- Find $(f^{-1})'(a)$. $f(x) = x^3 + 3\sin x + 2\cos x, \quad a = 2$

By inspection, $f(0) = 0 + 3\sin 0 + 2\cos 0 = 2$, so $f^{-1}(2) = 0$.

$$f'(x) = 3x^2 + 3\cos x - 2\sin x.$$

So

$$f'(0) = 0 + 3\cos 0 - 2\sin 0 = 0 + 3 - 0 = 3.$$

Therefore

$$\left(f^{-1}\right)'(2) = \frac{1}{f'(0)} = \frac{1}{3}.$$

- Use the Laws of Logarithms to expand the quantity.

  $\ln \sqrt[3]{\dfrac{x-1}{x+1}}$

$$\begin{aligned}\ln \sqrt[3]{\frac{x-1}{x+1}} &= \ln\left[\left(\frac{x-1}{x+1}\right)^{\frac{1}{3}}\right] \\ &= \frac{1}{3}\ln\left(\frac{x-1}{x+1}\right) \\ &= \frac{1}{3}\left(\ln(x-1) - \ln(x+1)\right).\end{aligned}$$

- Express the quantity as a single logarithm.

  $\ln 3 + \frac{1}{3}\ln 8$

$$\begin{aligned}\ln 3 + \frac{1}{3}\ln 8 &= \ln 3 + \ln\left(8^{\frac{1}{3}}\right) \\ &= \ln 3 + \ln 2 \\ &= \ln(3 \cdot 2) \\ &= \ln 6.\end{aligned}$$

- Differentiate the function.

  $y = \ln|2 - x - 5x^2|$

$$y' = \left[\ln|2-x-5x^2|\right]' = \frac{(2-x-5x^2)'}{2-x-5x^2}$$

$$= \frac{-1-10x}{2-x-5x^2}$$

$g(x) = \ln\left(x\sqrt{x^2-1}\right)$

$$g'(x) = \left[\ln\left(x\sqrt{x^2-1}\right)\right]'$$

$$= \left[\ln x + \ln\sqrt{x^2-1}\right]'$$

$$= \left[\ln x + \frac{1}{2}\ln(x^2-1)\right]'$$

$$= \frac{1}{x} + \frac{1}{2}\cdot\frac{(x^2-1)'}{x^2-1}$$

$$= \frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2-1}$$

$$= \frac{1}{x} + \frac{x}{x^2-1}$$

- Use logarithmic differentiation to find the derivative of the function.

$$y = \sqrt{\frac{x-1}{x^4+1}}$$

$$y = \sqrt{\frac{x-1}{x^4+1}} = \left(\frac{x-1}{x^4+1}\right)^{\frac{1}{2}}$$

$$\ln y = \frac{1}{2}\ln\left(\frac{x-1}{x^4+1}\right) = \frac{1}{2}\left[\ln(x-1) - \ln(x^4+1)\right]$$

$$\frac{y'}{y} = \frac{1}{2}\left[\frac{(x-1)'}{x-1} - \frac{(x^4+1)'}{x^4+1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{x-1} - \frac{4x^3}{x^4+1}\right]$$

$$\therefore y' = \frac{y}{2}\left[\frac{1}{x-1} - \frac{4x^3}{x^4+1}\right]$$

$$= \sqrt{\frac{x-1}{x^4+1}}\left[\frac{1}{x-1} - \frac{4x^3}{x^4+1}\right]$$

$$= \frac{1}{\sqrt{(x^4+1)(x-1)}} - \frac{4x^3\sqrt{x-1}}{(x^4+1)^{\frac{3}{2}}} \quad \#$$

- Evaluate the integral.

$$\int_1^2 \frac{dt}{8-3t}$$

$$\int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$$

$$\int_e^6 \frac{dx}{x \ln x}$$

$$\int_1^e \frac{x^2+x+1}{x}\, dx$$

Let $u = 8-3t$ $\therefore du = -3dt$

$$\int_1^2 \frac{dt}{8-3t} = \frac{-1}{3}\int_{8-3\cdot 1}^{8-3\cdot 2} \frac{du}{u}$$

$$= \frac{-1}{3}\int_5^2 \frac{du}{u}$$

$$= \frac{1}{3}\int_2^5 \frac{du}{u}$$

$$= \frac{1}{3}\left[\ln|u|\right]_2^5$$

$$= \frac{1}{3}(\ln 5 - \ln 2).$$

$$\int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$$

$$= \int_4^9 \left(x + 2 + \frac{1}{x}\right) dx$$

$$= \left[\frac{x^2}{2} + 2x + \ln|x|\right]_4^9$$

$$= \frac{1}{2}(9^2 - 4^2) + 2(9-4) + (\ln 9 - \ln 4)$$

$$= \frac{65}{2} + 10 + \ln\frac{9}{4}$$

$$= \frac{85}{2} + \ln\frac{9}{4}.$$

Let $u = \ln x$ $\therefore du = \frac{1}{x} dx$

$$\int_e^6 \frac{dx}{x \ln x} = \int_{\ln e}^{\ln 6} \frac{du}{u} = \int_1^{\ln 6} \frac{du}{u}$$

$$= \left[\ln|u|\right]_1^{\ln 6}$$

$$= \ln(\ln 6). \quad (\ln 6 > 0)$$

$$\int_1^e \frac{x^2 + x + 1}{x} dx$$

$$= \int_1^e \left(x + 1 + \frac{1}{x}\right) dx$$

$$= \left[\frac{x^2}{2} + x + \ln|x|\right]_1^e$$

$$= \left(\frac{e^2}{2} + e + \ln e\right) - \left(\frac{1^2}{2} + 1 + \ln 1\right) \quad (e > 0)$$

$$= \left(\frac{e^2}{2} + e + 1\right) - \left(\frac{1}{2} + 1 + 0\right)$$

$$= \frac{e^2}{2} + e - \frac{1}{2} \quad \#$$

- Differentiate the function.

