Ex 6.1

9 Let $u = 2x + 1$, so $du = 2dx$,

$$\begin{aligned}\int \ln(2x+1)dx =&\frac{1}{2}\int \ln u du\\ =&\frac{1}{2}\int (\ln u)u' du\\ =&\frac{1}{2}u\ln u - \frac{1}{2}\int u(\ln u)' du \qquad \text{(by parts)}\\ =&\frac{1}{2}u\ln u - \frac{1}{2}\int 1du\\ =&\frac{1}{2}u\ln u - u + C\\ =&\frac{2x+1}{2}\ln(2x+1) - \frac{1}{2}(2x+1) + C\\ =&\frac{2x+1}{2}\ln(2x+1) - x + C_1.\end{aligned}$$

12

$$\begin{aligned}\int \sin^{-1} x dx =& \int \sin^{-1} x \cdot x' dx\\ =&x\sin^{-1} x - \int x \cdot (\sin^{-1} x)' dx \qquad \text{(by parts)}\\ =&x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}dx\\ =&x\sin^{-1} x + \frac{1}{2}\int \frac{du}{\sqrt{u}}dx \qquad (u = 1 - x^2, du = -2xdx)\\ =&x\sin^{-1} x + u^{\frac{1}{2}} + C\\ =&x\sin^{-1} x + (1-x^2)^{\frac{1}{2}} + C.\end{aligned}$$

27 Let $u = \sqrt{x}$, then $du = \frac{1}{2\sqrt{x}}dx$, i.e. $dx = 2udu$,

$$\begin{aligned}\int \cos\sqrt{x}dx =& \int \cos u \cdot 2udu\\ =&2\int u\cos udu\\ =&2\int u(\sin u)' du\\ =&2u\sin u - 2\int \sin udu \qquad \text{(by parts)}\\ =&2u\sin u + 2\cos u + C\\ =&2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C.\end{aligned}$$

30 Let $u = \sqrt{x}$, so again $dx = 2udu$,

$$\begin{aligned}\int_1^4 e^{\sqrt{x}} dx =& \int_1^2 e^u \cdot 2udu \\ =&2\int_1^2 u\left(e^u\right)' du \\ =&2\left[ue^u\right]_1^2 - 2\int_1^2 e^u du \qquad \text{(by parts)} \\ =&4e^2 - 2e - 2\left[e^u\right]_1^2 \\ =&4e^2 - 2e - 2\left[e^2 - e\right] \\ =&2e^2.\end{aligned}$$

Ex 6.2

1

$$\begin{aligned}\int \sin^2 x \cos 3x dx =& \int \sin^2 x \cos^2 x (\sin x)' dx \\ =& \int \sin^2 x (1 - \sin^2 x)(\sin x)' dx \\ =& \int u^2(1-u^2) du \qquad (u = \sin x) \\ =& \int u^2 - u^4 du \\ =& \frac{u^3}{3} - \frac{u^5}{5} + C \\ =& \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.\end{aligned}$$

24 Let $u = \sec x$, so $du = \sec x \tan x dx$,

$$\begin{aligned}\int \tan^5 x \sec^3 x dx =& \int \tan^4 x \sec^2 x du \\ =& \int (\sec^2 x - 1)^2 u^2 du \\ =& \int (u^2-1)^2 u^2 du \\ =& \int (u^6 - 2u^4 + u^2) du \\ =& \frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3} + C \\ =& \frac{\sec^7 x}{7} - \frac{2\sec^5 x}{5} + \frac{\sec^3 x}{3} + C.\end{aligned}$$

39 Let $x = 2\sin\theta$ $(-\frac{\pi}{2} < \theta < \frac{\pi}{2})$, $dx = 2\cos\theta d\theta$

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{4-x^2}}dx = & \int \frac{2\cos\theta d\theta}{4\sin^2\theta \cdot 2\cos\theta} \\
= & \frac{1}{4}\int \csc^2\theta d\theta \\
= & -\frac{1}{4}\cot\theta + C \\
= & -\frac{\sqrt{4-x^2}}{4x} + C \qquad \text{(draw a triangle!)}
\end{aligned}$$

52 (You may do a substitution $x = \tan\theta$, but this is not the best way.) Let $u = 1 + x^2$, so $du = 2xdx$,

$$\begin{aligned}
\int \frac{x}{\sqrt{1+x^2}}dx = & \frac{1}{2}\int \frac{du}{\sqrt{u}} \\
= & \frac{1}{2} \cdot 2u^{\frac{1}{2}} + C \\
= & (1+x^2)^{\frac{1}{2}} + C.
\end{aligned}$$

Recommended

3) $\int x \cos 5x \, dx$

$= \int x \left(\frac{\sin 5x}{5}\right)' dx$

$= x \frac{\sin 5x}{5} - \int \frac{\sin 5x}{5} dx$ (b.p.)

$= \frac{1}{5} x \sin 5x - \frac{1}{5} \int \sin 5x \, dx$

$= \frac{1}{5} x \sin 5x + \frac{\cos 5x}{25} + C.$

4) $\int y e^{0.2y} dy$

$= \int y \left(\frac{e^{0.2y}}{0.2}\right)' dy$

$= y \frac{e^{0.2y}}{0.2} - \int \frac{e^{0.2y}}{0.2} dy$ (b.p.)

$= 5 y e^{0.2y} - 5 \int e^{0.2y} dy$

$= 5 y e^{0.2y} - 5 \frac{e^{0.2y}}{0.2} + c$

$= 5 y e^{0.2y} - 25 e^{0.2y} + C.$

5) $\int t e^{-3t} dt = \int t \left(\frac{e^{-3t}}{-3}\right)' dt$

$= -\frac{1}{3} t e^{-3t} + \frac{1}{3} \int e^{-3t} dt$

$= -\frac{1}{3} t e^{-3t} - \frac{1}{9} e^{-3t} + C.$

6) $\int (x-1)\sin \pi x \, dx$

$= \int x \sin \pi x \, dx - \int \sin \pi x \, dx$

$= \int x \sin \pi x \, dx + \frac{\cos \pi x}{\pi}$

Consider $\int x \sin \pi x \, dx = \int x \left(\frac{-\cos \pi x}{\pi} \right)' dx$

$= -\frac{x \cos \pi x}{\pi} + \int \frac{\cos \pi x}{\pi} \, dx$ (b.p.)

