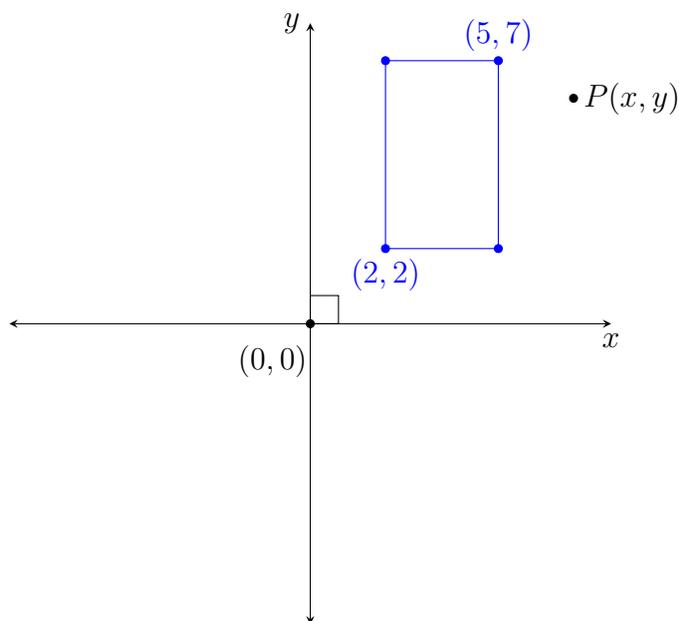


## 4 Linear Perspective

### 4.1 The Cartesian Coordinate System in Two Dimensions

We now begin our exploration of the space we live in – three-dimensional space. Let's recall from high school algebra *the Cartesian or Rectangular Coordinate System*, which was the discovery of the colorful French philosopher and mathematician René Descartes, who actually made his rather successful living as a soldier<sup>1</sup>.

Descartes's great (and in hindsight, simple) discovery is that we can specify any point  $P$  in the plane by an *ordered pair*,  $(x, y)$ , of numbers, which we call the *coordinates* of the point  $P$ . This simple observation was completely revolutionary as it brought algebra into geometry creating what has come to be known as *analytic geometry*. Here's a picture:



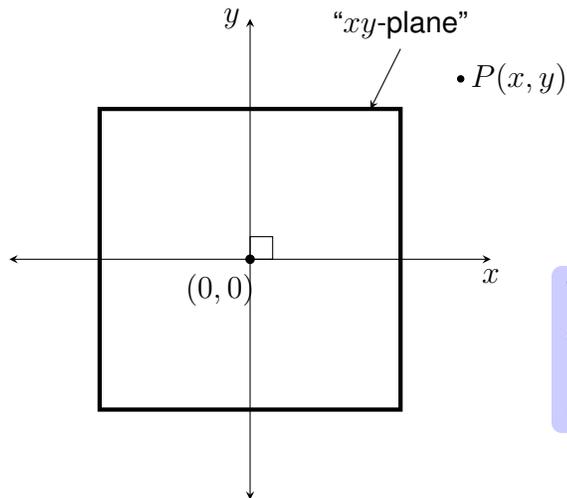
The  $xy$ -plane extends to infinity both vertically and horizontally. An important thing to notice is that the  $x$ -axis is perpendicular to the  $y$ -axis; together they split the plane into four quadrants (numbered counterclockwise starting with *Quadrant I* at the upper-right).

Suppose now that there is a blue rectangle in quadrant I with sides parallel to the  $x$ - and  $y$ -axes. If its opposite corners given by  $(2, 2)$  and  $(5, 7)$ , what would be the coordinates of the other two corners? With a little thought, it should be clear that the lower right corner of the blue rectangle should have the same  $y$ -value as  $(2, 2)$  because they have the same height; while at the same time it should have the same  $x$ -value as  $(5, 7)$  because they have the same “horizontal” distance from the vertical  $y$ -axis. Hence the lower right corner has coordinates  $(5, 2)$ . In the same manner, we can see that the upper left corner of the blue rectangle should have coordinates  $(2, 7)$ .

