

SYMMETRIC GROUP TILINGS

The Waldspurger and Mienrenken Decompositions for Type A

James McKeown

October 2016

University of Miami

- Background, Waldspurger and Meinrenken Theorems
- Geometry \Rightarrow Permutations
- Permutations \Rightarrow Alternating Sign Matrices
- Alternating Sign Matrices \Rightarrow Geometry

An element $g \in O(n)$ is a reflection if it sends some nonzero vector $\alpha \in \mathbb{R}^n$ to its negative and fixes the hyperplane orthogonal to α pointwise.

An element $g \in O(n)$ is a reflection if it sends some nonzero vector $\alpha \in \mathbb{R}^n$ to its negative and fixes the hyperplane orthogonal to α pointwise.

Let $G \subset O(n)$ be a finite group generated by reflections.

THE REFLECTION REPRESENTATION OF \mathfrak{S}_n , AND THE BRAID ARRANGEMENT

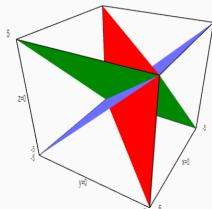


Figure: $\mathfrak{S}_3 \curvearrowright \mathbb{R}^3$

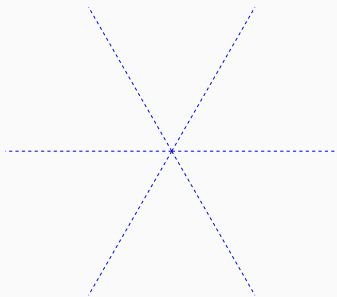


Figure: $\mathfrak{S}_3 \curvearrowright \mathbb{R}^2 = \mathbb{R}^3 / (1, 1, 1)$

THE REFLECTION REPRESENTATION OF \mathfrak{S}_n , AND THE BRAID ARRANGEMENT

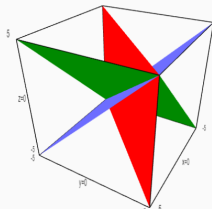


Figure: $\mathfrak{S}_3 \curvearrowright \mathbb{R}^3$

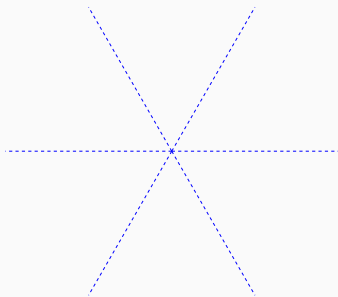


Figure: $\mathfrak{S}_3 \curvearrowright \mathbb{R}^2 = \mathbb{R}^3 / (1,1,1)$

Theorem (Coxeter): Let \mathcal{A} be an arrangement in \mathbb{R}^n of reflecting hyperplanes for the reflection group G . Then $G \curvearrowright (\mathbb{R}^n \setminus \cup_{H \in \mathcal{A}} H)$ freely and transitively on the chambers.

THE REFLECTION REPRESENTATION OF \mathfrak{S}_n , AND THE BRAID ARRANGEMENT

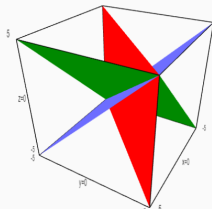


Figure: $\mathfrak{S}_3 \curvearrowright \mathbb{R}^3$

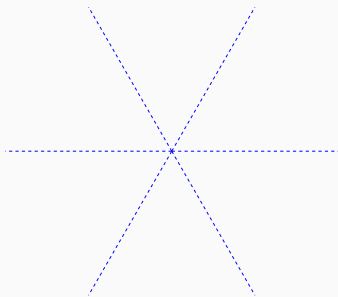
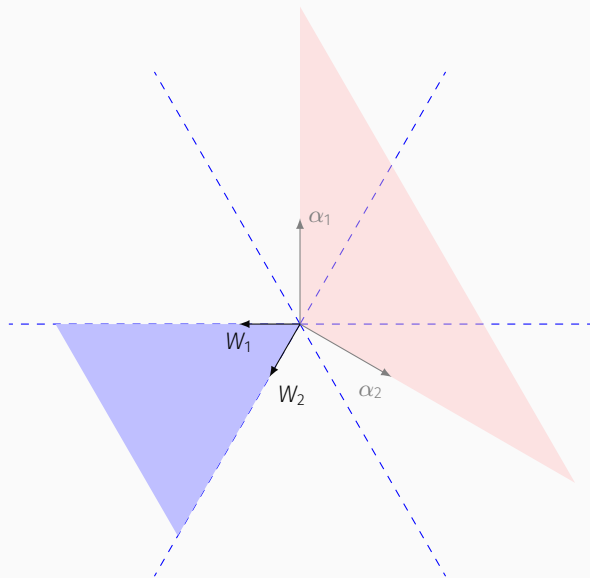


Figure: $\mathfrak{S}_3 \curvearrowright \mathbb{R}^2 = \mathbb{R}^3 / (1, 1, 1)$

Theorem (Coxeter): Let \mathcal{A} be an arrangement in \mathbb{R}^n of reflecting hyperplanes for the reflection group G . Then $G \curvearrowright (\mathbb{R}^n \setminus \cup_{H \in \mathcal{A}} H)$ freely and transitively on the chambers.

We pick one such chamber and call it the “weight cone” denoted C_W . The cone dual to C_W we call the “root cone” and denote C_R .

A FIRST EXAMPLE— WEIGHTS AND ROOTS



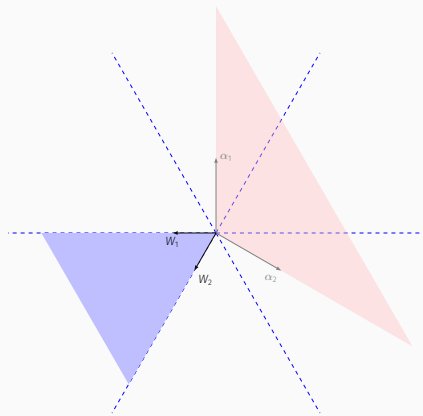
Fact (Coxeter): C_W is a simplicial cone.