$$y = \frac{e^u - e^{-u}}{e^u + e^{-u}}$$

$$\begin{aligned}
y' &= \frac{(e^u + e^{-u})(e^u - e^{-u})' - (e^u - e^{-u})(e^u + e^{-u})'}{(e^u + e^{-u})^2} \\
&= \frac{(e^u + e^{-u})(e^u + e^{-u}) - (e^u - e^{-u})(e^u - e^{-u})}{(e^u + e^{-u})^2} \\
&= \frac{(e^u + e^{-u})^2 - (e^u - e^{-u})^2}{(e^u + e^{-u})^2} \\
&= \frac{(e^{2u} + 2e^u e^{-u} + e^{-2u}) - (e^{2u} - 2e^u e^{-u} + e^{-2u})}{(e^u + e^{-u})^2} \quad (\because (e^u)^2 = e^{2u}, (e^{-u})^2 = e^{-2u}) \\
&= \frac{(e^{2u} + 2 + e^{-2u}) - (e^{2u} - 2 + e^{-2u})}{(e^u + e^{-u})^2} \qquad (e^u \cdot e^{-u} = 1) \\
&= \frac{4}{(e^u + e^{-u})^2}.
\end{aligned}$$

$$y = x^{\cos x}$$

Let $y = x^{\cos x}$, then $\ln y = \cos x \ln x$. Differentiating w.r.t. x,

$$\begin{aligned}
\frac{y'}{y} &= -\sin x \ln x + \frac{\cos x}{x} \\
y' &= y(-\sin x \ln x + \frac{\cos x}{x}) \\
&= x^{\cos x}\left(-\sin x \ln x + \frac{\cos x}{x}\right).
\end{aligned}$$

- Evaluate the integral.

$$\int_0^1 \frac{\sqrt{1 + e^{-x}}}{e^x}\, dx \qquad \int \frac{2^x}{2^x + 1}\, dx$$

Let $u = e^{-x} + 1$, so $du = -e^{-x}dx = -\frac{1}{e^x}dx$. So

$$\begin{aligned}
\int_0^1 \frac{\sqrt{e^{-x} + 1}}{e^x} dx &= -\int_{e^0+1}^{e^{-1}+1} \sqrt{u} du \\
&= -\int_2^{e^{-1}+1} u^{\frac{1}{2}} du \\
&= -\frac{2}{3}\left[u^{\frac{3}{2}}\right]_2^{e^{-1}+1} \\
&= -\frac{2}{3}\left((e^{-1} + 1)^{\frac{3}{2}} - 2\sqrt{2}\right).
\end{aligned}$$

Let $u = 2^x + 1$, then $du = 2^x \ln 2\, dx$, i.e. $2^x dx = \frac{1}{\ln 2} du$. So

$$\begin{aligned}\int \frac{2^x}{2^x+1} dx &= \frac{1}{\ln 2} \int \frac{1}{u} du \\ &= \frac{1}{\ln 2} \ln|u| + C \\ &= \frac{1}{\ln 2} \ln|2^x+1| + C \\ &= \frac{1}{\ln 2} \ln(2^x+1) + C \quad (\because 2^x + 1 \geq 1)\end{aligned}$$

- Find the derivative of the function.

  $y = \tan^{-1}(x^2)$

  $y = \sin^{-1}(2x+1)$

$$y' = \frac{1}{1+(x^2)^2}(x^2)' \qquad \text{(chain rule)}$$

$$= \frac{1}{1+x^4} \cdot 2x$$

$$= \frac{2x}{1+x^4}$$

$$y' = \frac{1}{\sqrt{1-(2x+1)^2}} \cdot (2x+1)'$$

$$= \frac{1}{\sqrt{1-(4x^2+4x+1)}} \cdot 2$$

$$= \frac{2}{\sqrt{-4x^2-4x}}$$

$$= \frac{2}{\sqrt{2^2(-x^2-x)}}$$

$$= \frac{1}{\sqrt{-x^2-x}} \qquad (-1 < x < 0 \text{ for } y \text{ to be well-defined})$$

Evaluate the integral.

