Chapter 4. Tensors Let V be a vectorspace over a field $\mathbb{F} = \mathbb{R}, \mathbb{C}$ of dimension n with frame $e_1, \ldots, e_n$.

Recall that $V^* = \{\lambda : V \to \mathbb{F}, \lambda \text{ linear } \}$ is the *dual space of* V. Call the dual frame $e^{1*}, \ldots, e^{n*}$. Note that $V^{**} = V$ (reflexivity) always holds when V is finite dimensional.

Let U, V, W be vector spaces. Let $f : V \to W$ be linear. Then we can define the dual map $f^* : W^* \to V^*$ by $f^* w^*(v) = w^*(fv), \forall w^* \in W^*, v \in V$. If $g : U \to V$ is linear we have $(f \circ g)^* = g^* \circ f^*$.

Let $\xi = \pi : E \to B$ be a vector bundle with fiber V. Then we can define the dual bundle $\xi^* = \pi' : E' \to B$ with fiber V^* by $E' = \cup_{p \in B} [\pi^{-1}(p)]^*$ and π' the corresponding projection.

Apply this to TM, the tangent bundle of M. Then we get T^*M, the *cotangent bundle*.

Note that $\mathcal{F} = C^\infty(M)$ is an abelian ring, the space of sections $C^\infty(\xi)$ is an abelian group and $\mathcal{F}$ acts on $C^\infty(\xi)$ by multiplication. Therefore $C^\infty(\xi)$ is a $\mathcal{F}-$module. Thus we can talk about linearity over $\mathcal{F}$!

Example. $s \in C^\infty(\xi^*)$ can be viewed as an $\mathcal{F}$-linear map $s : C^\infty(\xi) \to \mathcal{F}$ as follows: let $u_1, u_2 \in C^\infty(\xi)$ be sections of ξ and let $f \in \mathcal{F}$. Then $s(fu_1 + u_2) = fs(u_1) + s(u_2)$. Here $s(u_1)(p) = s(p)(u_1(p)) \in \mathbb{R}$. Thus the spaces of sections are dual in the natural sense.

Tensor product: Let $V_1, \ldots, V_k$ be vector spaces over $\mathbb{F}$. A map $T : V_1 \times \cdots V_k \to \mathbb{F}$ is called *multi linear* if $\forall v_1, \ldots v_k \in V_1 \times \cdots \times V_k, \forall i : 1 \leq i \leq k$, the map $v \mapsto T(v_1, \ldots, v_{i-1}, v, v_{i+1}, v_k)$ is linear, i.e. if T is linear in each argument separately. The set of all such multilinear maps is called the tensor product $V_1^* \otimes \cdots \otimes V_k^*$.

Let $S \in V_1^* \otimes \cdots \otimes V_k^*$, $T \in V_{k+1}^* \otimes \cdots \otimes V_{k+\ell}^*$, define $S \otimes T \in V_1^* \otimes \cdots \otimes V_{k+\ell}^*$ by

$$S \otimes T(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+\ell}) = S(v_1, \ldots, v_k) T(v_{k+1}, \ldots, v_{k+\ell}).$$

$\otimes$ is linear and associative by *not* commutative.

Example: $\mathcal{T}^k(V) = V^* \otimes \cdots \otimes V^*$ (k copies), har frame $\{e^{i_1 *} \otimes \cdots \otimes e^{i_k *}\}$, $1 \leq i_1, \ldots, i_k \leq n$, i.e. $\dim \mathcal{T}^k(V) = n^k$, $\dim \bigotimes_{i=1}^k V_i = \prod_{i=1}^k \dim V_i$.

Mixed tensor products $\mathcal{T}_\ell^k(V)$ are formed by tensor product of k V^*-factors and ℓ V-factors. If $S \in \mathcal{T}_\ell^k(V)$, $T \in \mathcal{T}_{\ell'}^{k'}(V)$, $S \oplus T \in T_{\ell+\ell'}^{k+k'}(V)$.

Example: $\mathcal{T}_1^1(V) = \mathrm{End}(V)$

Tensor product of bundles: Given bundles ξ_i, $i = 1, 2$ over M with fiber V_i one forms a new bundle $\xi_1 \otimes \xi_2$ over M with fiber $V_1 \otimes V_2$. This leads to tensor bundles over M, eg. $\mathcal{T}^k(TM)$ (covariant tensors of order k, fiber $\mathcal{T}^k(M_p)$) eller $\mathcal{T}_\ell^k(TM)$, fiber $\mathcal{T}_\ell^k(M_p)$. Note: $\mathcal{T}_1^1(TM) = \mathrm{End}(TM)$ (bundle map) and the identity map in $\mathrm{End}(TM)$ is $\sum \delta_j^i dx^i \otimes \frac{\partial}{\partial x^i}$.