Let w_1, w_2, \dots, w_n be vectors generating the rays of C_W [Jargon: called the “fundamental weights”] Then the dual cone is defined as

$$C_R := \{x \in \mathbb{R}^n : (x, y) \leq 0 \forall y \in C_W\}$$

Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be vectors which generate rays of C_R [Jargon: called the “simple roots”]

THE FIRST EXAMPLE A_2 (\mathfrak{S}_3)



$$3w_1 = -2\alpha_2 - 1\alpha_1$$

$$3w_2 = -1\alpha_2 - 2\alpha_1$$

$$-\alpha_1 = 2w_2 - 1w_1$$

$$-\alpha_2 = -1w_2 + 2w_1$$

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

In type A, it is conventional to let $\alpha_i = e_i - e_{i+1}$ so $(\alpha_i, \alpha_i) = 2 \forall i$

In type A, it is conventional to let $\alpha_i = e_i - e_{i+1}$ so $(\alpha_i, \alpha_i) = 2 \forall i$

Lengths of weights are then normalized so that

$$- \begin{pmatrix} | & | & | & | \\ | & | & | & | \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \\ | & | & | & | \\ | & | & | & | \end{pmatrix} = \begin{pmatrix} | & | & | & | \\ | & | & | & | \\ w_1 & w_2 & \dots & w_n \\ | & | & | & | \\ | & | & | & | \end{pmatrix}^{-1}$$

In type A, it is conventional to let $\alpha_i = e_i - e_{i+1}$ so $(\alpha_i, \alpha_i) = 2 \forall i$

Lengths of weights are then normalized so that

$$-\begin{pmatrix} | & | & | & | \\ | & | & | & | \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \\ | & | & | & | \\ | & | & | & | \end{pmatrix} = \begin{pmatrix} | & | & | & | \\ | & | & | & | \\ w_1 & w_2 & \dots & w_n \\ | & | & | & | \\ | & | & | & | \end{pmatrix}^{-1}$$

The matrix of α 's is called the Cartan Matrix. It gives the coordinates of the simple roots in the basis of fundamental weights.

Consider the reflection representation of the symmetric group

$$\phi : \mathfrak{S}_n \longrightarrow GL_{n-1}(\mathbb{R})$$

Let D be the matrix with columns the fundamental weights in basis of the simple roots (i.e. the $n - 1 \times n - 1$ inverse of the Cartan matrix).

$$\mathbf{W}(g) := [\phi(1) - \phi(g)]D$$

expressed in the coordinates of simple roots we will call the Waldspurger Matrix of g .

Waldspurger's Theorem (2005):

For G a finite reflection group acting on a Euclidean vector space V , C_R the (closed) cone over the positive roots, and $\mathring{C}_W \subset V$ the interior of a fundamental domain for the action of G (sometimes called the weight cone), one has the following decomposition:

$$C_R = \bigsqcup_{g \in G} (1 - g)\mathring{C}_W$$

Meinrenken's Theorem (2007):

Waldspurger's Theorem (2005):

For G a finite reflection group acting on a Euclidean vector space V , C_R the (closed) cone over the positive roots, and $\mathring{C}_W \subset V$ the interior of a fundamental domain for the action of G (sometimes called the weight cone), one has the following decomposition:

$$C_R = \bigsqcup_{g \in G} (1 - g)\mathring{C}_W$$

Take away: Waldspurger matrices give a tiling of the root cone!

Meinrenken's Theorem (2007):

Waldspurger's Theorem (2005):

Translation: Cones over columns of Waldspurger matrices give a tiling of the root cone!

Meinrenken's Theorem (2007):

Waldspurger's Theorem (2005):

Translation: Cones over columns of Waldspurger matrices give a tiling of the root cone!

Meinrenken's Theorem (2007):

For G an affine reflection group acting on a Euclidean vector space V , and $\mathring{A}_W \subset V$ the interior of a fundamental domain for the action of G (sometimes called a fundamental alcove) one has the following decomposition:

$$V = \bigsqcup_{g \in G} (1 - g)\mathring{A}_W$$

Waldspurger's Theorem (2005):

Translation: Cones over columns of Waldspurger matrices give a tiling of the root cone!

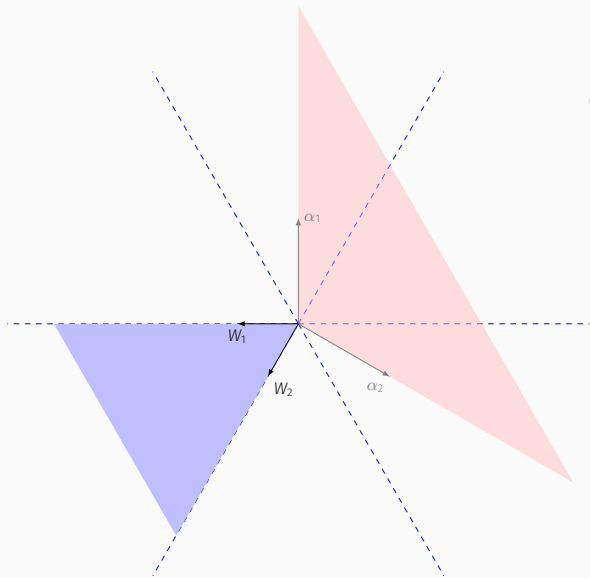
Meinrenken's Theorem (2007):

For G an affine reflection group acting on a Euclidean vector space V , and $\mathring{A}_W \subset V$ the interior of a fundamental domain for the action of G (sometimes called a fundamental alcove) one has the following decomposition:

$$V = \bigsqcup_{g \in G} (1 - g)\mathring{A}_W$$

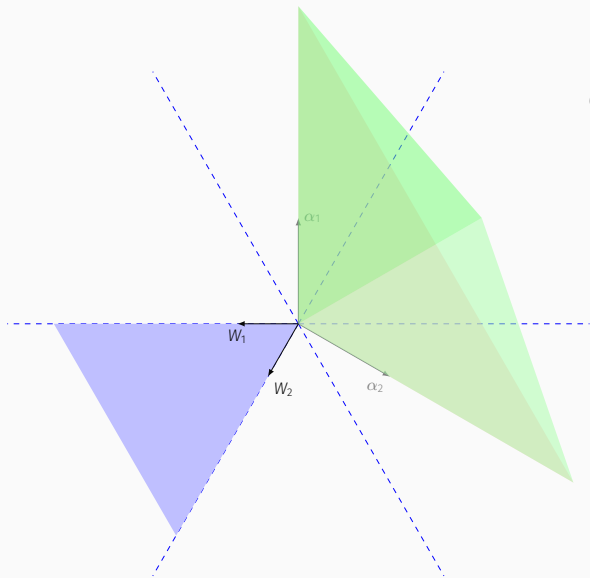
Take away: Convex hulls of columns of Waldspurger matrices (and the zero vector) give a tiling of the \mathbb{R}^n !

