3

$$\begin{aligned}\lim_{x\to(\frac{\pi}{2})^+}\frac{\cos x}{1-\sin x} &= \lim_{x\to(\frac{\pi}{2})^+}\frac{-\sin x}{-\cos x} \qquad (\text{L'Hosp: } \lim_{x\to(\frac{\pi}{2})^+}\cos x = 0, \lim_{x\to(\frac{\pi}{2})^+}(1-\sin x)=0)\\ &= \lim_{x\to(\frac{\pi}{2})^+}\tan x\\ &=\infty.\end{aligned}$$

4

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 4x}{\tan 5x} &= \lim_{x\to 0}\frac{4\cos 4x}{5\sec^2 5x} \qquad (\text{L'Hosp: } \lim_{x\to 0}\sin 4x = 0, \quad \lim_{x\to 0}\tan 5x = 0)\\ &=\frac{4}{5}.\end{aligned}$$

5

$$\begin{aligned}\lim_{t\to 0}\frac{e^{2t}-1}{\sin t} &= \lim_{t\to 0}\frac{2e^{2t}}{\cos t} \qquad (\text{L'Hosp: } \lim_{t\to 0}(e^{2t}-1) = 0, \quad \lim_{t\to 0}\sin t = 0)\\ &=2.\end{aligned}$$

6

$$\begin{aligned}\lim_{x\to 0}\frac{x^2}{1-\cos x} &= \lim_{x\to 0}\frac{2x}{\sin x} \qquad (\text{L'Hosp: } \lim_{x\to 0}x^2 = 0, \quad \lim_{x\to 0}(1-\cos x) = 0)\\ &=2. \qquad (\lim_{x\to 0}\frac{x}{\sin x} = 1)\end{aligned}$$

24

$$\begin{aligned}\lim_{x\to\infty}\sqrt{x}e^{-\frac{x}{2}} &= \lim_{x\to\infty}\frac{\sqrt{x}}{e^{\frac{x}{2}}}\\ &= \lim_{x\to\infty}\frac{\frac{1}{2\sqrt{x}}}{\frac{1}{2}e^{\frac{x}{2}}} \qquad (\text{L'Hosp: } \lim_{x\to\infty}\sqrt{x} = \infty, \quad \lim_{x\to\infty}e^{\frac{x}{2}} = \infty)\\ &= \lim_{x\to\infty}\frac{1}{\sqrt{x}e^{\frac{x}{2}}}\\ &=0.\end{aligned}$$

HW7

Ex.5.8

1) $\lim_{x \to 1} x^2-1 = 0$, $\lim_{x \to 1} x^2 - x = 0$

$$\therefore \lim_{x \to 1} \frac{x^2-1}{x^2-x} = \lim_{x \to 1} \frac{2x}{2x-1} \quad \text{(L'Hosp.)}$$

$$= 2 . \quad \left(\text{Alternative : } \frac{x^2-1}{x^2-x} = \frac{(x-1)(x+1)}{(x-1)x}\right)$$

2) $\lim_{x \to 2} x^2+x-6 = 2^2+2-6 = 0$

$\lim_{x \to 2} x-2 = 0$

$$\therefore \lim_{x \to 2} \frac{x^2+x-6}{x-2} = \lim_{x \to 2} \frac{2x+1}{1} \quad \text{(L'Hospital)}$$

$$= 5 .$$

$$\left(\text{Or : } \frac{x^2+x-6}{x-2} = \frac{(x-2)(x+3)}{x-2}\right)$$

※ From NOW on I'll omit checking if it's ok to use the L'Hospital Rule.