<sup>1</sup> If interested, you can read about his life in E.T. Bell's book, *Men of Mathematics*, and see how often and close Descartes came to not offering the world his fantastic discoveries; i.e., how close we all came to not having to study any of this stuff! Then again, we'd never have heard of his famous saying: “Cogito ergo sum” (I think, therefore I am), which really wouldn't be a huge loss since it's totally untrue.

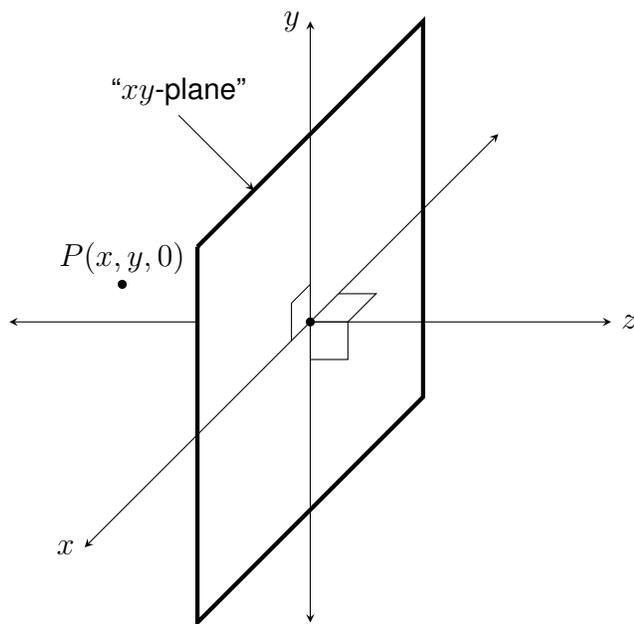
## 4.2 The Cartesian Coordinate System in Three Dimensions

Instead of dwelling in the two-dimensional space of the  $xy$ -plane (as is done in high school), we now move into the *third dimension*. To do this we introduce the  $z$ -axis, which must be perpendicular<sup>2</sup> to both the  $x$  and  $y$ -axes. The way to do this is to imagine the  $z$ -axis as going straight in and out of the page at the *origin*,  $(0, 0)$ , the point of intersection of the  $x$ - and  $y$ -axes. Let's look again at the  $xy$ -plane, but imagine the (negative part of the)  $z$ -axis pointing out at us. We draw a thick black square to help us visualize the next step.



The  $xy$ -plane extends to infinity both vertically and horizontally. The thick black square is drawn to help you visualize our next bold move.

In order for us to draw this, we need to use a little imagination and rotate the  $xy$ -plane so that the positive  $x$ -axis points out to us. Having done so, the  $z$ -axis is now a horizontal line running from left to right<sup>3</sup>.