$= -\frac{1}{\pi} x \cos \pi x + \frac{1}{\pi^2} \sin \pi x + C$

$\therefore \int (x-1)\sin \pi x \, dx = -\frac{1}{\pi} x \cos \pi x + \frac{1}{\pi^2} \sin \pi x + \frac{1}{\pi} \cos \pi x + C.$

7) $\int (x^2+2x)\cos x \, dx = \int x^2 \cos x \, dx + 2\int x \cos x \, dx$

Consider $\int x \cos x \, dx = \int x (\sin x)' \, dx$

$= x \sin x - \int \sin x \, dx$ (b.p.)

$= x \sin x + \cos x + C.$

Consider $\int x^2 \cos x \, dx = \int x^2 (\sin x)' \, dx$

$= x^2 \sin x - \int \sin x \cdot 2x \, dx$ (b.p.)

$= x^2 \sin x - 2\int x \sin x \, dx$

$= x^2 \sin x + 2\int x (\cos x)' \, dx$

$= x^2 \sin x + 2x\cos x - 2\int \cos x \, dx$

$= x^2 \sin x + 2x \cos x - 2\sin x + C.$

$\therefore \int (x^2+2x)\cos x \, dx = x^2 \sin x + 2x\cos x - 2\sin x + 2x \sin x + 2\cos x + C$

8) $\int t^2 \sin\beta t \, dt = \int t^2 \left(\frac{-\cos\beta t}{\beta}\right)' dt$ (assume $\beta \neq 0$)

$$= -\frac{1}{\beta} t^2 \cos\beta t + \frac{1}{\beta}\int (\cos\beta t)\cdot 2t \, dt \quad \text{(b.p.)}$$

$$= -\frac{1}{\beta} t^2 \cos\beta t + \frac{2}{\beta}\int t\left(\frac{\sin\beta t}{\beta}\right)' dt$$

$$= -\frac{1}{\beta} t^2 \cos\beta t + \frac{2}{\beta^2} t\sin\beta t - \frac{2}{\beta^2}\int \sin\beta t \, dt \quad \text{(b.p.)}$$

$$= -\frac{1}{\beta} t^2 \cos\beta t + \frac{2}{\beta^2} t\sin\beta t + \frac{2}{\beta^3}\cos\beta t + C.$$

10) $\int p^5 \ln p \, dp$

$$= \int \ln p \left(\frac{p^6}{6}\right)' dp$$

$$= \frac{1}{6} p^6 \ln p - \int \frac{p^6}{6} (\ln p)' dp \quad \text{(b.p.)}$$

$$= \frac{1}{6} p^6 \ln p - \frac{1}{6}\int p^5 \, dp$$

$$= \frac{1}{6} p^6 \ln p - \frac{1}{36} p^6 + C.$$

11) $\int \tan^{-1}(4t)\, dt$

$$= \int \tan^{-1}(4t)\cdot t' \, dt$$

$$= t\tan^{-1}(4t) - \int t\,(\tan^{-1} 4t)' dt \quad \text{(b.p.)}$$

$$= t\tan^{-1} 4t - \int \frac{4t}{1+16t^2} dt$$

$$= t\tan^{-1} 4t - \frac{1}{8}\int \frac{du}{u} \quad (u = 1+16t^2, \; du = 32t\,dt)$$

$$= t\tan^{-1} 4t - \frac{1}{8}\ln|u| + C$$

$$= t\tan^{-1} 4t - \frac{1}{8}\ln|1+16t^2| + C = t\tan^{-1} 4t - \frac{1}{8}\ln(1+16t^2) + C$$

13) $\int e^{2\theta} \sin 3\theta \, d\theta$

$= \int \sin 3\theta \left(\frac{e^{2\theta}}{2}\right)' d\theta$

$= \frac{1}{2} \sin 3\theta \cdot e^{2\theta} - \frac{1}{2}\int e^{2\theta} \cdot 3\cos 3\theta \, d\theta$ (b.p.)

$= \frac{1}{2} e^{2\theta} \sin 3\theta - \frac{3}{2}\int \cos 3\theta \left(\frac{e^{2\theta}}{2}\right)' d\theta$

$= \frac{1}{2} e^{2\theta} \sin 3\theta - \frac{3}{4} e^{2\theta} \cos 3\theta + \frac{3}{4}\int e^{2\theta} \cdot (-3\sin 3\theta) d\theta$

(b.p)

$= \frac{1}{2} e^{2\theta} \sin 3\theta - \frac{3}{4} e^{2\theta} \cos 3\theta - \frac{9}{4}\int e^{2\theta} \sin 3\theta \, d\theta$

$\therefore \left(1 + \frac{9}{4}\right) \int e^{2\theta} \sin 3\theta \, d\theta = \frac{1}{2} e^{2\theta} \sin 3\theta - \frac{3}{4} e^{2\theta} \cos 3\theta + C$

$\Rightarrow \int e^{2\theta} \sin 3\theta \, d\theta = \frac{4}{13}\left[\frac{1}{2} e^{2\theta} \sin 3\theta - \frac{3}{4} e^{2\theta} \cos 3\theta\right] + C_1$

$= \frac{2}{13} e^{2\theta} \sin 3\theta - \frac{3}{13} e^{2\theta} \cos 3\theta + C_1$.

14) $\int e^{-\theta} \cos 2\theta \, d\theta$

$= \int \cos 2\theta (-e^{-\theta})' d\theta$

$= -e^{-\theta} \cos 2\theta + \int e^{-\theta} \cdot (-2\sin 2\theta) d\theta$ (b.p.)

$= -e^{-\theta} \cos 2\theta - 2\int \sin 2\theta \cdot e^{-\theta} d\theta$

$= -e^{-\theta} \cos 2\theta + 2\int \sin 2\theta (e^{-\theta})' d\theta$

$= -e^{-\theta} \cos 2\theta + 2e^{-\theta} \sin 2\theta - 2\int e^{-\theta} \cdot 2\cos 2\theta \, d\theta$

$= -e^{-\theta} \cos 2\theta + 2e^{-\theta} \sin 2\theta - 4\int e^{-\theta} \cos 2\theta \, d\theta$

$$\therefore (1+4)\int e^{-\theta}\cos 2\theta \, d\theta = -e^{-\theta}\cos 2\theta + 2e^{-\theta}\sin 2\theta + C$$

$$\Rightarrow \int e^{-\theta}\cos 2\theta \, d\theta = -\frac{1}{5}e^{-\theta}\cos 2\theta + \frac{2}{5}e^{-\theta}\sin 2\theta + C_1$$

15) Let $u = 1+2x$. $\therefore du = 2dx$.

