Total: 30 pts (=15% of the final grade)   Time allowed: 50 minutes.

You are not allowed to use any electronic devices, such as calculators, laptops or phones, during the test. Please show your steps clearly.

1. (10 pts) Evaluate the integral.

   (a) (3 pts) $\int \sin^2(2x)\cos(2x)\, dx$.

   (b) (3 pts) $\int \sin^{-1} x\, dx$.

   (c) (4 pts) $\int \frac{1}{9+x^2} dx$.

   Sol. (a) Let $u = \sin(2x)$, then $du = 2\cos(2x)dx$, so

$$\int \sin^2(2x)\cos(2x)dx = \frac{1}{2}\int u^2 du = \frac{u^3}{6} + C = \frac{\sin^3(2x)}{6} + C.$$

   (b)

$$\begin{aligned}\int \sin^{-1} x dx =& x\sin^{-1} x - \int x(\sin^{-1} x)' dx \qquad \text{(by parts)}\\ =& x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx\\ =& x\sin^{-1} x + \frac{1}{2}\int \frac{du}{\sqrt{u}} \qquad (u = 1-x^2, du = -2xdx)\\ =& x\sin^{-1} x + \sqrt{u} + C\\ =& x\sin^{-1} x + \sqrt{1-x^2} + C.\end{aligned}$$

   (c) Let $x = 3\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, then $dx = 3\sec^2\theta d\theta$.

$$\begin{aligned}\int \frac{1}{9+x^2} dx =& \int \frac{3\sec^2\theta d\theta}{9(1+\tan^2\theta)}\\ =& \frac{1}{3}\int \frac{\sec^2\theta}{\sec^2\theta} d\theta \qquad (1+\tan^2\theta = \sec^2\theta)\\ =& \frac{1}{3}\theta + C\\ =& \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C.\end{aligned}$$

□

2. (10 pts)

(a) (3 pts) Compute $\frac{d}{dx} \sec x$ and express it in terms of $\sec x$ and $\tan x$.

(b) (3 pts) Compute $\displaystyle\int \tan x \, dx$.

(c) (4 pts) Compute $\displaystyle\int \tan^3 x dx$.

Sol. (a) $\dfrac{d}{dx} \sec x = \sec x \tan x$.

(b)

$$\begin{aligned}\int \tan x dx &= \int \frac{\sin x}{\cos x} dx \\ &= -\int \frac{d(\cos x)}{\cos x} \\ &= -\ln|\cos x| + C \\ &= \ln|(\cos x)^{-1}| + C \\ &= \ln|\sec x| + C.\end{aligned}$$

(c)

$$\begin{aligned}\int \tan^3 x dx &= \int \tan x(\tan^2 x) dx \\ &= \int \tan x(\sec^2 x - 1) \\ &= \int \tan x \sec^2 x dx - \int \tan x dx \\ &= \int \tan x \; d(\tan x) - \int \tan x dx \\ &= \frac{\tan^2 x}{2} - \ln|\sec x| + C \qquad \text{(by (b))}\end{aligned}$$

□

3. (10 pts)

(a) (3 pts) Write out the form of the partial fraction decomposition of the function. Do not determine the numerical values of the coefficients:

(i) $\dfrac{x^3 - x^2 + 1}{x^2(x^2+1)^3(x^2+4)^2}$ (ii) $\dfrac{x^3 + x^2 + 1}{(x+1)^2(x^2+9)(x^2-9)}$ (iii) $\dfrac{1}{\pi x^2 + e x^3}$

(b) (4 pts) Compute $\displaystyle\int \frac{1}{x^2 - 3x + 2} dx$.

(c) (3 pts) Is $\displaystyle\int_0^\infty e^{-2x} dx$ convergent or divergent? If it is convergent, compute its value.

Sol. (a) i. Note that x^2+1 and x^2+4 are irreducible.

$$\frac{x^3 - x^2 + 1}{x^2(x^2+1)^3(x^2+4)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2} + \frac{Gx+H}{(x^2+1)^3} + \frac{Ix+J}{x^2+4} + \frac{Kx+L}{(x^2+4)^2}.$$

ii. Note that x^2+9 is irreducible but x^2-9 is not: $x^2 - 9 = (x-3)(x+3)$.

$$\frac{x^3 + x^2 + 1}{(x+1)^2(x^2+9)(x^2-9)} = \frac{x^3 + x^2 + 1}{(x+1)^2(x^2+9)(x-3)(x+3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-3} + \frac{D}{x+3} + \frac{Ex+F}{x^2+9}$$

iii.

$$\frac{1}{\pi x^2 + e x^3} = \frac{1}{x^2(\pi + ex)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{\pi + ex}$$

(b) Note $x^2 - 3x + 2 = (x-1)(x-2)$. Let

$$\frac{1}{x^2 - 3x + 2} = \frac{A}{x-1} + \frac{B}{x-2}.$$

Then $1 = A(x-2) + B(x-1) = (A+B)x + (-2A - B)$. So

$$\begin{cases} A + B = 0 \\ -2A - B = 1 \end{cases}$$

Adding the two equations: $-A = 1$, i.e. $A = -1$, and so $B = -A = 1$. Therefore

$$\int \frac{1}{x^2 - 3x + 2} dx = \int \left(-\frac{1}{x-1} + \frac{1}{x-2} \right) dx = \ln|x-2| - \ln|x-1| + C.$$

$$
\begin{aligned}
\int_0^\infty e^{-2x} dx &= \lim_{t\to\infty} \int_0^t e^{-2x} dx \\
&= \lim_{t\to\infty} \left[\frac{e^{-2x}}{-2}\right]_0^t \\
&= \lim_{t\to\infty} \left(-\frac{e^{-2t}}{2} + \frac{1}{2}\right) \\
&= \frac{1}{2}.
\end{aligned}
$$

So the improper integral is convergent with value $\frac{1}{2}$. $\square$