7) $$\lim_{\theta \to \frac{\pi}{2}} \frac{1-\sin\theta}{1+\cos 2\theta} = \lim_{\theta \to \frac{\pi}{2}} \frac{-\cos\theta}{-2\sin 2\theta} \quad \text{(L'H)}$$

$$= \lim_{\theta \to \frac{\pi}{2}} \frac{\cos\theta}{2\sin 2\theta}$$

$$= \lim_{\theta \to \frac{\pi}{2}} \frac{-\sin\theta}{4\cos 2\theta} \quad \text{(L'H)}$$

$$= \frac{-\sin\frac{\pi}{2}}{4\cos(2\cdot\frac{\pi}{2})} = \frac{1}{4} .$$

8) $\lim_{\theta \to \frac{\pi}{2}} \frac{1-\sin\theta}{\csc\theta}$

$= \frac{1-\sin\frac{\pi}{2}}{\csc\frac{\pi}{2}}$

$= \frac{1-1}{1}$

$= 0$.

9) As $\lim_{x \to 0^+} \ln x = -\infty$ and $\lim_{x \to 0^+} x = 0$

$\therefore \lim_{x \to 0^+} \frac{\ln x}{x} = -\infty$. (L'Hospital rule can't apply here, as it's of the form "$\frac{-\infty}{0}$")

10) $\lim_{x \to \infty} \frac{\ln\sqrt{x}}{x^2} = \lim_{x \to \infty} \frac{\frac{1}{2}\ln x}{x^2}$

$= \frac{1}{2} \lim_{x \to \infty} \frac{1/x}{2x}$ (L'H)

$= \frac{1}{4} \lim_{x \to \infty} \frac{1}{x^2}$

$= 0$.

11) $\lim_{t \to 1} \frac{t^8-1}{t^5-1} = \lim_{t \to 1} \frac{8t^7}{5t^4}$ (L'H)

$= \lim_{t \to 1} \frac{8}{5} t^3$

$= \frac{8}{5}$.

12) $\lim_{t\to 0} \dfrac{8^t - 5^t}{t}$

$= \lim_{t\to 0} \dfrac{8^t \cdot \ln 8 - 5^t \cdot \ln 5}{1}$ (L'H)

$= \ln 8 - \ln 5$.

13) $\lim_{x\to 0} \dfrac{e^x - 1 - x}{x^2}$

$= \lim_{x\to 0} \dfrac{e^x - 1}{2x}$ (L'H)

$= \lim_{x\to 0} \dfrac{e^x}{2}$ (L'H)

$= \frac{1}{2}$.

14) $\lim_{u\to\infty} \dfrac{e^{u/10}}{u^3} = \lim_{u\to\infty} \dfrac{\frac{1}{10} e^{\frac{u}{10}}}{3u^2}$ (L'H)

$= \frac{1}{30} \lim_{u\to\infty} \dfrac{e^{\frac{u}{10}}}{u^2}$

$= \frac{1}{30} \lim_{u\to\infty} \dfrac{\frac{1}{10} e^{u/10}}{2u}$ (L'H)

$= \frac{1}{30\cdot 20} \lim_{u\to\infty} \dfrac{e^{u/10}}{u}$

$= \frac{1}{30\cdot 20} \lim_{u\to\infty} \dfrac{\frac{1}{10} e^{u}}{1}$ (L'H)

$= \infty$.

15) $\lim_{x \to 0} \dfrac{x 3^x}{3^x - 1}$

$= \lim_{x \to 0} \dfrac{3^x + x \cdot \ln 3 \cdot 3^x}{\ln 3 \cdot 3^x}$ (L'H)

$= \dfrac{3^0 + 0}{(\ln 3) 3^0}$

$= \dfrac{1}{\ln 3}$.

16) $\lim_{x \to 0} \dfrac{\cos mx - \cos nx}{x^2}$

$= \lim_{x \to 0} \dfrac{-m \sin mx + n \sin nx}{2x}$ (L'H)

$= \lim_{x \to 0} \dfrac{-m^2 \cos(mx) + n^2 \cos(nx)}{2}$ (L'H)

$= \dfrac{n^2 - m^2}{2}$.

17) $\lim_{x \to 1} \dfrac{1 - x + \ln x}{1 + \cos \pi x}$

$= \lim_{x \to 1} \dfrac{-1 + 1/x}{-\pi \sin \pi x}$ (L'H)

$= \lim_{x \to 1} \dfrac{-x + 1}{-\pi x \cdot \sin \pi x}$

$= \lim_{x \to 1} \dfrac{-1}{-\pi \sin(\pi x) - \pi^2 x \cos(\pi x)}$ (L'H)

$= \dfrac{-1}{-\pi^2 \cdot (-1)} = -\dfrac{1}{\pi^2}$.

