Ex. 6.2

56 Let $x = \tan\theta$, $(-\frac{\pi}{2} < \theta < \frac{\pi}{2})$, then $dx = \sec^2\theta d\theta$.

$$\begin{aligned}\int_0^1 \sqrt{x^2+1}dx &= \int_{\tan^{-1}(0)}^{\tan^{-1}(1)} \sqrt{\tan^2\theta + 1}\cdot \sec^2\theta d\theta \\ &= \int_0^{\frac{\pi}{4}} \sqrt{\sec^2\theta}\cdot\sec^2\theta d\theta \\ &= \int_0^{\frac{\pi}{4}} \sec\theta\cdot\sec^2\theta d\theta \qquad (\sec\theta \geq 0 \text{ on } (0, \frac{\pi}{4})) \\ &= \int_0^{\frac{\pi}{4}} \sec^3\theta d\theta\end{aligned}$$

Consider

$$\begin{aligned}\int \sec^3\theta d\theta &= \int \sec\theta(\tan\theta)' d\theta \\ &= \sec\theta\tan\theta - \int \sec\theta\tan^2\theta d\theta \qquad \text{(by parts)} \\ &= \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)d\theta \\ &= \sec\theta\tan\theta - \int \sec^3\theta d\theta + \int \sec\theta d\theta \\ &= \sec\theta\tan\theta - \int \sec^3\theta d\theta + \ln|\sec\theta + \tan\theta| + C.\end{aligned}$$

So $\int \sec^3\theta d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C_1$. Therefore

$$\begin{aligned}\int_0^1 \sqrt{x^2+1}dx &= \int_0^{\frac{\pi}{4}} \sec^3\theta d\theta \\ &= \left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|\right]_0^{\frac{\pi}{4}} \\ &= \frac{1}{2}\sec\frac{\pi}{4}\tan\frac{\pi}{4} + \frac{1}{2}\ln(\sec\frac{\pi}{4} + \tan\frac{\pi}{4}) \\ &= \frac{1}{\sqrt{2}} + \frac{1}{2}\ln\left(\frac{1}{\sqrt{2}} + 1\right).\end{aligned}$$

58 Let $x = \tan\theta$, then $dx = \sec^2\theta d\theta$.

$$\begin{aligned}\int_0^1 \frac{dx}{(x^2+1)^2} &= \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta d\theta}{(\sec^2\theta)^2} \\ &= \int_0^{\frac{\pi}{4}} \cos^2\theta d\theta \\ &= \int_0^{\frac{\pi}{4}} \frac{1}{2}(1 + \cos 2\theta)d\theta \\ &= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}} \\ &= \frac{\pi}{8} + \frac{1}{4}.\end{aligned}$$

Ex. 6.3

1 (a) $\frac{A}{4x-3} + \frac{B}{2x+5}$.

(b) $5x^2 - 2x^3 = x^2(5-2x)$, so

$$\frac{10}{5x^2-2x^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{5-2x}.$$

3 (a) $x^5 + 4x^3 = x^3(x^2+4)$. Note that x^2+4 is irreducible as $0^2 - 4\cdot 4 < 0$, so

$$\frac{x^4+1}{x^5+4x^3} = \frac{x^4+1}{x^3(x^2+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+4}.$$

(b) Note that x^2-9 is **not** irreducible. In fact $x^2-9 = (x-3)(x+3)$, so

$$\frac{1}{(x^2-9)^2} = \frac{1}{(x-3)^2(x+3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{x+3} + \frac{D}{(x+3)^2}.$$

6 (a) $t^6+t^3 = t^3(t^3+1) = t^3(t+1)(t^2-t+1)$. Note that t^2-t+1 is irreducible as $(-1)^2-4\cdot1\cdot1<0$. So

$$\frac{t^6+1}{t^6+t^3} = \frac{t^6+1}{t^3(t+1)(t^2-t+1)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t^3} + \frac{D}{t+1} + \frac{Et+F}{t^2-t+1}.$$

(b) We have $x^2-x = x(x-1)$ and $x^4+2x^2+1 = (x^2+1)^2$, where x^2+1 is irreducible ($0^2-4\cdot1\cdot1<0$), so

$$\frac{x^5+1}{(x^2-x)(x^4+2x^2+1)} = \frac{x^5+1}{x(x-1)(x^2+1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2}.$$

16 (i) (Long division)

$$x^3 - 4x - 10 = (x+1)(x^2-x-6) + (3x-4).$$

$$\begin{array}{r|rrrr}
 & & x & +1 & \\
\hline
x^2-x-6 & x^3 & +0x^2 & -4x & -10 \\
 & x^3 & -x^2 & -6x & \\
\hline
 & & x^2 & +2x & -10 \\
 & & x^2 & -x & -6 \\
\hline
 & & & 3x & -4
\end{array}$$

Therefore $\frac{x^3-4x-10}{x^2-x-6} = x+1+\frac{3x-4}{x^2-x-6}$.

