Total: 30 pts (=15% of the final grade ) Time allowed: 50 minutes.

You are not allowed to use any electronic devices, such as calculators, laptops or phones, during the test. Please show your steps clearly.

1. (10 pts) Evaluate the integral.

   (a) (2 pts) $\int \sin x \cos x \, dx$.

   (b) (3 pts) $\int \tan^2(2x) \, dx$.

   (c) (5 pts) $\int_0^1 \tan^{-1} x \, dx$.

*Sol.* (a)

$$\begin{aligned}\int \sin x \cos x dx = \int \sin x (\sin x)' dx = \int u du \qquad (u = \sin x)\\ = \frac{u^2}{2} + C\\ = \frac{\sin^2 x}{2} + C.\end{aligned}$$

(b)

$$\begin{aligned}\int \tan^2(2x) dx &= \int \tan^2 u \left(\frac{du}{2}\right) \qquad (u = 2x \quad \therefore dx = \frac{du}{2})\\ &= \frac{1}{2} \int \left(\sec^2 u - 1\right) du \qquad (\tan^2 u = \sec^2 u - 1)\\ &= \frac{1}{2} \tan u - \frac{u}{2} + C\\ &= \frac{\tan 2x}{2} - x + C.\end{aligned}$$

(c)

$$\begin{aligned}\int_0^1 \tan^{-1} x dx &= \left[x \tan^{-1} x\right]_0^1 - \int_0^1 x (\tan^{-1} x)' dx\\ &= \left(1 \cdot \frac{\pi}{4} - 0 \cdot 0\right) - \int_0^1 \frac{x}{1 + x^2} dx\\ &= \frac{\pi}{4} - \frac{1}{2} \int_0^1 \frac{(1 + x^2)'}{1 + x^2} dx\\ &= \frac{\pi}{4} - \frac{1}{2} \int_1^2 \frac{du}{u} dx \qquad (u = x^2 + 1)\\ &= \frac{\pi}{4} - \frac{1}{2} \left[\ln |u|\right]_1^2\\ &= \frac{\pi}{4} - \frac{1}{2} \ln 2.\end{aligned}$$

□

2. (10 pts)

(a) (3 pts) Compute $\frac{d}{dx}\sec x$ and express it in terms of $\sec x$ and $\tan x$.

(b) (2 pts) Prove that $\displaystyle\int \sec x \, dx = \ln|\sec x + \tan x| + C$.

(c) (5 pts) Compute $\displaystyle\int \sec^3 x dx$.

*Sol.* (a) $\frac{d}{dx}(\sec x) = \sec x \tan x$. (You can write the answer directly, or derive it using the derivative of $\cos x$.)

(b)

$$\frac{d}{dx}(\ln|\sec x + \tan x|) = \frac{(\sec x + \tan x)'}{\sec x + \tan x} = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \sec x.$$

So $\int \sec x dx = \ln|\sec x + \tan x| + C$.

(c)

$$\begin{aligned}\int \sec^3 x dx &= \int \sec x (\tan x)' dx \\ &= \sec x \tan x - \int \tan x (\sec x)' dx \quad \text{(by parts)} \\ &= \sec x \tan x - \int \tan x \cdot (\sec x \tan x) dx \\ &= \sec x \tan x - \int \sec x \tan^2 x dx \\ &= \sec x \tan x - \int \sec x (\sec^2 x - 1) dx \\ &= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx\end{aligned}$$

So

$$\begin{aligned}2\int \sec x^3 dx &= \sec x \tan x + \int \sec x dx \\ &= \sec x \tan x + \ln|\sec x + \tan x| + C \qquad \text{(by (b))}.\end{aligned}$$

i.e.

$$\int \sec^3 x dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C_1.$$

□

3. (10 pts)

(a) (3 pts) Write out the form of the partial fraction decomposition of the function. Do not determine the numerical values of the coefficients:

$$\text{(i)}\ \frac{x^3+1}{x^2(x^2+1)^2} \qquad \text{(ii)}\ \frac{x}{(x^2+1)(x^2-1)} \qquad \text{(iii)}\ \frac{1}{5x^2-2x^3}$$

(b) (4 pts) Compute $\int \frac{1}{x^2+x}dx$.

(c) (3 pts) Is $\int_1^\infty \frac{1}{(x+1)^2}dx$ convergent or divergent? If it is convergent, compute its value.

*Sol.* (a) i. $\frac{x^3+1}{x^2(x^2+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2}$ (note that $x^2+1$ is irreducible)

ii.

$$\begin{aligned} \frac{x}{(x^2+1)(x^2-1)} &= \frac{x}{(x^2+1)(x-1)(x+1)} \qquad \text{(note } x^2-1 \text{ is \textbf{not} irreducible)} \\ &= \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1} \end{aligned}$$

iii.

$$\frac{1}{5x^2-2x^3} = \frac{1}{x^2(5-2x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{5-2x}.$$

(b) As $x^2+x = x(x+1)$, let $\frac{1}{x^2+x} = \frac{A}{x} + \frac{B}{x+1}$. Then $1 = A(x+1) + Bx = (A+B)x + A$. So

$$\begin{cases} A+B=0 \\ A=1 \end{cases}$$

Therefore $A=1, B=-1$, i.e.

$$\frac{1}{x^2+x} = \frac{1}{x} - \frac{1}{x+1}.$$

$$\begin{aligned} \therefore \int \frac{1}{x^2+x}dx &= \int \left(\frac{1}{x} - \frac{1}{x-1}\right) dx \\ &= \ln|x| - \ln|x-1| + C. \end{aligned}$$

(c)

$$\begin{aligned} \int_1^\infty \frac{1}{(x+1)^2}dx &= \lim_{t\to\infty} \int_1^t \frac{1}{(x+1)^2}dx \\ &= \lim_{t\to\infty} \int_2^{t+1} \frac{1}{u^2}du \qquad (u = x+1) \\ &= \lim_{t\to\infty} \left[-\frac{1}{u}\right]_2^{t+1} \\ &= \lim_{t\to\infty} \left[-\frac{1}{t+1} + \frac{1}{2}\right] \\ &= \frac{1}{2}. \end{aligned}$$

Therefore the improper integral $\int_1^\infty \frac{1}{(x+1)^2}dx$ is convergent with value $\frac{1}{2}$.

□