18) $\lim_{x\to 0} \frac{x}{\tan^{-1} 4x} = \lim_{x\to 0} \frac{1}{4/1+(4x)^2}$ (L'H)

$= \lim_{x\to 0} \frac{1+(4x)^2}{4}$

$= \frac{1}{4}$.

19) $\lim_{x\to 1} \frac{x^a - ax + a - 1}{(x-1)^2}$

$= \lim_{x\to 1} \frac{a x^{a-1} - a}{2(x-1)}$ (L'H)

$= \lim_{x\to 1} \frac{a(a-1)x^{a-2}}{2}$ (L'H)

$= \frac{a(a-1)}{2}$.

20) $\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$

$= \lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1 - \cos x}$ (L'H)

$= \lim_{x\to 0} \frac{e^x - e^{-x}}{\sin x}$ (L'H)

$= \lim_{x\to 0} \frac{e^x + e^{-x}}{\cos x}$ (L'H)

$= 2$.

21) Omitted.

22) We first claim that

$$\lim_{x\to a^+} \frac{\ln(x-a)}{\ln(e^x-e^a)} = 1$$

Consider $\lim_{x\to a^+} \frac{\ln(x-a)}{\ln(e^x-e^a)}$

$$= \lim_{x\to a^+} \frac{1/x-a}{e^x/e^x-e^a} \quad (L'H)$$

$$= \lim_{x\to a^+} \frac{e^x-e^a}{e^x(x-a)}$$

$$= \frac{1}{e^a} \cdot \lim_{x\to a^+} \frac{e^x-e^a}{x-a}$$

$$= \frac{1}{e^a} \lim_{x\to a^+} \frac{e^x}{1} \quad (L'H)$$

$$= 1.$$

$$\therefore \lim_{x\to a^+} \frac{\cos x \cdot \ln(x-a)}{\ln(e^x-e^a)} = \lim_{x\to a^+} \cos x \cdot \lim_{x\to a^+} \frac{\ln(x-a)}{\ln(e^x-e^a)}$$

$$= \cos a \cdot 1$$

$$= \cos a$$

23) $\lim_{x\to 0} \dfrac{\sin 6x}{\tan 2x}$

$= \lim_{x\to 0} \left(\dfrac{\sin 6x}{\sin 2x} \cdot \cos 2x \right)$

$= \left(\lim_{x\to 0} \dfrac{\sin 6x}{\sin 2x} \right) \cos(2 \cdot 0)$

$= \lim_{x\to 0} \dfrac{\sin 6x}{\sin 2x}$

$= \lim_{x\to 0} \dfrac{6\cos 6x}{2\cos 2x}$ (L'H)

$= 3$

25) $\lim_{x\to\infty} x^3 e^{-x^2} = \lim_{x\to\infty} \dfrac{x^3}{e^{x^2}}$

$= \lim_{x\to\infty} \dfrac{3x^2}{2xe^{x^2}}$ (L'H)

$= \lim_{x\to\infty} \dfrac{6x}{2e^{x^2}+4x^2e^{x^2}}$ (L'H)

$= \lim_{x\to\infty} \dfrac{6}{4xe^{x^2}+8xe^{x^2}+8x^3e^{x^2}}$ (L'H)

$= 0.$

26) $\lim_{x \to 0^+} \sin x \ln x$

$= \lim_{x \to 0^+} \frac{\ln x}{\csc x}$

$= \lim_{x \to 0^+} \frac{1/x}{-\csc x \cot x}$ (L'Hosp.)