$$\int_0^{\sqrt{3}/4} \frac{dx}{1+16x^2}$$

Let $u = 4x$, then $du = 4dx$, so

$$\begin{aligned}\int_0^{\frac{\sqrt{3}}{4}} \frac{dx}{1+16x^2} =& \frac{1}{4}\int_{4\cdot 0}^{4\cdot\frac{\sqrt{3}}{4}} \frac{du}{1+u^2} \\ =& \frac{1}{4}\int_0^{\sqrt{3}} \frac{du}{1+u^2} \\ =& \frac{1}{4}\left[\tan^{-1} u\right]_0^{\sqrt{3}} \\ =& \frac{1}{4}\left[\frac{\pi}{3} - 0\right] \\ =& \frac{\pi}{12}.\end{aligned}$$

$$\int \frac{t^2}{\sqrt{1-t^6}}\, dt$$

Let $u = t^3$, then $du = 3t^2dt$, so

$$\begin{aligned}\int \frac{t^2}{\sqrt{1-t^6}}dt =& \frac{1}{3}\int \frac{du}{\sqrt{1-u^2}} \\ =& \frac{1}{3}\sin^{-1} u + C \\ =& \frac{1}{3}\sin^{-1}(t^3) + C.\end{aligned}$$

$$\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx$$

Let $u = \sin^{-1} x$, so $du = \frac{1}{\sqrt{1-x^2}}dx$, so

$$\begin{aligned}\int_0^{\frac{1}{2}} \frac{\sin^{-1} x}{\sqrt{1-x^2}}dx =& \int_{\sin^{-1} 0}^{\sin^{-1}\left(\frac{1}{2}\right)} u\, du \\ =& \int_0^{\frac{\pi}{6}} u\, du \\ =& \left[\frac{u^2}{2}\right]_0^{\frac{\pi}{6}} \\ =& \frac{\pi^2}{72}.\end{aligned}$$

- Find the derivative.
  $\cosh(\ln x)$

$$g'(x) = \sinh(\ln x)\cdot\frac{1}{x} = \frac{\sinh(\ln x)}{x}.$$

$\sinh^{-1}(\tan x)$

$$y' = \frac{1}{\sqrt{1+\tan^2 x}} \cdot \sec^2 x = \frac{1}{\sqrt{\sec^2 x}} \cdot \sec^2 x = \frac{1}{|\sec x|} \cdot |\sec x|^2$$
$$= |\sec x|.$$

- Evaluate the integral.
  $$\int \frac{\cosh x}{\cosh^2 x - 1}\, dx$$

$$\int \frac{\cosh x}{\cosh^2 x - 1} dx = \int \frac{\cosh x}{\sinh^2 x} dx \qquad (\cosh^2 x - 1 = \sinh^2 x)$$
$$= \int \frac{du}{u^2} \qquad (u = \sinh x, \quad du = \cosh x dx)$$
$$= -\frac{1}{u} + C$$
$$= -\frac{1}{\sinh x} + C.$$

$$\int_4^6 \frac{1}{\sqrt{t^2-9}}\, dt$$

$$\int_4^6 \frac{1}{\sqrt{t^2-9}} dt = \int_4^6 \frac{1}{3\sqrt{\frac{t^2}{9}-1}} dt$$
$$= \frac{1}{3}\int_4^6 \frac{1}{\sqrt{(\frac{t}{3})^2-1}} dt$$
$$= \int_{\frac{4}{3}}^{\frac{6}{3}} \frac{du}{\sqrt{u^2-1}} \qquad (u = \frac{t}{3}), du = \frac{dt}{3}$$
$$= \int_{\frac{4}{3}}^{2} \frac{du}{\sqrt{u^2-1}}$$
$$= \cosh^{-1}(2) - \cosh^{-1}\left(\frac{4}{3}\right).$$

$$\int \frac{e^x}{1-e^{2x}}\, dx$$

Let $u = e^x$, then $du = e^x dx$. So

$$\begin{aligned}\int \frac{e^x}{1-e^{2x}} dx &= \int \frac{du}{1-u^2} \\ &= -\int \frac{du}{u^2-1} \\ &= -\tanh^{-1} u + C \\ &= -\tanh^{-1}(e^x) + C.\end{aligned}$$

- Find the limit.

$$\lim_{x \to (\pi/2)^+} \frac{\cos x}{1 - \sin x}$$

$$\begin{aligned}\lim_{x\to(\frac{\pi}{2})^+} \frac{\cos x}{1-\sin x} &= \lim_{x\to(\frac{\pi}{2})^+} \frac{-\sin x}{-\cos x} \qquad (\text{L'Hosp: } \lim_{x\to(\frac{\pi}{2})^+} \cos x = 0,\ \lim_{x\to(\frac{\pi}{2})^+}(1-\sin x) = 0) \\ &= \lim_{x\to(\frac{\pi}{2})^+} \tan x \\ &=\infty.\end{aligned}$$

$$\lim_{t \to 0} \frac{e^{2t} - 1}{\sin t}$$

$$\begin{aligned}\lim_{t\to 0} \frac{e^{2t}-1}{\sin t} &= \lim_{t\to 0} \frac{2e^{2t}}{\cos t} \qquad (\text{L'Hosp: } \lim_{t\to 0}(e^{2t}-1) = 0, \quad \lim_{t\to 0} \sin t = 0) \\ &=2.\end{aligned}$$

$$\lim_{x \to \infty} \sqrt{x}\, e^{-x/2}$$

$$\begin{aligned}\lim_{x\to\infty} \sqrt{x}e^{-\frac{x}{2}} &= \lim_{x\to\infty} \frac{\sqrt{x}}{e^{\frac{x}{2}}} \\ &= \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{\frac{1}{2}e^{\frac{x}{2}}} \qquad (\text{L'Hosp: } \lim_{x\to\infty} \sqrt{x} = \infty, \quad \lim_{x\to\infty} e^{\frac{x}{2}} = \infty) \\ &= \lim_{x\to\infty} \frac{1}{\sqrt{x}e^{\frac{x}{2}}} \\ &=0.\end{aligned}$$

