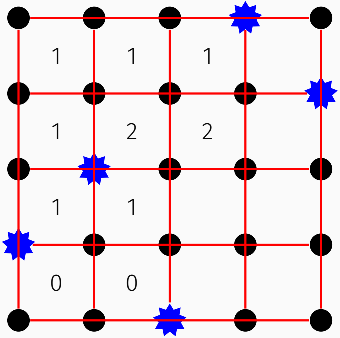

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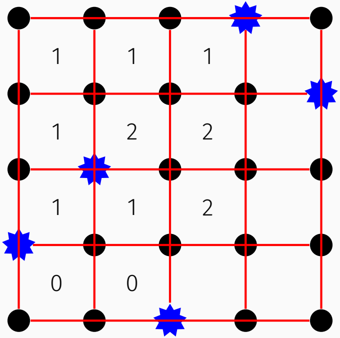

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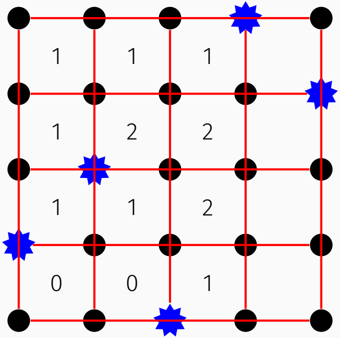

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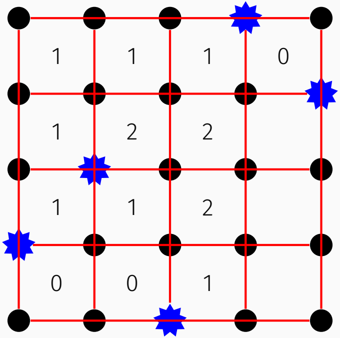

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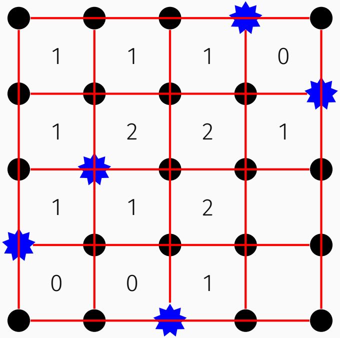

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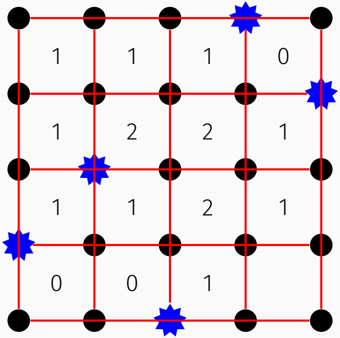

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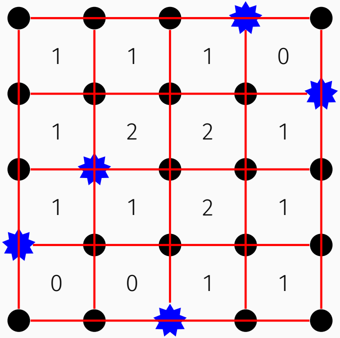

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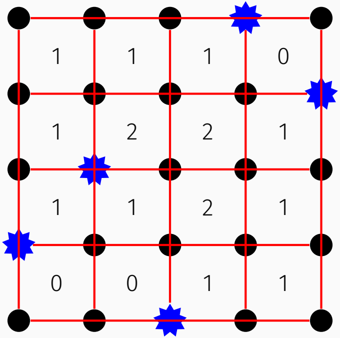
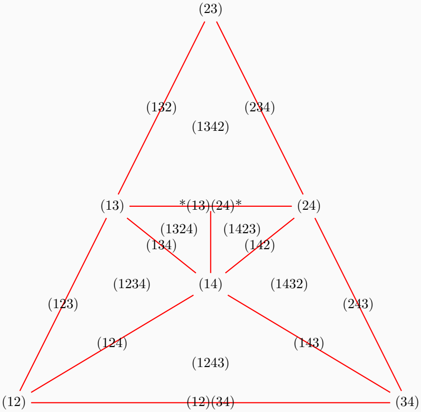


