Math 113 Finite Math with a Special Emphasis on Math & Art
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2 Symmetry notes

2.1 Different Ideas of Symmetry

Our Lowe Museum tour has set the stage for the next topic of the course–*Symmetry*.

We started class by talking about the different ideas that artsy and mathy people have about symmetry. Our museum guide, Jodi Sypher, is the curator of education at the Lowe and is trained in the arts. Some adjectives that she might use to describe *symmetry* might be: balanced, well-distributed, well-proportioned, beautiful, harmonious, ordered, and even. On the other hand, the words that I use when talking about symmetry are bilateral, rotational, and translational[1]. Let's explore this difference, if indeed there is one.

I want to note here that whatever our idea of symmetry is, the sensation that we get when looking at something symmetrical is a rather calm or pleasant one. However, when we look at something that isn't symmetrical, there is a sense of tension. This could convey the feeling of movement (as in a couple of the sculptures we saw on our walk to the museum the other day), or a feeling of unease like we are being challenged.

The point is that artists, consciously or unconsciously, employ this interplay of *symmetry* and *asymmetry* to evoke different reactions and emotions from the viewer, but let's save that for later.

Now, we give the general (and very precise) mathematical definition of what it means for something to be *symmetric* and what a *symmetry* is. These are very powerful definitions that include all that we have mentioned so far (and actually a lot more)!

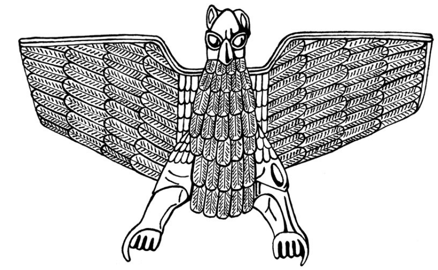


fig.1 Probably the Anzu, an eagle-like bird with lion's head and out-stretched wings. Detail of a relief plaque from Girsu, Mesopotamia. Stone. Around 2450 BCE.

fig.2 Initially, this fierce relief of Assurnasipal appears to be symmetric. But upon closer inspection, can you see that it isn't strictly so?

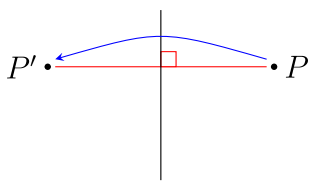
[1] There's actually another term that I didn't mention because it requires a little more explanation–a *glide reflection*.

2.2 The Mathematical Definition of Symmetry

Definition 2.1. An object (shape, graph, equation, etc.) is symmetric with respect to a given transformation if this transformation when applied to this object results in something that appears to be exactly the same.

Definition 2.2. A symmetry of an object is a transformation that does not change the appearance of the object.

When we say something is bilaterally symmetric, we mean that there is some line that divides the thing so that it looks the same on the right and on the left as we were holding a mirror along this line. Let's take a closer look at this transformation of *reflection*.



The transformation of *reflection* along a line takes every point in the plane and sends it to its "mirror image." For example, it takes the point P to the point P' that has the same perpendicular distance from the line as P, but is on the other side of the line. Note that the only points that are unaffected by the reflection are those that are on the "axis" of reflection.

Usually, the transformation of reflection along a line causes things to look different–to be reversed, or moved around. However, in the special instance that this transformation results in exactly the same appearance, we say that the object is bilaterally symmetric[2].

fig.3 The bilateral symmetry of an orchid flower. See how under reflection, any point on the red line is sent to its mirror image on the other side leaving the appearance of the line unchanged?

In class, I showed you several images of symmetrical things. First, all the images had *bilateral* symmetry. Then we encountered objects that have *rotational symmetry*, which means that for each such object there exists a point C and an angle θ such that rotation by θ about C results in something identical.

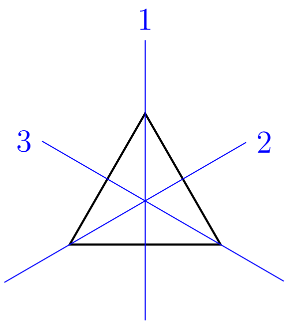
Example 2.3. Consider the letters of our alphabet[3].
1) Find all symmetric *uppercase* letters and list their individual symmetries.
2) Find all symmetric *lowercase* letters and list their individual symmetries.