- Evaluate the integral.

$$\int \cos \sqrt{x}\, dx$$

Let $u=\sqrt{x}$ so that $du=\frac{1}{2\sqrt{x}}dx$, i.e. $dx=2udu$, then

$$\begin{aligned}\int \cos\sqrt{x}dx=&\int \cos u\cdot 2udu\\ =&\int 2u(\sin u)'du\\ =&2u\sin u-2\int \sin udu\\ =&2u\sin u+2\cos u+C\\ =&2\sqrt{x}\sin(\sqrt{x})+\cos(\sqrt{x})+C\end{aligned}$$

$$\int \sin^{-1}x\,dx$$

$$\begin{aligned}\int \sin^{-1}xdx=&x\sin^{-1}x-\int x\left(\sin^{-1}x\right)'dx\\ =&x\sin^{-1}x-\int\frac{x}{\sqrt{1-x^2}}dx\\ =&x\sin^{-1}x+\frac{1}{2}\int\frac{1}{\sqrt{u}}du\quad (u=1-x^2,du=-2xdx)\\ =&x\sin^{-1}x+\sqrt{u}+C\\ =&x\sin^{-1}x+\sqrt{1-x^2}+C.\end{aligned}$$

$$\int_1^4 e^{\sqrt{x}}\,dx$$

Let $u=\sqrt{x}$, so again $dx=2udu$,

$$\begin{aligned}\int_1^4 e^{\sqrt{x}}dx=&\int_1^2 e^u\cdot 2udu\\ =&2\int_1^2 u\left(e^u\right)'du\\ =&2\left[ue^u\right]_1^2-2\int_1^2 e^udu\qquad \text{(by parts)}\\ =&4e^2-2e-2\left[e^u\right]_1^2\\ =&4e^2-2e-2\left[e^2-e\right]\\ =&2e^2.\end{aligned}$$

$$\int \sin^2x\,\cos^3x\,dx$$

$$\int \sin^2 x \cos^3 x \, dx$$

$$
\begin{aligned}
&= \int \sin^2 x \cos^2 x (\sin x)' dx \\
&= \int \sin^2 x (1 - \sin^2 x)(\sin x)' dx \\
&= \int u^2 (1 - u^2) du \qquad (u = \sin x) \\
&= \int u^2 - u^4 du \\
&= \frac{u^3}{3} - \frac{u^5}{5} + C \\
&= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.
\end{aligned}
$$

- Evaluate the integral using trigonometric substitution.

$$\int \frac{dx}{x^2\sqrt{4 - x^2}}$$

Let $x = 2\sin\theta$ $(-\frac{\pi}{2} < \theta < \frac{\pi}{2})$, $dx = 2\cos\theta d\theta$

$$
\begin{aligned}
\int \frac{dx}{x^2\sqrt{4-x^2}} dx &= \int \frac{2\cos\theta d\theta}{4\sin^2\theta \cdot 2\cos\theta} \\
&= \frac{1}{4}\int \csc^2\theta d\theta \\
&= -\frac{1}{4}\cot\theta + C \\
&= -\frac{\sqrt{4-x^2}}{4x} + C \qquad \text{(draw a triangle!)}
\end{aligned}
$$

$$\int \frac{x}{\sqrt{1 + x^2}} dx$$ (You may need: (sec x)' = sec x tan x)

(You may do a substitution $x = \tan\theta$, but this is not the best way.) Let $u = 1 + x^2$, so $du = 2x dx$,

$$
\begin{aligned}
\int \frac{x}{\sqrt{1+x^2}} dx &= \frac{1}{2}\int \frac{du}{\sqrt{u}} \\
&= \frac{1}{2} \cdot 2u^{\frac{1}{2}} + C \\
&= (1 + x^2)^{\frac{1}{2}} + C.
\end{aligned}
$$

$$\int_0^1 \sqrt{x^2 + 1} \, dx$$

Let $x = \tan\theta$, $(-\frac{\pi}{2} < \theta < \frac{\pi}{2})$, then $dx = \sec^2\theta d\theta$.

$$\begin{aligned}
\int_0^1 \sqrt{x^2+1}dx &= \int_{\tan^{-1}(0)}^{\tan^{-1}(1)} \sqrt{\tan^2\theta+1}\cdot\sec^2\theta d\theta \\
&= \int_0^{\frac{\pi}{4}} \sqrt{\sec^2\theta}\cdot\sec^2\theta d\theta \\
&= \int_0^{\frac{\pi}{4}} \sec\theta\cdot\sec^2\theta d\theta \qquad (\sec\theta \geq 0 \text{ on } (0,\frac{\pi}{4})) \\
&= \int_0^{\frac{\pi}{4}} \sec^3\theta d\theta
\end{aligned}$$