THE WALDSPURGER DECOMPOSITION FOR $A_2(\mathfrak{S}_3)$



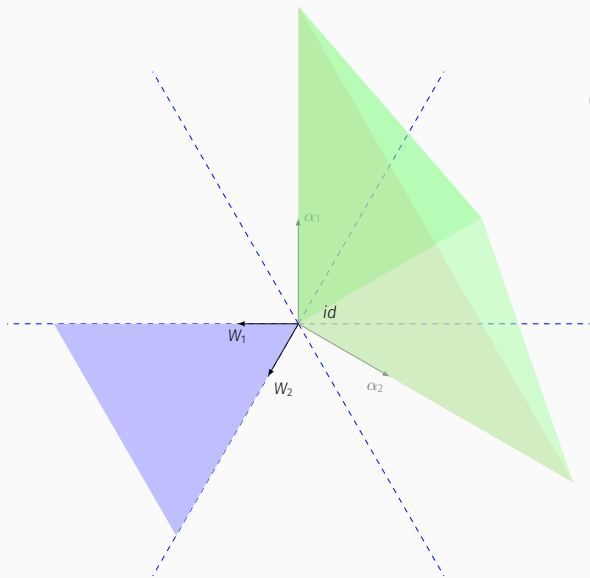
$$C_R = \bigsqcup_{g \in G} (1-g)\dot{C}_W$$

THE WALDSPURGER DECOMPOSITION FOR $A_2 (\mathfrak{S}_3)$



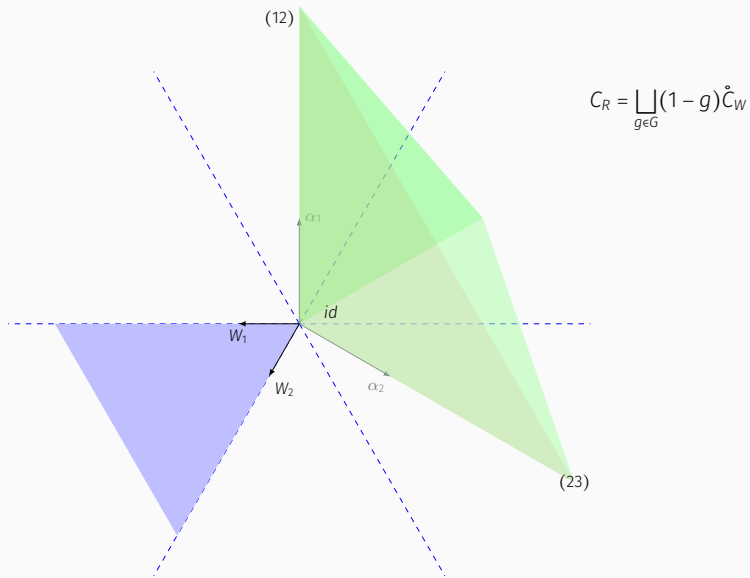
$$C_R = \bigsqcup_{g \in G} (1-g) \dot{C}_W$$

THE WALDSPURGER DECOMPOSITION FOR A_2 (\mathfrak{S}_3)

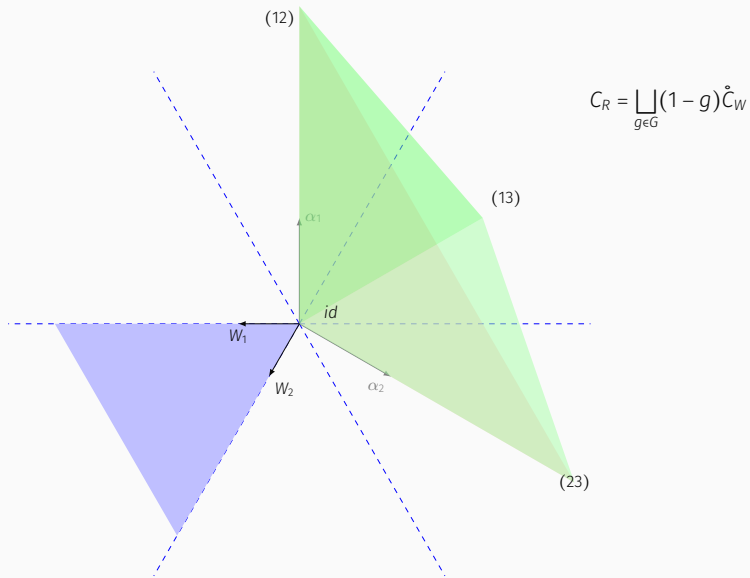


$$C_R = \bigsqcup_{g \in G} (1-g)\dot{C}_W$$

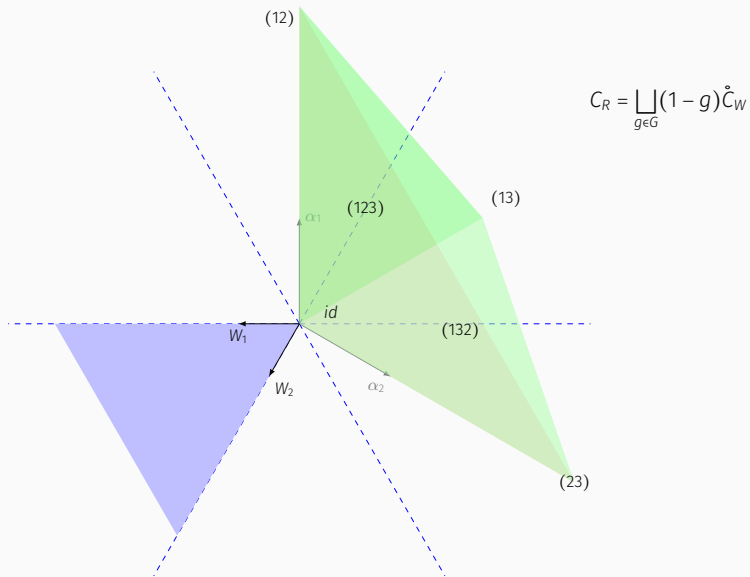
THE WALDSPURGER DECOMPOSITION FOR $A_2(\mathfrak{S}_3)$



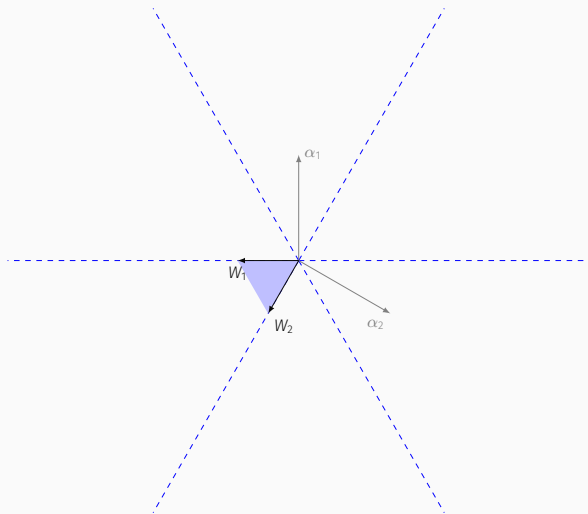
THE WALDSPURGER DECOMPOSITION FOR $A_2(\mathfrak{S}_3)$