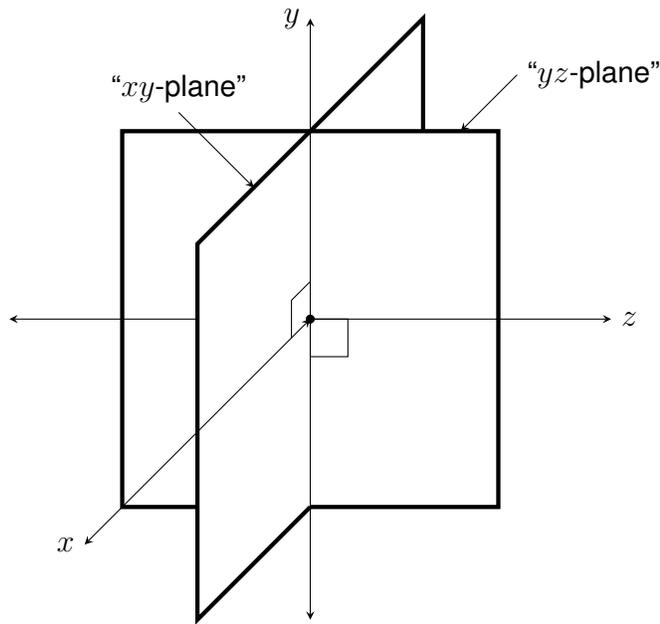


Our next bold move consists of rotating the  $xy$ -plane so that the (positive)  $x$ -axis points out of the page. Obviously, it's not possible to actually draw this in air, so we draw at an angle to suggest this. The thick black square is similarly drawn to suggest that it too is sticking out of the page. See how the  $z$ -axis is now horizontal coming out perpendicularly from  $xy$ -plane? Note also that our previous point  $(x, y)$  is now  $(x, y, 0)$ , this is because points in 3-dimensional space need *three* coordinates  $(x, y, z)$  and any point on the  $xy$ -plane has  $z = 0$ .

<sup>2</sup>Actually, the correct word is *orthogonal*.

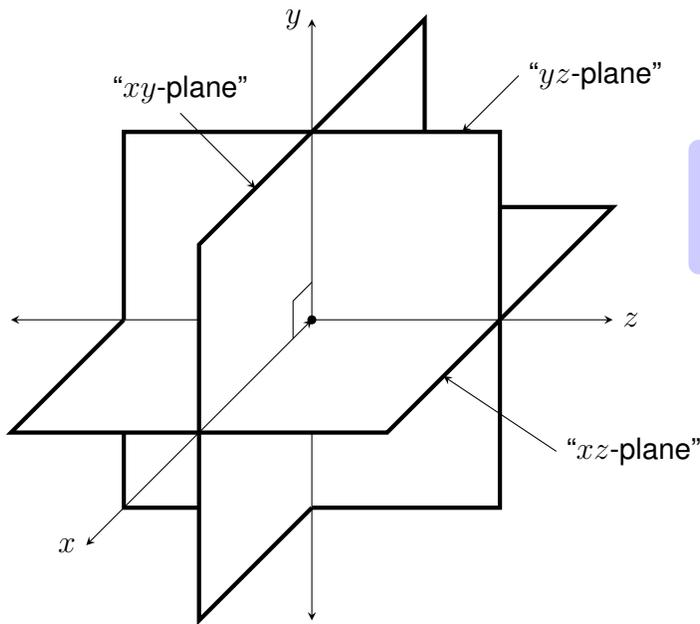
<sup>3</sup>Usually, like in calculus, the  $z$ -axis is vertical, but we do it this way, because it gives us a more convenient set-up for perspective. But it makes no real difference how we orient the  $xyz$ -coordinate system.

By introducing the  $z$ -axis, we actually create two additional *coordinate planes* giving us a total of three: our familiar  $xy$ -plane, which is now in a new position coming out of the page; a new  $yz$ -plane; and a new  $xz$ -plane. Let's see the position of these coordinate planes by drawing them. First, the  $yz$ -plane.



Here we introduce the  $yz$ -plane represented by a square. See how it sits flat on the page?

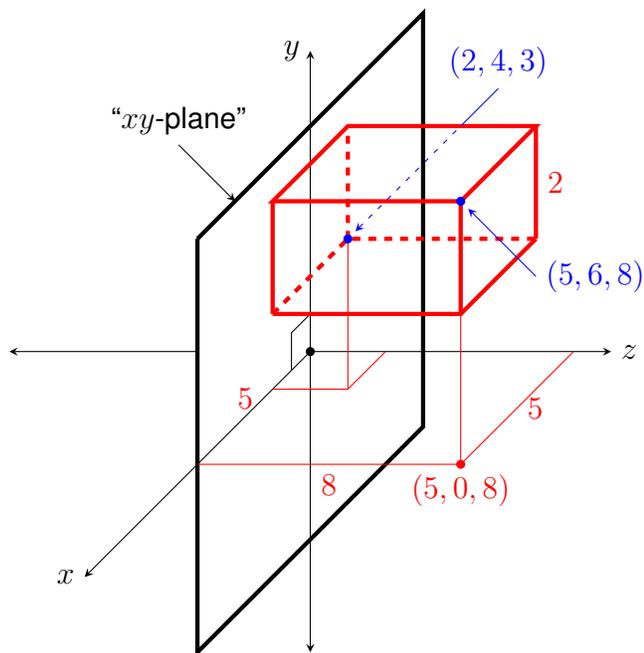
Finally, we introduce the  $xz$ -plane, which is horizontal and sticks out of the page.



