

1. MODELS AND COORDINATES FOR THE HYPERBOLIC PLANE

The models:

- Poincaré disc: $\{|z| < 1\}$ with metric

$$(1.1) \quad ds^2 = \frac{4dzd\bar{z}}{(1 - |z|^2)^2}$$

- Upper half plane: $\{\Im z > 0\}$ with metric

$$(1.2) \quad ds^2 = \frac{dzd\bar{z}}{(\Im z)^2}$$

- Minkowski model: $\{(x_0, x_1, x_2) \in \mathbb{R}^{2,1} : -x_0^2 + x_1^2 + x_2^2 = 1\}$ with metric induced from the Minkowski metric $-dx_0^2 + dx_1^2 + dx_2^2$.

$$\bullet \tanh(u) = \frac{1}{2} \ln(1 + u) - \frac{1}{2} \ln(1 - u)$$

1.1. Hyperbolic polar coordinates in the Poincaré disc.

$$(1.3) \quad \tanh(r/2) = |z|, \quad \tan(\phi) = y/x$$

Then

$$ds^2 = dw^2 + \sinh^2(w)d\phi^2$$

$$(1.4a) \quad x_0 = \cosh(r)$$

$$(1.4b) \quad x_1 = \cos(\phi) \sinh(r)$$

$$(1.4c) \quad x_2 = \sin(\phi) \sinh(r)$$

gives the isometry to the hyperboloid $-x_0^2 + x_1^2 + x_2^2 = -1$ with metric $-dx_0^2 + dx_1^2 + dx_2^2$.

1.2. Upper half plane \rightarrow Poincaré. A Moebius transformation of \mathbb{C} is a map

$$T(z) = \frac{az + b}{cz + d}, \quad ad - bc = 1$$

T is uniquely given by its values at 3 points. Let $T(0) = 1, T(i) = 0, T(\infty) = -1$.

If $\Im z > 0$, then $|T(z)| < 1$ so T maps the upper half plane $\{\Im z > 0\}$ to the Poincaré disc $|z| < 1$.

The conditions on T give $a = (i/2)^{1/2}, b = d = -ia, c = -a$, and after simplifying we have

$$T(z) = \frac{i - z}{i + z}$$

Let

$$w = T(z)$$

Then in terms of real coordinates $z = x + iy$, we have

$$\begin{aligned} w &= \frac{(i - z)\overline{(i + z)}}{|i + z|^2} \\ &= \frac{1 - x^2 - y^2 + 2ix}{x^2 + (1 + y)^2} \end{aligned}$$

and

$$(1.5a) \quad |w|^2 = \frac{1 - |z|^2 - 2\Im(z)}{1 + |z|^2 + 2\Im(z)} = \frac{x^2 + (1 - y)^2}{x^2 + (1 + y)^2}$$

$$(1.5b) \quad \Re(w) = \frac{1 - |z|^2}{|i + z|^2} = \frac{1 - x^2 - y^2}{x^2 + (1 + y)^2}$$

$$(1.5c) \quad \Im(w) = \frac{2\Re(z)}{|i + z|^2} = \frac{2x}{x^2 + (1 + y)^2}$$

With $w = T(z)$ we have

$$\frac{4dw d\bar{w}}{(1 - w\bar{w})^2} = \frac{dz d\bar{z}}{(\Im z)^2}$$

Thus T is an isometric isomorphism from the the upper half plane with metric (1.2) to the Poincare disc with metric (1.1).

1.3. Poincaré \rightarrow upper half plane. Let $z = T^{-1}(w)$. Then with a, b, c, d as above,

$$z = \frac{dw - b}{-cw + a} = i \frac{1 - w}{1 + w}$$

and

$$\frac{dz d\bar{z}}{(\Im z)^2} = \frac{4dw d\bar{w}}{(1 - w\bar{w})^2}$$

1.4. Poincaré \rightarrow Minkowski. Equation (1.3) gives

$$\sinh(r) = 2 \frac{|z|}{1 - |z|^2}$$

$$\cosh(r) = \frac{1 + |z|^2}{1 - |z|^2}$$

Therefore using $\cos(\phi) = x/|z|$, $\sin(\phi) = y/|z|$ and (1.4) we are able to write the Minkowski coordinates x_0, x_1, x_2 in terms of the Poincare coordinate z as

$$(1.6a) \quad x_0 = \frac{1 + |z|^2}{1 - |z|^2}$$

$$(1.6b) \quad x_1 = \frac{2x}{1 - |z|^2}$$

$$(1.6c) \quad x_2 = \frac{2y}{1 - |z|^2}$$

1.5. Upper half plane to Minkowski. We now compose the map T from the upper half plane to the Poincaré disc with the map (1.6) from the Poincaré disc to the Minkowski model. Let w be the coordinate in the Poincaré disc and

Then (1.6) is

$$\begin{aligned}x_0 &= \frac{1 + |w|^2}{1 - |w|^2} \\x_1 &= \frac{2\Re(w)}{1 - |w|^2} \\x_2 &= \frac{2\Im(w)}{1 - |w|^2}\end{aligned}$$

which using (1.5) gives

$$(1.7a) \quad x_0 = \frac{1 + x^2 + y^2}{2y}$$

$$(1.7b) \quad x_1 = \frac{(1 - x^2 - y^2)}{2y}$$

$$(1.7c) \quad x_2 = \frac{x}{y}$$

From this we get

$$(1.8) \quad x_0 + x_2 = \frac{(1 + x)^2 + y^2}{2y}$$

$$(1.9) \quad x_0 - x_2 = \frac{(1 - x)^2 + y^2}{2y}$$

$$(1.10) \quad x_0 + x_1 = \frac{1}{y}$$

$$(1.11) \quad x_0 - x_1 = \frac{x^2 + y^2}{y}$$