We spent some time on part one of this exercise and found that it was easy to find the letters with bilateral symmetry. But initially, a lot of people overlooked letters that have rotational symmetry (and no bilateral symmetry). For example, "N", "Z", or "S". We also noticed that for some letters, the symmetries depend

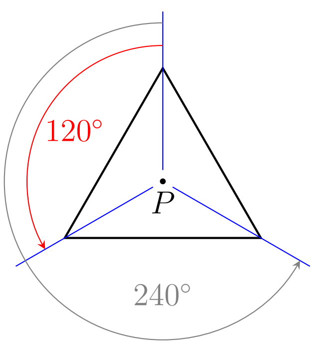
[2]with respect to a transformation of reflection about a particular line.
[3]It is assumed that we are looking at some sans serif font.

on how you write the letters, e.g., if you write your "K" so that the two diagonals meet at the midpoint of the vertical, then the "K" is symmetric; if your "Q" is a perfect circle with the tail a straight line segment pointing towards the center of the circle, then your "Q" has bilateral symmetry along the axis determined by the line segment; or if your "X" were like a multiplication sign "$\times$", then it would have four axes of reflection and three (non-identity) rotations, otherwise you'd only have two axes of reflection and one rotation by $180°$.

Example 2.4. Let's list the *symmetries* of the equilateral triangle.



Beginning with the bilateral symmetries, we just need to find all the lines across which we can reflect (and not change the appearance or placement of the triangle). Clearly, reflection across any blue line is a symmetry. Let's use F_1, F_2, and F_3 to denote reflection across lines 1, 2, and 3, respectively.



Now we find the rotational symmetries, i.e., the rotations about some point P that will not change the appearance or placement of the triangle. Clearly, a rotation through an angle of $120°$[a] about the center P of the triangle is one such symmetry. Likewise a rotation through an angle of $240°$ is another one. Of course, we can't forget rotation through $360°$, but that's just the identity symmetry, which we often write using I. Let's use $R_{120°}$ and $R_{240°}$ to denote rotation through $120°$ and $240°$, respectively.

[a] In math, there is the (perhaps odd) convention that a positive rotation is in the *counterclockwise* direction.

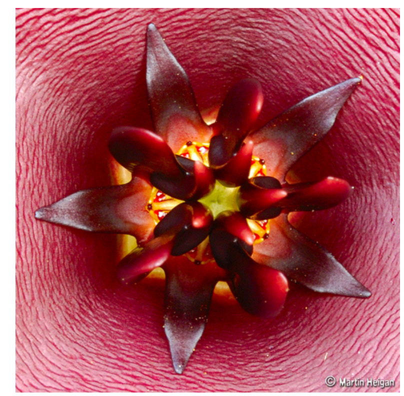


fig.4 Check out the beautiful Stapelia Schinzii and its rotational symmetry.

Thus, our list of symmetries of the equilateral triangle consists of the set $\{I, R_{120°}, R_{240°}, F_1, F_2, F_3\}$, which is a set of six distinct transformations.

At this point, there may be some confusion regarding the notation that we are using. What do all these capital letters really mean, anyway? They're not numbers, so what are they?

The thing to remember is that we are trying to find a way to describe *precisely* the "symmetry" of an object. Ok, an equilateral triangle isn't the most interesting thing in the world, but it is simple and provides us with a good place to start. We need to translate our vague idea of "symmetry" into something more concrete and that we can work with instead of waxing poetic about. For mathematicians, when we say an object is *symmetric* we need to know what transformations we are talking about, i.e., what are the transformations that are leaving the appearance of our object unchanged?

And so we need to have a way to write down each such transformation[4], which we now call a *symmetry*. Here's where the letters come in; each letter we've written down represents a particular symmetric transformation.

2.3 The Definiton of a Mathematical Entity Known as a *Group*

Now, we introduce the "space" within which these letters or transformations reside. This is the very profound and fruitful notion of a mathematical object known as a *group*.[5]

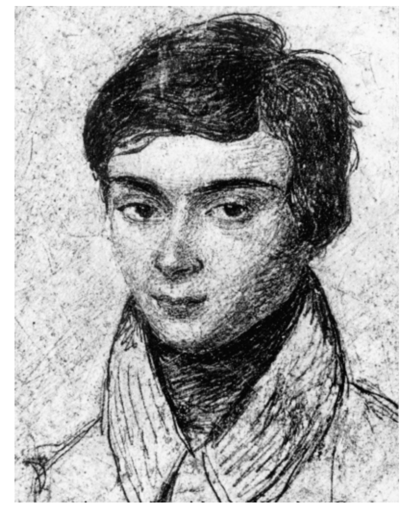


fig.5 Évariste Galois (1811-1832), the innovator of group theory at age 15.