Here we introduce the  $xz$ -plane represented by a square that sits horizontally and sticks out of the page.

In the above “3-D” diagram, it should be easy to see how we represent *three-dimensional space* on paper. However, we will not usually use this diagram as it has too much information on it and gets in the way. What you need to do is try to keep it in your mind so that you can refer to it when necessary.

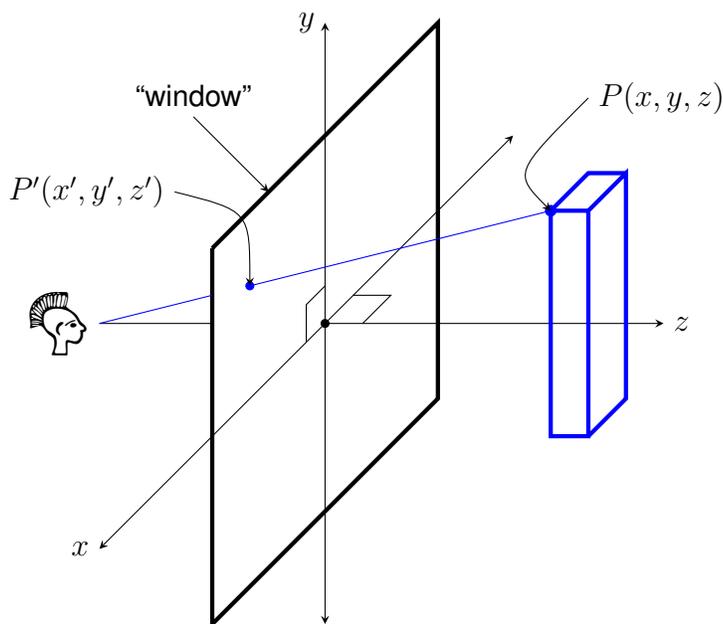




The thing to realize is that a box is made up of rectangles! And that all the corners are on any two opposite rectangles. Let's consider the bottom and the top of the box. As  $y$  measures the height, the four bottom corners must have the same  $y$ , because they are all the same height off the ground, i.e.,  $y$  must be 4. Similarly, the four top corners must have  $y = 6$ . Now, think of front and back. The four front corners must have the same  $x$ -value, because  $x$  measures the distance "out" of the page. Similarly, the back four corners. Now, think of "left" and "right" sides. The right four corners must have the same  $z$ -value, because  $z$  measures the distance right and left. Similarly, the left four corners. In this way, you can find the missing coordinates of the corners. To find the dimensions of the box, find the difference in each corresponding coordinate of any pair of opposite corners.

#### 4.4 The Point of the Window Taping Activity

Using our three-dimensional coordinate system as a model for the reality we experience, we can now make the connection with our Window Taping Activity. As you recall, in this activity, there was an "art director" whose job was to direct the "artists" to tape on a window lines that they saw on the other side of the window, in the "real world." Now, the position on the window directly in front of the director's eye was the origin of the " $xy$ -plane" of the window. The  $z$ -axis was the line from the director's eye through this origin point and into the real world on the other side of the window. Here's the picture:

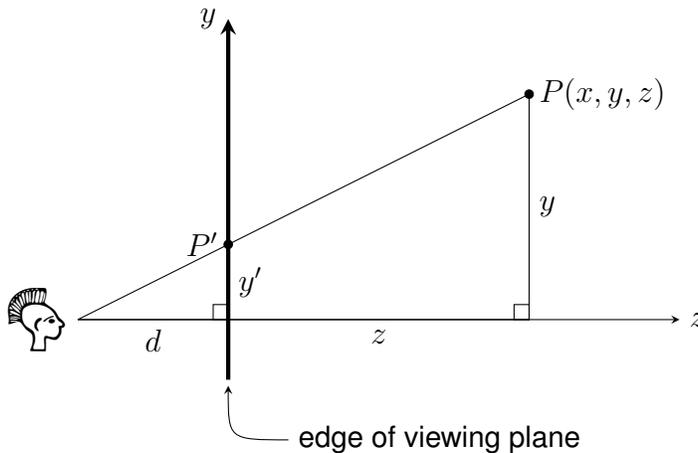


In the window taping activity, the "art director" (here pictured with mohawk) stood a fixed distance from the window; the point directly in front of her eye was the origin of the window ( $xy$ -plane). Suppose the director wants to direct an "artist" to tape a point  $P(x, y, z)$  at the top of a column that she sees. Then she needs to direct her "artist" to a point  $P'(x', y', z')$  on the window.

So the big question is: *Given any point  $P(x, y, z)$  in the real world what are the coordinates of its projection  $P'(x', y', z')$  on the window?* It should be easy to realize that  $z'$  of  $P'$  must be 0, because all points on the window are on the  $xy$ -plane, which has  $z = 0$ , thus  $z' = 0$ . In the next section, we will derive formulas for  $x'$  and  $y'$ ; it turns out that we've seen the calculation before when we were studying similar triangles in our section on geometry.

## 4.5 The Perspective Theorem

Let's take the side view of the viewer at a  $d$  from the viewing plane ( $xy$ -plane). Since we are looking directly from the side, we do not see the  $x$ -axis because it is sticking straight out at us. For this reason, we only have the  $y$ - and  $z$ -axes.



Here we have our nested similar triangles! Recall that similar triangles have equal ratios of their corresponding sides. Thus, we have:

$$\frac{y'}{y} = \frac{d}{z + d}$$

and so,

$$y' = \frac{d \cdot y}{z + d}$$

Similarly, by taking the top view, we get nested similar triangles that give us the formula for  $x'$ :

$$x' = \frac{d \cdot x}{z + d}$$

The most amazing thing is that these super simple formulas give us in a nutshell all there is to translating points in the real world to points on the viewing plane! As you'll see in the homework, the problem is getting (or making up) the coordinates of points in the real world. Even something as simple as a house can be quite confusing initially, but with time and practice (and research about sizes of things in the real world, e.g., how tall was a Tyrannosaurus Rex?), you can create an image on the  $xy$ -plane like the ones they made for *Jurassic Park*.

Of course that's just the tip of the iceberg, because not only is everything in the world in motion, but so is the viewer and his or her viewing plane! This means the equations we have for  $x'$  and  $y'$  have a bunch of continuously changing variables and our simple problem becomes very complex very fast! If you're interested in exploring these things, consider taking multidimensional calculus and linear algebra for starters.