that is, $43512 \mapsto \left\{ a\begin{pmatrix}1\\1\\1\\0\end{pmatrix} + b\begin{pmatrix}1\\2\\1\\0\end{pmatrix} + c\begin{pmatrix}1\\2\\2\\1\end{pmatrix} + d\begin{pmatrix}0\\1\\1\\1\end{pmatrix} \middle| a, b, c, d \in \mathbb{R}_{\geq 0} \right\}$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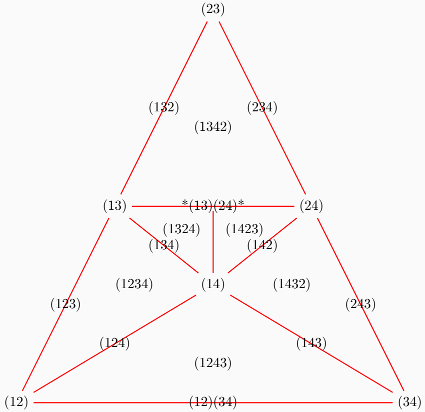
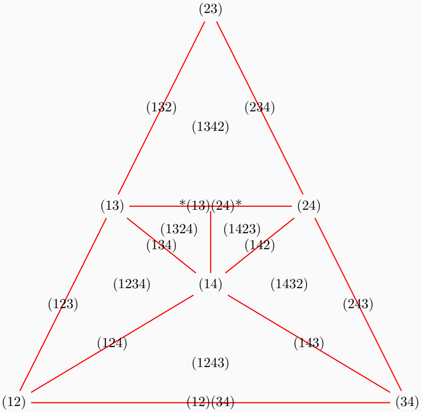


- four copies of $\mathfrak{S}_3$ picture

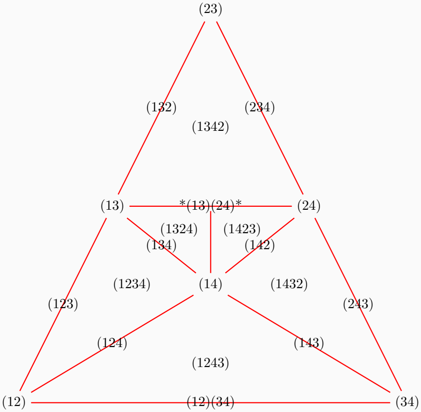


- four copies of $\mathfrak{S}_3$ picture

- Codimension

$$\updownarrow$$

\# cycles $(c(n, k))$



- four copies of $\mathfrak{S}_3$ picture
  - Codimension

    $\updownarrow$

    # cycles ($c(n,k)$)
- NOT A CW- complex!

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There are exactly 2^n possible columns of Waldspurger matrices of type A_n.
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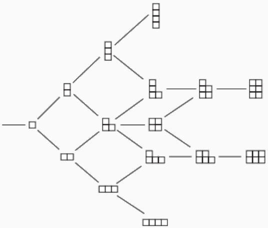
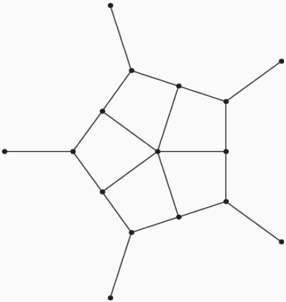
Interesting Bijections:

- Unimodal Motzkin Paths.
- Elements of the root lattice inside a certain polytope
- Abelian ideals in the nilradical of $\mathfrak{sl}_n$
- Young diagrams with hooklength less than n.





From an $n \times n$ matrix M, define the $n-1 \times n-1$ matrix, $\mathcal{WT}(M)$ where

$$\mathcal{WT}(M)_{i,j} := \begin{cases} \sum\limits_{\substack{a \leq i \\ b > j}} M_{a,b} & i \leq j \\ \sum\limits_{\substack{a > i \\ b \leq j}} M_{a,b} & i \geq j \end{cases}.$$

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Warning: Note that $\mathcal{WT}(M)$ may be "over-determined" on the diagonal. In the case where M is a permutation matrix, but in general this need not be the case.

