Total: 30 pts (=15% of the final grade)   Time allowed: 50 minutes.

You are not allowed to use any electronic devices, such as calculators, laptops or phones, during the test. Please show your steps clearly.

1. (10 pts) Let $f(x) = \sqrt{x-2}$, defined on $[2, \infty)$.
   (a) (3 pts) Find the inverse of f.
   (b) (3 pts) Find $f^{-1}(2)$.
   (c) (4 pts) Compute $(f^{-1})'(2)$.

Sol. (a) Let $y = \sqrt{x-2}$.

$$\begin{aligned} \therefore y^2 &= x - 2 \\ x &= y^2 + 2. \end{aligned}$$

So $f^{-1}(y) = y^2 + 2$.
(You may convert y to x and write $f^{-1}(x) = x^2 + 2$. I don't think it's necessary.)

(b) From (a), $f^{-1}(2) = 2^2 + 2 = 6$. (You can easily check this.)

(c) We compute $f'(x) = \left((x-2)^{\frac{1}{2}}\right)' = \frac{1}{2}(x-2)^{-\frac{1}{2}}$. So

$$f'(6) = \frac{1}{2}(6-2)^{-\frac{1}{2}} = \frac{1}{2} \cdot \frac{1}{4^{\frac{1}{2}}} = \frac{1}{4}.$$

Therefore

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(6)} = 4.$$

□

2. (10 pts)

(a) (2 pts) Compute $\int \frac{1}{x} dx$ and $\int 7^x dx$.

(b) (4 pts) Expand the quantity $\ln \sqrt{\frac{x+1}{(x-1)^3}}$.

(c) (4 pts) Use logarithmic differentiation to find the derivative of $\sqrt{\frac{x+1}{(x-1)^3}}$. (Your answer must be exact and fully simplified.)

Sol. (a)

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int 7^x dx = \frac{7^x}{\ln 7} + C$$

Warning: don't omit dx, the absolute sign $|\ |$ and $+C$.

(b)

$$\begin{aligned}
\ln \sqrt{\frac{x+1}{(x-1)^3}} &= \frac{1}{2} \ln \left(\frac{x+1}{(x-1)^3} \right) \\
&= \frac{1}{2} \left(\ln(x+1) - \ln((x-1)^3) \right) \\
&= \frac{1}{2} \left[\ln(x+1) - 3\ln(x-1) \right] \\
&= \frac{1}{2} \ln(x+1) - \frac{3}{2} \ln(x-1).
\end{aligned}$$

(c) Let $y = \sqrt{\frac{x+1}{(x-1)^3}}$, so by (a),

$$\begin{aligned}
\ln y &= \ln \sqrt{\frac{x+1}{(x-1)^3}} \\
&= \frac{1}{2} \ln(x+1) - \frac{3}{2} \ln(x-1)
\end{aligned}$$

Differentiating both sides w.r.t. x,

$$\begin{aligned}
\frac{y'}{y} &= \frac{1}{2} \cdot \frac{1}{x+1} - \frac{3}{2} \cdot \frac{1}{x-1} \\
\therefore y' &= y \left(\frac{1}{2(x+1)} - \frac{3}{2(x-1)} \right) \\
&= \sqrt{\frac{x+1}{(x-1)^3}} \left(\frac{1}{2(x+1)} - \frac{3}{2(x-1)} \right) \quad \text{(this is an acceptable answer)} \\
&= \frac{(x+1)^{\frac{1}{2}}}{(x-1)^{\frac{3}{2}}} \left(\frac{1}{2(x+1)} - \frac{3}{2(x-1)} \right) \\
&= \frac{1}{2(x-1)^{\frac{3}{2}}(x+1)^{\frac{1}{2}}} - \frac{3(x+1)^{\frac{1}{2}}}{(x-1)^{\frac{5}{2}}}.
\end{aligned}$$

□

3. (10 pts)

(a) (3 pts) Differentiate the function $f(x) = e^{x^2+1}$.

(b) (3 pts) Compute $\int \frac{e^{\frac{1}{x}}}{x^2} dx$.

(c) (4 pts) Compute $\frac{d}{dx}\left(x^{\sin x}\right)$.

Sol. (a) By chain rule, $\left(e^{x^2+1}\right)' = e^{x^2+1} \cdot 2x = 2xe^{x^2+1}$.

(b) Let $u = \frac{1}{x}$, so $du = -\frac{1}{x^2} dx$. Therefore

$$\int \frac{e^{\frac{1}{x}}}{x^2} dx = -\int e^u du = -e^u + C = -e^{\frac{1}{x}} + C.$$

(c) Let $y = x^{\sin x}$. So

$$\ln y = \sin x \ln x.$$

Differentiating w.r.t. x,

$$\begin{aligned} \frac{y'}{y} &= \cos x \ln x + \frac{\sin x}{x} \\ y' &= y(\cos x \ln x + \frac{\sin x}{x}) \\ &= x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right) \end{aligned}$$

i.e.

$$\frac{d}{dx}\left(x^{\sin x}\right) = x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right).$$

□