

**Chapter 4. Tensors** Let  $V$  be a vectorspace over a field  $\mathbb{F} = \mathbb{R}, \mathbb{C}$  of dimension  $n$  with frame  $e_1, \dots, e_n$ .

Recall that  $V^* = \{\lambda : V \rightarrow \mathbb{F}, \lambda \text{ linear}\}$  is the *dual space of  $V$* . Call the dual frame  $e^{1*}, \dots, e^{n*}$ . Note that  $V^{**} = V$  (reflexivity) always holds when  $V$  is finite dimensional.

Let  $U, V, W$  be vector spaces. Let  $f : V \rightarrow W$  be linear. Then we can define the dual map  $f^* : W^* \rightarrow V^*$  by  $f^*w^*(v) = w^*(fv), \forall w^* \in W^*, v \in V$ . If  $g : U \rightarrow V$  is linear we have  $(f \circ g)^* = g^* \circ f^*$ .

Let  $\xi = \pi : E \rightarrow B$  be a vector bundle with fiber  $V$ . Then we can define the dual bundle  $\xi^* = \pi' : E' \rightarrow B$  with fiber  $V^*$  by  $E' = \cup_{p \in B} [\pi^{-1}(p)]^*$  and  $\pi'$  the corresponding projection.

Apply this to  $TM$ , the tangent bundle of  $M$ . Then we get  $T^*M$ , the *cotangent bundle*.

Note that  $\mathcal{F} = C^\infty(M)$  is an abelian ring, the space of sections  $C^\infty(\xi)$  is an abelian group and  $\mathcal{F}$  acts on  $C^\infty(\xi)$  by multiplication. Therefore  $C^\infty(\xi)$  is a  $\mathcal{F}$ -module. Thus we can talk about linearity over  $\mathcal{F}$ !

Example.  $s \in C^\infty(\xi^*)$  can be viewed as an  $\mathcal{F}$ -linear map  $s : C^\infty(\xi) \rightarrow \mathcal{F}$  as follows: let  $u_1, u_2 \in C^\infty(\xi)$  be sections of  $\xi$  and let  $f \in \mathcal{F}$ . Then  $s(fu_1 + u_2) = fs(u_1) + s(u_2)$ . Here  $s(u_1)(p) = s(p)(u_1(p)) \in \mathbb{R}$ . Thus the spaces of sections are dual in the natural sense.

**Tensor product:** Let  $V_1, \dots, V_k$  be vector spaces over  $\mathbb{F}$ . A map  $T : V_1 \times \dots \times V_k \rightarrow \mathbb{F}$  is called *multi linear* if  $\forall v_1, \dots, v_k \in V_1 \times \dots \times V_k, \forall i : 1 \leq i \leq k$ , the map  $v \mapsto T(v_1, \dots, v_{i-1}, v, v_{i+1}, v_k)$  is linear, i.e. if  $T$  is linear in each argument separately. The set of all such multilinear maps is called the tensor product  $V_1^* \otimes \dots \otimes V_k^*$ .

Let  $S \in V_1^* \otimes \dots \otimes V_k^*, T \in V_{k+1}^* \otimes \dots \otimes V_{k+\ell}^*$ , define  $S \otimes T \in V_1^* \otimes \dots \otimes V_{k+\ell}^*$  by

$$S \otimes T(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+\ell}) = S(v_1, \dots, v_k)T(v_{k+1}, \dots, v_{k+\ell}).$$

$\otimes$  is linear and associative by *not* commutative.

Example:  $\mathcal{T}^k(V) = V^* \otimes \dots \otimes V^*$  ( $k$  copies), has frame  $\{e^{i_1*} \otimes \dots \otimes e^{i_k*}\}$ ,  $1 \leq i_1, \dots, i_k \leq n$ , i.e.  $\dim \mathcal{T}^k(V) = n^k$ ,  $\dim \bigotimes_{i=1}^k V_i = \prod_{i=1}^k \dim V_i$ .

Mixed tensor products  $\mathcal{T}_\ell^k(V)$  are formed by tensor product of  $k$   $V^*$ -factors and  $\ell$   $V$ -factors. If  $S \in \mathcal{T}_\ell^k(V), T \in \mathcal{T}_{\ell'}^{k'}(V), S \oplus T \in \mathcal{T}_{\ell+\ell'}^{k+k'}(V)$ .

Example:  $\mathcal{T}_1^1(V) = \text{End}(V)$

**Tensor product of bundles:** Given bundles  $\xi_i, i = 1, 2$  over  $M$  with fiber  $V_i$  one forms a new bundle  $\xi_1 \otimes \xi_2$  over  $M$  with fiber  $V_1 \otimes V_2$ . This leads to tensor bundles over  $M$ , eg.  $\mathcal{T}^k(TM)$  (covariant tensors of order  $k$ , fiber  $\mathcal{T}^k(M_p)$ ) eller  $\mathcal{T}_\ell^k(TM)$ , fiber  $\mathcal{T}_\ell^k(M_p)$ . Note:  $\mathcal{T}_1^1(TM) = \text{End}(TM)$  (bundle map) and the identity map in  $\text{End}(TM)$  is  $\sum \delta_j^i dx^i \otimes \frac{\partial}{\partial x^j}$ .