THE WALDSPURGER DECOMPOSITION FOR $A_2(\mathfrak{S}_3)$

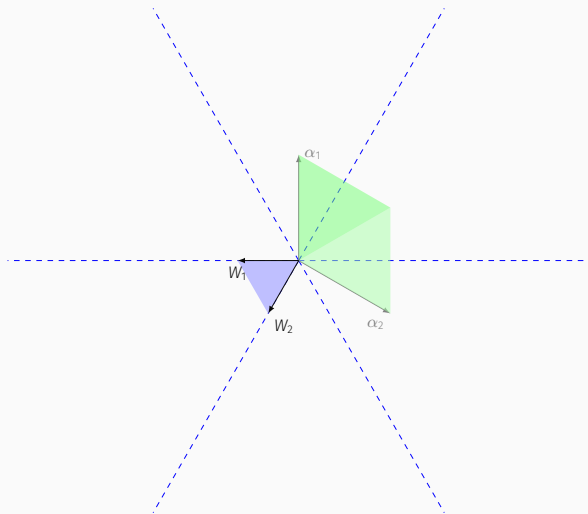


THE MIENRENKEN DECOMPOSITION FOR $A_2(\mathfrak{S}_3)$



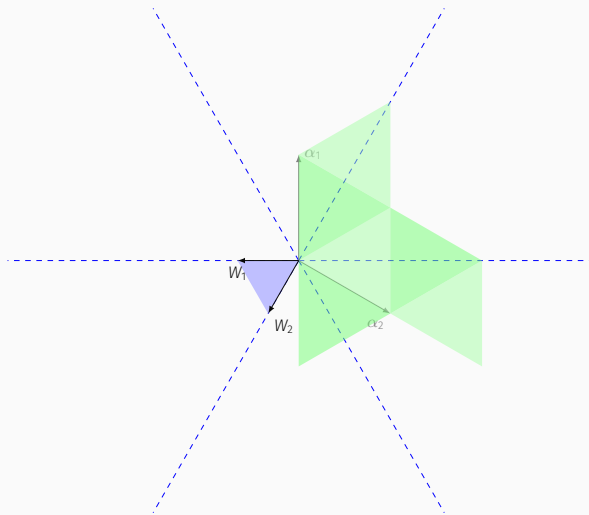
$$V = \bigsqcup_{g \in G} (1 - g) \dot{A}_W$$

THE MIENRENKEN DECOMPOSITION FOR $A_2(\mathfrak{S}_3)$



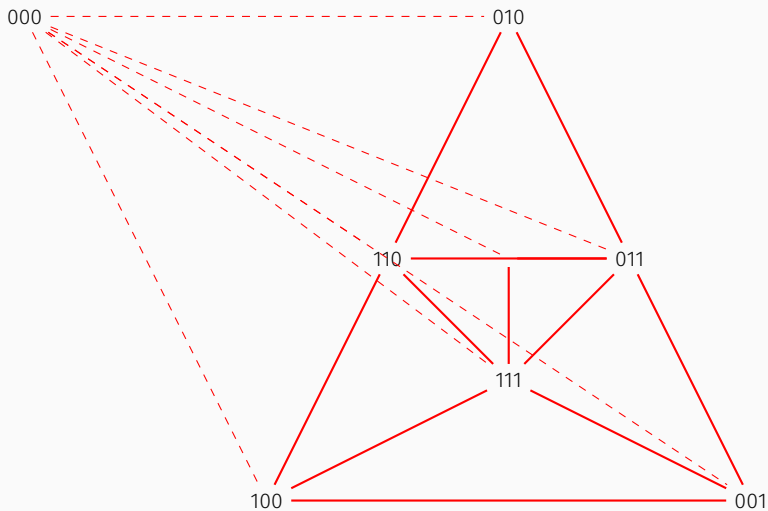
$$V = \bigsqcup_{g \in G} (1-g)\dot{A}_W$$

THE MIENRENKEN DECOMPOSITION FOR $A_2(\mathfrak{S}_3)$

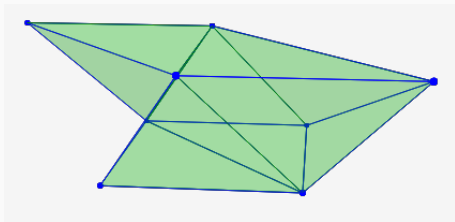
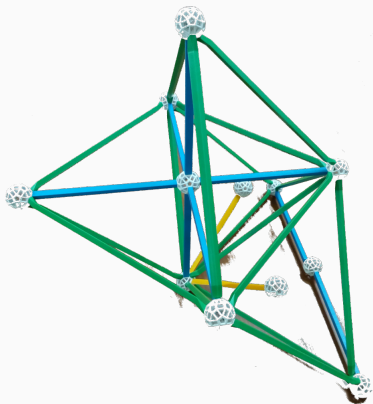


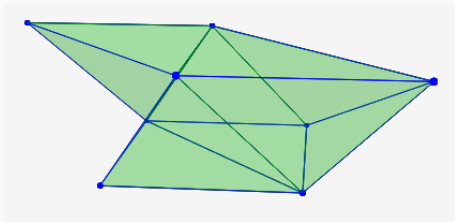
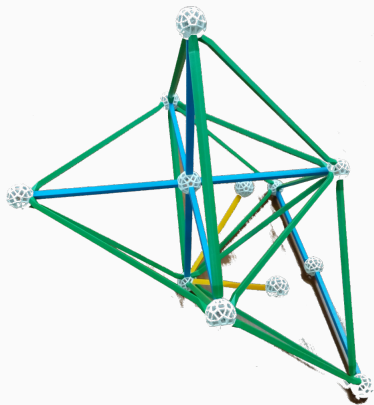
$$V = \bigsqcup_{g \in G} (1 - g) \mathring{A}_W$$

THE WALDSPURGER DECOMPOSITION FOR A_3 (\mathfrak{S}_4)