Consider

$$\begin{aligned}
\int \sec^3\theta d\theta &= \int \sec\theta(\tan\theta)' d\theta \\
&= \sec\theta\tan\theta - \int \sec\theta\tan^2\theta d\theta \qquad \text{(by parts)} \\
&= \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)d\theta \\
&= \sec\theta\tan\theta - \int \sec^3\theta d\theta + \int \sec\theta d\theta \\
&= \sec\theta\tan\theta - \int \sec^3\theta d\theta + \ln|\sec\theta+\tan\theta| + C.
\end{aligned}$$

So $\int \sec^3\theta d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + C_1$. Therefore

$$\begin{aligned}
\int_0^1 \sqrt{x^2+1}dx &= \int_0^{\frac{\pi}{4}} \sec^3\theta d\theta \\
&= \left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta|\right]_0^{\frac{\pi}{4}} \\
&= \frac{1}{2}\sec\frac{\pi}{4}\tan\frac{\pi}{4} + \frac{1}{2}\ln(\sec\frac{\pi}{4}+\tan\frac{\pi}{4}) \\
&= \frac{1}{\sqrt{2}} + \frac{1}{2}\ln\left(\frac{1}{\sqrt{2}}+1\right).
\end{aligned}$$

$\displaystyle\int_0^1 \frac{dx}{(x^2+1)^2}$ [You may need the formula: $\cos^2 x = \frac{1}{2}(1+\cos 2x)$]

Let $x = \tan\theta$, then $dx = \sec^2\theta d\theta$.

$$\begin{aligned}
\int_0^1 \frac{dx}{(x^2+1)^2} &= \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta d\theta}{(\sec^2\theta)^2} \\
&= \int_0^{\frac{\pi}{4}} \cos^2\theta d\theta \\
&= \int_0^{\frac{\pi}{4}} \frac{1}{2}(1+\cos 2\theta)d\theta \\
&= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}} \\
&= \frac{\pi}{8} + \frac{1}{4}.
\end{aligned}$$

- Write out the form of the partial fraction decomposition of the function . Do not determine the numerical values of the coefficients..

(a) $\dfrac{t^6+1}{t^6+t^3}$ (b) $\dfrac{x^5+1}{(x^2-x)(x^4+2x^2+1)}$

(a) $t^6+t^3 = t^3(t^3+1) = t^3(t+1)(t^2-t+1)$. Note that t^2-t+1 is irreducible as $(-1)^2-4\cdot1\cdot1<0$. So

$$\frac{t^6+1}{t^6+t^3} = \frac{t^6+1}{t^3(t+1)(t^2-t+1)} = \frac{A}{t}+\frac{B}{t^2}+\frac{C}{t^3}+\frac{D}{t+1}+\frac{Et+F}{t^2-t+1}.$$

(b) We have $x^2-x = x(x-1)$ and $x^4+2x^2+1 = (x^2+1)^2$, where x^2+1 is irreducible ($0^2-4\cdot1\cdot1<0$), so

$$\frac{x^5+1}{(x^2-x)(x^4+2x^2+1)} = \frac{x^5+1}{x(x-1)(x^2+1)^2} = \frac{A}{x}+\frac{B}{x-1}+\frac{Cx+D}{x^2+1}+\frac{Ex+F}{(x^2+1)^2}.$$

- Evaluate the integral.

$$\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,dx$$

(i) (Long division)

$$x^3-4x-10 = (x+1)(x^2-x-6)+(3x-4).$$

$$\begin{array}{r|rrrr}
 & & & x & +1 \\ \hline
x^2-x-6 \,\big) & x^3 & +0x^2 & -4x & -10 \\
 & x^3 & -x^2 & -6x & \\ \hline
 & & x^2 & +2x & -10 \\
 & & x^2 & -x & -6 \\ \hline
 & & & 3x & -4
\end{array}$$

Therefore $\frac{x^3-4x-10}{x^2-x-6} = x+1+\frac{3x-4}{x^2-x-6}$.

(ii) (Factorization) By inspection, $x^2-x-6 = (x-3)(x+2)$.

(iii) (Partial fraction) Let

$$\frac{3x-4}{x^2-x-6} = \frac{3x-4}{(x-3)(x+2)} = \frac{A}{x-3}+\frac{B}{x+2}$$

Then

$$\begin{aligned} 3x-4 =& A(x+2)+B(x-3) \\ =& (A+B)x+(2A-3B). \end{aligned}$$

So

$$\begin{cases} A+B=3 & (1) \\ 2A-3B=-4 & (2) \end{cases}$$

(1) $\times 2 -$ (2) gives $5B = 10$, so $B = 2$ and hence $A = 3-2 = 1$. Therefore

$$\begin{aligned}
\int_0^1 \frac{x^3-4x-10}{x^2-x-6}dx &= \int_0^1 \left(x+1+\frac{3x-4}{x^2-x-6}\right)dx \\
&= \int_0^1 \left(x+1+\frac{1}{x-3}+\frac{2}{x+2}\right)dx \\
&= \left[\frac{x^2}{2}+x+\ln|x-3|+2\ln|x+2|\right]_0^1 \\
&= \frac{3}{2}+\ln 2-\ln 3.
\end{aligned}$$

- Use the Comparison Theorem to determine whether the integral is convergent or divergent.