(ii) (Factorization) By inspection, $x^2-x-6 = (x-3)(x+2)$.

(iii) (Partial fraction) Let

$$\frac{3x-4}{x^2-x-6} = \frac{3x-4}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$

Then

$$\begin{aligned} 3x-4 =& A(x+2) + B(x-3) \\ =& (A+B)x + (2A-3B). \end{aligned}$$

So

$$\begin{cases} A + B = 3 & (1) \\ 2A - 3B = -4 & (2) \end{cases}$$

$(1) \times 2 - (2)$ gives $5B = 10$, so $B = 2$ and hence $A = 3 - 2 = 1$. Therefore

$$\begin{aligned} \int_0^1 \frac{x^3 - 4x - 10}{x^2 - x - 6} dx &= \int_0^1 \left(x + 1 + \frac{3x - 4}{x^2 - x - 6} \right) dx \\ &= \int_0^1 \left(x + 1 + \frac{1}{x - 3} + \frac{2}{x + 2} \right) dx \\ &= \left[\frac{x^2}{2} + x + \ln|x - 3| + 2\ln|x + 2| \right]_0^1 \\ &= \frac{3}{2} + \ln 2 - \ln 3. \end{aligned}$$

<u>Recommended</u>

<u>Ex 6.3</u> 2(a) $x^2+x-2=(x+2)(x-1)$

$$\therefore \frac{x}{x^2+x-2} = \frac{x}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}.$$

(b) x^2+x+2 is irreducible $(1^2-4\cdot1\cdot2=1-8=-7<0$

By long division, $x^2 = 1\cdot(x^2+x+2)-x-2$

$$\begin{array}{r} 1 \\ x^2+x+2 \,\overline{)\, x^2+0x+0} \\ x^2+x+2 \\ \hline -x-2 \end{array}$$

$$\therefore \frac{x^2}{x^2+x+2} = 1 - \frac{x+2}{x^2+x+2}.$$

4(a) $x^2-2x+1=(x-1)^2$.

Long division:

$$\begin{array}{r} x^2 \\ x^2-2x+1 \,\overline{)\, x^4-2x^3+x^2+2x-1} \\ x^4-2x^3+x^2 \\ \hline 2x-1 \end{array}$$

$$\therefore \frac{x^4-2x^3+x^2+2x-1}{x^2-2x+1} = x^2 + \frac{2x-1}{x^2-2x+1}$$

$$= x^2 + \frac{2x-1}{(x-1)^2}$$

$$= x^2 + \frac{A}{x-1} + \frac{B}{(x-1)^2}.$$

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4(b) $x^3+x^2+x = x(x^2+x+1)$

Easy to see that x^2+x+1 is irred. (Ex)

$$\therefore \frac{x^2-1}{x^3+x^2+x} = \frac{x^2-1}{x(x^2+x+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+x+1}.$$

5(a) By long division:

$$\begin{array}{r|l} & x^4 \qquad\quad +4x^2 \qquad\quad +16 \\ \hline x^2-4 & x^6+0x^5+0x^4+0x^3+0x^2+0x+0 \\ & x^6 \qquad\quad -4x^4 \\ \hline & \qquad\quad 4x^4 \\ & \qquad\quad 4x^4 \qquad -16x^2 \\ \hline & \qquad\qquad\qquad 16x^2 \\ & \qquad\qquad\qquad 16x^2-64 \\ \hline & \qquad\qquad\qquad\qquad 64 \end{array}$$

$$x^6 = (x^4+4x^2+16)(x^2-4)+64$$

$$\therefore \frac{x^6}{x^2-4} = x^4+4x^2+16+\frac{64}{x^2-4}$$

$$= x^4+4x^2-16+\frac{64}{(x-2)(x+2)}$$

$$= x^4+4x^2-16+\frac{A}{x-2}+\frac{B}{x+2}.$$

(b) Easy to see: x^2-x+1 is irred. (Ex) and
x^2+2 is irred.

$$\therefore \frac{x^4}{(x^2-x+1)(x^2+2)^2} = \frac{Ax+B}{x^2-x+1} + \frac{Cx+D}{x^2+2} + \frac{Ex+F}{(x^2+2)^2}.$$

7) $x^4 = (x^3 + x^2 + x + 1)(x-1) + 1$ by long division (Ex.)