MEINRENKEN TILE FOR $A_3 = \mathfrak{S}_4$

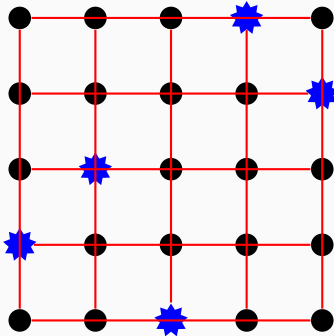




Part 2: Geometry \Rightarrow Permutations

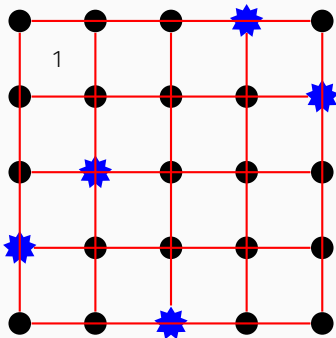
THE FUNDAMENTAL TRANSFORMATION

Consider $43512 \in \mathfrak{S}_5$



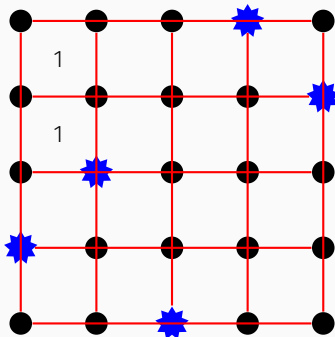
THE FUNDAMENTAL TRANSFORMATION

Consider $43512 \in \mathfrak{S}_5$



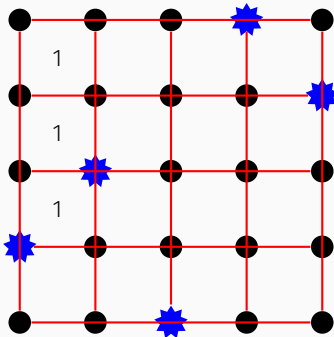
THE FUNDAMENTAL TRANSFORMATION

Consider $43512 \in \mathfrak{S}_5$



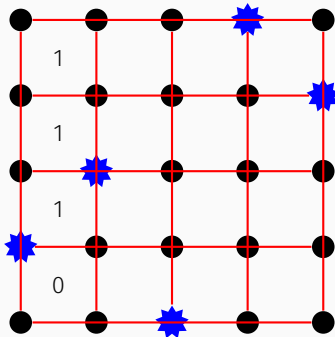
THE FUNDAMENTAL TRANSFORMATION

Consider $43512 \in \mathfrak{S}_5$



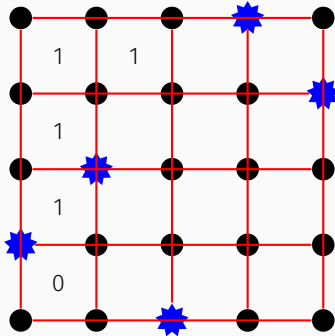
THE FUNDAMENTAL TRANSFORMATION

Consider $43512 \in \mathfrak{S}_5$



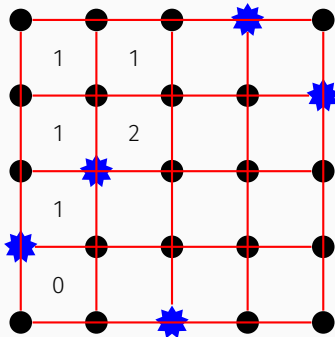
THE FUNDAMENTAL TRANSFORMATION

Consider $43512 \in \mathfrak{S}_5$



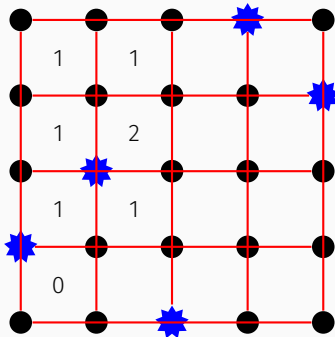
THE FUNDAMENTAL TRANSFORMATION

Consider $43512 \in \mathfrak{S}_5$



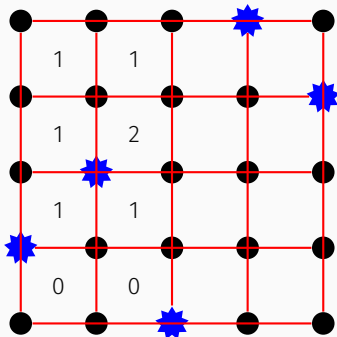
THE FUNDAMENTAL TRANSFORMATION

Consider $43512 \in \mathfrak{S}_5$



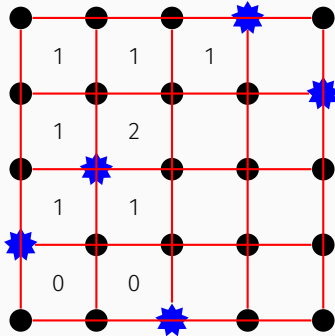
THE FUNDAMENTAL TRANSFORMATION

Consider $43512 \in \mathfrak{S}_5$



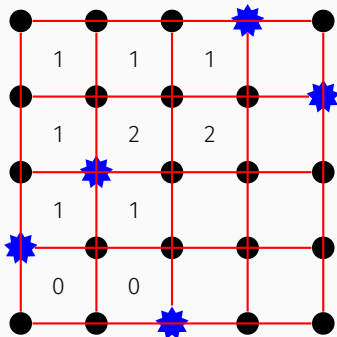
THE FUNDAMENTAL TRANSFORMATION

Consider $43512 \in \mathfrak{S}_5$



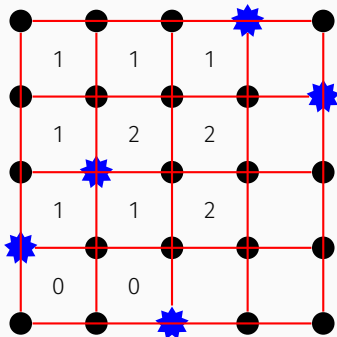
THE FUNDAMENTAL TRANSFORMATION

Consider $43512 \in \mathfrak{S}_5$



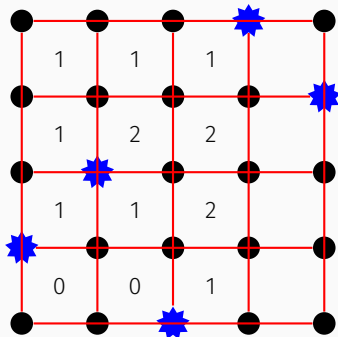
THE FUNDAMENTAL TRANSFORMATION

Consider $43512 \in \mathfrak{S}_5$



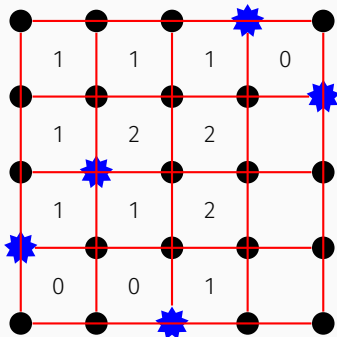
THE FUNDAMENTAL TRANSFORMATION

Consider $43512 \in \mathfrak{S}_5$



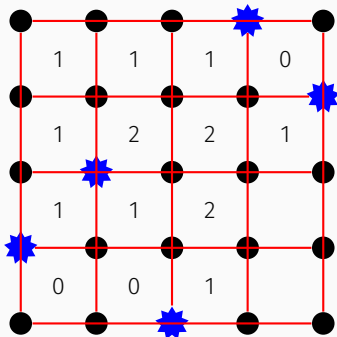
THE FUNDAMENTAL TRANSFORMATION

Consider $43512 \in \mathfrak{S}_5$



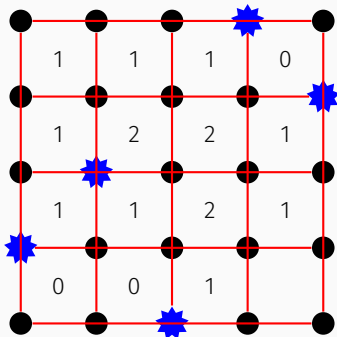
THE FUNDAMENTAL TRANSFORMATION

Consider $43512 \in \mathfrak{S}_5$



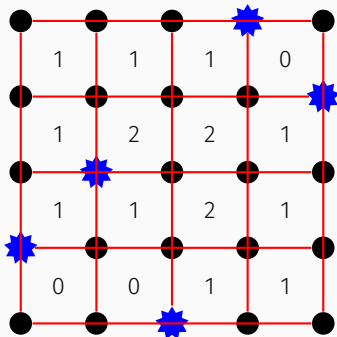