Let  $(x, U)$  be a chart at  $p \in M$ . A frame for  $\mathcal{T}^k(M_p)$  is given by  $dx^{i_1}(p) \otimes \dots \otimes dx^{i_k}(p), 1 \leq i_1, \dots, i_k \leq n$ . That is to say, a section  $A$  of  $\mathcal{T}^k(TM)$  over  $U$  can be written as

$$A = \sum A_{i_1, \dots, i_k} dx^{i_1} \otimes \dots \otimes dx^{i_k}$$

where  $A_{i_1, \dots, i_k}$  is a collection of  $n^k$  functions.

Let  $(x', V)$  be a new chart at  $p$  with  $A = \sum A'_{i_1, \dots, i_k} dx'^{i_1} \otimes \dots \otimes dx'^{i_k}$ . Due to the chain rule,  $dx^i = \sum_j \frac{\partial x^i}{\partial x'^j} dx'^j$  so

$$A'_{i_1, \dots, i_k} = \sum A_{j_1, \dots, j_k} \frac{\partial x^{j_1}}{\partial x'^{i_1}} \dots \frac{\partial x^{j_k}}{\partial x'^{i_k}}$$

Let  $\mathcal{V} = C^\infty(TM)$  denote the space of vector fields, this is a module over  $\mathcal{F}$ . A covariant tensor field of order  $k$  corresponds precisely to multilinear (over  $\mathcal{F}$ ) maps  $\mathcal{V} \times \dots \times \mathcal{V} \rightarrow \mathcal{F}$  (Thm. 2).

Thus we have several different descriptions of tensor fields:

- (1) section of a bundle, eg.  $\mathcal{T}^k(TM)$ .
- (2) multilinear map (over  $\mathcal{F}$ ).
- (3) a collection of functions,  $A_{i_1, \dots, i_k}$  def. w.r.t. a chart  $(x, U)$  with transformation rules for passing to a new chart.

**Operations on tensors:** We have already looked at tensor product. Contraction  $C : \mathcal{T}_\ell^k(V) \rightarrow \mathcal{T}_{\ell-1}^{k-1}(V)$  is defined by

$$CT(v_1, \dots, v_{k-1}, v_1^*, \dots, v_{\ell-1}^*) = \sum T(v_1, \dots, e_i, \dots, v_{k-1}, v_1^*, \dots, e^{i*}, \dots, v_{\ell-1}^*).$$

the notation is deliberately vague here, in order to completely define the contraction, one must denote exactly which positions  $e_i$  and  $e^{i*}$  have. This is often most easily done using index notation.

Remark:  $A \in \mathcal{T}_1^1(V) = \text{End}(V)$ ,  $CA = \text{tr}A =$  the trace of  $A$  considered as a linear endomorphism of  $V$ .

**Pullback of a covariant tensor:** Let  $f : M \rightarrow N$  be  $C^\infty$ , let  $T$  be a section of  $\mathcal{T}^k(TN)$ . Then we can define a section  $f^*T$  of  $\mathcal{T}^k(TM)$  by  $f^*T(X_1, \dots, X_k) = T(f_*X_1, \dots, f_*X_k)$ .

**Symmetry and anti symmetry of tensors:**  $T \in \mathcal{T}^k(V)$  is symmetric if

$$T(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = T(v_1, \dots, v_k)$$

for every permutation  $\sigma$  of  $1, \dots, k$ . Analogously for tensor fields.

Example: A symmetric tensor  $g \in C^\infty(\mathcal{T}^2(TM))$  (symmetric covariant 2-tensor) is called a Riemannian metric if  $g = \sum_{ij} g_{ij} dx^i \otimes dx^j$  where  $g_{ij}$  is a positive definite matrix (this is independent of the coordinate system). This is equivalent (for symmetric  $g$ ) with  $\forall X \in \mathcal{V}, p \in M, g(X, X)_p \geq 0$  and  $g(X, X)_p = 0 \Rightarrow X(p) = 0$ . The notation  $\langle X, Y \rangle$  is often used instead of  $g(X, Y)$ .

$T \in \mathcal{T}^k(V)$  is anti symmetric (alternating, skew symmetric) if

$$T(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \text{sign}(\sigma)T(v_1, \dots, v_k)$$

for every permutation  $\sigma$  av  $1, \dots, k$ . The bundle of antisymmetric covariant  $k$ -tensors is denoted by  $\Omega^k(TM)$ . Sections of this bundle are called differential forms.

**Problems, Chapter 4:** 1,5,7,8,10