Total: 30 pts (=15% of the final grade) Time allowed: 50 minutes.

You are not allowed to use any electronic devices, such as calculators, laptops or phones, during the test. Please show your steps clearly.

1. (10 pts)

   (a) (4 pts) Compute the followings:

$$\frac{d}{dx}(2\sin^{-1}x), \quad \frac{d}{dx}\left(\tan^{-1}(-x)\right), \quad \int \frac{2}{x^2-1}dx, \quad \int \frac{2}{\sqrt{x^2-1}}dx$$

   (b) (1 pts) Expand $(x+2)^2-4$.

   (c) (5 pts) Compute

$$\int \frac{1}{\sqrt{x^2+4x}}dx$$

Sol. (a)

$$\frac{d}{dx}(2\sin^{-1}x) = \frac{2}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1}(-x)) = \frac{1}{1+(-x)^2}\cdot(-1) = -\frac{1}{1+x^2}.$$

$$\int \frac{2}{x^2-1}dx = -2\int \frac{1}{1-x^2}dx = -2\tanh^{-1}x + C$$

$$\int \frac{2}{\sqrt{x^2-1}}dx = 2\cosh^{-1}x + C.$$

(b)

$$(x+2)^2-4 = (x^2+4x+4)-4 = x^2+4x.$$

(c) By (b),

$$\int \frac{1}{\sqrt{x^2+4x}}dx = \int \frac{1}{\sqrt{(x+2)^2-4}}dx$$

Let $u = x+2$, so $du = dx$, and

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+4x}}dx &= \int \frac{1}{\sqrt{u^2-4}}du \\ &= \int \frac{du}{2\sqrt{\frac{u^2}{4}-1}} \\ &= \int \frac{dv}{\sqrt{v^2-1}} \qquad (v = \frac{u}{2}, \quad \therefore dv = \frac{du}{2}) \\ &= \cosh^{-1}v + C \\ &= \cosh^{-1}(\frac{u}{2}) + C \\ &= \cosh^{-1}\left(\frac{x+2}{2}\right) + C.\end{aligned}$$

Check: Exercise. Use **different** symbols (e.g. u, v, w) for different substitutions!

□

2. (10 pts)

(a) (3 pts) Compute $\lim_{x\to 0} \frac{e^{2x}-1}{\sin x}$.

(b) (3 pts) Compute $\lim_{x\to 0^+} x^2(\ln x)$.

(c) (4 pts) Compute $\lim_{x\to\infty} \sqrt{x}e^{-\frac{x}{2}}$.

Sol. (a)

$$\begin{aligned}\lim_{x\to 0} \frac{e^{2x}-1}{\sin x} &= \lim_{x\to 0} \frac{2e^{2x}}{\cos x} \qquad (\text{L'Hospital rule, note that } \lim_{x\to 0} e^{2x}-1=0, \lim_{x\to 0} \sin x = 0)\\ &= \frac{2e^0}{\cos 0}\\ &= 2.\end{aligned}$$

(b)

$$\begin{aligned}\lim_{x\to 0^+} x^2 \ln x &= \lim_{x\to 0^+} \frac{\ln x}{1/x^2}\\ &= \lim_{x\to 0^+} \frac{1/x}{-2x^{-3}} \qquad (\text{L'Hospital rule, note that } \lim_{x\to 0^+} \ln x = -\infty, \lim_{x\to 0^+} \frac{1}{x^2} = \infty)\\ &= \lim_{x\to 0^+} -\frac{x^2}{2}\\ &= 0.\end{aligned}$$

(c)

$$\begin{aligned}\lim_{x\to\infty} \sqrt{x}e^{-\frac{x}{2}} &= \lim_{x\to\infty} \frac{\sqrt{x}}{e^{\frac{x}{2}}}\\ &= \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{\frac{1}{2}e^{\frac{x}{2}}} \qquad (\text{L'Hospital rule, note that } \lim_{x\to\infty} \sqrt{x} = \infty, \lim_{x\to\infty} e^{\frac{x}{2}} = \infty)\\ &= \lim_{x\to\infty} \frac{1}{\sqrt{x}e^{\frac{x}{2}}}\\ &= 0. \qquad (\lim_{x\to\infty} \sqrt{x} = \infty, \lim_{x\to\infty} e^{\frac{x}{2}} = \infty)\end{aligned}$$

□

3. (10 pts)

(a) (3 pts) State the definitions of $\sinh x$, $\cosh x$ and $\operatorname{sech} x$.

(b) (2 pts) Compute $\sin^{-1}(\frac{1}{\sqrt{2}})$ and $\tan^{-1}(\sqrt{3})$.

(c) (5 pts) Are the following true or false? (No need to give explanation.)

i. $\sin(-x) = -\sin x$.

ii. $\cos(-x) = -\cos x$.

iii. $2\pi = 180°$.

iv. $\sin^{-1}(\sin 3\pi) = 3\pi$.

v. $\sin^{-1} 0 = 0$.

vi. $\cos^{-1} 0 = 0$.

vii. $0 \leq \sin x \leq 1$.

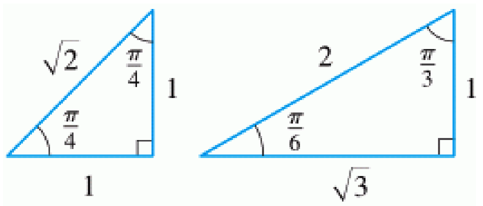
viii. $\sin(2x) = 2\sin x$.

ix. $\cosh^2 x + \sinh^2 x = 1$.

x. $\frac{d}{dx}\cosh x = -\sinh x$.

xi. You like this course.

Sol. (a) $\sinh x = \dfrac{e^x - e^{-x}}{2}$, $\quad \cosh x = \dfrac{e^x + e^{-x}}{2}$, $\quad \operatorname{sech} x = \dfrac{1}{\cosh x}$.



(b) $\sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right) = \dfrac{\pi}{4}$, $\quad \tan^{-1}(\sqrt{3}) = \dfrac{\pi}{3}$. (See the fig. on the right.)

(c) (T=True. F=False. You don't really need to give any explanation in the test.)

i. $\sin(-x) = -\sin x$.
T.

ii. $\cos(-x) = -\cos x$.
F. Indeed $\cos(-x) = \cos x$. e.g. $\cos 0 = 1$, $\cos(-0) = 1$.

iii. $2\pi = 180°$.
F. $\pi = 180°$

iv. $\sin^{-1}(\sin 3\pi) = 3\pi$.
F. The range of $\sin^{-1}$ is only $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

v. $\sin^{-1} 0 = 0$.
T.

vi. $\cos^{-1} 0 = 0$.
F. $\cos^{-1}(0) = \frac{\pi}{2}$.

vii. $0 \leq \sin x \leq 1$.
F. $-1 \leq \sin x \leq 1$.

viii. $\sin(2x) = 2\sin x$.
F. e.g. $\sin\frac{\pi}{2} = 1$, $\sin\pi = 0$.

ix. $\cosh^2 x + \sinh^2 x = 1$.
F. $\cosh^2 x - \sinh^2 x = 1$. (But $\cos^2 x + \sin^2 x = 1$.)

x. $\frac{d}{dx}\cosh x = -\sinh x$.
F. $\frac{d}{dx}\cosh x = \sinh x$

xi. You like this course.
T. Obviously.

□