THE FUNDAMENTAL TRANSFORMATION

Consider $43512 \in \mathfrak{S}_5$



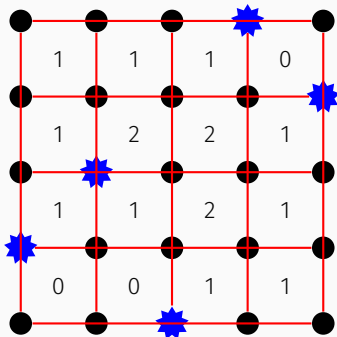
THE FUNDAMENTAL TRANSFORMATION

Consider $43512 \in \mathfrak{S}_5$



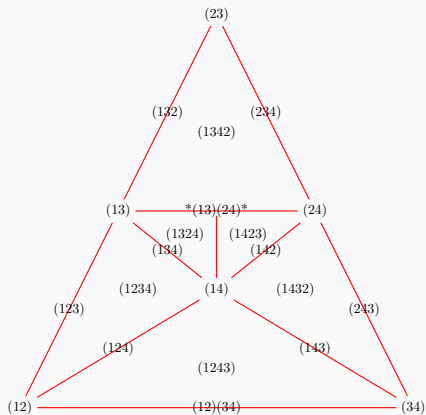
THE FUNDAMENTAL TRANSFORMATION

Consider $43512 \in \mathfrak{S}_5$

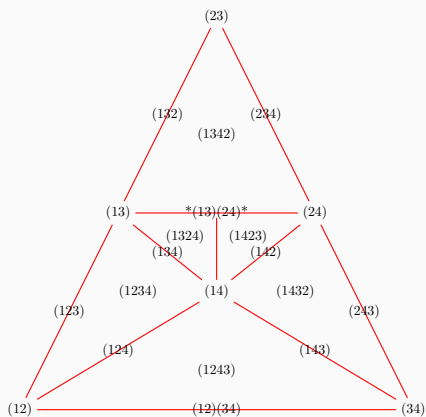


$$\text{that is, } 43512 \mapsto \left\{ a \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix} + d \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \mid a, b, c, d \in \mathbb{R}_{\geq 0} \right\}$$

CLASSICAL PERMUTATION STATISTICS, SOMETHING WEIRD...

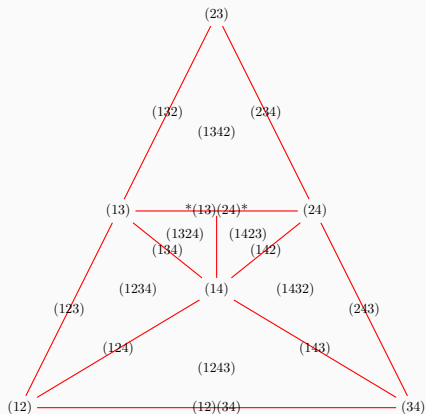


CLASSICAL PERMUTATION STATISTICS, SOMETHING WEIRD...



- four copies of \mathfrak{S}_3 picture

CLASSICAL PERMUTATION STATISTICS, SOMETHING WEIRD...



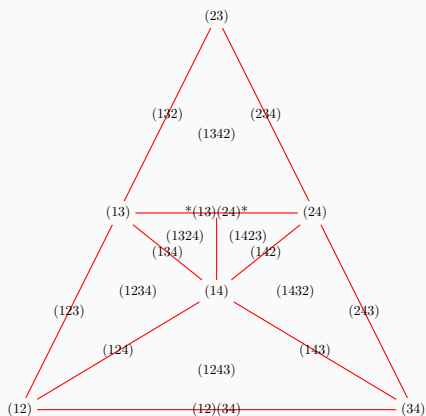
· four copies of S_3 picture

· Codimension



cycles $(c(n, k))$

CLASSICAL PERMUTATION STATISTICS, SOMETHING WEIRD...



- four copies of \mathfrak{S}_3 picture

· Codimension
 \updownarrow
cycles ($c(n, k)$)

- NOT A CW- complex!

Theorem:

There are exactly 2^n possible columns of Waldspurger matrices of type A_n .

We will call them UM vectors.

Theorem:

There are exactly 2^n possible columns of Waldspurger matrices of type A_n .
We will call them UM vectors.

Interesting Bijections:

Theorem:

There are exactly 2^n possible columns of Waldspurger matrices of type A_n .
We will call them UM vectors.

Interesting Bijections:

- Unimodal Motzkin Paths.

Theorem:

There are exactly 2^n possible columns of Waldspurger matrices of type A_n .
We will call them UM vectors.

Interesting Bijections:

- Unimodal Motzkin Paths.
- Elements of the root lattice inside a certain polytope

Theorem:

There are exactly 2^n possible columns of Waldspurger matrices of type A_n .
We will call them UM vectors.