$= \lim_{x \to 0^+} -\frac{\sin x}{x} \cdot \tan x$

$= -\lim_{x \to 0^+} \frac{\sin x}{x} \cdot \lim_{x \to 0^+} \tan x$

$= 0.$

27) $\lim_{x \to 1^+} \ln x \tan\left(\frac{\pi x}{2}\right)$

$= \lim_{x \to 1^+} \frac{\ln x}{\cot\left(\frac{\pi x}{2}\right)}$

$= \lim_{x \to 1^+} \frac{1/x}{-\csc^2\left(\frac{\pi x}{2}\right) \cdot \frac{\pi}{2}}$ (L'H)

$= \lim_{x \to 1^+} -\frac{2}{\pi} \cdot \frac{\sin^2\left(\frac{\pi x}{2}\right)}{x}$

$= -\frac{2}{\pi}.$

28) $\lim_{x\to\infty} x \tan\frac{1}{x}$

$= \lim_{y\to 0^+} \frac{1}{y}\tan y$ $\quad\left(y=\frac{1}{x}\to 0^+ \text{ as } x\to\infty\right)$

$= \lim_{y\to 0^+} \frac{\tan y}{y}$

$= \lim_{y\to 0^+} \frac{\sec^2 y}{1}$ (L'H)

$= 1$.

29) $\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{e^x-1}\right)$

$= \lim_{x\to 0^+} \frac{e^x-1-x}{x(e^x-1)}$

$= \lim_{x\to 0^+} \frac{e^x-1}{e^x-1+xe^x}$ (L'H)

$= \lim_{x\to 0^+} \frac{e^x}{e^x+e^x+xe^x}$ (L'H)

$= \frac{1}{2}$.

30) $\lim_{x\to 0} (\csc x - \cot x)$

$= \lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right)$

$= \lim_{x\to 0} \frac{1-\cos x}{\sin x}$

$= \lim_{x\to 0} \frac{\sin x}{\cos x}$ (L'H)

$= 0$.

31) Let $y = e^{x - \ln x}$ so that

$$x - \ln x = \ln y.$$

Consider $\lim_{x\to\infty} y$

$$= \lim_{x\to\infty} e^{x-\ln x}$$

$$= \lim_{x\to\infty} \frac{e^x}{x}$$

$$= \lim_{x\to\infty} \frac{e^x}{1} \qquad (L'H)$$

$$= \infty.$$

$$\therefore \lim_{x\to\infty} (x - \ln x) = \lim_{x\to\infty} \ln y$$

$$= \infty \qquad \text{as } \lim_{x\to\infty} y = \infty.$$

32) $\ln(x^7-1) - \ln(x^5-1)$

$$= \ln\left(\frac{x^7-1}{x^5-1}\right)$$

Let $y = \dfrac{x^7-1}{x^5-1}$, $\therefore \ln(x^7-1) - \ln(x^5-1) = \ln y$.

Consider $\lim_{x\to 1^+} \dfrac{x^7-1}{x^5-1} = \lim_{x\to 1^+} \dfrac{7x^6}{5x^4}$ (L'H)

$$= \lim_{x\to 1^+} \frac{7}{5}x^2$$

$$= \frac{7}{5}$$

$$\therefore \lim_{x\to 1^+}\left[\ln(x^7-1) - \ln(x^5-1)\right] = \lim_{x\to 1^+} \ln y$$

