Examination problems for "Elementary Differentialgeometry", part 1

Problem 1: Let $U \subset \mathbb{R}^2$ be an open domain, let $f : U \to \mathbb{R}^3$ be a C^∞ immersion, then $M = f(U)$ is an immersed (hyper–)surface in $\mathbb{R}^3$.

For $p \in M$, $p = f(s,t)$ for some $(s,t) \in U$ we can consider the tangent space M_p as a subspace of $\mathbb{R}^3_p$ given by the range $\mathcal{R}Df(s,t)$ of Df. Let $\langle \cdot, \cdot \rangle$ be the standard Euclidean metric (inner product) on $\mathbb{R}^3$. Then we can view this as defining an inner product $\langle \cdot, \cdot \rangle_p$ on $\mathbb{R}^3_p$. At p we can choose a unit normal vector $N(p)$ so that $\langle N(p), X \rangle = 0$ for all $X \in M_p$ and $||N(p)|| = (\langle N(p), N(p) \rangle)^{1/2} = 1$. There is at least locally a smooth choice of N so that we get a *normal vectorfield* N on M, with $N(p) \in \mathbb{R}^3$.

Since $||N|| \equiv 1$, we can view $N(p)$ as a point in S^2 (the standard $S^2 \subset \mathbb{R}^3$). This defines the Gauss map $G : M \to S^2$.

Let η denote the Euclidean metric on $\mathbb{R}^3$ viewed as a Riemannian metric. Since M is a submanifold of $\mathbb{R}^n$, it gets an induced metric which in terms of the immersion f is given by $g = f^*\eta$.

Let x^1, x^2 be Cartesian coordinates on $U \subset \mathbb{R}^2$, let w^1, w^2, w^3 be Cartesan coordinates on $\mathbb{R}^3$ and let $N = \sum_{i=1}^3 N^i \partial_{w^i}$ and let X, Y be vectors in M_p. Then we can define the second fundamental form K of M in $\mathbb{R}^3$ as the tensor given by

$$K(X,Y) = \langle X, D_Y N \rangle.$$

Here $D_Y N = \sum_{ij} Y^j D_j N^i \partial_{w^i}$ is the flat covariant derivative defined by the standard partial derivative on $\mathbb{R}^3$. Note that we are differentiating N only in the tangent direction of M.

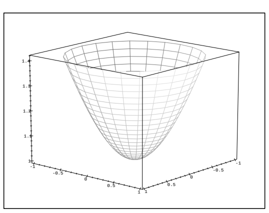
Let e_1, e_2 be an ON frame w.r.t. the induced metric g. The mean curvature of M is defined as

$$H = \frac{1}{2} \sum_{i=1}^{2} K(e_i, e_i)$$

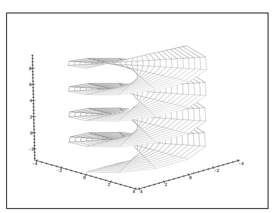
If $H = 0$, M is said to be minimal.

Consider the following surfaces:

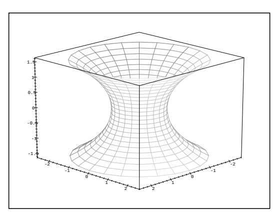
(a) The Hyperboloid. $U = \mathbb{R}^2$, $f(s,t) = (s, t, \sqrt{1 + s^2 + t^2})$.
(b) The Helicoid. $U = \mathbb{R}^2$. $f(s,t) = (t \cos s, t \sin s, s)$
(c) The Catenoid. $U = \mathbb{R}^2$. $f(s,t) = (\cosh s \cos t, \cosh s \sin t, s)$



hyperboloid



helicoid



catenoid

For the explicitely given surfaces above compute the following:
(a) The normal field
(b) The image of the Gauss map in S^2.
(c) The induced Riemannian metric
(d) The mean curvature H, decide if M is minimal.

Problem 2: Let M^n, N^k be C^∞ manifolds of dimension n, k respectively. A C^∞ mapping $f : M \to N$ with constant rank k is called a submersion. Prove that if $f : M \to N$ is a submersion, then for $q \in N$, $f^{-1}(q)$ is a submanifold of dimension $n - k$.

Problem 3: Spivak I, 2.29

Problem 4: Show how to construct a C^∞ structure on the Mobius band M (see eg. problem 2.14.c or construct an atlas). Prove that the Moebius band can be viewed as the total space of a $\mathbb{R}^1$ bundle over S^1. Show that this bundle is not trivial.

Problem 5: Let $\xi = \pi : B \to M$ be an $\mathbb{R}^k$ bundle over a manifold M. Prove that ξ is trivial if and only if there are k linearly independent sections of ξ, i.e. sections $s_1, \ldots, s_k$ so that $\forall p \in M : s_1(p), \ldots, s_k(p)$ are linearly independent in $\pi^{-1}(p)$.

Problem 6: Spivak I, 3.31

Lars Andersson