$\therefore \int \frac{x^4}{x-1}\,dx = \int \left(x^3 + x^2 + x + 1 + \frac{1}{x-1}\right)dx$

$= \frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + x + \ln|x-1| + C.$

8) $\int \frac{3t-2}{t+1}\,dt = \int \frac{3t+3-5}{t+1}\,dt$

$= \int \left(3 - \frac{5}{t+1}\right)dt$

$= 3t - 5\ln|t+1| + C.$

9) Let $\frac{5x+1}{(2x+1)(x-1)} = \frac{A}{2x+1} + \frac{B}{x-1}$

$\therefore 5x+1 = A(x-1) + B(2x+1)$

Put $x=1$: $6 = 3B$ $\quad \therefore B = 2$

Put $x = -\frac{1}{2}$: $-\frac{3}{2} = -\frac{3}{2}A$ $\quad \therefore A = 1$

$\therefore \int \frac{5x+1}{(2x+1)(x-1)}\,dx = \int \left(\frac{1}{2x+1} + \frac{1}{x-1}\right)dx$

$= \frac{1}{2}\ln|2x+1| + \ln|x-1| + C.$

10) Let $\frac{y}{(y+4)(2y-1)} = \frac{A}{y+4} + \frac{B}{2y-1}$

$\therefore y = A(2y-1) + B(y+4).$

Put $y=-4$: $-4 = -9A$ $\quad \therefore A = 4/9$

Put $y = \frac{1}{2}$: $\frac{1}{2} = \frac{9}{2}B$ $\quad \therefore B = \frac{1}{9}$

$\therefore \int \frac{y}{(y+4)(2y-1)}\,dy = \frac{1}{9}\int \left(\frac{4}{y+4} + \frac{1}{2y-1}\right)dy = \frac{1}{9}\left[\ln|y+4| + \frac{1}{2}\ln|2y-1|\right] + C$

(11) $2x^2+3x+1 = (2x+1)(x+1)$.

$$\therefore \frac{2}{2x^2+3x+1} = \frac{2}{(2x+1)(x+1)}$$

$$= \frac{A}{2x+1} + \frac{B}{x+1}$$

$\therefore 2 = A(x+1) + B(2x+1)$

Put $x=-1$: $2 = -B$ $\therefore B = -2$

Put $x = -\frac{1}{2}$: $2 = \frac{A}{2}$ $\therefore A = 4$

$$\therefore \int_0^1 \frac{2}{2x^2+3x+1}\,dx = \int_0^1 \left(\frac{4}{2x+1} - \frac{2}{x+1}\right)dx$$

$$= \Big[2\ln|2x+1| - 2\ln|x+1|\Big]_0^1$$

$$= 2\ln 3 - 2\ln 2.$$

(12) $x^2-5x+6 = (x-2)(x-3)$.

$$\therefore \frac{x-4}{x^2-5x+6} = \frac{x-4}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$$

$\therefore x-4 = A(x-3) + B(x-2)$

Put $x=2$: $-2 = -A$ $\therefore A = 2$

Put $x=3$: $-1 = B$ $\therefore B = -1$

$$\therefore \int_0^1 \frac{x-4}{x^2-5x+6}\,dx = \int_0^1 \frac{2}{x-2} - \frac{1}{x-3}\,dx$$

$$= \Big[2\ln|x-2| - \ln|x-3|\Big]_0^1$$

$$= -3\ln 2 + \ln 3.$$

13) $\int \frac{ax}{x^2-bx}\,dx = \int \frac{a}{x-b}\,dx$

$= a\ln|x-b| + C.$

14) Let $\frac{1}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b}$

$\therefore 1 = A(x+b) + B(x+a)$

Put $x=-a$: $1 = (b-a)A \quad \therefore A = \frac{1}{b-a}.$

Put $x=-b$: $1 = (a-b)B \quad \therefore B = \frac{1}{a-b}.$ $\left(\begin{matrix}\text{assume} \\ a\neq b\end{matrix}\right)$

$\therefore \int \frac{1}{(x+a)(x+b)}dx = \frac{1}{b-a}\int\left(\frac{1}{x+a} - \frac{1}{x+b}\right)dx$

$= \frac{1}{b-a}\left(\ln|x+a| - \ln|x+b|\right) + C.$

if $a \neq b$.