$$\int_0^\infty \frac{x}{x^3+1}\,dx$$

$$\begin{aligned}
\int_0^\infty \frac{x}{x^3+1}dx &= \int_0^1 \frac{x}{x^3+1}dx + \int_1^\infty \frac{x}{x^3+1}dx \\
&\le \int_0^1 \frac{x}{x^3+1}dx + \int_1^\infty \frac{x}{x^3}dx \\
&= \int_0^1 \frac{x}{x^3+1}dx + \int_1^\infty \frac{1}{x^2}dx.
\end{aligned}$$

Note that $\int_0^1 \frac{x}{x^3+1}dx$ and $\int_1^\infty \frac{1}{x^2}dx = 1$ are both finite. So by comparison, $\int_0^\infty \frac{x}{x^3+1}dx$ is convergent.

$$\int_1^\infty \frac{2+e^{-x}}{x}\,dx$$

$$\begin{aligned}
\int_1^\infty \frac{2+e^{-x}}{x}dx &\ge \int_1^\infty \frac{2}{x}dx \\
&= \lim_{t\to\infty}\int_1^t \frac{2}{x}dx \\
&= \lim_{t\to\infty}\left[2\ln|x|\right]_1^t \\
&= \infty.
\end{aligned}$$

So by comparison, $\int_1^\infty \frac{2+e^{-x}}{x}dx$ is divergent.

- Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

$$\int_1^{\infty} \frac{\ln x}{x}\, dx$$

$$\begin{aligned}\int_1^{\infty} \frac{\ln x}{x} dx &= \lim_{t\to\infty} \int_1^t \frac{\ln x}{x} dx \\ &= \lim_{t\to\infty} \int_1^t \ln x (\ln x)' dx \\ &= \lim_{t\to\infty} \int_0^{\ln t} u du \quad (u = \ln x) \\ &= \lim_{t\to\infty} \frac{1}{2}(\ln t)^2 \\ &= \infty.\end{aligned}$$

So the improper integral $\int_1^{\infty} \frac{\ln x}{x} dx$ is divergent.

$$\int_0^{\infty} \frac{e^x}{e^{2x}+3}\, dx$$

$$\begin{aligned}\int_0^{\infty} \frac{e^x}{e^{2x}+3} dx &= \lim_{t\to\infty} \int_0^t \frac{e^x}{e^{2x}+3} dx \\ &= \lim_{t\to\infty} \int_1^{e^t} \frac{1}{u^2+3} du \quad (u = e^x) \\ &= \lim_{t\to\infty} \left[\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{u}{\sqrt{3}}\right)\right]_1^{e^t} \\ &= \frac{1}{\sqrt{3}} \lim_{s\to\infty} \tan^{-1}(s) - \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \quad \left(\frac{e^t}{\sqrt{3}} \to \infty \text{ as } t \to \infty\right) \\ &= \frac{1}{\sqrt{3}} \cdot \frac{\pi}{2} - \frac{1}{\sqrt{3}} \cdot \frac{\pi}{6} \\ &= \frac{\pi}{3\sqrt{3}}.\end{aligned}$$

$$\int_0^{5} \frac{w}{w-2}\, dw$$

$$\int_0^5 \frac{w}{w-2}dw = \int_0^2 \frac{w}{w-2}dw + \int_2^5 \frac{w}{w-2}dw$$
$$= \lim_{t\to 2^-} \int_0^t \frac{w}{w-2}dw + \lim_{t\to 2^+} \int_t^5 \frac{w}{w-2}dw.$$

Consider

$$\lim_{t\to 2^-} \int_0^t \frac{w}{w-2}dw = \lim_{t\to 2^-} \int_0^t \left(1+\frac{2}{w-2}\right) dw$$
$$= \lim_{t\to 2^-} [w + 2\ln|w-2|]_0^t$$
$$= 2 + \lim_{t\to 2^-} \ln(2-t)$$
$$= -\infty.$$

Therefore the improper integral $\int_0^5 \frac{w}{w-2}dw$ is divergent.

- Find a power series representation for the function

$$f(x) = \frac{1}{x+10}$$

$$f(x) = \frac{1}{x+10} = \frac{1}{10} \cdot \frac{1}{1+\frac{x}{10}}$$
$$= \frac{1}{10} \cdot \frac{1}{1-(-\frac{x}{10})}$$
$$= \frac{1}{10}\left(1 + (-\frac{x}{10}) + (\frac{-x}{10})^2 + (\frac{-x}{10})^3 + \dots\right)$$
$$= \frac{1}{10} - \frac{x}{10^2} + \frac{x^2}{10^3} - \frac{x^3}{10^4} + \dots$$

- Find the Taylor Series of f centered at x=0:

$f(x) = \sinh x$

$f(x) = \sinh x$

$f'(x) = \cosh x \qquad \Rightarrow \qquad f(0) = 0$

$f''(x) = \sinh x \qquad f'(0) = 1$

$f^{(3)}(x) = \cosh x \qquad f''(0) = 0$

$\vdots \qquad f^{(3)}(0) = 1$

$\therefore$ Maclaurin Series of $\sinh x$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

- Find the Taylor Series of f centered at x=a:

  $f(x) = \cos x, \quad a = \pi$

$f(x) = \cos x$ $\qquad \therefore f(\pi) = -1$

$f'(x) = -\sin x$ $\qquad f'(\pi) = 0$

$f''(x) = -\cos x$ $\qquad f''(\pi) = 1$

$f^{(3)}(x) = \sin x$ $\qquad f^{(3)}(\pi) = 0$

$f^{(4)}(x) = \cos x = f^{(0)}(x) \ldots$ $\qquad f^{(4)}(\pi) = -1$ etc.

