Math 162
Fall 2017
Exam 2
11/16/2017
Time Limit: 75 Minutes

Name: KEY

C# ____________

This exam contains 6 pages (including this cover page) and 6 questions. Total of points is 25.

No graphing calculators are allowed. If I see you using one, I take your test and you get a zero. No wandering eyes are allowed. If I see your eyes wandering, I take your test and you get a zero. No cell phones, computers or other technological aids are allowed. If I see your tech out ... you get the picture.

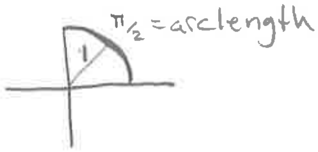
Grade Table (for teacher use only)

Question	Points	Score
1	8	
2	3	
3	8	
4	6	
5	0	
6	0	
Total:	25	

1. (8 points) (a) (2 points) Find the *arclength* of the curve defined by $f(x) = \sqrt{1-x^2}$ when x goes from zero to one.

$f'(x) = \frac{2x}{2\sqrt{1-x^2}} = \frac{x}{\sqrt{1-x^2}}$

$$\int_0^1 \sqrt{1+(f'(x))^2}\,dx = \int_0^1 \sqrt{1+\frac{x^2}{1-x^2}}\,dx$$



$$= \int_0^1 \sqrt{\frac{1-x^2}{1-x^2}+\frac{x^2}{1-x^2}}\,dx = \int_0^1 \sqrt{\frac{1}{1-x^2}}\,dx$$

$$= \sin^{-1}(x)\Big|_0^1 = \boxed{\pi/2.}$$

(b) (2 points) Find the *surface area* of the solid obtained by rotating the region bounded by this curve (in the first quadrant) about the y axis.

$$2\pi\int_0^1 x\sqrt{1+(f'(x))^2}\,dx = 2\pi\int_0^1 x\sqrt{\frac{1}{1-x^2}}\,dx = 2\pi\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$$

$u = 1-x^2$ when $x=0$ $u=1$

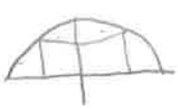
$du = -2x\,dx$ when $x=1$ $u=0$

$$= \pi\int_1^0 \frac{-1}{\sqrt{u}}\,du = -\pi\int_1^0 u^{-1/2}\,du = -2\pi u^{1/2}\Big|_1^0 = \boxed{2\pi}$$

(c) (3 points) Find the *volume* of the solid obtained by rotating the region bounded by this curve (in the first quadrant) about the y axis IN TWO WAYS: Cylindrical shell method, and disc method.

Cylinder: $\int_0^1 2\pi x\sqrt{1-x^2}\,dx$

Disc: $\int_0^1 -\pi(y^2-1)\,dy$



$u = 1-x^2$ when $x=0, u=1$

$du = -2x\,dx$ when $x=1, u=0$

$$\pi\int_1^0 -\sqrt{u}\,du = -\pi\frac{2}{3}u^{3/2}\Big|_1^0 = \boxed{\frac{2}{3}\pi}$$

$$\pi\left[\frac{y^3}{3}-y\right]_0^1 = \frac{2}{3}\pi$$

(d) (1 point) All of the integrals you just computed lead to formulas in boxes in middle school textbooks.

$2\pi r$ Circumference of the unit circle= 2π (four times part a)

$4\pi r^2$ Surface area of the unit sphere= 4π (twice part b)

$\frac{4}{3}\pi r^3$ Volume of the unit sphere= $\frac{4}{3}\pi$ (twice part c)

2. (3 points) Determine whether the following series converge or diverge. State the test you are using along with the hypothesis necessary to use that test!

(a) (1 point) $\displaystyle\sum_{n=0}^{\infty} \frac{-9n}{(2n+3)^2}$ $= -9 \underbrace{\sum_{n=0}^{\infty} \frac{n}{4n^2+12n+9}}_{(*)}$

(*) diverges by limit comparison to $\frac{1}{n}$.

Indeed, both $\frac{n}{4n^2+12n+9}$ and $\frac{1}{n}$ have all positive terms, and

$\lim_{n\to\infty} \frac{n}{4n^2+12n+9} \cdot \frac{n}{1} = \frac{1}{4}$ which is a nonzero constant.

(b) (1 point) $\displaystyle\sum_{n=0}^{\infty} \frac{(-5)^n n^2}{(2n+3)}$

Diverges by the test for divergence

since $\lim_{n\to\infty} \frac{(-5)^n n^2}{2n+3}$ DNE

(c) (1 point) $\displaystyle\sum_{n=0}^{\infty} \frac{5n}{(-1)^n((2n)^2-3)}$

Converges by alternating series test since the terms alternate, are decreasing in absolute value, and $\lim_{n\to\infty} \frac{5n}{(2n)^3-3} = 0$.

3. (8 points) Euler's Formula

(a) (1 point) Write the Maclaurin series of $\sin(x)$ and list the first five non-zero terms.

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$

(b) (1 point) Write the Maclaurin series of $\cos(x)$ and list the first five non-zero terms.

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$$

(c) (1 point) Write the Maclaurin series of e^x and list the first five non-zero terms.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

(d) (2 points) Recall that $i := \sqrt{-1}$ and thus $i^2 = -1, i^3 = -i, i^4 = 1$, etc. Use this to write a power series expression for e^{ix} with no higher powers of i in it. Write (at least) the first five non-zero terms. (*hint*: you can leave your answer with two sigma sums.)