$$= \ln\frac{7}{5} = \ln 7 - \ln 5$$

33). Let $y = \ln(x^{\sqrt{x}})$

$= \sqrt{x} \ln x$ and $x^{\sqrt{x}} = e^y$

$$\lim_{x\to 0^+} y = \lim_{x\to 0^+} \sqrt{x} \ln x$$

$$= \lim_{x\to 0^+} \frac{\ln x}{x^{-\frac{1}{2}}}$$

$$= \lim_{x\to 0^+} \frac{1/x}{-\frac{1}{2} x^{-3/2}} \quad (L'H)$$

$$= -2 \lim_{x\to 0^+} x^{1/2} \quad (\text{sure?})$$

$$= 0.$$

$$\therefore \lim_{x\to 0^+} x^{\sqrt{x}} = \lim_{x\to 0^+} e^y$$

$$= e^{\lim_{x\to 0^+} y}$$

$$= e^0$$

$$= 1.$$

34) Let $y = \ln[(\tan 2x)^x] = x \ln(\tan 2x)$

$$\therefore \lim_{x\to 0^+} y = \lim_{x\to 0^+} x \ln(\tan 2x)$$

$$= \lim_{x\to 0^+} \frac{\ln(\tan 2x)}{1/x}$$

$$= \lim_{x\to 0^+} \frac{2\sec^2 2x / \tan 2x}{-1/x^2}$$

$$= \lim_{x\to 0^+} -2 \cdot \sec^2 2x \cdot \frac{x^2}{\tan 2x}$$

$$= -2 \lim_{x\to 0^+} \sec^2 2x \cdot \lim_{x\to 0^+} \frac{x^2}{\tan 2x}$$

$$= -2 \cdot 1 \cdot \lim_{x\to 0^+} \frac{x^2}{\tan 2x}$$

$$= -2 \cdot \lim_{x\to 0} \frac{2x}{2\sec^2 2x} \qquad (L'H)$$

$$= -2 \cdot 0$$

$$= 0.$$

$$\therefore \lim_{x\to 0^+} (\tan 2x)^x = \lim_{x\to 0^+} e^y$$

$$= e^{\lim_{x\to 0^+} y}$$

$$= e^0$$

$$= 1.$$

35) Omitted.

36) Let $y = \ln\left[\left(1+\frac{a}{x}\right)^{bx}\right] = bx \ln\left(1+\frac{a}{x}\right)$.

$\therefore \left(1+\frac{a}{x}\right)^{bx} = e^y$.

Consider $\lim_{x\to\infty} y = \lim_{x\to\infty} bx \ln\left(1+\frac{a}{x}\right)$

$$= b \lim_{x\to\infty} \frac{\ln\left(1+\frac{a}{x}\right)}{1/x}$$

$$= b \lim_{x\to\infty} \frac{\frac{1}{1+a/x} \cdot (-ax^{-2})}{-x^{-2}}$$

$$= ab \lim_{x\to\infty} \frac{1}{1+a/x} = ab.$$

$\therefore \lim_{x\to\infty} \left(1+\frac{a}{x}\right)^{bx}$

$= \lim_{x\to\infty} e^{y}$

$= e^{ab}$.

(Alternative: $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} = \lim_{x\to\infty}\left[\left(1+\frac{1}{x/a}\right)^{\frac{x}{a}}\right]^{ab} = e^{ab}$)

37) let $y = \ln\left(x^{\frac{1}{1-x}}\right) = \frac{1}{1-x}\ln x$

$\therefore x^{\frac{1}{1-x}} = e^{y}$ and

$\lim_{x\to 1^+} y = \lim_{x\to 1^+} \frac{\ln x}{1-x}$

$= \lim_{x\to 1^+} \frac{1/x}{-1}$ (L'H)

$= -1$

$\therefore \lim_{x\to 1^+} x^{\frac{1}{1-x}} = e^{-1}$.

38) Let $y = \ln\left[(e^x+x)^{\frac{1}{x}}\right]$ $\therefore (e^x+x)^{\frac{1}{x}} = e^{y}$

Consider $\lim_{x\to\infty} y = \lim_{x\to\infty} \ln\left[(e^x+x)^{\frac{1}{x}}\right]$

$= \lim_{x\to\infty} \frac{\ln(e^x+x)}{x}$

$= \lim_{x\to\infty} \frac{e^x+1}{e^x+x}$ (L'H)

$$= \lim_{x\to\infty} \frac{e^x}{e^x+1} \qquad (L'H)$$

$$= \lim_{x\to\infty} \frac{1}{1+e^{-x}} \qquad \left(\text{multiply } \frac{e^{-x}}{e^{-x}}\right)$$

$$= \frac{1}{1+0}$$

$$= 1.$$

$$\therefore \lim_{x\to\infty} (e^x + x)^{\frac{1}{x}} = \lim_{x\to\infty} e^y$$

$$= e^{\lim_{x\to\infty} y}$$

$$= e^1.$$

$$= e$$