$$\therefore \int \frac{xe^{2x}}{(1+2x)^2}dx = \frac{1}{4}\int \frac{(u-1)e^{u-1}}{u^2}du$$

$$= \frac{1}{4e}\left(\int \frac{e^u}{u}du - \int \frac{e^u}{u^2}du\right)$$

Consider $\int \frac{e^u}{u^2}du = \int e^u\left(-\frac{1}{u}\right)' du$

$$= -\frac{e^u}{u} + \int \frac{e^u}{u}du \quad \text{(b.p.)}$$

$$\therefore \int \frac{xe^{2x}}{(1+2x)^2}dx = \frac{1}{4e}\left(\int \frac{e^u}{u}du - \left(-\frac{e^u}{u} + \int \frac{e^u}{u}du\right)\right) + c$$

$$= \frac{1}{4e} \cdot \frac{e^u}{u} + c$$

$$= \frac{e^{2x}}{4(1+2x)} + C.$$

16) $\int t^3e^t dt = t^3e^t - \int e^t \cdot 3t^2 dt \quad$ (b.p.)

$$= t^3e^t - 3\int t^2(e^t)' dt$$

$$= t^3e^t - 3t^2e^t + 3\int e^t \cdot 2t\,dt \quad \text{(b.p)}$$

$$= t^3e^t - 3t^2e^t + 6\int t(e^t)' dt$$

$$= t^3e^t - 3t^2e^t + 6te^t - 6\int e^t\,dt \quad (\text{b.p.})$$
$$= t^3e^t - 3t^2e^t + 6te^t - 6e^t + C.$$

17) $$\int_0^{\frac{1}{2}} x\cos\pi x\,dx = \int_0^{\frac{1}{2}} x\left(\frac{\sin\pi x}{\pi}\right)'dx$$
$$= \left[\frac{x\sin\pi x}{\pi}\right]_0^{\frac{1}{2}} - \frac{1}{\pi}\int_0^{\frac{1}{2}}\sin\pi x\,dx \quad (\text{b.p.})$$
$$= \frac{1}{2\pi} + \frac{1}{\pi}\left[\frac{\cos\pi x}{\pi}\right]_0^{\frac{1}{2}}$$
$$= \frac{1}{2\pi} + \frac{1}{\pi^2}\left[0 - 1\right]$$
$$= \frac{1}{2\pi} - \frac{1}{\pi^2}.$$

18) $$\int_0^1 (x^2+1)e^{-x}\,dx = \int_0^1 x^2e^{-x}\,dx + \int_0^1 e^{-x}\,dx$$
$$= \int_0^1 x^2e^{-x}\,dx + \left[-e^{-x}\right]_0^1$$
$$= \int_0^1 x^2e^{-x} - e^{-1} + 1$$

Consider $$\int_0^1 x^2e^{-x}\,dx = \left[x^2e^{-x}\right]_0^1 - \int_0^1(-e^{-x})\cdot 2x\,dx \quad (\text{b.p.})$$
$$= e^{-1} + 2\int_0^1 e^{-x}x\,dx$$
$$= e^{-1} + 2\int_0^1 x(-e^{-x})'\,dx$$
$$= e^{-1} - \left[2xe^{-x}\right]_0^1 + 2\int_0^1 e^{-x}\,dx$$
$$= e^{-1} - 2e^{-1} + 2\left[-e^{-x}\right]_0^1$$
$$= -e^{-1} - 2e^{-1} + 2 = -3e^{-1} + 2.$$

$$\therefore \int_0^1 (x^2+1)e^{-x}\,dx = (-3e^{-1}+2) - e^{-1} + 1$$
$$= -4e^{-1} + 3.$$

19) $\displaystyle\int_1^3 r^3 \ln r\,dr = \int_1^3 \ln r\left(\frac{r^4}{4}\right)' dr$

$$= \left[\ln r \cdot \frac{r^4}{4}\right]_1^3 - \frac{1}{4}\int_1^3 r^4 \cdot \frac{1}{r}\,dr \quad (\text{i.p.})$$
$$= \frac{81}{4}\ln 3 - \frac{1}{4}\int_1^3 r^3\,dr$$
$$= \frac{81}{4}\ln 3 - \frac{1}{4}\left[\frac{r^4}{4}\right]_1^3$$
$$= \frac{81\ln 3}{4} - 5.$$

20) $\displaystyle\int_4^9 \frac{\ln y}{\sqrt{y}}\,dy = \int_4^9 \ln y\,(2\sqrt{y})'\,dy$

$$= \left[2\sqrt{y}\ln y\right]_4^9 - 2\int_4^9 \sqrt{y}\cdot\frac{1}{y}\,dy \quad (\text{i.p.})$$
$$= 13\ln 3 - 8\ln 2 - 2\int_4^9 y^{-\frac{1}{2}}\,dy$$
$$= 13\ln 3 - 8\ln 2 - 4\left[\sqrt{y}\right]_4^9$$
$$= 13\ln 3 - 8\ln 2 - 4.$$

21) $\displaystyle\int_0^1 t\cosh t\,dt = \int_0^1 t(\sinh t)'\,dt$

$$= \left[t\sinh t\right]_0^1 - \int_0^1 \sinh t\,dt$$
$$= \sinh 1 - \left[\cosh t\right]_0^1$$
$$= \sinh 1 - \cosh 1 + 1$$

22) $\int_1^{\sqrt{3}} \tan^{-1}\left(\frac{1}{x}\right) dx$

$= \int_1^{\sqrt{3}} \tan^{-1}\left(\frac{1}{x}\right) x' \, dx$

$= \left[x \tan^{-1} \frac{1}{x}\right]_1^{\sqrt{3}} - \int_1^{\sqrt{3}} x \cdot \frac{1}{1+\frac{1}{x^2}} \left(\frac{-1}{x^2}\right) dx$

$= \sqrt{3} \cdot \frac{\pi}{6} - \frac{\pi}{4} + \int_1^{\sqrt{3}} \frac{x}{x^2+1} \, dx$

$= \frac{\sqrt{3}\pi}{6} - \frac{\pi}{4} + \frac{1}{2} \int_1^{\sqrt{3}} \frac{(x^2+1)'}{x^2+1} dx$

$= \frac{\sqrt{3}\pi}{6} - \frac{\pi}{4} + \frac{1}{2} \left[\ln|x^2+1|\right]_1^{\sqrt{3}}$

$= \frac{\sqrt{3}\pi}{6} - \frac{\pi}{4} + \frac{1}{2} \ln 2.$

23) Similar to 22.