If $b=a$, then $\int \frac{1}{(x+a)(x+b)}dx = \int \frac{1}{(x+a)^2}dx$

$= \int \frac{1}{u^2}\,du \quad (u = x+a)$

$= -\frac{1}{u} + C$

$= -\frac{1}{x+a} + C.$

15) Let $\frac{2x+3}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \quad \therefore 2x+3 = A(x+1) + B$

Put $x=-1$: $1 = B.$ Put $x=0.$ $3 = A+B = A+1 \therefore A=2$

$\therefore \int_0^1 \frac{2x+3}{(x+1)^2} = \int_0^1\left(\frac{2}{x+1} + \frac{1}{(x+1)^2}\right)dx = \left[2\ln|x+1| - \frac{1}{x+1}\right]_0^1 = 2\ln 2 + \frac{1}{2}$

17) Let $\dfrac{4y^2-7y-12}{y(y+2)(y-3)} = \dfrac{A}{y} + \dfrac{B}{y+2} + \dfrac{C}{y-3}$

$\therefore 4y^2-7y-12 = A(y+2)(y-3) + By(y-3) + Cy(y+2)$

Put $y=0$: $-12 = -6A \quad \therefore A = 2$

Put $y=-2$: $4(-2)^2-7(-2)-12 = 10B$

$10B = 18$

$\therefore B = \frac{9}{5}$

Put $y=3$: $4\cdot 3^2 - 7\cdot 3 - 12 = 3 = 15C$

$\therefore C = \frac{1}{5}$.

$$\therefore \int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}\,dy = \int_1^2 \left[\frac{2}{y} + \frac{9}{5(y+2)} + \frac{1}{5(y-3)}\right]$$

$$= \left[2\ln|y| + \frac{9}{5}\ln|y+2| + \frac{1}{5}\ln|y$$

$$= \frac{27}{5}\ln 2 - \frac{9}{5}\ln 3.$$

18) $\dfrac{x^2+2x-1}{x^3-x} = \dfrac{x^2+2x-1}{x(x-1)(x+1)} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{x+1}$

Direct computation gives: $A=1,\ B=1,\ C=-1$

$$\therefore \int \frac{x^2+2x-1}{x^3-x}\,dx = \int\left(\frac{1}{x} + \frac{1}{x-1} - \frac{1}{x+1}\right)dx$$

$$= \ln|x| + \ln|x-1| - \ln|x+1| + C_1.$$

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From now on, I will omit some computations.
(for the partial fractions)

19) $$\frac{x^2+1}{(x-3)(x-2)^2} = \frac{A}{x-3} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$\Rightarrow A = 10, \quad B = -9, \quad C = -5$ (Ex.)

$$\therefore \int \frac{x^2+1}{(x-3)(x-2)^2}\,dx = 10\ln|x-3| - 9\ln|x-2| + \frac{5}{x-2} + C_1$$

20) $$\frac{x^2-5x+16}{(2x+1)(x-2)^2} = \frac{A}{2x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$\Rightarrow A = \frac{11}{5}, \quad B = -\frac{3}{5}, \quad C = 6$ (Ex.)

$$\Rightarrow \int \frac{x^2-5x+16}{(2x+1)(x-2)^2} = \frac{11}{10}\ln|2x+1| - \frac{3}{5}\ln|x-2| - \frac{6}{x-2} + C_1$$

21) $$\frac{x^3+4}{x^2+4} = x - \frac{4(x-1)}{x^2+4}$$ by long division.

$$\therefore \int \frac{x^3+4}{x^2+4}\,dx = \int\left(x - \frac{4x}{x^2+4} + \frac{4}{x^2+4}\right)dx$$
$$= \frac{x^2}{2} - 2\ln|x^2+4| + 2\tan^{-1}\frac{x}{2} + C.$$

22) Omitted : $-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+2\ln|x|-\frac{1}{2}\ln(x^2+3)+C$.