Interesting Bijections:

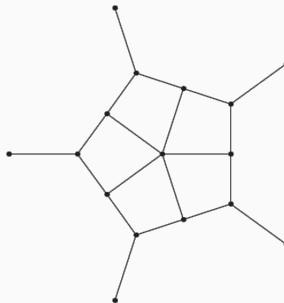
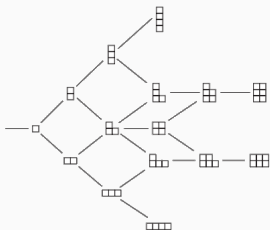
- Unimodal Motzkin Paths.
- Elements of the root lattice inside a certain polytope
- Abelian ideals in the nilradical of \mathfrak{sl}_n

Theorem:

There are exactly 2^n possible columns of Waldspurger matrices of type A_n .
We will call them UM vectors.

Interesting Bijections:

- Unimodal Motzkin Paths.
- Elements of the root lattice inside a certain polytope
- Abelian ideals in the nilradical of \mathfrak{sl}_n
- Young diagrams with hooklength less than n .



From an $n \times n$ matrix M , define the $(n-1) \times (n-1)$ matrix, $\mathcal{WT}(M)$ where

$$\mathcal{WT}(M)_{i,j} := \begin{cases} \sum_{\substack{a \leq i \\ b > j}} M_{a,b} & i \leq j \\ \sum_{\substack{a > i \\ b \leq j}} M_{a,b} & i \geq j \end{cases}.$$

THE WALDSPURGER TRANSFORM APPLIED TO OTHER MATRICES

From an $n \times n$ matrix M , define the $(n-1) \times (n-1)$ matrix, $\mathcal{WT}(M)$ where

$$\mathcal{WT}(M)_{i,j} := \begin{cases} \sum_{\substack{a \leq i \\ b > j}} M_{a,b} & i \leq j \\ \sum_{\substack{a > i \\ b \leq j}} M_{a,b} & i \geq j \end{cases}.$$

Warning: Note that $\mathcal{WT}(M)$ may be “over-determined” on the diagonal. In the case where M is a permutation matrix, but in general this need not be the case.

THE WALDSPURGER TRANSFORM APPLIED TO OTHER MATRICES

From an $n \times n$ matrix M , define the $(n-1) \times (n-1)$ matrix, $\mathcal{WT}(M)$ where

$$\mathcal{WT}(M)_{i,j} := \begin{cases} \sum_{\substack{a \leq i \\ b > j}} M_{a,b} & i \leq j \\ \sum_{\substack{a > i \\ b \leq j}} M_{a,b} & i \geq j \end{cases}.$$

Proposition: $\mathcal{WT}(M)$ is well-defined if and only if the i th row sum of M equals the i th column sum of M , for $1 \leq i \leq n$.

THE WALDSPURGER TRANSFORM APPLIED TO OTHER MATRICES

From an $n \times n$ matrix M , define the $(n-1) \times (n-1)$ matrix, $\mathcal{WT}(M)$ where

$$\mathcal{WT}(M)_{i,j} := \begin{cases} \sum_{\substack{a \leq i \\ b > j}} M_{a,b} & i \leq j \\ \sum_{\substack{a > i \\ b \leq j}} M_{a,b} & i \geq j \end{cases}.$$

Proposition: $\mathcal{WT}(M)$ is well-defined if and only if the i th row sum of M equals the i th column sum of M , for $1 \leq i \leq n$.

If an $n \times n$ matrix M has this property, we will say it is **sum-symmetric**

$$M \in SS_n$$

THE WALDSPURGER TRANSFORM APPLIED TO OTHER MATRICES

From an $n \times n$ matrix M , define the $(n-1) \times (n-1)$ matrix, $\mathcal{WT}(M)$ where

$$\mathcal{WT}(M)_{i,j} := \begin{cases} \sum_{\substack{a \leq i \\ b > j}} M_{a,b} & i \leq j \\ \sum_{\substack{a > i \\ b \leq j}} M_{a,b} & i \geq j \end{cases}.$$

Proposition: $\mathcal{WT}(M)$ is well-defined if and only if the i th row sum of M equals the i th column sum of M , for $1 \leq i \leq n$.

If an $n \times n$ matrix M has this property, we will say it is **sum-symmetric**

$$M \in SS_n$$

The map is linear and surjective, with kernel the diagonal matrices.

$$\mathcal{WT} : SS_n \rightarrow \text{Mat}_{n-1}$$

WALDSPURGER, MEINRENKEN AND ???





Part 2: Permutations \Rightarrow Alternating Sign
Matrices

An alternating sign matrix (or ASM) is a square matrix of 0s, 1s, and -1 s such that the sum of each row and column is 1 and the nonzero entries in each row and column alternate in sign.

An alternating sign matrix (or ASM) is a square matrix of 0s, 1s, and -1 s such that the sum of each row and column is 1 and the nonzero entries in each row and column alternate in sign.

These matrices generalize permutation matrices and arise naturally when using Dodgson condensation to compute a determinant.

An alternating sign matrix (or ASM) is a square matrix of 0s, 1s, and -1 s such that the sum of each row and column is 1 and the nonzero entries in each row and column alternate in sign.

These matrices generalize permutation matrices and arise naturally when using Dodgson condensation to compute a determinant.

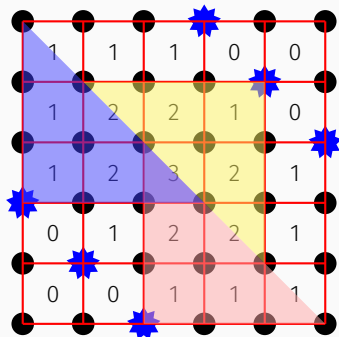
$(n-1) \times (n-1)$ matrices with UM columns and rows with their maxes on the diagonal are in bijection with $n \times n$ ASMs via the WT map!

Theorem:

$$\sum_{1 \leq i, j \leq n-1} \mathcal{WT}(g) = \frac{1}{2} \sum_{i=1}^n (g(i) - i)^2$$

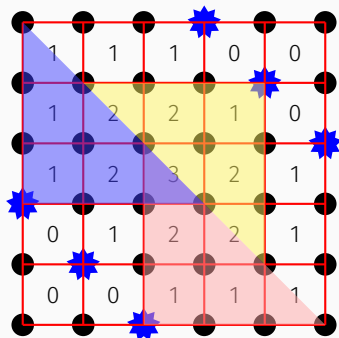
Theorem:

$$\sum_{1 \leq i, j \leq n-1} \mathcal{WT}(g) = \frac{1}{2} \sum_{i=1}^n (g(i) - i)^2$$