Definition 2.5. A group is a set G of transformations together with a "multiplication," $\cdot$, on G which satisfies four axioms:
0) **Closure.** For all S, T in G, $S \cdot T$ is also an element of G;
i) **Associativity.** For all S, T, U in G, $(S \cdot T) \cdot U = S \cdot (T \cdot U)$;
ii) **Identity.** There is a (unique) element I in G such that for all S in G, $S \cdot I = I \cdot S = S$; and
iii) **Inverses.** For each S in G, there exists a (unique) T in G such that $S \cdot T = T \cdot S = I$.

Remark 2.6. Let's talk about the operation of "multiplication" that we will be using.[6] It is also known as *composition*. This is how it works: when we write $S \cdot T$ we mean the transformation that results by first applying T and then applying S. Suppose we have $S = R$ and $T = F$, then $R \cdot F$ means first doing the reflection transformation F and then doing the rotation R. At first, it seems to be odd that we do the operation on the right first and then the one on the left. The reason for this is that we often consider the effect of $R \cdot F$ on a particular point P, and we write $R \cdot F(P)$. This notation then makes sense as we should naturally do F first (since it's next to P) and then do R.

[4]There's an even more precise name for the kind of transformations that we are looking at; they're called *isometries*. An *isometry* is a transformation that preserves distance.

[5]There is a great story behind the person who first truly recognized the importance and usefulness of this idea. His name was Évariste Galois and he died in a duel at the age of 20. The story is that he knew he couldn't win the shoot out the next morning, so he pulled an all-nighter and wrote down an outline of his mathematical ideas. Hermann Weyl, a great 20th century mathematician (who also wrote the little symmetry book I suggested), had this to say of what Galois wrote down that night: "This letter, if judged by the novelty and profundity of ideas it contains, is perhaps the most substantial piece of writing in the whole literature of mankind."

[6]There are actually many different kinds of multiplications, but for our purposes we will only consider *composition*.

Notation 2.7. Since our "multiplication" will always be *composition* of functions in the order just mentioned, we will often write $S \cdot T$ as ST, eliminating the "$\cdot$" between S and T. Furthermore, as *composition* is always associative[7], we will usually not bother with parentheses to write $(ST)U$ and instead write STU to mean the transformation that applies U then T then S.

Let's look back at our notation for the symmetries of the equilateral triangle, $\{I, R_{120^\circ}, R_{240^\circ}, F_1, F_2, F_3\}$. Let $R = R_{120^\circ}$, then clearly $R^2 = R_{240^\circ}$, and $R^3 = I$. In fact, $\{I, R, R^2\}$ form a group; this group is known as C_3, the *cyclic group of order 3*. We can characterize a group by writing out its "multiplication table."

Table1 Can you fill rest of the table assuming the left most column is S and the top most row is T?

$S \cdot T$	I	R	R^2
I		R	
R	R		I
R^2			

Now, in addition to the (symmetric) rotations of the equilateral triangle, we include the reflections to get the so-called *dihedral group* $D_3 = \{I, R, R^2, F_1, F_2, F_3\}$.[8] The question is: does this actually form a group? For example, is Axiom 0) satisfied? In other words, are all products of elements in D_3 actually in D_3? What's F_1R, F_1R^2? To answer these questions, we need to write down the "multiplication table" for D_3.

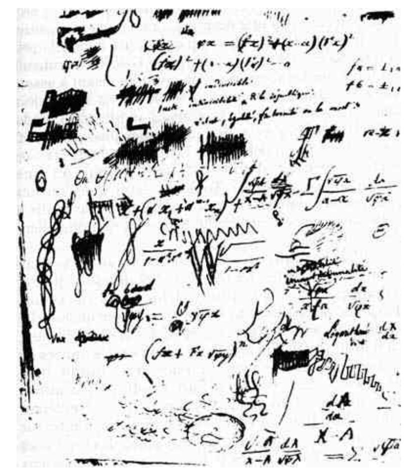


fig.6 Galois's mathematical notes.

As a group work in class (Tuesday, 2/9), you were able to show that $F_1R = F_2$ and $F_1R^2 = F_3$.[9] So, if we let $F = F_1$, then $FR = F_2$ and $FR^2 = F_3$; and we can rewrite the *dihedral group* as $D_3 = \{I, R, R^2, F, FR, FR^2\}$.

[7]This is a freebie as by the definition of *composition*, we have: $[(ST)U](P) = (ST)(U(P)) = ST(U(P)) = S(T(U(P))) = S((TU)(P)) = [S(TU)](P)$.