## 4.6 Lines in the “Real World” and Their Images on the Viewing Plane

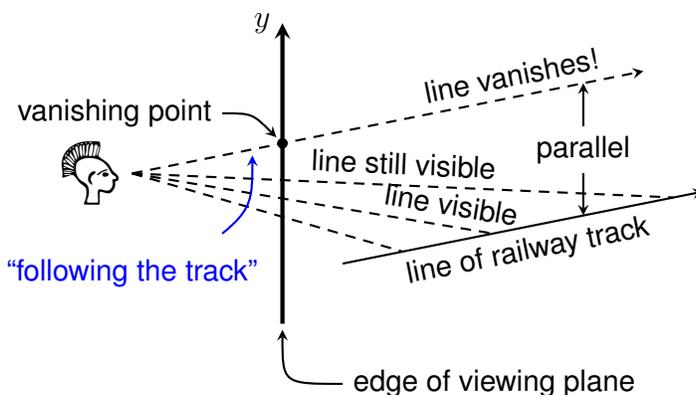
Suppose again that a viewer is standing at a distance  $d$  from a window and looking out onto the “real world”; and that the point on the window directly in front of her dominant eye is the origin  $(0, 0, 0)$ . Let’s consider a line that is perpendicular (orthogonal) to the window or viewing plane (i.e., the  $xy$ -plane). Say it intersects the viewing plane at  $(a, b, 0)$ , then all other points of the line are given by  $(a, b, z)$ . According to the perspective theorem, the image of any point of the line  $(a, b, z)$  onto the window is  $(x', y', 0)$  where  $x' = \frac{da}{z+d}$  and  $y' = \frac{db}{z+d}$ .

As lines are infinite, let’s see what happens when we follow this perpendicular line  $(a, b, z)$  as  $z$  goes to infinity. Since  $x' = \frac{d \cdot a}{z+d}$  and  $y' = \frac{d \cdot b}{z+d}$ , as  $z$  gets larger and larger, both  $x'$  and  $y'$  get smaller and smaller, which means that  $(x', y') \rightarrow (0, 0)$  as  $z \rightarrow \infty$ . In other words, the perpendicular line  $(a, b, z)$  has image on the window  $(x', y', 0)$  that has a *vanishing point* at the origin  $(0, 0, 0)$ . This is the mathematical explanation of the existence of the *vanishing point*  $(0, 0, 0)$  for a line that is perpendicular (orthogonal) to the viewing plane.

### 4.6.1 Vanishing Points

Let’s now consider this idea of the *vanishing point* of a line. The first and most important thing to note is that the *vanishing point* is **a point on the  $xy$ -plane**, i.e., a point on the window. In fact, most lines in space<sup>7</sup> have a vanishing point (on the viewing plane). Let’s see why this is true.

Imagine you’re looking through a drinking straw at a line in the real world, say a line made by a railway track (not necessarily perpendicular to your viewing plane). You follow the line of track with your straw until the moment it disappears; the exact point where this happens is the vanishing point of that line. The reason this happens is that your *line of sight* given by the direction of your straw becomes parallel to the line in the real world.



As the viewer follows the line of track with his (imaginary) straw, his *lines of sight* shown by the dashes shift until suddenly the line disappears. The point on the viewing plane at which this happens is called the vanishing point of the line of track. In this side view, we actually get the  $y'$ -coordinate of the vanishing point. The  $x'$ -coordinate is gotten in the same way using the top view.

<sup>7</sup>We’ll soon qualify this by saying any line in space not parallel to the viewing plane has a vanishing point.

## 4.6.2 The Vanishing Point Theorem

**Remark.** It is important to note that a single line in the real world has a vanishing point. This is contrary to common knowledge, which presumes that the vanishing point is the intersection of two or more lines.<sup>8</sup>

However, if we have two or more lines that are parallel in the real world, we can see from our picture that they will share the same vanishing point.

There are lines in the real world that do not have vanishing points. We know this from our window taping activity – they are lines that are parallel to the viewing plane. Let's see why this is true.

We say that a line is *parallel to a plane* if it never intersects that plane. If a line is parallel to the viewing plane ( $xy$ -plane) it must have a fixed  $z$ -value. But if  $z$  is fixed, then it never recedes off to infinity and so the line cannot have a vanishing point. In our window taping activity these were the vertical lines from the sides of columns or horizontal lines on the ground or ceiling.

We summarize our discussion with the following theorem.

### The Vanishing Point Theorem.

If two or more lines in the real world are parallel to each other, but not parallel to the viewing plane ( $xy$ -plane), then they have the same vanishing point (on the viewing plane). The images of these lines on the viewing plane will not be parallel but intersect at the vanishing point.