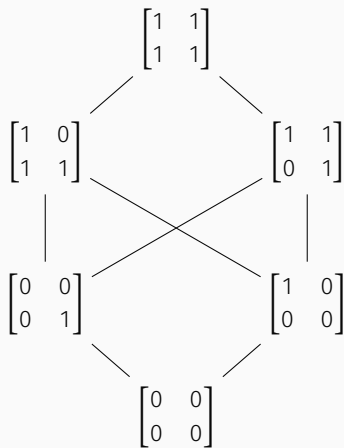
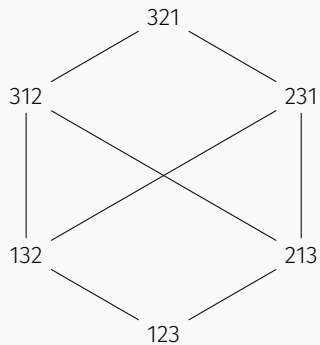
Theorem:

$$\sum_{1 \leq i, j \leq n-1} \mathcal{WT}(g) = \frac{1}{2} \sum_{i=1}^n (g(i) - i)^2$$

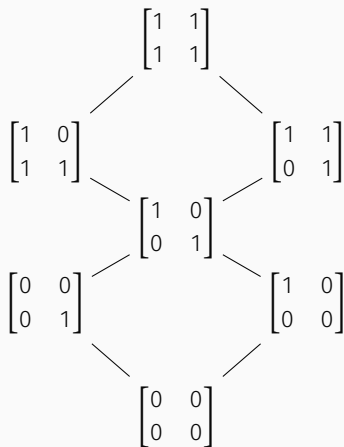
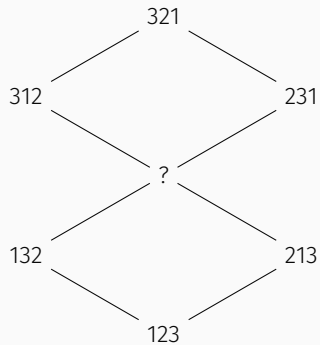


Fact (A. Lascoux, M. Schützenberger): Half the entropy of a permutation is its rank in the MacNeille completion of the Bruhat order— a distributive lattice with elements the alternating sign matrices, or ASMs.

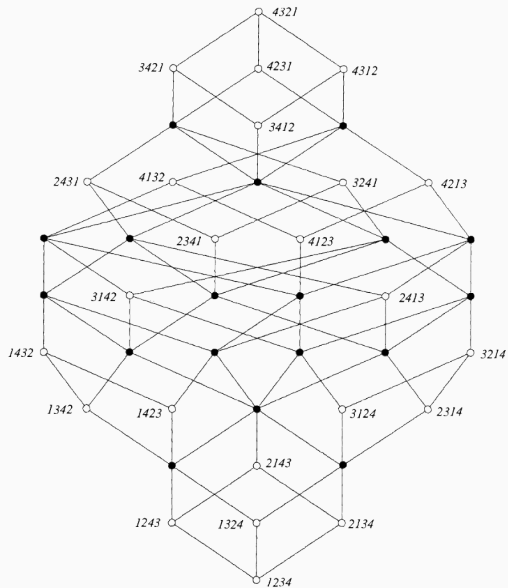
THE BRUHAT ORDER



THE MACNIELLE COMPLETION OF THE BRUHAT ORDER

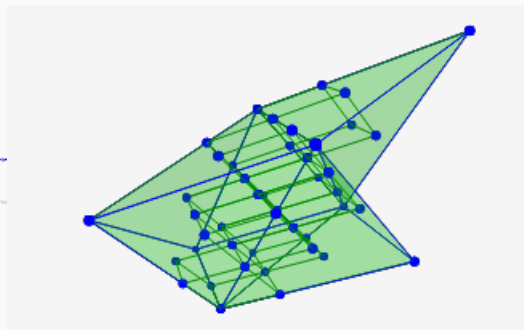
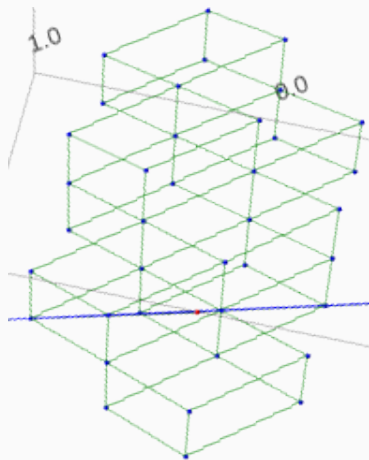


THE MACNIELLE COMPLETION FOR \mathfrak{S}_4



Part 3: ASMs \Rightarrow Geometry

A GEOMETRIC REALIZATION OF THE HASSE DIAGRAM OF AMS LATTICE



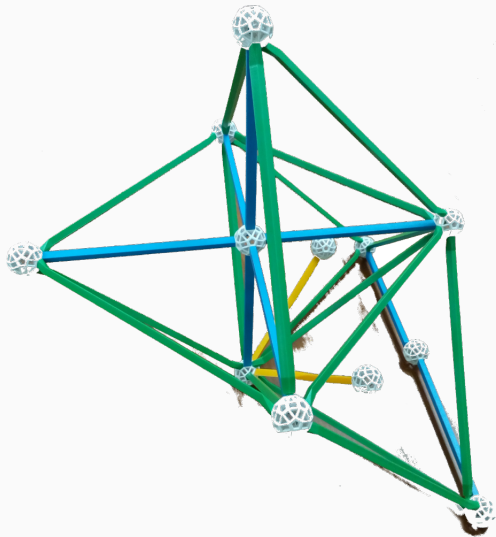
- Does this embedding of the Hasse diagram extend to other types?

- Does this embedding of the Hasse diagram extend to other types?
- Is there a good way to study the Meinrenken tile as a polytopal complex?

- Does this embedding of the Hasse diagram extend to other types?
- Is there a good way to study the Meinrenken tile as a polytopal complex?
- Is there always a way to rearrange the pieces in the Mienrenken tile to get a convex polytopal complex?

- Does this embedding of the Hasse diagram extend to other types?
- Is there a good way to study the Meinrenken tile as a polytopal complex?
- Is there always a way to rearrange the pieces in the Mienrenken tile to get a convex polytopal complex?
- The domain of the \mathcal{WT} map is SS_n . If we restrict ourselves to Gl_n what is the image? What is its topology?

- Does this embedding of the Hasse diagram extend to other types?
- Is there a good way to study the Meinrenken tile as a polytopal complex?
- Is there always a way to rearrange the pieces in the Mienrenken tile to get a convex polytopal complex?
- The domain of the \mathcal{WT} map is SS_n . If we restrict ourselves to Gl_n what is the image? What is its topology?
- In experimentation, the \mathcal{WT} map seems to preserve both the Birkhoff polytope and the ASM polytope. Is this true in general?



Thank You!