[8]In class, we addressed the question of why the subset $\{F_1, F_2, F_3\}$ does not constitute a group like the set of rotations $\{I, R, R^2\}$ did to make the cyclic group. The answer we found was that it was not closed; e.g., what's F_1^2? or F_1F_2?

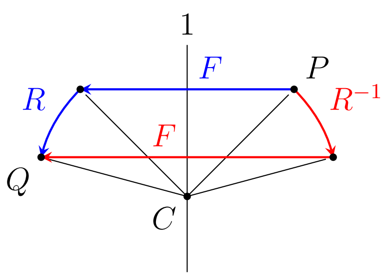
[9]You had to "label" or "tag" your triangle and then remember that you first apply the transformation on the right and then the one on the left; and don't forget that the rotation R is in the counterclockwise direction (as $R = R_{120^\circ}$ is rotation by a positive number). Finally, the center of rotation and the axes of reflection are fixed–the transformations don't affect them.

Table2 Now, let's try to write down its multiplication table keeping in mind that $R^3 = I$ and $FF = I$.

$S \cdot T$	I	R	R^2	F	FR	FR^2
I	I	R	R^2	F	FR	FR^2
R	R	R^2	I			
R^2	R^2	I	R			
F	F	FR	FR^2	I	R	R^2
FR	FR	FR^2	F			
FR^2	FR^2	F	FR			

We were able to fill up a lot of the table, but now we've run into a bit of a problem, because we are getting expressions with RF in them and aren't sure what to do with them. For example, one of the blanks is $R \cdot FR = RFR$. We might be tempted to just do this: $(RF)R = (FR)R = FR^2$, which means we are assuming that the products in the parentheses are equal, $RF = FR$, but that's totally NOT true. Recall that $FR = F_2$, but $RF = F_3$.[10]

To get around this RF, we are going to need a little tool (formula) that will help us big time. The general diagram below is the key to the formula. In it R^{-1} refers to "the inverse of" R; and C is the center of rotation.



From this picture, we can see that the transformation RF takes the point P to the point Q. Likewise, the transformation FR^{-1} also takes P to Q. And since P was chosen arbitrarily, this works for all points P. Thus we have $RF = FR^{-1}$. In our case, since we are dealing with the equilateral triangle, R^{-1} (the inverse of R) is R^2; and we have the formula: $RF = FR^2$.

The great thing about this formula is that it allows us to change the order of the R's and F's to put products in a form that we can use our other formulas ($R^3 = I, F^2 = I$) to get an element in our list for D_3. In so doing, we can populate the remainder of the table.

[10] You can show this like we did in class with a labeled triangle. Just remember that RF means apply F first then R.

For example, let's do the product $R \cdot FR^2$ in excruciating detail. Again, we don't need to write the dot $\cdot$ for the multiplication:

$$R \cdot FR^2 = R(FR^2) = (RF)R^2 = (FR^2)R^2 = F(R^2R^2) = FR^4 = FR$$

And so, our answer is FR. The second equality uses the associative property; the third equality uses our formula to switch the order of the letters; the fourth equality is the associative property again; and finally we use $R^4 = R$ to get FR.

Table3 I've inserted several (in red) to give you an idea. Try to fill in the rest.

$S \cdot T$	I	R	R^2	F	FR	FR^2
I	I	R	R^2	F	FR	FR^2
R	R	R^2	I	FR^2		FR
R^2	R^2	I	R			
F	F	FR	FR^2	I	R	R^2
FR	FR	FR^2	F	R^2		
FR^2	FR^2	F	FR			

Once you have filled out the multiplication table (correctly), you will have everything that you need to know about the group of all symmetries of the equilateral triangle, a.k.a., the *dihedral group* D_3. You will have shown that it satisfies all the axioms of a *group*–closure, associativity[11], identity, and inverses[12].

Now, if you're starting to like this sort of thing try to calculate the *dihedral group* of the symmetries of the square. It turns out by the same kind of work we did for D_3 that $D_4 = \{I, R, R^2, R^3, F, FR, FR^2, FR^3\}$. We still have that cool and useful formula $RF = FR^{-1}$, but now $R^{-1} = R^3$. Enjoy.

fig.7 The dihedral symmetry of the Pentagon is super visible from the sky, which makes it a great target. Not too bright, huh?

[11] We already mention and showed (in a previous footnote) that this is always true when the multiplication is *composition*.

[12] To find *the* inverse of any element all you have to do is pick the row corresponding to the element and then find I in it and go up to get the inverse.