23) Partial fraction (p.f.): $\frac{1}{x-1} - \frac{x+1}{x^2+9}$

$$\int\frac{10}{(x-1)(x^2+9)}dx = \int\left(\frac{1}{x-1} - \frac{x}{x^2+9} - \frac{1}{x^2+9}\right)dx$$

$$= \ln|x-1| - \frac{1}{2}\ln(x^2+9) - \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+C.$$

24) P.f.: $\frac{1}{x-1} - \frac{1}{(x-1)^2} - \frac{x+1}{x^2+1}$.

$$\int\frac{x^2-2x-1}{(x-1)^2(x^2+1)}dx = \int\left(\frac{1}{x-1} - \frac{1}{(x-1)^2} - \frac{x}{x^2+1} - \frac{1}{x^2+1}\right)dx$$

$$= \ln|x-1| + \frac{1}{x-1} - \frac{1}{2}\ln(x^2+1) - \tan^{-1}x + C$$

25) p.f.: $\frac{x}{x^2+1} + \frac{1}{x^2+2}$

$$\int\frac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)}dx = \frac{1}{2}\ln(x^2+1) + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+C.$$

26, 27, 28) omitted.

29) $x^3-1 = (x-1)(x^2+x+1)$.

$$\therefore \frac{1}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

Solving: $A=\frac{1}{3}$ $B=\frac{-1}{3}$, $C=-\frac{2}{3}$

$$\therefore \int\frac{1}{x^3-1}dx = \int\left(\frac{1}{3(x-1)} - \frac{x+2}{3(x^2+x+1)}\right)dx$$

..

Consider $\dfrac{x+2}{x^2+x+1} = \dfrac{x+\frac{1}{2}}{x^2+x+1} + \dfrac{3}{2(x^2+x+1)}$

$$\therefore -\frac{1}{3}\int \frac{x+2}{x^2+x+1}\,dx = -\frac{1}{3}\cdot\frac{1}{2}\ln|x^2+x+1|$$
$$-\frac{1}{2}\int \frac{1}{x^2+x+1}\,dx$$

Now consider $x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4}$

$$\therefore \int \frac{1}{x^2+x+1}\,dx = \int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}$$
$$= \int \frac{du}{u^2+\frac{3}{4}} \qquad \left(u = x+\tfrac{1}{2}\right)$$
$$= \frac{\sqrt{3}}{2}\tan^{-1}\frac{2u}{\sqrt{3}} + C$$
$$= \frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C.$$

$$\therefore \int \frac{1}{x^3-1}\,dx = \int\left[\frac{1}{3(x-1)} - \frac{1}{3}\,\frac{x+2}{x^2+x+1}\right]dx$$
$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln|x^2+x+1| - \frac{\sqrt{3}}{4}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

30) Long division gives:

$$\frac{x^5+x-1}{x^3+1} = x^2 - \frac{x^2-x+1}{x^3+1} = x^2 - \frac{x^2-x+1}{(x+1)(x^2-x+1)}$$
$$= x^2 - \frac{1}{x+1}$$

$$\therefore \int \frac{x^5+x-1}{x^3+1}\,dx = \int\left(x^2 - \frac{1}{x+1}\right)dx = \frac{x^3}{3} - \ln|x+1| + C.$$

31) $$\frac{1}{x(x^2+4)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}$$

Solving: $$\frac{1}{x(x^2+4)^2} = \frac{1}{4x} - \frac{x}{4(x^2+4)} \quad (\text{Ex.})$$

$$\therefore \int \frac{1}{x(x^2+4)^2}\,dx = \int\left(\frac{1}{4x} - \frac{x}{4(x^2+4)}\right)dx$$

$$= \frac{1}{4}\ln|x| - \frac{1}{8}\ln(x^2+4) + C.$$

32) $$\int \frac{x^4+3x^2+1}{x^5+5x^3+5x}\,dx$$

$$= \frac{1}{5}\int \frac{5x^4+15x^2+5}{x^3+5x^3+5x}\,dx$$

$$= \frac{1}{5}\int \frac{(x^5+5x^3+5x)'}{x^5+5x^3+5x}\,dx$$

$$= \frac{1}{5}\ln|x^5+5x^3+5x| + C.$$

33) Note that $\frac{x-3}{(x^2+2x+4)^2}$ is already in the form of partial fraction (x^2+2x+4 is irred.)

Similar to (29), the integral is

$$-\frac{4x+7}{6(x^2+2x+4)} - \frac{2}{3\sqrt{3}}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C.$$

34) Omitted.