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The map is linear and surjective, with kernel the diagonal matrices.

$$\mathcal{WT} : SS_n \twoheadrightarrow \mathrm{Mat}_{n-1}$$

  

  

Part 2: Permutations ⇒ Alternating Sign Matrices

An alternating sign matrix (or ASM) is a square matrix of 0s, 1s, and −1s such that the sum of each row and column is 1 and the nonzero entries in each row and column alternate in sign.

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$n - 1 \times n - 1$ matrices with UM columns and rows with their maxes on the diagonal are in bijection with $n \times n$ ASMs via the $\mathcal{WT}$ map!

Theorem:

$$\sum_{1 \le i,j \le n-1} \mathcal{WT}(g) = \frac{1}{2}\sum_{i=1}^{n}(g(i) - i)^2$$

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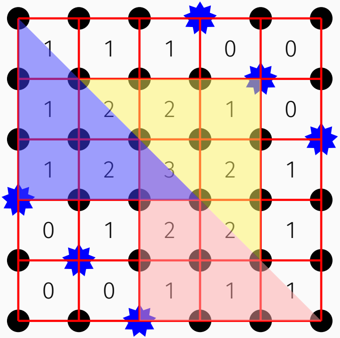

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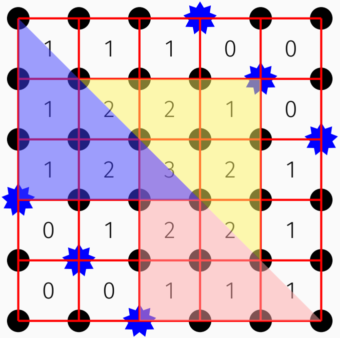
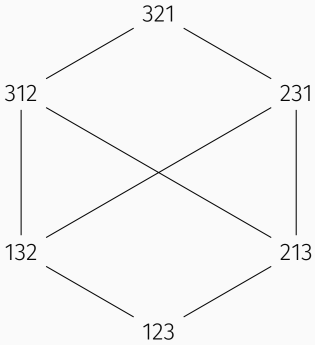
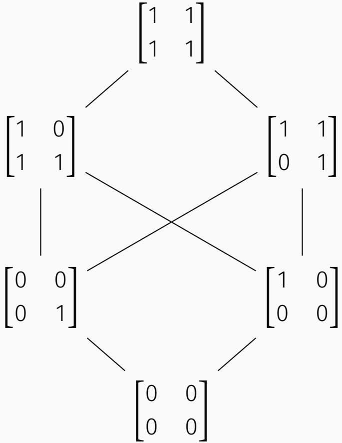
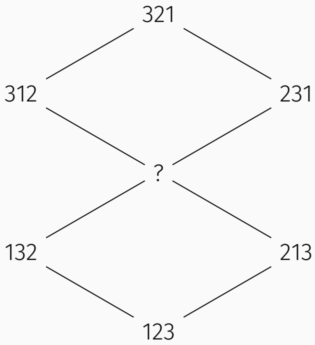


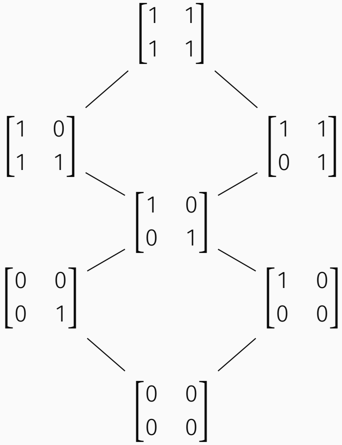
Fact (A. Lascoux, M. Schützenberger): Half the entropy of a permutation is its rank in the MacNeille completion of the Bruhat order– a distributive lattice with elements the alternating sign matrices, or ASMs.

