Total: 30 pts (=15% of the final grade) Time allowed: 50 minutes.

You are not allowed to use any electronic devices, such as calculators, laptops or phones, during the test. Please show your steps clearly.

1. (10 pts) Let $f(x) = x^3 - 1$.
   (a) (3 pts) Find the inverse of f.
   (b) (3 pts) Find $f^{-1}(0)$.
   (c) (4 pts) Compute $(f^{-1})'(0)$.

Sol. (a) Let $y = x^3 - 1$, so

$$\begin{aligned} x^3 &= y + 1 \\ x &= (y+1)^{\frac{1}{3}}. \\ \therefore f^{-1}(y) &= (y+1)^{\frac{1}{3}}. \end{aligned}$$

(b)

$$f^{-1}(0) = (0+1)^{\frac{1}{3}} = 1.$$

(c)

$$(f^{-1})'(0) = \frac{1}{f'(f^{-1}(0))} = \frac{1}{f'(1)}.$$

As $f(x) = x^3 - 1$, $f'(x) = 3x^2$. So $f'(0) = 3$ and

$$(f^{-1})'(0) = \frac{1}{3}.$$

□

2. (10 pts)

(a) (2 pts) Compute $\int \frac{2}{x} dx$ and $\frac{d}{dx} 2^x$.

(b) (4 pts) Expand the quantity $\ln \sqrt[3]{\frac{(x+e)^2}{x}}$. ($\sqrt[3]{\ }$ is the cube root.)

(c) (4 pts) Use logarithmic differentiation to find the derivative of $\sqrt[3]{\frac{(x+e)^2}{x}}$.

Sol. (a)

$$\int \frac{2}{x} dx = 2 \int \frac{1}{x} dx = 2 \ln |x| + C.$$

$$\frac{d}{dx} 2^x = (\ln 2) 2^x.$$

(b)

$$\begin{aligned} \ln \sqrt[3]{\frac{(x+e)^2}{x}} &= \frac{1}{3} \ln \frac{(x+e)^2}{x} \\ &= \frac{1}{3} \left(2 \ln(x+e) - \ln x \right) \\ &= \frac{2}{3} \ln(x+e) - \frac{1}{3} \ln x. \end{aligned}$$

(c) Let $y = \sqrt[3]{\frac{(x+e)^2}{x}}$, then by (a), $\ln y = \frac{2}{3} \ln(x+e) - \frac{1}{3} \ln x$. So differentiating gives

$$\begin{aligned} \frac{y'}{y} &= \frac{2}{3} \cdot \frac{1}{x+e} - \frac{1}{3x} \\ \therefore y' &= \sqrt[3]{\frac{(x+e)^2}{x}} \left(\frac{2}{3(x+e)} - \frac{1}{3x} \right). \end{aligned}$$

□

3. (10 pts)

(a) (3 pts) Differentiate the function $f(x) = \ln|\cos x|$. (| | is the absolute value)

(b) (3 pts) Compute $\int \frac{(\ln x)^2}{x} dx$.

(c) (4 pts) Differentiate the function $f(x) = x^{2x}$.

Sol. (a) By the chain rule,

$$(\ln|\cos x|)' = \frac{(\cos x)'}{\cos x} = -\frac{\sin x}{\cos x} = -\tan x.$$

(b) Let $u = \ln x$, so $du = \frac{1}{x}dx$

$$\int \frac{(\ln x)^2}{x} dx = \int u^2 du = \frac{u^3}{3} + C = \frac{(\ln x)^3}{3} + C.$$

Check: Exercise.

(c) Let $y = x^{2x}$, then $\ln y = 2x \ln x$. Differentiate this w.r.t. x,

$$\begin{aligned} \frac{y'}{y} &= 2\ln x + \frac{2x}{x} = 2\ln x + 2 \\ y' &= y(2\ln x + 2) = x^{2x}(2\ln x + 2) = 2x^{2x}(\ln x + 1). \end{aligned}$$

□