24) $\int_0^1 \frac{r^3}{\sqrt{4+r^2}} dr.$ Let $r = 2\tan\theta \quad \left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$

$\therefore dr = 2\sec^2\theta \, d\theta$

$\int_0^1 \frac{r^3}{\sqrt{4+r^2}} dr = \int_0^{\tan^{-1}\frac{1}{2}} \frac{8\tan^3\theta \cdot 2\sec^2\theta \, d\theta}{\sqrt{4(1+\tan^2\theta)}}$

$= 8 \int_0^{\tan^{-1}\frac{1}{2}} \frac{\tan^3\theta \sec^2\theta \, d\theta}{\sec\theta}$ $\quad \left(1+\tan^2\theta = \sec^2\theta, \ \sec\theta > 0 \text{ on } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

$= 8 \int_0^{\tan^{-1}\frac{1}{2}} \tan^3\theta \sec\theta \, d\theta$

Let $u = \sec\theta$, then $\int_0^{\tan^{-1}\frac{1}{2}} \tan^3\theta \sec\theta \, d\theta$

$$= \int_1^{\sec\alpha} \tan^2\theta \, du$$

$du = \sec\theta \tan\theta \, d\theta$

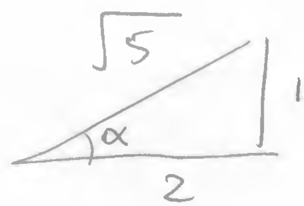
$\alpha = \tan^{-1}\frac{1}{2}$

$$= \int_1^{\sec\alpha} (\sec^2\theta - 1)\, du$$

$$= \int_1^{\sec\alpha} (u^2 - 1)\, du$$

$$= \left[\frac{u^3}{3} - u\right]_1^{\sec\alpha}$$

$$= \frac{\sec^3\alpha}{3} - \sec\alpha + \frac{2}{3}.$$



$\alpha = \tan^{-1}\frac{1}{2}.$

$$= \frac{5\sqrt{5}}{6} - \frac{\sqrt{5}}{2} + \frac{2}{3}$$

$\cos\alpha = \frac{2}{\sqrt{5}}$

$\sec\alpha = \frac{\sqrt{5}}{2}$

25) $\int_1^2 (\ln x)^2 \, dx$

$$= \left[x(\ln x)^2\right]_1^2 - \int_1^2 x \cdot 2\ln x \cdot \frac{1}{x}\, dx \qquad \text{(b.p.)}$$

$$= 2(\ln 2)^2 - 2\int_1^2 \ln x \, dx$$

$$= 2(\ln 2)^2 - 2\left[x\ln x\right]_1^2 + 2\int_1^2 x \cdot \frac{1}{x}\, dx \qquad \text{(b.p.)}$$

$$= 2(\ln 2)^2 - 4\ln 2 + 2\cdot(2-1)$$

$$= 2(\ln 2)^2 - 4\ln 2 + 2$$

26) $\int_0^t e^s \sin(t-s)\, ds = -\int_0^1 e^{s-t} \cdot e^t \cdot \sin(s-t)\, ds$

$$= -e^t \int_{-t}^{1-t} e^u \sin u \, du \qquad (u = s-t)$$

Consider the indefinite integral

$$\int e^u \sin u\, du = e^u \sin u - \int e^u (\sin u)' du \quad \text{(b.p.)}$$
$$= e^u \sin u - \int (e^u)' \cos u\, du$$
$$= e^u \sin u - e^u \cos u + \int e^u (\cos u)' du \quad \text{(b.p.)}$$
$$= e^u \sin u - e^u \cos u - \int e^u \sin u\, du$$

$$\therefore\ 2\int e^u \sin u\, du = e^u \sin u - e^u \cos u + C$$
$$\Rightarrow \int e^u \sin u\, du = \frac{1}{2} e^u \sin u - \frac{1}{2} e^u \cos u + C_1$$

$$\therefore \int_0^t e^s \sin(t-s)\, ds$$
$$= -e^t \int_{-t}^{1-t} e^u \sin u\, du$$
$$= -\frac{e^t}{2} \Big[e^u \sin u - e^u \cos u \Big]_{-t}^{1-t}$$
$$= -\frac{e^t}{2} e^{1-t} (\sin(1-t) - \cos(1-t)) + \frac{e^t}{2} \cdot e^{-t} (\sin(-t) - \cos(-t))$$
$$= -\frac{e}{2} \Big[\sin(1-t) - \cos(1-t) \Big] - \frac{1}{2} (\sin t + \cos t).$$

28) let $u = t^2$ $\therefore du = 2t\,dt$

$$\int t^3 e^{-t^2} dt = \int t^2 e^{-t^2} \cdot t\,dt$$
$$= \int u e^{-u} \frac{du}{2}$$
$$= \frac{1}{2} \int u e^{-u} du$$
$$= \frac{1}{2} \int u(-e^{-u})' du$$
$$= -\frac{1}{2} u e^{-u} + \frac{1}{2} \int e^{-u} du$$
$$= -\frac{1}{2} u e^{-u} - \frac{1}{2} e^{-u} + C$$
$$= -\frac{t^2 e^{-t^2}}{2} - \frac{1}{2} e^{-t^2} + C.$$

29) Similar to 28.

<u>Ex 6.2</u> (1) $\int \sin^2 x \cos^3 x\,dx$

$$= \int \sin^2 x \cos^2 x\, d(\sin x)$$
$$= \int u^2 (1-u^2)\, du \qquad (u = \sin x)$$
$$= \int (u^2 - u^4)\, du$$
$$= \frac{u^3}{3} - \frac{u^5}{5} + C$$
$$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.$$

2) $\int \sin^3\theta \cos^4\theta \, d\theta$

$= -\int \sin^2\theta \cos^4\theta \, d(\cos\theta)$

$= -\int (1-u^2)\, u^4 \, du \qquad (u = \cos\theta)$

$= \int (u^6 - u^4)\, du$

$= \frac{u^7}{7} - \frac{u^5}{5} + C$

$= \frac{\cos^7\theta}{7} - \frac{\cos^5\theta}{5} + C.$

3) $\int_0^{\pi/2} \sin^7\theta \cos^5\theta \, d\theta$

$= \int_0^{\frac{\pi}{2}} \sin^6\theta \cos^4\theta \cdot \sin\theta\cos\theta \, d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^6\theta \cos^4\theta \sin 2\theta \, d\theta \qquad (\sin 2\theta = 2\sin\theta\cos\theta)$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (\sin^2\theta)^3 (\cos^2\theta)^2 \sin 2\theta \, d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{(1-\cos 2\theta)^3}{2^3} \cdot \frac{(1+\cos 2\theta)^2}{2^2} \cdot \sin 2\theta \, d\theta \qquad \left(\begin{array}{l}\sin^2\theta = \frac{1}{2}(1-\cos 2\theta) \\ \cos^2\theta = \frac{1}{2}(1+\cos 2\theta)\end{array}\right)$