The Bruhat Order






321
312
231
?
132
213
123


$\begin{bmatrix}1&1\\1&1\end{bmatrix}$
$\begin{bmatrix}1&0\\1&1\end{bmatrix}$
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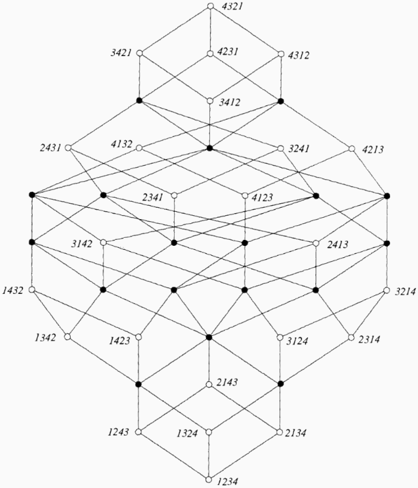
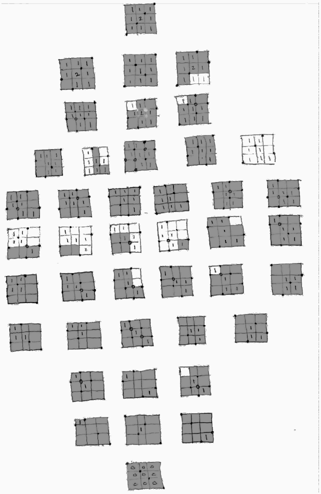
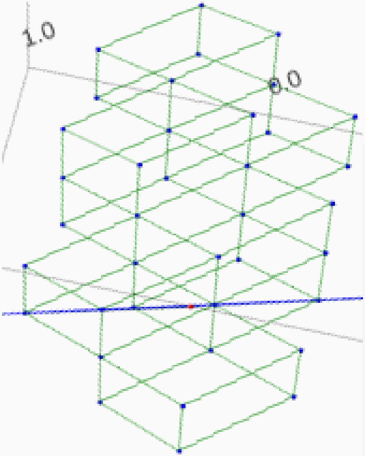
The MacNielle Completion for $\mathfrak{S}_4$

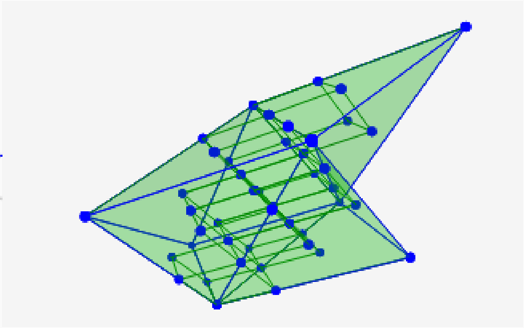




Part 3: ASMs $\Rightarrow$ Geometry

A Geometric Realization of the Hasse Diagram of AMS lattice





Open Questions

- $\prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!} = \frac{1!4!7!\cdots(3n-2)!}{n!(n+1)!\cdots(2n-1)!} = \sum_{\text{labeled forests on } [n]} \text{mult of baricenters?}$

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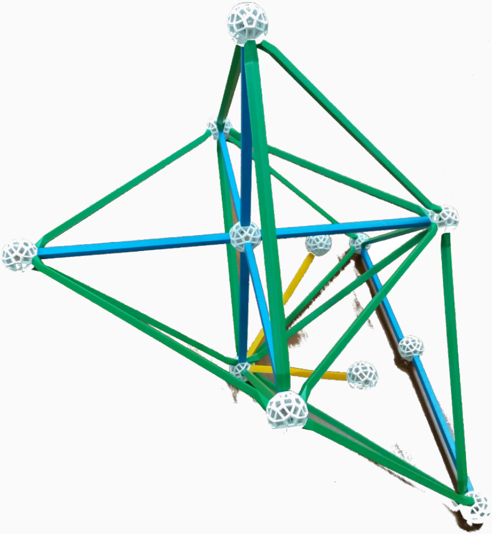
- In experimentation, the $\mathcal{WT}$ map seems to preserve both the Birkhoff polytope and the ASM polytope. Is this true in general?



Thank You!