Total: 30 pts (=15% of the final grade) Time allowed: 50 minutes.

You are not allowed to use any electronic devices, such as calculators, laptops or phones, during the test. Please show your steps clearly.

1. (10 pts)

   (a) (4 pts) Compute the following (simplify where possible):

$$\frac{d}{dx}\left(2\tanh^{-1}(-x)\right), \quad \frac{d}{dx}(\cos^{-1}(3x)), \quad \int \frac{\pi}{\sqrt{x^2+1}}dx, \quad \int \frac{2dx}{\sqrt{1-(2x)^2}}$$

   (b) (1 pts) Expand $1-(x-1)^2$.

   (c) (5 pts) Compute

$$\int \frac{1}{x^2-2x}dx$$

Sol. (a)

$$\frac{d}{dx}\left(2\tanh^{-1}(-x)\right) = 2\cdot\frac{1}{1-(-x)^2}\cdot(-x)' = -\frac{2}{1-x^2}.$$

$$\frac{d}{dx}\left(\cos^{-1}(3x)\right) = -\frac{1}{\sqrt{1-(3x)^2}}\cdot(3x)' = -\frac{3}{\sqrt{1-9x^2}}.$$

$$\int \frac{\pi}{\sqrt{x^2+1}}dx = \pi\int\frac{1}{\sqrt{1+x^2}}dx = \pi\sinh^{-1}x + C.$$

$$\begin{aligned}\int \frac{2dx}{\sqrt{1-(2x)^2}} &= \int\frac{du}{\sqrt{1-u^2}} \quad (\text{let } u = 2x)\\ &= \sin^{-1}u + C\\ &= \sin^{-1}(2x) + C.\end{aligned}$$

(b)

$$1-(x-1)^2 = 1-\left(x^2-2x+1\right) = 2x-x^2.$$

(c) By (b),

$$\int\frac{1}{x^2-2x}dx = -\int\frac{1}{1-(x-1)^2}dx.$$

Let $u = x-1$, so $du = dx$, then

$$\begin{aligned}\int\frac{1}{x^2-2x}dx &= -\int\frac{1}{1-(x-1)^2}dx\\ &= -\int\frac{1}{1-u^2}du\\ &= -\tanh^{-1}u + C\\ &= -\tanh^{-1}(x-1) + C.\end{aligned}$$

Check: Exercise

□

2. (10 pts)

(a) (3 pts) Compute $\lim_{x\to 0} \frac{\cos x - 1}{x^2}$.

(b) (3 pts) Compute ~~$\lim_{x\to 0^+} x \tan\left(\frac{1}{x}\right)$.~~ $\lim_{x\to\infty} x \tan\left(\frac{1}{x}\right)$.

(c) (4 pts) Compute $\lim_{x\to\infty} x^3 e^{-2x}$.

Sol. (a)

$$\begin{aligned}\lim_{x\to 0} \frac{\cos x - 1}{x^2} &= \lim_{x\to 0} \frac{-\sin x}{2x} && \text{(L'Hospital rule: } \lim_{x\to 0}(\cos x - 1) = 0, \lim_{x\to 0} x^2 = 0) \\ &= \lim_{x\to 0} \frac{-\cos x}{2} && \text{(L'Hospital rule: } \lim_{x\to 0} -\sin x = 0, \lim_{x\to 0} 2x = 0) \\ &= -\frac{1}{2}.\end{aligned}$$

(b) I am sorry for making a mistake, I am going to give full marks for this part to everyone.

The question I want to ask is $\lim_{x\to\infty} x \tan\left(\frac{1}{x}\right)$, and the answer is 1:

$$\begin{aligned}\lim_{x\to\infty} x \tan\left(\frac{1}{x}\right) &= \lim_{x\to\infty} \frac{\tan\left(\frac{1}{x}\right)}{1/x} \\ &= \lim_{x\to\infty} \frac{\sec^2\left(\frac{1}{x}\right)\cdot -\frac{1}{x^2}}{-1/x^2} && \text{(L'Hospital rule: } \lim_{x\to\infty} \tan\left(\frac{1}{x}\right) = 0, \lim_{x\to\infty} \frac{1}{x} = 0) \\ &= \lim_{x\to\infty} \sec^2\left(\frac{1}{x}\right) \\ &= \sec^2(0) \\ &= 1.\end{aligned}$$

(Indeed, $\lim_{x\to 0^+} x \tan\left(\frac{1}{x}\right)$ does not exist.)

(c)

$$\begin{aligned}\lim_{x\to\infty} x^3 e^{-2x} &= \lim_{x\to\infty} \frac{x^3}{e^{2x}} \\ &= \lim_{x\to\infty} \frac{3x^2}{2e^{2x}} && \text{(L'Hospital rule: } \lim_{x\to\infty} x^3 = \lim_{x\to\infty} e^{2x} = \infty) \\ &= \lim_{x\to\infty} \frac{6x}{4e^{2x}} && \text{(L'Hospital rule: } \lim_{x\to\infty} 3x^2 = \lim_{x\to\infty} 2e^{2x} = \infty) \\ &= \lim_{x\to\infty} \frac{3x}{2e^{2x}} \\ &= \lim_{x\to\infty} \frac{3}{4e^{2x}} && \text{(L'Hospital rule: } \lim_{x\to\infty} 3x = \lim_{x\to\infty} 2e^{2x} = \infty) \\ &= 0.\end{aligned}$$

□

3. (10 pts)

(a) (2 pts) Compute $\cos^{-1}(\frac{\sqrt{3}}{2})$ and $\tan^{-1}(1)$.

(b) (3 pts) State the definitions of $\sinh x$, $\cosh x$ and $\csc x$ ($\neq$ **csch x!**)

(c) (5 pts) Are the following true or false? (No need to give explanation.)

i. $\sinh(-x) = -\sinh x$.

ii. $\sin 2\pi = 0$.

iii. $\frac{d}{dx}\ln(\cos x) = \tan x$.

iv. $\cosh 0 = 0$.

v. $\cosh^2 x - \sinh^2 x = 1$.

vi. $\frac{\pi}{6} = 30°$.

vii. $\sin(x+y) = \sin x + \sin y$.

viii. $\cosh(-x) = -\cosh x$.

ix. $-1 \leq -\sin x \leq 0$ (for all x).

x. $\cos(x+2\pi) = \cos x$.

Sol. (a) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$, $\tan^{-1}(1) = \frac{\pi}{4}$.

(b) $\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$, $\csc x = \frac{1}{\sin x}$.

(c) i. $\sinh(-x) = -\sinh x$.
T

ii. $\sin 2\pi = 0$.
T

iii. $\frac{d}{dx}\ln(\cos x) = \tan x$.
F

iv. $\cosh 0 = 0$.
F

v. $\cosh^2 x - \sinh^2 x = 1$.
T

vi. $\frac{\pi}{6} = 30°$.
T

vii. $\sin(x+y) = \sin x + \sin y$.
F

viii. $\cosh(-x) = -\cosh x$.
F

ix. $-1 \leq -\sin x \leq 0$ (for all x).
F

x. $\cos(x+2\pi) = \cos x$.
T

□