$= \frac{1}{2\cdot 64}\int_0^{\frac{\pi}{2}} -(-\cos^5 2\theta + \cos^4 2\theta + 2\cos^3 2\theta - 2\cos^2 2\theta - \cos 2\theta + 1)(\cos 2\theta)' d\theta$

$= \frac{1}{128}\int_1^0 (u^5 - u^4 - 2u^3 + 2u^2 + u - 1)\, du$

$= \frac{1}{128}\left[\frac{u^6}{6} - \frac{u^5}{5} - \frac{u^4}{2} + \frac{2u^3}{3} + \frac{u^2}{2} - u\right]_1^0$

$= \frac{11}{3840}.$

4) $\int_0^{\frac{\pi}{2}} \sin^5 x \, dx$

$= -\int_0^{\frac{\pi}{2}} \sin^4 x \, (\cos x)' \, dx$

$= -\int_0^{\frac{\pi}{2}} (1-\cos^2 x)^2 (\cos x)' \, dx$

$= -\int_1^0 (1-u^2)^2 \, du \qquad (u = \cos x)$

$= \int_0^1 (1 - 2u^2 + u^4) \, du$

$= \left[u - \frac{2u^3}{3} + \frac{u^5}{5} \right]_0^1$

$= \frac{8}{15}.$

15) $\int \frac{1-\sin x}{\cos x} \, dx = \int (\sec x - \tan x) \, dx$

$= \ln|\sec x + \tan x| - \ln|\sec x| + C$

$= \ln \left| \frac{\sec x + \tan x}{\sec x} \right| + C$

$= \ln|1 + \sin x| + C.$

16) $\int \cos^2 x \sin 2x \, dx = \frac{1}{2} \int (1 + \cos 2x) \sin 2x \, dx$

$= \frac{1}{2} \int (\sin 2x + \sin 2x \cos 2x) dx$

$= \frac{1}{2} \left(\frac{-\cos 2x}{2} \right) + \frac{1}{2} \int \sin 2x \, \frac{d(\sin 2x)}{2}$

$= -\frac{1}{4} \cos 2x + \frac{1}{8} \sin^2 2x + C$

17) $\int \tan x \sec^3 x \, dx$

$= \int \sec^2 x \, d(\sec x)$

$= \frac{\sec^3 x}{3} + C$.

18) $\int \tan^2\theta \sec^4\theta \, d\theta$

$= \int \tan^2\theta \sec^2\theta \, d(\tan\theta)$

$= \int u^2 (1+u^2) \, du$ $\qquad (u = \tan\theta , 1+\tan^2\theta = \sec^2\theta)$

$= \frac{u^3}{3} + \frac{u^5}{5} + C$

$= \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} + C$.

19) $\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx$

$= \tan x - x + C$.

20) $\int (\tan^2 x + \tan^4 x) \, dx$

$= \int \tan^2 x \, (1+\tan^2 x) \, dx$

$= \int \tan^2 x \sec^2 x \, dx$

$= \int u^2 \, du$ $\qquad (u = \tan x)$

$= \frac{u^3}{3} + C$

$= \frac{\tan^3 x}{3} + C$.

21) $\int \tan^4 x \sec^6 x \, dx$

$= \int \tan^4 x \sec^4 x \, d(\tan x)$

$= \int u^4 (1+u^2)^2 \, du \qquad (u = \tan x)$

$= \int (u^4 + 2u^6 + u^8) \, du$

$= \frac{u^5}{5} + \frac{2u^7}{7} + \frac{u^9}{9} + C$

$= \frac{\tan^5 x}{5} - \frac{2\tan^7 x}{7} + \frac{\tan^9 x}{9} + C.$

22) $\int_0^{\frac{\pi}{4}} \sec^4\theta \tan^4\theta \, d\theta$

$= \int_0^{\frac{\pi}{4}} \tan^4\theta \sec^2\theta \, (\tan\theta)' \, d\theta$

$= \int_0^1 u^4 (1+u^2) \, du \qquad (u = \tan\theta)$

$= \left[\frac{u^5}{5} + \frac{u^7}{7}\right]_0^1$

$= \frac{12}{35}.$

23) $\int_0^{\frac{\pi}{3}} \tan^5 x \sec^4 x \, dx$

$= \int_0^{\frac{\pi}{3}} \tan^5 x \, (1+\tan^2 x) (\tan x)' \, dx$

$= \int_0^{\sqrt{3}} u^5 (1+u^2) \, du \qquad (u = \tan x)$

$= \left[\frac{u^6}{6} + \frac{u^8}{8}\right]_0^{\sqrt{3}} = \frac{351}{24}.$

25) $\int \tan^3 x \sec x \, dx$

$= \int \tan^2 x \, d(\sec x)$

$= \int (\sec^2 x - 1) \, d(\sec x)$

$= \dfrac{\sec^3 x}{3} - \sec x + C$

28) $\int \tan^2 x \sec x \, dx$

$= \int \tan x \, (\sec x)' dx$

$= \tan x \sec x - \int \sec^3 x \, dx$ (b.p.)

$= \tan x \sec x - \int \sec x (1 + \tan^2 x) \, dx$

$= \tan x \sec x - \int \sec x \, dx - \int \tan^2 x \sec x \, dx$

$\therefore 2 \int \tan^2 x \sec x \, dx = \tan x \sec x - \int \sec x \, dx$

$= \tan x \sec x - \ln|\sec x + \tan x| + C$

$\Rightarrow \int \tan^2 x \sec x \, dx = \frac{1}{2} \tan x \sec x - \frac{1}{2} \ln|\sec x + \tan x| + C.$

27) $\int \tan^5 x \, dx = \int \tan^3 x (\sec^2 x - 1) \, dx$

$= \int \tan^3 x \, d(\tan x) - \int \tan^3 x \, dx$

$= \dfrac{\tan^4 x}{4} - \int \tan x \, (\sec^2 x - 1) \, dx$

$= \dfrac{\tan^4 x}{4} - \int \tan x \, d(\tan x) + \int \tan x \, dx$

$= \dfrac{\tan^4 x}{4} - \dfrac{\tan^2 x}{2} + \ln|\sec x| + C.$

26) $\int_0^{\frac{\pi}{4}} \tan^4 t \, dt$

$= \int_0^{\frac{\pi}{4}} \tan^2 t \, (\sec^2 t - 1) dt$

$= \int_0^{\frac{\pi}{4}} \tan^2 t \, (\tan t)' dt - \int_0^{\frac{\pi}{4}} \tan^2 t \, dt$

$= \int_0^1 u^2 du - \int_0^{\frac{\pi}{4}} (\sec^2 t - 1) \, dt$ $\quad (u = \tan t)$

$= \left[\frac{u^3}{3}\right]_0^1 - \left[\tan t - t\right]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{4} - \frac{2}{3}$.

29) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot^2 x \, dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\csc^2 x - 1) \, dx$

$= \left[-\cot x - x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$ $\quad ((\cot x)' = -\csc^2 x)$

$= \left(0 - \frac{\pi}{2}\right) - \left(-\sqrt{3} - \frac{\pi}{6}\right)$

$= -\frac{\pi}{3} + \sqrt{3}$.

30) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^3 x \, dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x \, (\csc^2 x - 1) dx$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x \, (-\cot x)' dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x \, dx$

$= -\int_1^0 u \, du - \left[\ln|\sin x|\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$$= \left[\frac{u^2}{2}\right]_0^1 + \ln\left(\frac{1}{\sqrt{2}}\right)$$

$$= 1 - \frac{1}{2}\ln 2.$$

31) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^5\phi \csc^3\phi \, d\phi$

$$= -\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^4\phi \csc^2\phi \, (\csc\phi)' d\phi \qquad ((\csc\phi)' = -\csc\phi\cot\phi)$$

$$= -\int_{\frac{\pi}{4}}^{\pi/2} (\csc^2\phi - 1)^2 \csc^2\phi \, (\csc\phi)' d\phi \quad \left(\begin{array}{c} 1+\cot^2\phi = \\ \csc^2\phi \end{array}\right)$$

$$= -\int_{\sqrt{2}}^{1} (u^2-1)^2 u^2 \, du \qquad (u = \csc\phi)$$

$$= \int_1^{\sqrt{2}} (u^6 - 2u^4 + u^2) du$$

$$= \left[\frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3}\right]_1^{\sqrt{2}}$$

$$= \frac{22\sqrt{2}}{105} - \frac{8}{105}.$$

32) $\displaystyle\int \csc^4 x \cot^6 x \, dx$ (Similar to 21).

33) $\displaystyle\int \csc x \, dx = \int \frac{\csc x(\csc x + \cot x)}{\csc x + \cot x} dx$

$$= -\int \frac{(\cot x)' + (\csc x)'}{\csc x + \cot x} dx$$

$$= -\ln|\csc x + \cot x| + C.$$

34) $\int \frac{1-\tan^2 x}{\sec^2 x}\,dx$

$= \int (\cos^2 x - \sin^2 x)\,dx$ $\quad \left(\tan x = \frac{\sin x}{\cos x}\right)$

$= \int \cos 2x \;dx$ $\quad (\cos 2x = \cos^2 x - \sin^2 x)$

$= \frac{1}{2}\sin 2x + C.$

35) $\int_0^{\frac{\pi}{6}} \sqrt{1+\cos 2x}\;dx$

$= \int_0^{\frac{\pi}{6}} \sqrt{1+(2\cos^2 x - 1)}\;dx$ $\quad (\cos 2x = 2\cos^2 x - 1)$

$= \sqrt{2}\int_0^{\frac{\pi}{6}} \cos x \;dx$ $\quad \left(\cos x \geq 0 \text{ on } \left[0, \frac{\pi}{6}\right]\right)$

$= \sqrt{2}\left[\sin x\right]_0^{\frac{\pi}{6}}$

$= 1.$

36) $\int \frac{dx}{\cos x - 1} = \int \frac{dx}{(1-2\sin^2\frac{x}{2})-1}$ $\quad \left(\cos x = 1 - 2\sin^2\frac{x}{2}\right)$

$= -\frac{1}{2}\int \csc^2\frac{x}{2}\;dx$

$= -\int \csc^2\frac{x}{2}\;d\left(\frac{x}{2}\right)$

$= \cot\frac{x}{2} + C.$ $\quad ((\cot\theta)' = -\csc^2\theta)$

42) Let $x = \sin\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\therefore \int_0^1 x^3\sqrt{1-x^2}\,dx$

$= \int_0^{\frac{\pi}{2}} \sin^3\theta \cos\theta \cdot \cos\theta \, d\theta$

$= -\int_0^{\frac{\pi}{2}} (1-\cos^2\theta)\cos^2\theta\,(\cos\theta)'\,d\theta$

$= \int_1^0 (u^4 - u^2)\,du \qquad (u = \cos\theta)$

$= \left[\frac{u^5}{5} - \frac{u^3}{3}\right]_1^0$

$= \frac{2}{15}.$

43) Let $t = \sec\theta$, $0 < \theta < \frac{\pi}{2}$ or $\pi < \theta < \frac{3\pi}{2}$.

$\therefore dt = \sec\theta\tan\theta\,d\theta$

$\int_{\sqrt{2}}^{2} \frac{1}{t^3\sqrt{t^2-1}}\,dt = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec\theta\tan\theta\,d\theta}{\sec^3\theta\sqrt{\sec^2\theta-1}}$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^2\theta\,d\theta \qquad (\sec^2\theta - 1 = \tan^2\theta)$

$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (1+\cos 2\theta)\,d\theta$

$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$

$= \frac{\pi}{24} + \frac{\sqrt{3}}{8} - \frac{1}{4}.$

44) Let $x = 2\tan\theta$ $\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\therefore dx = 2\sec^2\theta \, d\theta$

$$\int_0^2 x^3 \sqrt{x^2+4}\, dx$$

$$= \int_0^{\frac{\pi}{4}} 8\tan^3\theta \sqrt{4\sec^2\theta} \cdot 2\sec^2\theta \, d\theta$$

$$= 32\int_0^{\frac{\pi}{4}} \tan^3\theta \sec^3\theta \, d\theta$$

$$= 32\int_0^{\frac{\pi}{4}} \tan^2\theta \sec^2\theta (\sec\theta)' \, d\theta$$

$$= 32\int_0^{\frac{\pi}{4}} (\sec^2\theta - 1)(\sec^2\theta)(\sec\theta)' \, d\theta$$

$$= 32\int_1^{\sqrt{2}} (u^2-1)u^2 \, du \qquad (u = \sec\theta)$$

$$= 32\left[\frac{u^5}{5} - \frac{u^3}{3}\right]_1^{\sqrt{2}}$$

$$= \frac{64}{15}(\sqrt{2}+1).$$

45) Let $x = a\tan\theta$, $\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

$$\int_0^a \frac{dx}{(a^2+x^2)^{3/2}} = \int_0^{\pi/4} \frac{a\sec^2\theta \, d\theta}{(a^2\sec^2\theta)^{3/2}}$$

$$= \int_0^{\frac{\pi}{4}} \frac{a\sec^2\theta}{a^3\sec^3\theta} d\theta \qquad (\sec\theta \geq 0)$$

$$= \frac{1}{a^2}\int_0^{\frac{\pi}{4}} \cos\theta \, d\theta = \frac{1}{a^2}\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}a^2}.$$

46) Let $t = 4\sec\theta$, $\quad 0 < \theta < \frac{\pi}{2}$ or $\pi < \theta < \frac{3\pi}{2}$.