## 4.6.3 One-point Perspective

If a line in the real world is perpendicular (orthogonal) to our viewing plane, then not only does it have a vanishing point, but its vanishing point will be directly in front of the viewer's eye. If it happens that all the lines are perpendicular to the viewing plane (and hence parallel to each other), then we say the view or picture is in true one-point perspective as there would only be one vanishing point.

In fact, this happens to be a very popular perspective for photographers and painters especially of those of the more classical variety. Check out this very representative work of the Venetian painter *Canaletto*, 1697 - 1768.



fig.1 *Piazza San Marco*.

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<sup>8</sup>The reason for this erroneous impression is that the vanishing point is on the viewing plane (for artists, *the picture plane*) and is more often than not the intersection point of many lines on that plane.

## 4.7 The Viewing Distance Formula (for one-point perspective)

### 4.7.1 The Viewing Position

When a picture or painting is in *one-point perspective*, then there is a simple way to find the correct viewing position. In other words, we can figure out from the picture where the artist (think: “art director” from the window taping project) was standing when they made the picture. As you may have realized, the image you made during our window taping activity was in *one-point perspective*. And I showed you that parallel lines not parallel to the window (viewing plane) shared a point of intersection that we now know to be their common *vanishing point*. Furthermore, this *vanishing point* happened to be directly in front of the art director’s eye<sup>9</sup>. The reason for this is that those parallel lines were orthogonal (perpendicular) to the window. You might ask: “But why is it that a line orthogonal to the viewing plane should have vanishing point in front of the viewer’s eye?” And the answer is that such a line is parallel to the viewer’s line of sight when they’re looking straight out and this line of sight has vanishing point  $(0, 0)$ . Consequently, as parallel lines share a common vanishing point, any line orthogonal to the viewing plane must have vanishing point  $(0, 0)$ .

Therefore, for a picture in *one-point perspective*, position yourself (i.e., your dominant eye) directly in front of the *vanishing point*. Now the question is: At what viewing distance should you be at?

### 4.7.2 The Viewing Distance Formula

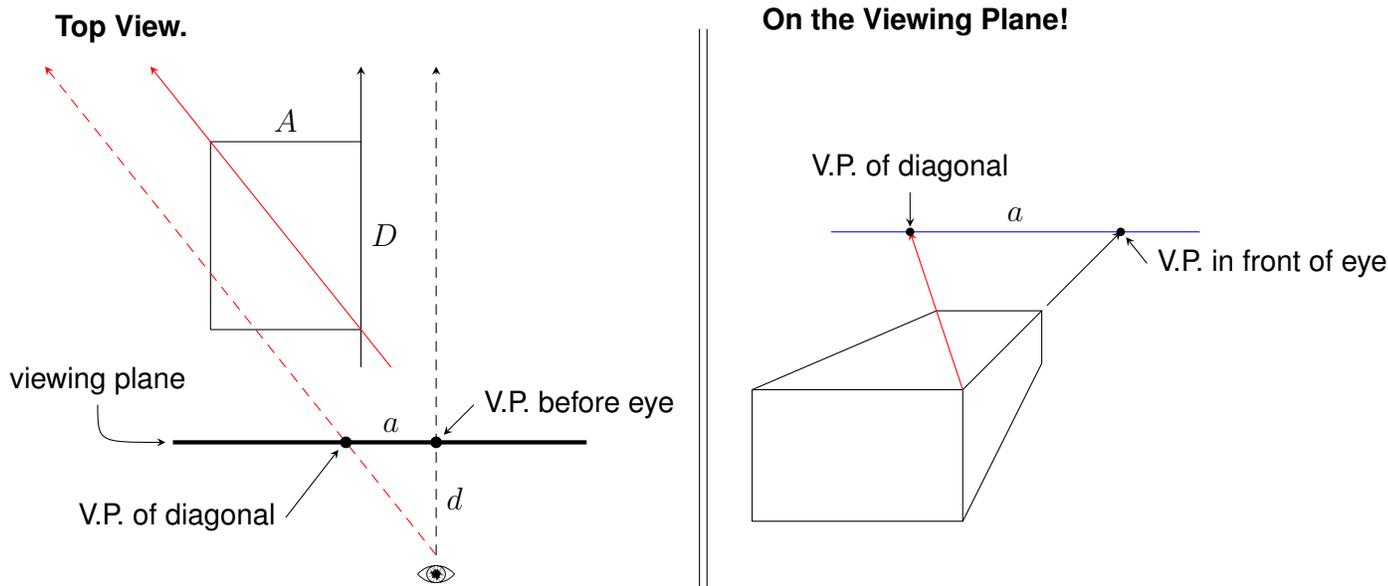
Getting the correct viewing distance is bit trickier than the viewing position. Here’s the way to think of the problem: after doing the window taping, you love the work so much that you decide leave the tape on and take the window home. At home you paint the window all different colors making sure to keep all those taped lines. Then you decide to exhibit the window/painting in a local gallery. Once installed in the gallery, viewers will have nothing to match the lines up with, so they will have no idea where to stand except a couple of them might know to position themselves in front of the vanishing point.

