28

$$g'(x) = \sinh(\ln x) \cdot \frac{1}{x} = \frac{\sinh(\ln x)}{x}.$$

36

$$\begin{aligned} y' = \frac{1}{\sqrt{1+\tan^2 x}} \cdot \sec^2 x = \frac{1}{\sqrt{\sec^2 x}} \cdot \sec^2 x = & \frac{1}{|\sec x|} \cdot |\sec x|^2 \\ = & |\sec x|. \end{aligned}$$

57

$$\begin{aligned} \int \frac{\cosh x}{\cosh^2 x - 1} dx = & \int \frac{\cosh x}{\sinh^2 x} dx \qquad (\cosh^2 x - 1 = \sinh^2 x) \\ = & \int \frac{du}{u^2} \qquad (u = \sinh x, \quad du = \cosh x dx) \\ = & -\frac{1}{u} + C \\ = & -\frac{1}{\sinh x} + C. \end{aligned}$$

59

$$\begin{aligned} \int_4^6 \frac{1}{\sqrt{t^2-9}} dt = & \int_4^6 \frac{1}{3\sqrt{\frac{t^2}{9}-1}} dt \\ = & \frac{1}{3} \int_4^6 \frac{1}{\sqrt{(\frac{t}{3})^2 - 1}} dt \\ = & \int_{\frac{4}{3}}^{\frac{6}{3}} \frac{du}{\sqrt{u^2-1}} \qquad (u = \frac{t}{3}), du = \frac{dt}{3} \\ = & \int_{\frac{4}{3}}^{2} \frac{du}{\sqrt{u^2-1}} \\ = & \cosh^{-1}(2) - \cosh^{-1}\left(\frac{4}{3}\right). \end{aligned}$$

61 Let $u = e^x$, then $du = e^x dx$. So

$$\begin{aligned} \int \frac{e^x}{1-e^{2x}} dx = & \int \frac{du}{1-u^2} \\ = & -\int \frac{du}{u^2-1} \\ = & -\tanh^{-1} u + C \\ = & -\tanh^{-1}(e^x) + C. \end{aligned}$$

(To be rigorous, this solution is only valid for $x < 0$, where $e^x < 1$ and $\tanh^{-1}(e^x)$ makes sense. For $x > 0$, the solution is $-\coth^{-1}(e^x) + C$.)

HW 6

Ex. 5.7

26) $f'(x) = \operatorname{sech}^2(1+e^{2x}) \cdot (1+e^{2x})'$ (chain rule)

$= \operatorname{sech}^2(1+e^{2x}) \cdot 2e^{2x}$

$= 2e^{2x} \operatorname{sech}^2(1+e^{2x}).$

27) $f'(x) = \sinh x + x\cosh x - \sinh x$

29) $h'(x) = \dfrac{1}{\cosh x} \cdot (\cosh x)'$

$= \dfrac{\sinh x}{\cosh x}$

$= \tanh x.$

30) $y' = \coth(1+x^2) - x\operatorname{csch}^2(1+x^2) \cdot [1+x^2]'$ $(\coth' = -\operatorname{csch}^2)$

$= \coth(1+x^2) - x\cosh^2(1+x^2) \cdot [2x]$

$= \coth(1+x^2) - 2x^2\cosh^2(1+x^2).$

31) $y' = e^{\cosh 3x} \cdot 3\sinh 3x$

$= 3e^{\cosh 3x}\sinh 3x.$

32) $f'(t) = -\operatorname{csch} t \coth t \cdot (1 - \ln(\operatorname{csch} t)) - \dfrac{\operatorname{csch} t}{\operatorname{csch} t}(\operatorname{csch} t)'$

$= -\operatorname{csch} t \coth t \cdot (1 - \ln(\operatorname{csch} t)) + \operatorname{csch} t \coth t$ $(\csc' = -\csc\cot)$

$= \operatorname{csch} t \coth t \ln(\operatorname{csch} t).$

33) $f'(t) = 2\operatorname{sech}(e^t) \cdot (-\operatorname{sech}(e^t)\tanh(e^t)) \cdot e^t$

$= -2e^t \operatorname{sech}^2(e^t)\tanh(e^t)$

34) $y' = \cosh(\cosh x) \cdot \sinh x$

$= \sinh x \cosh(\cosh x)$.