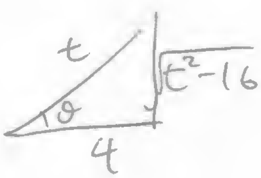
$\therefore dt = 4\sec\theta\tan\theta\, d\theta$

$$\int \frac{dt}{t^2\sqrt{t^2-16}} = \int \frac{4\sec\theta\tan\theta\, d\theta}{16\sec^2\theta \cdot \sqrt{16(\sec^2\theta - 1)}}$$

$$= \frac{1}{16}\int \frac{\sec\theta\tan\theta\, d\theta}{\sec^2\theta\ \tan\theta}$$

$$= \frac{1}{16}\int \cos\theta\, d\theta$$

$$= \frac{1}{16}\sin\theta + C$$

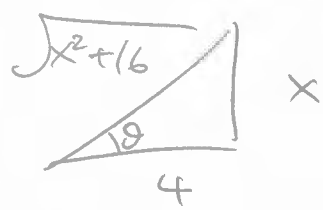


$$= \frac{1}{16} \cdot \frac{\sqrt{t^2-16}}{t} + C.$$

47) Let $x = 4\tan\theta$ $\quad \therefore dx = 4\sec^2\theta\, d\theta$

$\left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$

$$\int \frac{dx}{\sqrt{x^2+16}} = \int \frac{4\sec^2\theta\, d\theta}{4\sqrt{1+\tan^2\theta}}$$



$$= \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{x^2+16}}{4} + \frac{x}{4}\right| + C$$

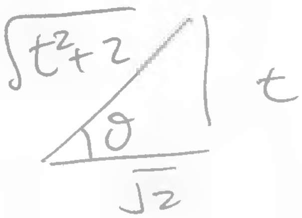
(Alternatively, use the formula $\int \frac{1}{\sqrt{x^2+1}}\,dx = \sinh^{-1}x + C$)

48) Let $t = \sqrt{2}\tan\theta$ $\quad \left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$

$\therefore dt = \sqrt{2}\sec^2\theta \, d\theta$

$$\int \frac{t^5}{\sqrt{t^2+2}}\,dt = \int \frac{4\sqrt{2}\tan^5\theta \cdot \sqrt{2}\sec^2\theta \, d\theta}{\sqrt{2(1+\tan^2\theta)}}$$

$$= 4\sqrt{2}\int \tan^5\theta \sec\theta \, d\theta$$



$$= 4\sqrt{2}\int \tan^4\theta \; d(\sec\theta)$$

$$= 4\sqrt{2}\int (\sec^2\theta - 1)\; d(\sec\theta)$$

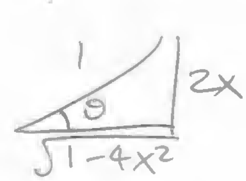
$$= 4\sqrt{2}\left(\frac{\sec^3\theta}{3} - \sec\theta\right) + C$$

$$= 4\sqrt{2}\left[\frac{1}{3}\sqrt{\frac{t^2+2}{2}}^{\,3} - \sqrt{\frac{t^2+2}{2}}\right] + C$$

$$= \frac{2t^2-8}{3}\sqrt{t^2+2} + C$$

49) Let $x = \frac{1}{2}\sin\theta$ $\quad \therefore dx = \frac{1}{2}\cos\theta \, d\theta$ $\quad \left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$

$$\int \sqrt{1-4x^2}\,dx = \int \sqrt{1-\sin^2\theta} \cdot \frac{1}{2}\cos\theta \, d\theta$$



$$= \frac{1}{2}\int \cos\theta \cos\theta \, d\theta$$

$$= \frac{1}{4}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{4}\theta + \frac{\sin 2\theta}{8} + C$$

$$= \frac{1}{4}\sin^{-1}(2x) + \frac{2\sin\theta\cos\theta}{8} + C \qquad \left(\begin{array}{l}\sin 2\theta \\ = 2\sin\theta\cos\theta\end{array}\right)$$

$$= \frac{1}{4}\sin^{-1}2x + \frac{1}{4}\cdot 2x\sqrt{1-4x^2} + C$$

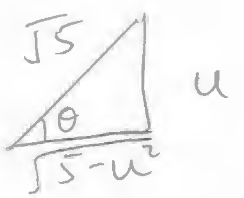
$$= \frac{\sin^{-1}2x}{4} + \frac{x\sqrt{1-4x^2}}{2} + C.$$

50) let $u = \sqrt{5}\sin\theta \qquad \left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$

$\therefore du = \sqrt{5}\cos\theta\, d\theta$

$$\int \frac{du}{u\sqrt{5-u^2}} = \int \frac{\sqrt{5}\cos\theta\, d\theta}{\sqrt{5}\sin\theta\sqrt{5(1-\sin^2\theta)}}$$

$$= \int \frac{1}{\sqrt{5}}\csc\theta\, d\theta$$



$$= -\frac{1}{\sqrt{5}}\ln|\csc\theta + \cot\theta| + C \qquad (\text{see } (33))$$

$$= -\frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{5}}{u} + \frac{\sqrt{5-u^2}}{u}\right| + C$$

$$= -\frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{5}+\sqrt{5-u^2}}{u}\right| + C.$$

51) let $x = 3\sec\theta \qquad \left(0 < \theta < \frac{\pi}{2} \text{ or } \pi < \theta < \frac{3\pi}{2}\right)$

$\therefore dx = 3\sec\theta\tan\theta\, d\theta$

$$\int \frac{\sqrt{x^2-9}}{x^3}\,dx = \int \frac{\sqrt{9(\sec^2\theta - 1)}\cdot 3\sec\theta\tan\theta\, d\theta}{27\sec^3\theta}$$