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^n x^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + \sum_{n=0}^{i} \frac{(-1)^n i x^{2n+1}}{(2n+1)!}$$

$$= 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \frac{i^6x^6}{6!} + \frac{i^7x^7}{7!} + \dots$$

$$= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \dots = \underbrace{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots}_{\cos x} + \underbrace{ix - \frac{ix^3}{3!} + \frac{ix^5}{5!} - \frac{ix^7}{7!}}_{i\sin(x)}$$

(e) (1 point) Write the power series expression for $\cos(x) + i\sin(x)$.

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + i\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

(f) (2 points) Congratulations! (Hopefully) you just proved what Richard Feynman called "the greatest formula in mathematics!" State it below. Use it (or just what you got in part d) to find $e^{i\pi}$.

$$e^{ix} = \cos x + i \sin x$$

$$e^{i\pi} = \cos\pi + i\sin\pi = -1 + 0 = -1$$

4. (6 points) Find the sum of each of the following series:

(a) (2 points) $\sum_{n=4}^{\infty} \frac{1}{3^{n-2}} = \sum_{n=2}^{\infty} \frac{1}{3^n} = \left(\sum_{n=0}^{\infty} \frac{1}{3^n}\right) - 1 - \frac{1}{3}$

$= \frac{1}{1-1/3} - 1 - 1/3$

$= 3/2 - 1 - 1/3 = \boxed{1/6}$

(b) (2 points) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^{3n}} = -1 \sum_{n=1}^{\infty} \frac{(-1)^n}{27^n} = -1\left[\sum_{n=0}^{\infty} \left(\frac{-1}{27}\right)^n - 1\right]$

$= -1\left[\frac{1}{1+1/27} - 1\right]$

$= -1\left[\frac{1}{28/27} - 1\right] = -1\left[\frac{27}{28} - 1\right]$

$= \boxed{\frac{1}{28}}$

(c) (2 points) $\sum_{n=0}^{\infty} \frac{(-1)^n(\pi)^{2n+1}}{(2n+1)!} = \sin(\pi)$

$= 0$

5. (extra credit 2 points)

For each of the following find a (hopefully elementary) functions $f(x) : \mathbb{R} \to \mathbb{R}$ that when restricted to the $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ gives the following sequence:

(a) (1 point) $\overset{0}{0}, -\overset{1}{1}, 0, \overset{2}{2}, 0, -\overset{3}{3}, 0, \overset{4}{4}, 0, -\overset{5}{5}, \dots$

$$a_n = -\left(\frac{n+1}{2}\right)\sin\left(n\frac{\pi}{2}\right)$$

(b) (1 point) $-5, -6, -5, 4, 27, 70, 139, 240, 379, 562, \dots$

$-1 \quad 1 \quad 9 \quad 23 \quad 43 \quad 69 \quad 101$

$2 \quad 8 \quad 14 \quad 20 \quad 26 \quad 32$

$6 \quad 6 \quad 6 \quad 6 \quad 6$

$$\boxed{a_n = n^3 - 2n^2 - 5}$$

$$a_n = a \cdot n^3 + bn^2 + cn + d$$

$a_0 = a(0)^3 + b(0)^2 + c(0) + d = -5 \Rightarrow d = -5$

$a_1 = a(1)^3 + b(1)^2 + c(1) + (-5) = -6 \Rightarrow a + b + c = -1$

$a_2 = 8a + 4b + 2c - 5 = -5 \Rightarrow 8a + 4b + 2c = 0$

$a_3 = 27a + 9b + 3c - 5 = 4 \Rightarrow 27a + 9b + 3c = 9$

$\Rightarrow a = 1, \quad b = -2, \quad c = 0$

6. (extra credit 2 points) Find the Taylor series the following polynomial centered at $x = 1$. You should get a new polynomial in the variable $(x - 1)$. View its coefficients as a sequence. It is sequence A104101 in the OEIS and was featured in a hit ABC drama that ran between 2004 and 2010. What is the sequence and what is that show?

→ 4, 8, 15, 16, 23, 42 LOST!

$$f(x) = 4x^5 - 12x^4 + 23x^3 - 21x^2 + 24x + 24.$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!}$$

$f(1) = 42$

$f'(1) = 23 \qquad f'(x) = 20x^4 - 48x^3 + 69x^2 - 42x + 24$

$f''(1) = 32 \qquad f''(x) = 80x^3 - 144x^2 + 138x - 42$

$f'''(1) = 90 \qquad f'''(x) = 240x^2 - 288x + 138$

$f^{(4)}(1) = 192 \qquad f^{(4)}(x) = 480x - 288$

$f^{(5)}(1) = 480 \qquad f^{(5)}(x) = 480$

$f^{(6)}(x) = 0$

$\Downarrow$

$$f(x) = 42 + 23(x-1) + \frac{32}{2!}(x-1)^2 + \frac{90}{3!}(x-1)^3 + \frac{192}{4!}(x-1)^4 + \frac{480}{5!}(x-1)^5$$

$\Downarrow$

$$f(x) = 42 + 23(x-1) + 16(x-1)^2 + 15(x-1)^3 + 8(x-1)^4 + 4(x-1)^5$$