Now remember how during the window taping activity there was a fixed distance that the viewer had to stand from the window? That’s what needs to be figured out from the clues in the window/painting in the art gallery.

Let’s do this by working backwards. Suppose there is an object (in this case a box) in the real world and we are looking at it so that it’s in *one-point perspective*; this means all its edges are either parallel to our viewing plane or orthogonal to it. If the edges that are orthogonal to the viewing plane were extended, then their image on the viewing plane would be lines that intersect at the vanishing point  $(0, 0)$ . The right side of the following diagram show how this is happening on the viewing plane.

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<sup>9</sup>Don’t make the mistake of thinking that all vanishing points are in front of the viewer’s eye! Only the vanishing point of lines orthogonal to the viewing plane is in front of the viewer’s eye.



The key is to take a look at the **top view** of the situation. Here the rectangle is the top of the box and it has length  $D$  and width  $A$ . The right side of the rectangle corresponds to the top right edge of the box. If we extend this edge, we get a line that is parallel to our line of sight that goes straight out and through  $(0, 0)$ .

Now suppose we construct the line (in solid red) that goes from the lower right corner of the rectangle to the upper left. (We can obviously do this because any two points define a line!) If we extend this, we get a line in the real world; since this line isn't parallel to the viewing plane, it must have (by our previous discussion) a vanishing point *on the viewing plane*. This vanishing point is the point given by our line of sight (in dashed red) that is parallel to the red diagonal line.

Notice how our two lines of sight (dashed red and dashed black) define a triangle with one corner at the eye of the viewer and the other two corners at the two vanishing points. This triangle is similar to the top right triangle of the rectangle cut by the red diagonal. If we let the distance between the two vanishing points be  $a$  and  $d$  be our *viewing distance*<sup>10</sup>, then we have the following relationship

$$\frac{d}{a} = \frac{D}{A}$$

Solving for  $d$ , we get our **Viewing Distance Formula**:

$$d = \frac{a \cdot D}{A}.$$

But how do gallery/museum goers use this formula when they see a one-point perspective picture (like your window/painting)? Basically, as they'll be looking at an image on the viewing plane, they'll have to find some sort of clue (like the top of a box) that will give them a second vanishing point and using that they can find  $a$ , which (as it usually happens) is equal to  $d$  because the top of the box is a square (so,  $D = A$ , and hence  $\frac{D}{A} = 1$ ). As we'll see in class, artists usually leave such clues to help the viewer along!

<sup>10</sup>Recall that the distance between the eye and the vanishing point in front of the eye  $(0, 0)$  is our viewing distance,  $d$ .