Let (x, U) be a chart at $p \in M$. A frame for $\mathcal{T}^k(M_p)$ is given by $dx^{i_1}(p) \otimes \cdots \otimes dx^{i_k}(p)$, $1 \leq i_1, \ldots, i_k \leq n$. That is to say, a section A of $\mathcal{T}^k(TM)$ over U can be written as

$$A = \sum A_{i_1, \ldots, i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k}$$

where $A_{i_1, \ldots, i_k}$ is a collection of n^k functions.

Let (x', V) be a new chart at p with $A = \sum A'_{i_1,\dots,i_k} dx'^{i_1} \otimes \cdots \otimes dx'^{i_k}$. Due to the chain rule, $dx^i = \sum_j \frac{\partial x^i}{\partial x'^j} dx'^j$ so

$$A'_{i_1,\dots,i_k} = \sum A_{j_1,\dots,j_k} \frac{\partial x^{j_1}}{\partial x'^{i_1}} \cdots \frac{\partial x^{j_k}}{\partial x'^{i_k}}$$

Let $\mathcal{V} = C^\infty(TM)$ denote the space of vector fields, this is a module over $\mathcal{F}$. A covariant tensor field of order k corresponds precisely to multilinear (over $\mathcal{F}$) maps $\mathcal{V} \times \cdots \times \mathcal{V} \to \mathcal{F}$ (Thm. 2).
Thus we have several different descriptions of tensor fields:

1. section of a bunlde, eg. $\mathcal{T}^k(TM)$.
2. multilinear map (over $\mathcal{F}$).
3. a collection of functions, $A_{i_1,\dots,i_k}$ def. w.r.t. a chart (x, U) with transformation rules for passing to a new chart.

Operations on tensors: We have already looked at tensor product. Contraction $C : \mathcal{T}^k_\ell(V) \to \mathcal{T}^{k-1}_{\ell-1}(V)$ is defined by

$$CT(v_1, \dots, v_{k-1}, v_1^*, \dots, v_{\ell-1}^*) = \sum T(v_1, \dots, e_i, \dots, v_{k-1}, v_1^*, \dots, e^{i*}, \dots, v_{\ell-1}^*).$$

the notation is deliberately vague here, in order to completely define the contration, one must denote exactly which positions e_i and e^{i*} have. This is often most easily done using index notation.
Remark: $A \in \mathcal{T}^1_1(V) = \mathrm{End}(V)$, $CA = \mathrm{tr}A =$ the trace of A considered as a linear endomorphism of V.
Pullback of a covariant tensor: Let $f : M \to N$ be C^∞, let T be a section of $\mathcal{T}^k(TN)$. Then we can define a section f^*T of $\mathcal{T}^k(TM)$ by $f^*T(X_1, \dots, X_k) = T(f_*X_1, \dots, f_*X_k)$.
Symmetry och anti symmetry of tensors: $T \in \mathcal{T}^k(V)$ is symmetric if

$$T(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = T(v_1, \dots, v_k)$$

for every permutation σ of $1, \dots, k$. Analogously for tensor fields.
Example: A symmetric tensor $g \in C^\infty(\mathcal{T}^2(TM))$ (symmetric covariant 2-tensor) is called a Riemannian metric if $g = \sum_{ij} g_{ij} dx^i \otimes dx^j$ where g_{ij} is a positive definite matrix (this is independent of the coordinate system). This is equivalent (for symmetric g) with $\forall X \in \mathcal{V}, p \in M$, $g(X, X)_p \geq 0$ and $g(X, X)_p = 0 \Rightarrow X(p) = 0$. The notation $\langle X, Y \rangle$ is often used instead of $g(X, Y)$.
$T \in \mathcal{T}^k(V)$ is anti symmetric (alternating, skew symmetric) if

$$T(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \mathrm{sign}(\sigma) T(v_1, \dots, v_k)$$

for every permutation σ av $1, \dots, k$. The bundle of antisymmetric covariant k–tensors is denoted by $\Omega^k(TM)$. Sections of this bundle are called differential forms.

Problems, Chapter 4: 1,5,7,8,10