$\therefore$ Taylor series at π

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi)}{n!}(x-\pi)^n$$

$$= -\frac{1}{0!} + \frac{(x-\pi)^2}{2!} - \frac{(x-\pi)^4}{4!} + \frac{(x-\pi)^6}{6!} - \cdots$$

- **Evaluate the indefinite integral as an infinite series.**

  $$\int \frac{e^x - 1}{x}\, dx$$

$$e^x - 1 = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) - 1$$

$$= \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\frac{e^x - 1}{x} = \frac{1}{1!} + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots$$

$$\int \frac{e^x - 1}{x}dx = \int\left(\frac{1}{1!} + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \dots\right)dx$$

$$= C + \frac{x}{1 \cdot 1!} + \frac{x^2}{2 \cdot 2!} + \frac{x^3}{3 \cdot 3!} + \frac{x^4}{4 \cdot 4!} + \frac{x^5}{5 \cdot 5!} + \dots$$

- Use the first four non-zero terms of the Taylor series to find an approximate value of the integral:

  $$\int_0^1 x \cos(x^3)\,dx$$

47) $\displaystyle\int_0^1 x \cos x^3 \, dx$

$$= \int_0^1 x\left(1 - \frac{(x^3)^2}{2!} + \frac{(x^3)^4}{4!} - \frac{(x^3)^6}{6!} + \dots\right)dx$$

$$= \int_0^1 \left(x - \frac{x^7}{2!} + \frac{x^{13}}{4!} - \frac{x^{19}}{6!} + \dots\right)dx$$

$$= \left[\frac{x^2}{2} - \frac{x^8}{8 \cdot 2!} + \frac{x^{14}}{14 \cdot 4!} - \frac{x^{20}}{20 \cdot 6!} + \dots\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{8 \cdot 2!} + \frac{1}{14 \cdot 4!} + \frac{1}{20 \cdot 6!} + \dots$$

$$\approx 0.440.$$

- Use series to evaluate the limit.

  $$\lim_{x \to 0} \frac{x - \ln(1 + x)}{x^2}$$

) $\lim_{x\to 0} \frac{x - \ln(1+x)}{x^2}$

$= \lim_{x\to 0} \frac{x - (x - \frac{x^2}{2} + \frac{x^3}{3} - \dots)}{x^2}$

$= \lim_{x\to 0} \frac{\frac{x^2}{2} - \frac{x^3}{3} + \dots}{x^2}$

$= \lim_{x\to 0} \left(\frac{1}{2} - \frac{x}{3} + \dots\right)$

$= \frac{1}{2}$.

$$\lim_{x\to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5}$$

) $\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5}$

$= \lim_{x\to 0} \frac{(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots) - x + \frac{x^3}{6}}{x^5}$

$= \lim_{x\to 0} \frac{\frac{x^5}{5!} - \frac{x^7}{7!} + \dots}{x^5}$

$= \lim_{x\to 0} \left(\frac{1}{5!} - \frac{x^2}{7!} + \dots\right)$

$= \frac{1}{5!}$.

$$\lim_{x\to 0} \frac{1 - \cos x}{1 + x - e^x}$$

2) $\lim_{x\to 0} \frac{1 - \cos x}{1 + x - e^x}$

$= \lim_{x\to 0} \frac{1 - (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots)}{1 + x - (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots)}$

$= \lim_{x\to 0} \frac{\frac{x^2}{2!} - \frac{x^4}{4!} + \dots}{-\frac{x^2}{2!} - \frac{x^3}{3!} + \dots}$

$= \lim_{x\to 0} \frac{\frac{1}{2!} - \frac{x^2}{4!} + \dots}{-\frac{1}{2!} - \frac{x}{3!} + \dots}$

$= \frac{1}{2!} \Big/ -\frac{1}{2!}$

$= -1$.

- Find an equation of the tangent to the curve at the point corresponding to the given value of the parameter.

  $x = \sin^3\theta, \quad y = \cos^3\theta; \quad \theta = \pi/6$

$$\frac{dy}{dx} = \frac{d}{d\theta}(\cos^3\theta) \Big/ \frac{d}{d\theta}(\sin^3\theta)$$

$$= -3\cos^2\theta \sin\theta \,/\, 3\sin^2\theta\cos\theta$$

$$= -\frac{\cos\theta}{\sin\theta}.$$

At $\theta = \frac{\pi}{6}$, $\frac{dy}{dx} = -\frac{\sqrt{3}/2}{1/2} = -\sqrt{3}$.

At $\theta = \frac{\pi}{6}$, $(x, y) = \left(\sin^3\frac{\pi}{6}, \cos^3\frac{\pi}{6}\right)$

$$= \left(\frac{1}{2^3}, \left(\frac{\sqrt{3}}{2}\right)^3\right)$$

$$= \left(\frac{1}{8}, \frac{3\sqrt{3}}{8}\right)$$

$\therefore$ Tangent:

$$y - \frac{3\sqrt{3}}{8} = -\sqrt{3}\left(x - \frac{1}{8}\right).$$

- Find the area enclosed by the x-axis and the curve $x = 1 + e^t, y = t - t^2$.