35) $G(x) = \dfrac{1 - \cosh x}{1 + \cosh x} = \dfrac{-(1+\cosh x) + 2}{1 + \cosh x}$

$= -1 + \dfrac{2}{1 + \cosh x}$

$\therefore G'(x) = -\dfrac{2}{(1+\cosh x)^2} \cdot (1 + \cosh x)'$

$= -\dfrac{2 \sinh x}{(1+\cosh x)^2}$.

37) $y' = \dfrac{1}{\sqrt{\sqrt{x}^2 - 1}} \cdot (\sqrt{x})'$

$= \dfrac{1}{\sqrt{x^2 - 1}} \cdot \dfrac{1}{2\sqrt{x}}$

$= \dfrac{1}{2\sqrt{x^3 - x}}$.

38) $y' = \tanh^{-1} x + \dfrac{x}{1 - x^2} + \left[\frac{1}{2} \ln(1 - x^2)\right]'$

$= \tanh^{-1} x + \dfrac{x}{1-x^2} + \dfrac{1}{2} \cdot \dfrac{-2x}{1-x^2}$

$= \tanh^{-1} x + \dfrac{x}{1-x^2} - \dfrac{x}{1-x^2}$

$= \tanh^{-1} x$.

39) $$y' = \sinh^{-1}\frac{x}{3} + \frac{x}{\sqrt{1+\left(\frac{x}{3}\right)^2}}\cdot\frac{1}{3} - \frac{2x}{2\sqrt{9+x^2}}$$

$$= \sinh^{-1}x + \frac{x}{\sqrt{3^2\left[1+\left(\frac{x}{3}\right)^2\right]}} - \frac{x}{\sqrt{9+x^2}}$$

$$= \sinh^{-1}x + \frac{x}{\sqrt{9+x^2}} - \frac{x}{\sqrt{9+x^2}}$$

$$= \sinh^{-1}x.$$

40) $$y' = -\frac{1}{e^{-x}\sqrt{1-(e^{-x})^2}}\cdot\left(e^{-x}\right)' \qquad \left((\operatorname{sech}^{-1})'(x) = \frac{-1}{x\sqrt{1-x^2}}\right)$$

$$= -\frac{1}{e^{-x}\sqrt{1-e^{-2x}}}\cdot\left(-e^{-x}\right)$$

$$= \frac{1}{\sqrt{1-e^{-2x}}}$$

41) $$y' = \frac{1}{1-\sec^2 x}\cdot(\sec x)' \qquad \left((\coth^{-1})'(x) = \frac{1}{1-x^2}\right)$$

$$= \frac{1}{1-\sec^2 x}\cdot\sec x\tan x \qquad \left((\sec x)' = \sec x\tan x\right)$$

$$= \frac{\sec x\tan x}{1-\sec^2 x}$$

$$= \frac{\sec x\tan x}{-\tan^2 x} \qquad \left(1+\tan^2 x = \sec^2 x\right)$$

$$= -\frac{\sec x}{\tan x} = -\frac{1/\cos x}{\sin x/\cos x} = -\csc x.$$

53) Let $u = \cosh x$ $\quad \therefore du = \sinh x \, dx$

$$\therefore \int \sinh x \cosh^2 x \, dx = \int u^2 \, du$$

$$= \frac{u^3}{3} + C$$

$$= \frac{\cosh^3 x}{3} + C. \qquad (\text{check: Ex.})$$

54) Let $u = 1 + 4x$ $\quad \therefore du = 4dx$

$$\therefore \int \sinh(1+4x)dx = \frac{1}{4}\int \sinh u \, du$$

$$= \frac{1}{4}\cosh u + C$$

$$= \frac{1}{4}\cosh(1+4x) + C.$$

Ex: check (Same for the rest)