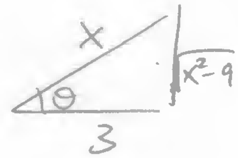
$$= \frac{1}{3}\int \frac{\tan^2\theta}{\sec^2\theta}\, d\theta$$

$$= \frac{1}{3}\int \sin^2\theta \, d\theta$$

$$= \frac{1}{6}\int (1-\cos 2\theta)\, d\theta \qquad \left(\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)\right)$$

$$= \frac{1}{6}\left[\theta - \frac{\sin 2\theta}{2}\right] + C$$

$$= \frac{1}{6}\cos^{-1}\left(\frac{3}{x}\right) - \frac{1}{12}\cdot 2\sin\theta\cos\theta + C \qquad \left(\sin 2\theta = 2\sin\theta\cos\theta\right)$$



$$= \frac{1}{6}\cos^{-1}\frac{3}{x} - \frac{1}{6}\,\frac{\sqrt{x^2-9}}{x}\cdot\frac{3}{x} + C$$

$$= \frac{1}{6}\cos^{-1}\frac{3}{x} - \frac{\sqrt{x^2-9}}{2x^2} + C$$

53) Let $x = \frac{3}{5}\sin\theta$ $\therefore dx = \frac{3}{5}\cos\theta\, d\theta$ $\left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$

$$\int_0^{0.6} \frac{x^2}{\sqrt{9-25x^2}}\, dx = \int_0^{\frac{\pi}{2}} \frac{\frac{9}{25}\sin^2\theta}{\sqrt{9-9\sin^2\theta}}\cdot\frac{3}{5}\cos\theta\, d\theta$$

$$= \frac{9}{125}\int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta$$

$$= \frac{9}{125}\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\, d\theta$$

$$= \frac{9}{250}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{9\pi}{500}$$

54) Let $x = \frac{b}{a}\sec\theta \quad \left(0<\theta<\frac{\pi}{2} \text{ or } \pi<\theta<\frac{3\pi}{2}\right)$

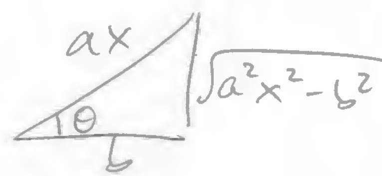
$\therefore dx = \frac{b}{a}\sec\theta\tan\theta\, d\theta$

$$\int \frac{dx}{\left((ax)^2-b^2\right)^{3/2}} = \int \frac{\frac{b}{a}\sec\theta\tan\theta\, d\theta}{\left[b^2(\sec^2\theta-1)\right]^{3/2}}$$

$$= \int \frac{\frac{b}{a}\sec\theta\tan\theta\, d\theta}{b^3\tan^3\theta}$$

$$= \frac{1}{ab^2}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta$$

$$= \frac{1}{ab^2}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta$$



$$= \frac{1}{ab^2}\int \frac{d(\sin\theta)}{\sin^2\theta}$$

$$= \frac{-1}{ab^2}\cdot\frac{1}{\sin\theta} + C$$

$$= \frac{-1}{ab^2}\cdot\frac{ax}{\sqrt{a^2x^2-b^2}} + C$$

$$= \frac{-x}{b^2\sqrt{a^2x^2-b^2}} + C.$$

55) Similar to 52.

56) Similar to 49.

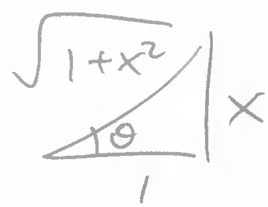
57) Let $x = \tan\theta \qquad (-\frac{\pi}{2} < \theta < \frac{\pi}{2})$

$$dx = \sec^2\theta \, d\theta$$

$$\therefore \int \frac{\sqrt{1+x^2}}{x} dx$$

$$= \int \frac{\sec\theta \cdot \sec^2\theta \, d\theta}{\tan\theta}$$

$$= \int \left(\frac{\sec\theta \, (1+\tan^2\theta)}{\tan\theta} \right) d\theta$$



$$= \int \left(\frac{\sec\theta}{\tan\theta} + \sec\theta\tan\theta \right) d\theta$$

$$= \int (\csc\theta + \sec\theta\tan\theta) \, d\theta$$

$$= -\ln|\csc\theta + \cot\theta| + \sec\theta + C \qquad \text{(see (33))}$$

$$= -\ln\left|\sqrt{1+x^2} + \frac{1}{x}\right| + \sqrt{1+x^2} + C$$

58) Let $x = \tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

$$\int_0^1 \frac{dx}{(x^2+1)^2} = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sec^4\theta} d\theta$$

$$= \int_0^{\frac{\pi}{4}} \cos^2\theta \, d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{4}} (1+\cos 2\theta) \, d\theta$$

$$= \frac{1}{2} \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{8} + \frac{1}{4}.$$

59) Let $u = x^2$ $\quad \therefore du = 2x\,dx$

$$\therefore \int x\sqrt{1-x^4}\,dx$$

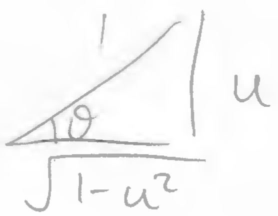
$$= \frac{1}{2}\int \sqrt{1-u^2}\,du$$

Let $u = \sin\theta$ $\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$\therefore \int x\sqrt{1-x^4}\,dx = \frac{1}{2}\int \cos\theta \cdot \cos\theta\,d\theta$$

$$= \frac{1}{2}\int \cos^2\theta\,d\theta$$

$$= \frac{1}{4}\int (1+\cos 2\theta)\,d\theta$$



$$= \frac{1}{4}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{1}{4}\sin^{-1}u + \frac{1}{8}\cdot 2\sin\theta\cos\theta + C$$

$$= \frac{1}{4}\sin^{-1}u + \frac{1}{4}\sin\theta\cos\theta + C$$

$$= \frac{1}{4}\sin^{-1}u + \frac{1}{4}\cdot u\cdot\sqrt{1-u^2} + C$$

$$= \frac{1}{4}\sin^{-1}(x^2) + \frac{1}{4}x^2\cdot\sqrt{1-x^4} + C$$

60) Let $u = \sin t$ $\therefore du = \cos t \, dt$

$$\int_0^{\frac{\pi}{2}} \frac{\cos t \, dt}{\sqrt{1+\sin^2 t}}$$

$$= \int_0^1 \frac{du}{\sqrt{1+u^2}}$$

$$= \left[\sinh^{-1} u\right]_0^1$$

$$= \sinh^{-1} 1 .$$