At the intersection points:

$$0 = y = t - t^2$$

$$\therefore t(t-1) = 0 \quad \text{i.e. } t = 0 \text{ or } 1.$$

$$\text{Area} = \int_0^1 y \frac{dx}{dt}\, dt = \int_0^1 (t - t^2)e^t\, dt$$

$$= \left[(t - t^2)e^t\right]_0^1 - \int_0^1 e^t(1 - 2t)\, dt$$

$$= -\int_0^1 e^t\, dt + 2\int_0^1 e^t \cdot t\, dt$$

$$= \left[-e^t\right]_0^1 + 2\left[e^t \cdot t\right]_0^1 - 2\int_0^1 e^t\, dt$$

$$= (-e+1) + 2e - 2\left[e^t\right]_0^1$$

$$= e+1-2(e-1)$$

$$= 3-e$$

- Find the exact length of the curve.

  $x = 1 + 3t^2, \quad y = 4 + 2t^3, \quad 0 \leqslant t \leqslant 1$

$$\text{Length} = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^1 \sqrt{(6t)^2 + (6t^2)^2}\, dt$$

$$= 6\int_0^1 t\sqrt{1+t^2}\, dt$$

$$= 3\int_1^2 \sqrt{u}\, du \qquad \begin{array}{l} u = 1+t^2 \\ du = 2t\,dt \end{array}$$

$$= 2\left[u^{\frac{3}{2}}\right]_1^2$$

$$= 2(2\sqrt{2}-1)$$

$$= 4\sqrt{2}-2$$

- Identify the curve by finding a Cartesian equation for the curve.

  $r = 2\cos\theta$

$$\begin{aligned} r = 2\cos\theta &= \frac{2x}{r} \\ r^2 &= 2x \\ x^2 + y^2 &= 2x \\ (x^2 - 2x + 1) + y^2 &= 1 \\ (x-1)^2 + y^2 &= 1^2. \end{aligned}$$

The curve is a circle centered at $(1,0)$ with radius 1.

- Find a polar equation for the curve represented by the given Cartesian equation.

  $y = 1 + 3x$

$$\begin{aligned} y &= 1 + 3x \\ r\sin\theta &= 1 + 3r\cos\theta \\ r(\sin\theta - 3\cos\theta) &= 1. \end{aligned}$$

- Find the slope of the tangent line to the given polar curve at the point specified by the value of θ.

  $r = 2 - \sin\theta, \quad \theta = \pi/3$

$$\begin{aligned} \text{Slope} = \frac{dy}{dx} &= \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} \\ &= \frac{\frac{d}{d\theta}((2-\sin\theta)\sin\theta)}{\frac{d}{d\theta}((2-\sin\theta)\cos\theta)} \\ &= \frac{2\cos\theta - 2\sin\theta\cos\theta}{-2\sin\theta - \cos^2\theta + \sin^2\theta} \end{aligned}$$

So at $\theta = \frac{\pi}{3}$,

$$\begin{aligned} \text{Slope} &= \frac{2\cos\frac{\pi}{3} - 2\sin\frac{\pi}{3}\cos\frac{\pi}{3}}{-2\sin\frac{\pi}{3} - \cos^2\frac{\pi}{3} + \sin^2\frac{\pi}{3}} \\ &= \frac{2\cdot\frac{1}{2} - 2\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2}}{-2\cdot\frac{\sqrt{3}}{2} - \frac{1}{4} + \left(\frac{\sqrt{3}}{2}\right)^2} \\ &= \frac{2-\sqrt{3}}{1-2\sqrt{3}} \\ &= \frac{4-3\sqrt{3}}{11} \quad \text{(by rationalization)} \end{aligned}$$

- Find the exact length of the polar curve.

  $r = e^{2\theta}, \quad 0 \leqslant \theta \leqslant 2\pi$

$$\begin{aligned} \text{Length} &= \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{e^{4\theta} + (2e^{2\theta})^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{5e^{4\theta}}\, d\theta \\ &= \sqrt{5}\int_0^{2\pi} e^{2\theta}\, d\theta \\ &= \sqrt{5}\left[\frac{e^{2\theta}}{2}\right]_0^{2\pi} \\ &= \frac{\sqrt{5}}{2}\left(e^{4\pi} - 1\right). \end{aligned}$$

- Find the area of the region enclosed by one loop of the curve.
  $r = 4 \cos 3\theta$

$$\text{Area} = \frac{1}{2}\int_0^{2\pi} 16\cos^2 3\theta \, d\theta$$

$$= 8\int_0^{2\pi} \cos^2 3\theta \, d\theta$$

$$= 4\int_0^{2\pi} (1+\cos 6\theta)\, d\theta$$

$$= 4\left[\theta + \frac{\sin 6\theta}{6}\right]_0^{2\pi}$$

$$= 8\pi.$$

- Find the area of the region enclosed by one loop of the curve.

  $r^2 = \sin 2\theta$

$$\text{Area} = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2\theta \, d\theta$$

$$= \frac{1}{2}\left[-\frac{\cos 2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}.$$