1. Models and coordinates for the hyperbolic plane

The models:

- Poincaré disc: $\{|z| < 1\}$ with metric

$$ds^2 = \frac{4dzd\bar{z}}{(1-|z|^2)^2} \tag{1.1}$$

- Upper half plane: $\{\Im z > 0\}$ with metric

$$ds^2 = \frac{dzd\bar{z}}{(\Im z)^2} \tag{1.2}$$

- Minkowski model: $\{(x_0, x_1, x_2) \in \mathbb{R}^{2,1} : -x_0^2 + x_1^2 + x_2^2 = 1\}$ with metric induced from the Minkowski metric $-dx_0^2 + dx_1^2 + dx_2^2$.

- $\tanh(u) = \frac{1}{2}\ln(1+u) - \frac{1}{2}\ln(1-u)$

1.1. Hyperbolic polar coordinates in the Poincaré disc.

$$\tanh(r/2) = |z|, \quad \tan(\phi) = y/x \tag{1.3}$$

Then

$$ds^2 = dw^2 + \sinh^2(w)d\phi^2$$

$$x_0 = \cosh(r) \tag{1.4a}$$

$$x_1 = \cos(\phi)\sinh(r) \tag{1.4b}$$

$$x_2 = \sin(\phi)\sinh(r) \tag{1.4c}$$

gives the isometry to the hyperboloid $-x_0^2 + x_1^2 + x_2^2 = -1$ with metric $-dx_0^2 + dx_1^2 + dx_2^2$.

1.2. Upper half plane → Poincaré.

A Moebius transformation of $\mathbb{C}$ is a map

$$T(z) = \frac{az+b}{cz+d}, \qquad ad - bc = 1$$

T is uniquely given by its values at 3 points. Let $T(0) = 1$, $T(i) = 0$, $T(\infty) = -1$.

If $\Im z > 0$, then $|T(z)| < 1$ so T maps the upper half plane $\{\Im z > 0\}$ to the Poincaré disc $|z| < 1$.

The conditions on T give $a = (i/2)^{1/2}$, $b = d = -ia$, $c = -a$, and after simplifying we have

$$T(z) = \frac{i-z}{i+z}$$

Let

$$w = T(z)$$

Then in terms of real coordinates $z = x + iy$, we have

$$\begin{aligned} w &= \frac{(i-z)\overline{(i+z)}}{|i+z|^2} \\ &= \frac{1 - x^2 - y^2 + 2ix}{x^2 + (1+y)^2} \end{aligned}$$

and

$$|w|^2 = \frac{1 - |z|^2 - 2\Im(z)}{1 + |z|^2 + 2\Im(z)} = \frac{x^2 + (1-y)^2}{x^2 + (1+y)^2} \tag{1.5a}$$

$$\Re(w) = \frac{1 - |z|^2}{|i + z|^2} = \frac{1 - x^2 - y^2}{x^2 + (1+y)^2} \tag{1.5b}$$

$$\Im(w) = \frac{2\Re(z)}{|i + z|^2} = \frac{2x}{x^2 + (1+y)^2} \tag{1.5c}$$

With $w = T(z)$ we have

$$\frac{4dwd\bar{w}}{(1 - w\bar{w})^2} = \frac{dzd\bar{z}}{(\Im z)^2}$$

Thus T is an isometric isomorphism from the the upper half plane with metric (1.2) to the Poincare disc with metric (1.1).

1.3. **Poincaré → upper half plane.** Let $z = T^{-1}(w)$. Then with a, b, c, d as above,

$$z = \frac{dw - b}{-cw + a} = i\frac{1 - w}{1 + w}$$

and

$$\frac{dzd\bar{z}}{(\Im z)^2} = \frac{4dwd\bar{w}}{(1 - z\bar{z})^2}$$

1.4. **Poincaré → Minkowski.** Equation (1.3) gives

$$\sinh(r) = 2\frac{|z|}{1 - |z|^2}$$

$$\cosh(r) = \frac{1 + |z|^2}{1 - |z|^2}$$

Therefore using $\cos(\phi) = x/|z|$, $\sin(\phi) = y/|z|$ and (1.4) we are able to write the Minkowski coordinates x_0, x_1, x_2 in terms of the Poincare coordinate z as

$$x_0 = \frac{1 + |z|^2}{1 - |z|^2} \tag{1.6a}$$

$$x_1 = \frac{2x}{1 - |z|^2} \tag{1.6b}$$

$$x_2 = \frac{2y}{1 - |z|^2} \tag{1.6c}$$

1.5. **Upper half plane to Minkowski.** We now compose the map T from the upper half plane to the Poincaré disc with the map (1.6) from the Poincaré disc to the Minkowski model. Let w be the coordinate in the Poincaré disc and

Then (1.6) is

$$x_0 = \frac{1+|w|^2}{1-|w|^2}$$

$$x_1 = \frac{2\Re(w)}{1-|w|^2}$$

$$x_2 = \frac{2\Im(w)}{1-|w|^2}$$

which using (1.5) gives

$$x_0 = \frac{1+x^2+y^2}{2y} \tag{1.7a}$$

$$x_1 = \frac{(1-x^2-y^2)}{2y} \tag{1.7b}$$

$$x_2 = \frac{x}{y} \tag{1.7c}$$

From this we get

$$x_0 + x_2 = \frac{(1+x)^2+y^2}{2y} \tag{1.8}$$

$$x_0 - x_2 = \frac{(1-x)^2+y^2}{2y} \tag{1.9}$$

$$x_0 + x_1 = \frac{1}{y} \tag{1.10}$$

$$x_0 - x_1 = \frac{x^2+y^2}{y} \tag{1.11}$$