35) Let $u=\sqrt{x}$ $\therefore du=\frac{1}{2\sqrt{x}}dx \Rightarrow dx=2u\,du$

$$\int_9^{16}\frac{\sqrt{x}}{x-4}dx=\int_3^4\frac{u}{u^2-4}\cdot 2u\,du$$
$$=\int_3^4\frac{2u^2}{u^2-4}du$$
$$=\int_3^4\frac{2u^2-8+8}{u^2-4}du$$
$$=\int_3^4\left(2+\frac{8}{u^2-4}\right)du$$

Let $\frac{8}{u^2-4}=\frac{8}{(u-2)(u+2)}=\frac{A}{u-2}+\frac{B}{u+2}$

Solving: $A=2$, $B=-2$.

$$\therefore \int_9^{16}\frac{\sqrt{x}}{x-4}dx=\int_3^4\left(2+\frac{2}{u-2}-\frac{2}{u+2}\right)du$$
$$=\Big[2u+2\ln|u-2|-2\ln|u+2|\Big]_3^4$$
$$=2+2\ln 2-2\ln 6+2\ln 5$$
$$=2-2\ln 3+2\ln 5.$$

36) Let $u=x^{\frac{1}{3}}$ $\therefore du=\frac{1}{3}x^{-\frac{2}{3}}dx=\frac{1}{3u^2}dx$ ie $dx=3u^2du$

$$\therefore \int_0^1\frac{1}{1+x^{\frac{1}{3}}}dx=\int_0^1\frac{1}{1+u}\cdot 3u^2du.$$
$$=\int_0^1\left(3u-3+\frac{3}{u+1}\right)du$$
$$=\left[\frac{3u^2}{2}-3u+3\ln|u+1|\right]_0^1=-\frac{3}{2}+3\ln 2.$$

37) $\int \frac{x^3}{(x^2+1)^{1/3}}\,dx$

$= \int \frac{x^3+x-x}{(x^2+1)^{1/3}}\,dx$

$= \int \left[\frac{x(x^2+1)}{(x^2+1)^{\frac{1}{3}}} - \frac{x}{(x^2+1)^{1/3}}\right] dx$

$= \int x(x^2+1)^{2/3} - \frac{x}{(x^2+1)^{1/3}}\,dx$

$= \frac{1}{2}\int (x^2+1)^{2/3}\,d(x^2+1) - \frac{1}{2}\int \frac{d(x^2+1)}{(x^2+1)^{1/3}}$

$= \frac{3}{10}(x^2+1)^{\frac{5}{3}} - \frac{3}{4}(x^2+1)^{2/3} + C.$

38) let $u = \sqrt{x}$ $\therefore$ $x = u^2$ and $dx = 2u\,du$

$\int_{\frac{1}{3}}^{3} \frac{\sqrt{x}}{x^2+x}\,dx = \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{u \cdot 2u\,du}{u^4+u^2}$

$= 2\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{u^2}{u^2(u^2+1)}\,du$

$= 2\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{u^2+1}\,du$

$= 2\left[\tan^{-1}u\right]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}$

$= 2\left[\frac{\pi}{3} - \frac{\pi}{6}\right]$

$= \frac{\pi}{3}.$

39) Let $u = e^x \quad \therefore du = e^x dx$

$$\therefore \int \frac{e^{2x}}{e^{2x}+3e^x+2}dx = \int \frac{u\,du}{u^2+3u+2}$$

$$= \int \frac{u}{(u+1)(u+2)}du$$

$$= \int \left(\frac{-1}{u+1} + \frac{2}{u+2}\right)du \quad (p.f.)$$

$$= -\ln|u+1| + 2\ln|u+2| + C$$

$$= -\ln|e^x+1| + 2\ln|e^x+2| + C$$

$$= -\ln(e^x+1) + 2\ln(e^x+2) + C$$

40) Let $u = \cos x \quad \therefore du = -\sin x\,dx$

$$\int \frac{\sin x\,dx}{\cos^2 x - 3\cos x} = \int \frac{-du}{u^2-3u}$$

$$= \int \frac{-1}{u(u-3)}du$$

$$= \int \left[\frac{1}{3u} - \frac{1}{3(u-3)}\right]du$$

$$= \frac{1}{3}\ln|u| - \frac{1}{3}\ln|u-3| + C$$

$$= \frac{1}{3}\ln|\cos x| - \frac{1}{3}\ln|\cos x - 3| + C$$

$$= \frac{1}{3}\ln|\cos x| - \frac{1}{3}\ln(3-\cos x) + C$$

p.16