55) Let $u = \sqrt{x}$ $\quad \therefore du = \frac{1}{2\sqrt{x}} dx$

$$\int \frac{\sinh \sqrt{x}}{\sqrt{x}} dx = 2\int \sinh u \, du$$

$$= 2\cosh u + C$$

$$= 2\cosh \sqrt{x} + C.$$

56) $$\int \tanh x \, dx = \int \frac{\sinh x}{\cosh x} dx$$

$$= \int \frac{du}{u} \qquad (u = \cosh x)$$

$$= \ln|u| + C = \ln|\cosh x| + C = \ln(\cosh x) + C.$$

57) $\displaystyle\int \frac{\cosh x}{\cosh^2 x - 1}\,dx$

$= \displaystyle\int \frac{\cosh x}{\sinh^2 x}\,dx \qquad (\cosh^2 x - \sinh^2 x = 1)$

Let $u = \sinh x \quad \therefore du = \cosh x\,dx$

$\therefore \displaystyle\int \frac{\cosh x}{\cosh^2 x - 1}\,dx = \int \frac{du}{u^2}$

$= -\frac{1}{u} + C$

$= -\frac{1}{\sinh x} + C$

$= -\operatorname{csch} x + C$

58) Let $u = 2 + \tanh x \quad \therefore du = \operatorname{sech}^2 x\,dx$

$\therefore \displaystyle\int \frac{\operatorname{sech}^2 x}{2 + \tanh x}\,dx = \int \frac{du}{u}$

$= \ln|u| + C$

$= \ln|2 + \tanh x| + C$

$= \ln(2 + \tanh x) + C$

$(-1 \le \tanh x \le 1 \Rightarrow 1 \le \tanh x + 2 \le 3)$

60) Let $u = 4t \quad \therefore du = 4\,dt$

$$\therefore \int_0^1 \frac{1}{\sqrt{16t^2+1}}\,dt = \int_0^1 \frac{1}{\sqrt{(4t)^2+1}}\,dt$$

$$= \frac{1}{4}\int_0^4 \frac{du}{\sqrt{u^2+1}}$$

$$= \frac{1}{4}\left[\sinh^{-1} 4 - \sinh^{-1} 0\right]$$

$$= \frac{1}{4}\sinh^{-1} 4.$$

(There's an explicit formula for $\sinh^{-1}$ in terms of ln and $\sqrt{\ }$, but $\frac{1}{4}\sinh^{-1} 4$ is good enough.)

Challenging

42) You can differentiate directly.

The best way is to simplify first:

$$1 + \tanh x = 1 + \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{2e^x}{e^x + e^{-x}}$$

$$1 - \tanh x = 1 - \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{2e^{-x}}{e^x + e^{-x}}$$

$$\therefore \frac{1+\tanh x}{1-\tanh x} = \frac{2e^x}{2e^{-x}} = e^{2x}$$

$$\therefore \sqrt[4]{\frac{1-\tanh x}{1+\tanh x}} = e^{-\frac{x}{2}}.$$ The result is immediate.

43) $\frac{d}{dx} \tan^{-1}(\tanh x)$

$$= \frac{1}{1+\tanh^2 x} \cdot \operatorname{sech}^2 x$$

$$= \frac{1}{1+\frac{\sinh^2 x}{\cosh^2 x}} \cdot \frac{1}{\cosh^2 x}$$

$$= \frac{1}{\cosh^2 x + \sinh^2 x}$$

It suffices to show $\cosh^2 x + \sinh^2 x = \cosh 2x$:

$$\cosh^2 x + \sinh^2 x$$

$$= \frac{(e^x + e^{-x})^2}{4} + \frac{(e^x - e^{-x})^2}{4}$$

$$= \frac{1}{4}\left[(e^{2x} + 2 + e^{-2x}) + (e^{2x} - 2 + e^{-2x})\right]$$

$$= \frac{1}{4} \cdot 2(e^{2x} + e^{-2x})$$

$$= \frac{e^{2x} + e^{-2x}}{2}$$

$= \cosh 2x$. The result follows.